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Some estimates of the minimizing properties of web functions

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Abstract. We study a general class of functionals over $W_0^{1,1}$ and we approximate their infimum by means of the minimum in the space of web functions. We provide several tools for estimating this approximation and we study in detail some meaningful models.

1 Introduction

Let Ω be an open bounded convex domain of $\mathbb{R}^n$ ($n \geq 2$), let $f\colon \mathbb{R}^+ \to \overline{\mathbb{R}}$ be a lower semicontinuous (l.s.c.) function and consider the functional J defined by

$$J(u) = \int_\Omega [f(|\nabla u|) - u]\, dx\ .$$

This kind of functionals (with possibly nonconvex functions f) arises from various fields of mathematical physics and optimal design, see [3,5,23,24]. We study the following problem of existence of minima,

$$\min_{u \in W_0^{1,1}(\Omega)} J(u)\ .$$

It is well-known that if f is not convex and superlinear then the minimum may not exist. In such case, one usually introduces the relaxed functional and considers its minimum, which coincides with the minimum of J if the latter exists. In particular, we mention a problem from elasticity [5,22,23]: we wish to place two different linearly elastic materials (of different shear moduli) in the plane domain Ω so as to maximize the torsional rigidity of the resulting rod; moreover, the proportions of these materials are prescribed. Such a problem may not have a solution, therefore one may construct new composite materials by mixing them together on a microscopic scale. Mathematically, this corresponds to the introduction of the relaxed problem which does have a minimum. Hence there exists an optimal design if one is allowed to incorporate composites. However, the resulting design may not be

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so easy to manufacture and therefore one may try to find an optimal design in a simpler class of possible designs.

Motivated by this application, we will be concerned with the following minimization problem

$$\min_{u\in\mathcal{K}} J(u)\ , \tag{1}$$

where $\mathcal{K}$ is the subset of $W_0^{1,1}(\Omega)$ of functions depending only on the distance from the boundary $\partial\Omega$. As in [19], we call *web functions* the functions in $\mathcal{K}$ and we recall that when Ω is a ball, web functions are none other than radially symmetric functions. Web functions were introduced in order to approximate the infimum of the functional J over $W_0^{1,1}(\Omega)$. As proved in [14, 19], under very mild assumptions on f, the minimum of J over $\mathcal{K}$ always exists even for nonconvex functions f. More precisely, in [19] it is proved that if f is superlinear at infinity and Ω is a regular polygon in $\mathbb{R}^2$ then J admits a unique minimum over $\mathcal{K}$ whose explicit form is given in terms of f. This result was extended in [14] to general convex domains $\Omega\subset\mathbb{R}^n$ ($n\geq 2$) and to functions f with at least linear growth at infinity.

The aim of this paper is to give estimates of the "error" one makes by means of such approximation. More precisely, thanks to a "normalization" (see Proposition 1 below) we may always reduce our discussion to functions f satisfying $f(0)=0$ so that $J(0)=0$; then, by Proposition 2 below we exclude the trivial cases where the minimum of J over $W_0^{1,1}(\Omega)$ exists and is $u\equiv 0$ (which is a web function!), so that the following ratio is well-defined

$$\mathcal{E} = \frac{\min_{u\in\mathcal{K}} J(u)}{\inf_{u\in W_0^{1,1}(\Omega)} J(u)}\ . \tag{2}$$

Since $\mathcal{K}\subset W_0^{1,1}(\Omega)$ one has $\mathcal{E}\in[0,1]$ and $\mathcal{E}$ represents the *relative error* of the above mentioned approximation: the closer $\mathcal{E}$ is to 1, the better the approximation is. The level 0 for the functional is chosen as a reference level because $u\equiv 0$ is the rest function; the value of $\mathcal{E}$ yields the relative error with respect to the rest function.

The outline of this paper is as follows.

In next section we establish some general estimates of $\mathcal{E}$: we first obtain a result by a symmetrization method, namely we estimate the infimum over $W_0^{1,1}(\Omega)$ by means of the (explicit) minimum of the corresponding problem in the symmetrized ball; since we also obtain the explicit value of (1), this yields the estimate. We apply this result to some particular domains Ω and we show that for "thin" domains this estimate loses interest: then, we obtain a different estimate by "truncating and shifting" the function f and by applying previous results by Cellina [10]; we show that this estimate improves the first one for thin domains. We also study in more detail the case where $f(s)=s^p/p$ ($p>1$) and we obtain an estimate of $\mathcal{E}$ which involves optimal Sobolev embedding constants: in Sect. 4 we show that in some cases this estimate is sharper than the first two.

In Sect. 3 we consider some meaningful models and we apply our results: we consider the three cases where the function f is convex superlinear, convex asymptotically linear, nonconvex. The results obtained show that the approximation by means of web functions is in general satisfactory ($\mathcal{E}$ is close to 1). The proofs of all the results are given in Sect. 5.

2 Notation and general results

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded open convex set and let W_Ω denote its inradius, namely the supremum of the radii of the open balls contained in Ω. The Lebesgue measure and the m-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ will be denoted respectively by $\mathcal{L}(A)$ and $\mathcal{H}^m(A)$. We denote by $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ the Lebesgue measure of the unit ball B_1 in $\mathbb{R}^n$ and we recall that $\mathcal{H}^{n-1}(\partial B_1) = n\omega_n$.

Assume that

$$\begin{aligned} & f \not\equiv +\infty \quad \text{is a l.s.c. function s.t.} \quad \exists M > \frac{\mathcal{L}(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)} \\ & \exists b \in \mathbb{R}\,,\ f(s) \geq Ms - b \quad \forall s \geq 0\,. \end{aligned} \tag{3}$$

The second condition is certainly satisfied if f is superlinear at infinity or if $M > W_\Omega$, see [14].

We denote by f^* the polar function of f and by f^{**} the bipolar function of f, see [17]. It is well-known that f^{**} is the greatest convex l.s.c. function which is pointwise less or equal than f. Let

$$\sigma = \max\{s \geq 0;\ f^{**}(s) = \min f^{**}\}\,; \tag{4}$$

we also define the *normalized non-decreasing bipolar function* f_{**} of f by

$$f_{**}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \sigma \\ f^{**}(s) - f^{**}(\sigma) & \text{if } s \geq \sigma\,. \end{cases}$$

Obviously, if f^{**} is non-decreasing and $f^{**}(0) = 0$, then $f^{**} = f_{**}$. Finally, we denote by f_* the polar function of f_{**} which coincides with f^* if $f(0) = 0$ and f is non-decreasing and convex. Our first result states that any function f satisfying (3) may be normalized without altering the minimization problem:

Proposition 1. *Assume that f satisfies* (3)*, let σ be as in* (4) *and let f_{**} be the normalized non-decreasing bipolar function of f; then*

$$\inf_{u \in W_0^{1,1}(\Omega)} \int_\Omega [f(|\nabla u|) - u] = \min_{u \in W_0^{1,1}(\Omega)} \int_\Omega [f_{**}(|\nabla u|) - u] + \mathcal{L}(\Omega) f^{**}(\sigma)\,. \tag{5}$$

Define the set of web functions relative to Ω

$$\mathcal{K} = \{u \in W_0^{1,1}(\Omega);\ u(x) = u(d(x, \partial\Omega))\ \forall x \in \Omega\}\,,$$

where $d(\cdot, \partial\Omega)$ denotes the distance function from the boundary. We also consider the one-parameter family of subsets of Ω defined by

$$\Omega_t = \{x \in \Omega;\ d(x, \partial\Omega) > t\} \qquad \forall t \in [0, W_\Omega]$$

and their boundaries $\partial\Omega_t$; we clearly have $\Omega_0 = \Omega$ and $\Omega_{W_\Omega} = \emptyset$. In the sequel a major role is played by the functions

$$\nu(t) = \frac{\mathcal{L}(\Omega_t)}{\mathcal{H}^{n-1}(\partial\Omega_t)}\ , \qquad \alpha(t) = \mathcal{H}^{n-1}(\partial\Omega_t) \qquad t \in [0, W_\Omega]\ .$$

Consider the functionals

$$J(u) = \int_\Omega [f(|\nabla u|) - u] \qquad J_{**}(u) = \int_\Omega [f_{**}(|\nabla u|) - u]$$

and the corresponding values

$$I_\mathcal{K} = \min_{u\in\mathcal{K}} J_{**}(u) \qquad I_{**} = \min_{u\in W_0^{1,1}(\Omega)} J_{**}(u)\ .$$

Note that by the results in [14, 19] and by Proposition 1 we have

$$I_\mathcal{K} = \min_{u\in\mathcal{K}} J(u) + \mathcal{L}(\Omega) f^{**}(\sigma)\ .$$

We are concerned with the estimate of the relative error $\mathcal{E}$ defined in (2). With the above notations we have $\mathcal{E} = I_\mathcal{K}/I_{**}$. If $I_{**} = 0$, then the problem “degenerates” (it loses interest) because the web function $u \equiv 0$ minimizes J_{**}; in other words, we also have $I_\mathcal{K} = 0$ and there is no need to approximate the minimum! In order to avoid this trivial case, we have to take into account a necessary condition:

Proposition 2. *Assume that f satisfies* (3) *and that $I_{**} < 0$. Then $f_*(W_\Omega) > 0$.*

In order to have the relative error $\mathcal{E}$ well-defined, Proposition 2 tells us that we have to assume that $f_*(W_\Omega) > 0$; as we will see, in some cases more stringent assumptions are needed. Note that the condition $f_*(s) = 0$ for some $s > 0$ implies that the right derivative of f_{**} at 0 is greater or equal than s. Hence, if this derivative vanishes then we have $f_*(s) > 0$ for every $s > 0$.

The basic tool which will be used in this paper in order to estimate $\mathcal{E}$ is

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set and let $R = (\frac{\mathcal{L}(\Omega)}{\omega_n})^{1/n}$. Assume that f satisfies* (3) *and $f_*(\frac{R}{n}) > 0$, then*

$$\mathcal{E} \geq \frac{\int_0^{W_\Omega} \alpha(t) f_*(\nu(t))dt}{n\omega_n \int_0^R t^{n-1} f_*(\frac{t}{n})dt} = \mathcal{E}_1\ . \tag{6}$$

This result is obtained by means of a symmetrization method, see Sect. 5 below. In order to use Theorem 1, the explicit form of the functions $\alpha(t)$ and $\nu(t)$ is needed. In the next two corollaries we apply Theorem 1 to some particular cases. We first deal with cubes in $\mathbb{R}^n$:

Corollary 1. *Let $\ell > 0$, $\Omega = (0, \ell)^n$ and assume that f satisfies* (3) *and $f_*(\frac{\ell}{n\omega_n^{1/n}}) > 0$. Then,*

$$\mathcal{E} \geq \frac{\int_0^{\ell\omega_n^{-1/n}} t^{n-1} f_*(\frac{\omega_n^{1/n}}{2}\frac{t}{n})dt}{\int_0^{\ell\omega_n^{-1/n}} t^{n-1} f_*(\frac{t}{n})dt}\ .$$

Next, we deal with planar regular polygons as domains. As stated in [19, Theorem 2], when the number of sides tends to infinity, the corresponding sequence of minimizing web functions converges (in a suitable sense) to the unique radial minimum in the circumscribed ball. Here, we rephrase this result in a more precise fashion:

Corollary 2. *Let $\rho > 0$, $m \in \mathbb{N}$ ($m \geq 3$) and let Ω^m be a regular polygon of m sides inscribed in the ball $B_\rho \subset \mathbb{R}^2$; assume that f satisfies* (3) *and* $f_*(\frac{\rho}{4}\sqrt{\frac{3\sqrt{3}}{\pi}}) > 0$. *Let $\mathcal{E}_m$ be the corresponding relative error, then*

$$\mathcal{E}_m \geq \frac{\int_0^{\rho\cos\frac{\pi}{m}} t f_*(\frac{t}{2})dt}{\int_0^{\rho\cos\frac{\pi}{m}} t f_*(\frac{\sigma_m t}{2})dt}\,, \qquad \sigma_m = \sqrt{\frac{m}{\pi}\tan\frac{\pi}{m}}\,.$$

In particular, $\lim_{m\to\infty}\mathcal{E}_m = 1$.

In the previous result, if one merely wants the asymptotic behavior of $\mathcal{E}_m$, the assumption on f_* may be relaxed to $f_*(\frac{\rho}{2}) > 0$.

We now study the asymptotic behavior of $\mathcal{E}_1$ in (6) in the case of "vanishing domains". To this end, we make a further assumption on f:

$$\exists\gamma,\delta > 0 \qquad \text{s.t. } f_*(s) \sim \gamma s^\delta \quad \text{as } s \to 0\,; \tag{7}$$

by this we mean that $f_*(s) = \gamma s^\delta + o(s^\delta)$ as $s \to 0$. This assumption implies that $f_*(s) > 0$ for all $s > 0$ and no further positiveness requirement on f_* is needed. An interesting fact is that if we take Ω to be a cube, then $\mathcal{E}_1$ in (6) tends to a constant (depending only on n, δ but not on the particular function f considered) as the measure of the cube tends to 0:

Theorem 2. *Let $\ell > 0$, let $\Omega = (0,\ell)^n$ and assume that f satisfies* (3) *and* (7)*; let $\mathcal{E}_1$ be as in* (6)*, then*

$$\lim_{\ell\to 0}\mathcal{E}_1 = \left(\frac{\omega_n}{2^n}\right)^{\delta/n}\,. \tag{8}$$

Note that the term inside parenthesis in the limit in (8) is just the ratio between the measure of the unit ball and the measure of the circumscribed cube: of course, this is a consequence of the symmetrization method.

The next result shows that Theorem 1 loses interest for "thin" domains (i.e. domains Ω with a small inradius W_Ω):

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set and assume that f satisfies* (3) *and* $f_*(W_\Omega) > 0$. *Let M be as in* (3)*, let* $R = (\frac{\mathcal{L}(\Omega)}{\omega_n})^{1/n}$ *and assume that*

$$W_\Omega < \min\left\{M, \frac{R}{2(n+1)}\right\}\,.$$

Let $\mathcal{E}_1$ be as in (6)*, then*

$$\mathcal{E}_1 \leq \left(\frac{1}{n+1}\frac{R}{W_\Omega} - 1\right)^{-1}\,. \tag{9}$$

For fixed $\mathcal{L}(\Omega)$, as $W_\Omega \to 0$ the ratio R/W_Ω tends to infinity and (9) states that $\mathcal{E}_1$ approaches zero. Therefore, for thin domains the estimate (6) is not satisfactory. In particular, if $0 < \lambda < \ell$ and $\Omega = (0,\ell)^{n-1} \times (0,\lambda)$ and if we let $\lambda \to 0$, then $\mathcal{E}_1 \to 0$. Of course, this does not mean that the same is true for $\mathcal{E}$. On the contrary, we conjecture that the minimum over $\mathcal{K}$ well approximates the infimum over $W_0^{1,1}(\Omega)$ for a suitable class of thin domains, see Propositions 4 and 5 below.

When dealing with thin domains, the next result seems more useful:

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set such that $W_\Omega < M$, assume that f satisfies* (3) *and $f_*(W_\Omega) > 0$. Then*

$$\mathcal{E} \geq \frac{W_\Omega \int_0^{W_\Omega} \alpha(t) f_*(\nu(t))dt}{f_*(W_\Omega)\int_0^{W_\Omega} \mathcal{L}(\Omega_t)\,dt} = \mathcal{E}_2\ . \tag{10}$$

To see how Theorem 4 improves Theorem 1 for thin domains, consider the following example:

Example 1. Let $f(s) = s^2/2$ and let $\Omega = (0,1) \times (0, 2W_\Omega)$, $0 < W_\Omega \leq 1/2$. Let us denote by $\mathcal{E}_1(W_\Omega)$ and $\mathcal{E}_2(W_\Omega)$ respectively the r.h.s. of (6) and (10). We have that $\mathcal{E}_2$ is monotone decreasing on $[0,1/2]$, $\lim_{W_\Omega \to 0} \mathcal{E}_2(W_\Omega) = 2/3$ and $\mathcal{E}_2(1/2) = 3/8$. On the other hand, $\mathcal{E}_1$ is monotone increasing on $[0,1/2]$, approaches 0 as W_Ω tends to 0 and $\mathcal{E}_1(1/2) = \pi/4$. The explicit computation of $\mathcal{E}_1$ and $\mathcal{E}_2$ gives that $\mathcal{E}_2 > \mathcal{E}_1$ for $0 < W_\Omega < \frac{3\pi - \sqrt{9\pi^2 - 12\pi}}{4\pi} \approx 0.181$.

As a consequence of Theorem 4, we obtain

Corollary 3. *Under the assumptions of Theorem 4, one has*

$$\mathcal{E} \geq \frac{f_*(\zeta)/\zeta}{f_*(W_\Omega)/W_\Omega}, \qquad \forall\, 0 < \zeta \leq \frac{1}{\mathcal{L}(\Omega)} \int_0^{W_\Omega} \mathcal{L}(\Omega_t)\,dt\ . \tag{11}$$

In particular,

$$\mathcal{E} \geq (n+1)\frac{f_*\left(\frac{W_\Omega}{n+1}\right)}{f_*(W_\Omega)}\ . \tag{12}$$

Finally, in the particular case where $f(s) = \frac{s^p}{p}$ for some $p > 1$, in order to estimate $\mathcal{E}$, a method involving optimal embedding constants may also be used:

Theorem 5. *Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be an open bounded convex set and assume that $f(s) = \frac{s^p}{p}$. Let*

$$S = S(\Omega) = \inf_{\substack{u \in W_0^{1,p}(\Omega)\\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p}{\left(\int_\Omega |u|\right)^p}\ .$$

Then

$$\mathcal{E} \geq S^{\frac{1}{p-1}} \int_0^{W_\Omega} \alpha(t)\nu^{\frac{p}{p-1}}(t)\,dt = \mathcal{E}_3\ . \tag{13}$$

In spite of its simple and elegant form, the previous result may not be so easily applied. Besides the already mentioned problem of determining explicitly the functions α and ν, here a major problem is also how to determine the (possibly sharp) constant S. Nevertheless, in Sect. 4, we quote some examples of planar rectangles for which Theorem 5 improves Theorems 1 and 4.

3 Estimates and applications

In this section we discuss the general results given in the previous section and we apply them to some particularly interesting models.

3.1 Some convex superlinear problems

Consider first the case where $f(s) = \frac{s^2}{2}$ so that the functional J becomes

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - u \right) dx \ .$$

The minimization problem of J is associated to the Euler equation

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{14}$$

The unique solution of (14) is precisely the minimum of the functional J over the space $W_0^{1,1}(\Omega)$ (in fact the solution is smooth). This equation describes a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of cross section Ω, see [28]. In the two-dimensional context, (14) also has different interpretations, see e.g. [4]: consider the torsion problem of a long cylindrical beam in $\mathbb{R}^3$ whose axis is the x_3 axis and whose uniform cross-section Ω is a simply connected region of the plane x_1, x_2; the state of stress in the interior of the beam is determined by a warping function u which satisfies (14).

We restrict our attention to some particular domains Ω which allow us to obtain the “exact” value of $\mathcal{E}$ and, as a consequence, to evaluate how fine the estimate of Theorem 1 is.

We first consider the case of a square:

Proposition 3. *If* $f(s) = \frac{s^2}{2}$ *and* $\Omega = (0,1)^2$, *then* $\mathcal{E} \approx 0.889$.

Therefore, in this case the approximation by means of web functions is satisfactory since $\mathcal{E}$ is close to 1. In order to compare the value of $\mathcal{E}$ obtained in Proposition 3 with the lower bound for $\mathcal{E}$ obtained in Corollary 1, we refer to Proposition 6 below: in the case $n = p = 2$ it yields the estimate $\mathcal{E} > 0.785$ which is approximately the 88.3% of the value 0.889.

If we shrink the length of two parallel sides of the square to 0, then the approximation tends to become optimal:

Proposition 4. *Let* $f(s) = \frac{s^2}{2}$, *let* $\ell \in (0,1)$ *and let* $\Omega = (0,\ell) \times (0,1)$. *Then* $\mathcal{E} \to 1$ *as* $\ell \to 0$.

Proposition 4 does not come unexpected; indeed, when $\ell \to 0$ we can say in some sense that the problem becomes 1-dimensional.

Apart from circles and regular polygons, the simplest planar domains seem to be ellipses. However, as will be shown in the proof of the next result, the explicit forms of the functions α and ν are complicated and one has to proceed numerically. With the aid of Mathematica we obtain

Proposition 5. *Let $f(s) = \frac{s^2}{2}$, let $0 < b < 1$ and let*

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2;\ x^2 + \frac{y^2}{b^2} < 1 \right\} .$$

Then, we have the following approximate values of $\mathcal{E}$;

b	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\mathcal{E} \approx$	0.895	0.897	0.903	0.914	0.928	0.945	0.962	0.977	0.989	0.997

Moreover, $\lim_{b \to 1} \mathcal{E} = 1$.

We point out that even in the simple case $f(s) = \frac{s^2}{2}$ the behavior of $\mathcal{E}$ for thin rectangles (small ℓ in Proposition 4) and thin ellipses (small b in Proposition 5) *is not* the same. We believe that the non optimal behavior of thin ellipses is due to their curvature.

Next, we deal with a slightly more general class of functions f. We consider the case where $f(s) = s^p/p$ for some $p > 1$ so that the functional to minimize is

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^p}{p} - u \right) dx .$$

Here, the corresponding Euler equation is the degenerate elliptic problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{15}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$. This operator may be used for the description of some phenomena in glaciology [26,27] and for the study of non-Newtonian fluids in rheology [2]. We also refer to the introduction in [16] for further applications. The unique solution of (15) is the minimum of the functional J.

An application of Corollary 1 yields

Proposition 6. *Let $p > 1$. If $f(s) = \frac{s^p}{p}$ and $\Omega = (0,1)^n$, then*

$$\mathcal{E} \geq \left(\frac{\omega_n^{1/n}}{2} \right)^{\frac{p}{p-1}} .$$

Note that the lower bound for $\mathcal{E}$ in Proposition 6 tends to 0 as $p \to 1$ (i.e. as the functional "loses coercivity") and tends to

$$\frac{\sqrt{\pi}}{2^{(n-1)/n} n^{1/n} \Gamma^{1/n}(n/2)}$$

as $p \to \infty$: in particular, when $n = 2$ this limit is about 0.886.

3.2 A convex asymptotically linear problem

Consider the function $f(s) = \sqrt{1+s^2} - 1$ so that the functional to minimize is the following

$$J(u) = \int_\Omega \left(\sqrt{1+|\nabla u|^2} - 1 - u \right) dx \,.$$

Associated to the functional J we have the Euler equation

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{16}$$

whose solutions are the celebrated Delaunay surfaces of constant mean curvature, see [15]. We take $f(s) = \sqrt{1+s^2} - 1$ instead of the usual $f(s) = \sqrt{1+s^2}$ so that $f(0) = 0$ and $\mathcal{E}$ is well-defined.

In this case we have $M = 1$ and if we wish to fulfill (3) the set Ω must be “sufficiently small”, namely

$$\nu_\Omega(0) = \frac{\mathcal{L}(\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)} < 1 \,. \tag{17}$$

It is well-known [18,21] that if $\nu_\Omega(0)$ exceeds the limit value 1 then (16) admits no solutions, see also [20] for the blow up of the $W^{1,\infty}$-norms of the solutions of (16) as $\nu_\Omega(0) \to 1^-$. This shows that the restriction (17) is not purely technical. However, in order to apply Theorem 1, a further restriction is needed. Indeed, in the case where $\Omega = (0,\ell)^n$, (17) yields $\ell < 2n$ while in the next result we require a smaller upper bound for ℓ.

Proposition 7. *Let $f(s) = \sqrt{1+s^2} - 1$, let $\ell \le n\omega_n^{1/n}$ and let $\Omega = (0,\ell)^n$. Then*

$$\mathcal{E} \ge \frac{2^n n^{n+1} \int_0^{\ell/2n} r^{n-1}\sqrt{1-r^2}dr - \ell^n}{\omega_n n^{n+1} \int_0^{\ell/n\omega_n^{1/n}} r^{n-1}\sqrt{1-r^2}dr - \ell^n} \,. \tag{18}$$

Note that the integrals in the r.h.s. of (18) may be determined explicitly. In particular, if $n = 2$ then (18) becomes

$$\mathcal{E} \ge \frac{3\ell^2 + 32\left(1 - \frac{\ell^2}{16}\right)^{3/2} - 32}{3\ell^2 + 8\pi\left(1 - \frac{\ell^2}{4\pi}\right)^{3/2} - 8\pi}$$

so that the lower bound in (18) is a decreasing function of ℓ.

Note also that as $s \to 0$ we have $f_*(s) \sim \frac{s^2}{2}$ and therefore, according to Theorem 2, we have $\lim\limits_{\ell\to 0} \mathcal{E} \ge \frac{\omega_n^{2/n}}{4}$, $\forall n \ge 2$.

3.3 Some nonconvex problems

Let $h_1(t) = \alpha t^2$, $h_2(t) = \beta t^2 + \gamma$ (with $\alpha > \beta > 0$, $\gamma > 0$) and

$$f(t) = \min\{h_1(t), h_2(t)\}\ , \quad t \geq 0\ , \tag{19}$$

and consider the functional $J(u) = \int_\Omega [f(|\nabla u|) - u]\, dx$. The problem of minimizing J over the space $W_0^{1,1}(\Omega)$ arises from elasticity [5,22,23]. We wish to place two different linearly elastic materials (of shear moduli $\frac{1}{2\alpha}$ and $\frac{1}{2\beta}$) in the plane domain Ω so as to maximize the torsional rigidity of the resulting rod; moreover, the proportions of these materials are prescribed. By applying Corollary 1 we obtain

Proposition 8. *Let $\ell > 0$, $\Omega = (0,\ell)^2$ and assume that f is as in (19), let $a = 2\sqrt{\frac{\alpha\beta\gamma}{\alpha-\beta}}$, then*

$$\mathcal{E} \geq \frac{\pi}{4} \qquad \textit{if } \ell \leq 2\sqrt{\pi}a$$

$$\mathcal{E} \geq \frac{\pi\beta(\alpha-\beta)\ell^4}{4\alpha(\alpha-\beta)\ell^4 - 128\pi\alpha\beta\gamma(\alpha-\beta)\ell^2 + 1024\pi^2\alpha^2\beta^2\gamma^2} \qquad \textit{if } 2\sqrt{\pi}a < \ell < 4a$$

$$\mathcal{E} \geq \frac{\pi(\alpha-\beta)\ell^4 - 128\pi(\alpha-\beta)\beta\gamma\ell^2 + 4096\pi\alpha\beta^2\gamma^2}{4(\alpha-\beta)\ell^4 - 128\pi(\alpha-\beta)\beta\gamma\ell^2 + 1024\pi^2\alpha\beta^2\gamma^2} \qquad \textit{if } \ell \geq 4a\ .$$

If ℓ is small, then the minimum of $\overline{u}$ of J_{**} has small gradients $|\nabla\overline{u}|$ in the main part of the square Ω. Therefore we have $f \approx h_1$ and the problem reduces to that of Proposition 6 (with $p = 2$). This is why we find $\frac{\pi}{4}$ as a lower bound for $\mathcal{E}$. Similarly, if $\ell \to \infty$ then we have $f \approx h_2$ and the problem tends again to that of Proposition 6.

Finally, we consider the case where

$$f(t) = \begin{cases} 0 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \\ \infty & \text{elsewhere}\ . \end{cases} \tag{20}$$

This function was studied in [8] in an attempt to simplify the function f in (19) by retaining its essential feature of lacking convexity. For simplicity, we only deal with the case of squares $\Omega = (0,\ell)^2$ for $\ell > 0$. By [10, Theorem 1] we know that if $\ell \leq 2$ then $\mathcal{E} = 1$. On the other hand, the functional J is known to have no minimum in $W_0^{1,1}(\Omega)$ if $\ell \in (2, 2+2\varepsilon)$ for sufficiently small ε, see [8] (we write $2+2\varepsilon$ instead of $2+\varepsilon$ for later convenience). Therefore, we wish to prove that $\mathcal{E} \to 1$ as $\ell \to 2^+$. The next result also gives the rate of such convergence:

Proposition 9. *Assume that f is as in (20) and let $\varepsilon \in (0,1)$, $\Omega = (0, 2+2\varepsilon)^2$. Then $\mathcal{E} \geq \frac{1}{1+\varepsilon}$.*

We point out that this result is not proved by means of the general statements of Sect. 2.

4 Some applications of Theorem 5

Throughout this section $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are respectively the constants defined in (6), (10) and (13). We show that in some cases we have $\mathcal{E}_3 > \mathcal{E}_1$ or $\mathcal{E}_3 > \mathcal{E}_2$ so that Theorem 5 gives a finer estimate of $\mathcal{E}$ than Theorems 1 or 4.

In order to apply Theorem 5 we need to determine the constant S. We give a lower bound for such constant in planar rectangles:

Lemma 1. *Let* $0 < \ell < 1$ *and let* $\Omega = (0,1) \times (0,\ell)$. *Then* $S \geq \frac{2^{p/2}(1+\ell)^p}{\ell^{2p-1}}$.

Proof. By a density argument it suffices to prove that

$$\frac{\|\nabla v\|_p^p}{\|v\|_1^p} \geq \frac{2^{p/2}(1+\ell)^p}{\ell^{2p-1}} \qquad \forall v \in C_c^\infty(\Omega) \text{ s.t. } v \geq 0 \,. \tag{21}$$

So, take $v \in C_c^\infty(\Omega)$ such that $v(x) \geq 0$ in Ω and denote by $\partial_i = \partial/\partial x_i$ $(i = 1,2)$. Then, since $v = 0$ on $\partial\Omega$, if we denote $s^+ = \max[s,0]$ and $s^- = \min[s,0]$, we have

$$\begin{aligned} &\int_0^1 [\partial_1 v(\xi_1, x_2)]^+ d\xi_1 + \int_0^1 [\partial_1 v(\xi_1, x_2)]^- d\xi_1 \\ &= \int_0^1 \partial_1 v(\xi_1, x_2) d\xi_1 = v(1, x_2) - v(0, x_2) = 0 \quad \forall x_2 \,. \end{aligned}$$

Furthermore,

$$\int_0^1 [\partial_1 v(\xi_1, x_2)]^+ d\xi_1 - \int_0^1 [\partial_1 v(\xi_1, x_2)]^- d\xi_1 = \int_0^1 |\partial_1 v(\xi_1, x_2)| d\xi_1 \quad \forall x_2 \,.$$

These two equations show that

$$\int_0^1 [\partial_1 v(\xi_1, x_2)]^+ d\xi_1 = \frac{1}{2} \int_0^1 |\partial_1 v(\xi_1, x_2)| d\xi_1 \quad \forall x_2 \,. \tag{22}$$

If we proceed similarly with the other variable we obtain

$$\int_0^\ell [\partial_2 v(x_1, \xi_2)]^+ d\xi_2 = \frac{1}{2} \int_0^\ell |\partial_2 v(x_1, \xi_2)| d\xi_2 \quad \forall x_1 \,. \tag{23}$$

Since

$$v(x_1, x_2) = \int_0^{x_1} \partial_1 v(\xi_1, x_2) d\xi_1 = \int_0^{x_2} \partial_2 v(x_1, \xi_2) d\xi_2 \;\; \forall (x_1, x_2) \in \Omega \,,$$

by Fubini's Theorem and (22)-(23) we obtain

$$\begin{aligned} \|v\|_1 &= \int_\Omega v(x_1, x_2) dx_1 dx_2 \\ &= \frac{1}{1+\ell} \int_\Omega \left(\ell \int_0^{x_1} \partial_1 v(\xi_1, x_2) d\xi_1 + \int_0^{x_2} \partial_2 v(x_1, \xi_2) d\xi_2 \right) dx_1 dx_2 \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{1+\ell}\int_\Omega\left(\ell\int_0^{x_1}[\partial_1 v(\xi_1,x_2)]^+d\xi_1+\int_0^{x_2}[\partial_2 v(x_1,\xi_2)]^+d\xi_2\right)dx_1dx_2\\
&\le \frac{\ell}{1+\ell}\int_\Omega\left([\partial_1 v(x_1,x_2)]^+ + [\partial_2 v(x_1,x_2)]^+\right)dx_1dx_2\\
&= \frac{\ell}{2(1+\ell)}\int_\Omega\left(|\partial_1 v(x_1,x_2)| + |\partial_2 v(x_1,x_2)|\right)dx_1dx_2\ .
\end{aligned}$$

Hence, if we use Hölder's inequality and the inequality $(a+b)^p \le 2^{p/2}(a^2+b^2)^{p/2}$ (which holds for all $a,b\ge 0$) we get

$$\|v\|_1 \le \frac{\ell}{\sqrt{2}(1+\ell)}[\mathcal{L}(\Omega)]^{(p-1)/p}\|\nabla v\|_p = \frac{\ell^{(2p-1)/p}}{\sqrt{2}(1+\ell)}\|\nabla v\|_p\ ,$$

which proves (21). □

Now we prove that if $\Omega=(0,1)^2$ and p is "close" to 1, then $\mathcal{E}_3>\mathcal{E}_2$. Indeed, Lemma 1 and (13) yield

$$\mathcal{E}_3 \ge \frac{p-1}{3p-2}2^{(p-2)/(2p-2)}\ .$$

On the other hand, if $\mathcal{E}_2$ is as in (10), we get

$$\mathcal{E}_2 = \frac{p-1}{3p-2}\frac{3}{2^{1/(p-1)}}$$

and we have $\mathcal{E}_3>\mathcal{E}_2$ for p sufficiently close to 1.

Finally, for all $p>1$ we have $\lim_{\ell\to 0}\mathcal{E}_3 = \frac{p-1}{2p-1}2^{(2-3p)/(2p-2)} > 0$ which, together with Theorem 3, shows that $\mathcal{E}_3>\mathcal{E}_1$ for ℓ small enough.

5 Proofs of the results

Proof of Proposition 1. Any minimizing sequence $\{u_m\}$ of both the l.h.s. and the r.h.s. of (5) may be chosen so that $|\nabla u_m(x)|\ge\sigma$ for a.e. $x\in\Omega$. Hence, $\int f(|\nabla u_m|)=\int f_\sigma(|\nabla u_m|)$ and the result follows. □

Proof of Proposition 2. For contradiction, if $f_*(W_\Omega)=0$ then by the very definition of polar function we have that $f_{**}(s)\ge sW_\Omega$ for every $s\ge 0$. Since $f_{**}(0)=0$, from [10, Theorem 1] (see also [9,29]) we conclude that the function $u\equiv 0$ is a minimizer of J_{**} over $W_0^{1,1}(\Omega)$, contradiction. □

In order to prove Theorem 1, we first recall the basic definitions of the Schwarz symmetrization, see e.g. [4]. Given the set Ω, we denote by Ω^s the ball centered at the origin such that $\mathcal{L}(\Omega^s)=\mathcal{L}(\Omega)$. Let u be a real-valued function defined in Ω, then we define its symmetrized function $u^s\colon\Omega^s\to\mathbb{R}$ by $u^s(x)=\sup\{\mu;\ x\in D_\mu^s\}$, where $D_\mu=\{y\in\Omega;\ u(y)\ge\mu\}$.

In the sequel, we denote

$$\mathcal{S}_J(u)=\int_{\Omega^s}[f(|\nabla u|)-u]\ ,\qquad I_s=\min_{u\in W_0^{1,1}(\Omega^s)}\mathcal{S}_J(u)\ .$$

Explicit form of the minimizing web function.

Consider the functions ν and α defined in Sect. 2 then, the unique web function w which minimizes J over $\mathcal{K}$ is given by

$$w(x) = \phi(d(x,\partial\Omega)) = \int_0^{d(x,\partial\Omega)} (f_*)'(\nu(s))ds \, . \tag{24}$$

This follows from formula (6) in [14] and by taking into account that $(f_*)'(t) = (f_*)'_-(t)$ for a.e. t. Indeed, we recall that by Theorem 4.1 in [14] we know that the function ν is strictly decreasing and therefore $(f_*)'(\nu(s)) = (f_*)'_-(\nu(s))$ for a.e. $s \in [0, W_\Omega]$.

Moreover, by (7) and (8) in [14], the corresponding (minimum) value of the functional is given by

$$I_{\mathcal{K}} = J(w) = \int_0^{W_\Omega} \alpha(t)[f(\phi'(t)) - \phi(t)]dt \, .$$

Indeed, from (24), we see that $\phi'(t) \geq 0$ for a.e. t and we have $|\phi'(t)| = \phi'(t)$ for a.e. $t \in [0, W_\Omega]$.

Using again the strict monotonicity of ν we have

$$f((f_*)'(\nu(t))) = \nu(t)\,(f_*)'(\nu(t)) - f_*(\nu(t)) \qquad \text{for a.e. } t \in [0, W_\Omega] \, .$$

Then, integrating by parts the term in ϕ (see Lemma 5.6 in [14] where $A(t) = \alpha(t)\nu(t)$) we obtain

$$I_{\mathcal{K}} = -\int_0^{W_\Omega} \alpha(t) f_*(\nu(t))dt \, . \tag{25}$$

In the case of regular polygons in $\mathbb{R}^2$ an equivalent explicit form of w and $I_{\mathcal{K}}$ may be given, see [19].

The symmetrization method.

Thanks to well-known results in [1, 7] we obtain

Lemma 2. *Assume that f satisfies* (3) *and that f is convex and non-decreasing. Then*

$$\inf_{u \in W_0^{1,1}(\Omega)} \int_\Omega [f(|\nabla u|) - u] \geq \min_{u \in W_0^{1,1}(\Omega^s)} \int_{\Omega^s} [f(|\nabla u|) - u] \, .$$

Proof. We show that, for every $\overline{u} \in W_0^{1,1}(\Omega)$ we have $J(\overline{u}) \geq \mathcal{S}_J(\overline{u}^s)$. Since $J(|u|) \leq J(u)$ for all $u \in W_0^{1,1}(\Omega)$, we may assume that $\overline{u} \geq 0$. Then, by the properties of the symmetrized function (see [4]) we have

$$\int_\Omega \overline{u} = \int_{\Omega^s} \overline{u}^s \, . \tag{26}$$

Next, by Proposition 2.1 in [7] and by arguing as in the proof of Lemma 3.1 in [7], one has $\int_\Omega f(|\nabla \overline{u}|)\,dx \geq \int_{\Omega^s} f(|\nabla \overline{u}^s|)\,dx$. This, together with (26) concludes the proof. □

Proof of Theorem 1. For any bounded convex set $\Omega \subset \mathbb{R}^n$ we have $\Omega^s = B_R(0)$ with $R = (\frac{\mathcal{L}(\Omega)}{\omega_n})^{1/n}$ and $\alpha_B(t) = n\omega_n(R-t)^{n-1}$, $\nu_B(t) = \frac{R-t}{n}$. Therefore, according to (25) we have

$$I_s = -n\omega_n \int_0^R (R-t)^{n-1} f_* \left(\frac{R-t}{n}\right) dt = -n\omega_n \int_0^R t^{n-1} f_* \left(\frac{t}{n}\right) dt \,. \tag{27}$$

Since $f_*(\frac{R}{n}) > 0$ we have $I_s < 0$ so that, by Lemma 2, we also have $I_{**} < 0$. Moreover, Lemma 2 yields

$$\mathcal{E} = \frac{I_{\mathcal{K}}}{I_{**}} \geq \frac{I_{\mathcal{K}}}{I_s} \,. \tag{28}$$

Theorem 1 follows now directly from (25), (28) and (27). □

Remark 1. By Theorem 3.6 in [13] (see also [11]) the minimization problem

$$\min_{u \in W_0^{1,1}(\Omega^s)} \int_{\Omega^s} [f(|\nabla u|) - u]$$

admits a unique solution u_s which is radially symmetric (and decreasing). Moreover, by (11) in [13] we know that its derivative $u_s'(r)$ (with respect to $r = |x|$) satisfies the Euler-Lagrange inclusion $-\frac{r}{n} \in \partial f_{**}(u_s'(r))$. When $f \in C^1(\mathbb{R}_+)$ and f is strictly increasing and strictly convex the above inclusion simply becomes $u_s'(r) = -(f')^{-1}\left(\frac{r}{n}\right)$, see Remark 1 in [20] for related results concerning the corresponding Euler equation. □

Proof of Corollary 1. Take $\ell > 0$ and $\Omega = (0, \ell)^n$, then $\Omega^s = B_R$ and

$$\begin{aligned} W_\Omega &= \frac{\ell}{2} \quad R = \frac{\ell}{\omega_n^{1/n}} \\ \mathcal{L}(\Omega_t) &= (\ell - 2t)^n \quad \alpha(t) = \mathcal{H}^{n-1}(\partial\Omega_t) = 2n(\ell - 2t)^{n-1} \\ \nu(t) &= \frac{\ell - 2t}{2n} \quad t \in \left[0, \frac{\ell}{2}\right] \,. \end{aligned} \tag{29}$$

Then, by (25) and by the change of variables $s = \frac{\ell - 2t}{\omega_n^{1/n}}$, we infer

$$I_{\mathcal{K}} = -n\omega_n \int_0^{\ell/\omega_n^{1/n}} t^{n-1} f_* \left(\frac{\omega_n^{1/n}}{2} \frac{t}{n}\right) dt \,. \tag{30}$$

This, together with (28) and (27), yields the estimate of $\mathcal{E}$. □

Remark 2. A more elegant form of the estimate of $\mathcal{E}$ may be obtained by using the convexity of f_*. Indeed, by (27) we obtain

$$I_s - I_{\mathcal{K}} \leq \left(1 - \frac{\omega_n^{1/n}}{2}\right) \omega_n \int_0^{\ell/\omega_n^{1/n}} t^n (f_*)' \left(\frac{t}{n}\right) dt \,.$$

This, together with (28) and (27), yields

$$\mathcal{E} \geq \frac{I_{\mathcal{K}}}{I_s} \geq 1 - \frac{(1 - \frac{\omega_n^{1/n}}{2}) \int_0^{\ell/\omega_n^{1/n}} t^n (f_*)' \left(\frac{t}{n}\right) dt}{n \int_0^{\ell/\omega_n^{1/n}} t^{n-1} f_* \left(\frac{t}{n}\right) dt} .$$

However, the estimate in the statement of Corollary 1 is sharper and therefore we will not make use of the latter. □

Proof of Corollary 2. The constants and functions relative to the polygon Ω^m are given by $W_{\Omega^m} = \rho \cos \frac{\pi}{m}$, $R_m = \rho \sqrt{\frac{m \sin \frac{2\pi}{m}}{2\pi}}$, $\alpha_m(t) = 2m \left(\rho \cos \frac{\pi}{m} - t\right) \tan \frac{\pi}{m}$, $\nu_m(t) = \frac{1}{2} \left(\rho \cos \frac{\pi}{m} - t\right)$. Therefore, (25) and the change of variables $s = \rho \cos \frac{\pi}{m} - t$ yield

$$I_{\mathcal{K}} = -2m \tan \frac{\pi}{m} \int_0^{\rho \cos \frac{\pi}{m}} s f_* \left(\frac{s}{2}\right) \, ds .$$

Moreover, (27) and the change of variables $s = \sqrt{\frac{\pi}{m} \cot \frac{\pi}{m}} \, t$ yield

$$I_s = -2m \tan \frac{\pi}{m} \int_0^{\rho \cos \frac{\pi}{m}} s f_* \left(\sigma_m \frac{s}{2}\right) \, ds .$$

Since $f_*(\frac{\rho}{4} \sqrt{\frac{3\sqrt{3}}{\pi}}) > 0$, we have $I_s < 0$ (for all m) and the estimate of $\mathcal{E}_m$ follows from (28). □

Proof of Theorem 2. The constants W_Ω, R and the functions α, ν relative to Ω are given in (29). Then, by (7) and (27) we get

$$|I_s| \sim n\omega_n \int_0^{\ell/\omega_n^{1/n}} t^{n-1} \gamma \frac{t^\delta}{n^\delta} \, dt = \frac{\gamma}{(n+\delta)\omega_n^{\delta/n} n^{\delta-1}} \ell^{n+\delta} \qquad \text{as } \ell \to 0 .$$

Furthermore, by (7) and (30) we obtain

$$|I_{\mathcal{K}}| \sim n\omega_n \gamma \int_0^{\ell/\omega_n^{1/n}} t^{n-1} \frac{\omega_n^{\delta/n}}{2^\delta} \frac{t^\delta}{n^\delta} \, dt = \frac{\gamma}{(n+\delta) 2^\delta n^{\delta-1}} \ell^{n+\delta} \quad \text{as } \ell \to 0 .$$

Taking the ratio and applying (28) shows that $\mathcal{E}_1$ (defined in (6)) tends to the value in (8) as $\ell \to 0$. □

Proof of Theorem 3. Since $0 \leq \nu(t) \leq W_\Omega$ for every $t \in [0, W_\Omega]$, we have that $0 \leq f_*(\nu(t)) \leq f_*(W_\Omega)$ for every $t \in [0, W_\Omega]$, hence by (25) we get

$$\begin{aligned} |I_{\mathcal{K}}| &= \int_0^{W_\Omega} \alpha(t) f_*(\nu(t)) \, dt \leq f_*(W_\Omega) \int_0^{W_\Omega} \alpha(t) \, dt \\ &= f_*(W_\Omega) \mathcal{L}(\Omega) = f_*(W_\Omega) \omega_n R^n . \end{aligned} \tag{31}$$

By (3) and the assumptions $W_\Omega < M$ and $f_*(W_\Omega) > 0$, we may find $\beta > 0$ such that $\frac{f_{**}(\beta)}{\beta} = W_\Omega$. Since f_* is the polar function of f_{**}, we have $f_*(z) \geq \beta z - f_{**}(\beta)$ for every $z \geq 0$. Then, by (27) we get

$$\begin{aligned}|I_s| &= n\omega_n \int_0^R t^{n-1} f_* \left(\frac{t}{n}\right) dt \geq n\omega_n \int_0^R t^{n-1} \left[\beta \frac{t}{n} - f_{**}(\beta)\right] dt \\ &= \omega_n \beta \left(\frac{R^{n+1}}{n+1} - W_\Omega R^n\right) .\end{aligned}$$

From the definition of β we have that $\beta \geq f_*(W_\Omega)/W_\Omega$, hence

$$|I_s| \geq \omega_n \frac{f_*(W_\Omega)}{W_\Omega} \left(\frac{R^{n+1}}{n+1} - W_\Omega R^n\right) . \tag{32}$$

The conclusion now follows from (31) and (32). □

Proof of Theorem 4. By the very definition of polar function we have that $f_{**}(s) \geq sW_\Omega - f_*(W_\Omega)$ for every $s \geq 0$, hence $f_{**}(s) \geq \hat{f}(s) = \max\{0, sW_\Omega - f_*(W_\Omega)\}$ for every $s \geq 0$. Let us define the functional

$$\hat{J}(u) = \int_\Omega [\hat{f}(|\nabla u|) - u]\, dx, \qquad u \in W_0^{1,1}(\Omega).$$

By [9,10,29] we have that the function $\hat{u}(x) = \frac{f_*(W_\Omega)}{W_\Omega} d(x, \partial\Omega)$, $x \in \Omega$, is a minimizer of $\hat{J}$ on $W_0^{1,1}(\Omega)$. Since $f_{**} \geq \hat{f}$ we deduce that

$$\inf_{W_0^{1,1}(\Omega)} J \geq \hat{J}(\hat{u}) = -\frac{f_*(W_\Omega)}{W_\Omega} \int_0^{W_\Omega} \mathcal{L}(\Omega_t)\, dt,$$

where we have used (25) applied to $\hat{J}$ and the fact that $\hat{f}_*(s) = \frac{f_*(W_\Omega)}{W_\Omega} s$ for $s \in [0, W_\Omega]$. The estimate (10) then follows. □

Proof of Corollary 3. For every fixed $\beta > 0$ we have that $f_*(z) \geq \beta z - f_{**}(\beta)$, for every $z \geq 0$. Then

$$\int_0^{W_\Omega} \alpha(t) f_*(\nu(t))\, dt \geq \int_0^{W_\Omega} \alpha(t)[\beta\nu(t) - f_{**}(\beta)]\, dt \tag{33}$$

$$= \beta \int_0^{W_\Omega} \alpha(t)\nu(t)\, dt - f_{**}(\beta)\mathcal{L}(\Omega) . \tag{34}$$

From (10) and (33)–(34) we have that

$$\mathcal{E} \geq \frac{W_\Omega}{f_*(W_\Omega)} \left[\beta - \frac{f_{**}(\beta)\mathcal{L}(\Omega)}{\int_0^{W_\Omega} \alpha(t)\nu(t)\, dt}\right] . \tag{35}$$

The right hand side in (35) is maximized choosing $\beta \in \partial f_*(\zeta_0)$, with $\zeta_0 = \frac{1}{\mathcal{L}(\Omega)} \int_0^{W_\Omega} \alpha(t)\nu(t)\, dt$, obtaining

$$\mathcal{E} \geq \frac{f_*(\zeta_0)/\zeta_0}{f_*(W_\Omega)/W_\Omega} .$$

Since the map $z \mapsto f_*(z)/z$ is monotone increasing on $(0, +\infty)$, then (11) follows.

Concerning (12), by (11) it is enough to prove that

$$\frac{1}{\mathcal{L}(\Omega)} \int_0^{W_\Omega} \alpha(t)\nu(t)\, dt \geq \frac{W_\Omega\, \mathcal{L}(\Omega)}{n+1} \,. \tag{36}$$

We recall that $A(t) = \alpha(t)\nu(t) = \int_t^{W_\Omega} \alpha(s)\, ds = \mathcal{L}(\Omega_t)$, $t \in [0, W_\Omega]$. From the Brunn-Minkowski theorem (see [6]) we have that the function $\gamma(t) = A(t)^{1/n}$ is concave on $[0, W_\Omega]$. Since $\gamma(0) = \mathcal{L}(\Omega)^{1/n}$ and $\gamma(W_\Omega) = 0$ we deduce that $\gamma(t) \geq \frac{\mathcal{L}(\Omega)}{W_\Omega}(W_\Omega - t)$, $t \in [0, W_\Omega]$, hence

$$\int_0^{W_\Omega} A(t)\, dt = \int_0^{W_\Omega} \gamma^n(t)\, dt \geq \frac{\mathcal{L}(\Omega)}{{W_\Omega}^n} \int_0^{W_\Omega} (W_\Omega - t)^n\, dt = \frac{\mathcal{L}(\Omega) W_\Omega}{n+1},$$

and (36) is proved. □

Proof of Theorem 5. First note that

$$f^*(s) = f_*(s) = \frac{p-1}{p} s^{\frac{p}{p-1}} \,. \tag{37}$$

Next, we remark that if $\overline{u} \in W_0^{1,1}(\Omega)$ minimizes J, then $\overline{u}$ solves (15), $\overline{u} \in W_0^{1,p}(\Omega)$ and $\overline{u} > 0$ on Ω. By multiplying the equation in (15) by $\overline{u}$ and by integrating by parts we obtain

$$\int_\Omega |\nabla \overline{u}|^p = \int_\Omega \overline{u} \,, \tag{38}$$

so that

$$f(s) = \frac{s^p}{p} \,,\ \overline{u} \text{ minimizes } J \quad \Longrightarrow \quad J(\overline{u}) = \frac{1-p}{p} \int_\Omega \overline{u} \,. \tag{39}$$

Since $\overline{u} > 0$ in Ω, by (38) and by definition of S, we have $\int_\Omega \overline{u} \leq S^{\frac{1}{1-p}}$. By (39) this gives

$$I_{**} = J(\overline{u}) \geq \frac{1-p}{p} S^{\frac{1}{1-p}} \,.$$

Therefore, by definition of $\mathcal{E}$ and by (25) (with f_* given by (37)) we get

$$\mathcal{E} = \frac{I_{\mathcal{K}}}{I_{**}} \geq S^{\frac{1}{p-1}} \int_0^{W_\Omega} \alpha(t) \nu^{\frac{p}{p-1}}(t)\, dt \,. \square \tag{40}$$

Proof of Proposition 3. By separating the variables one finds that the unique solution $\overline{u}$ of (14) is given by

$$\overline{u}(x,y) = \frac{x - x^2}{2} - \frac{4}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\pi x]}{(2k+1)^3 (e^{(2k+1)\pi} + 1)}$$
$$\times \left\{ \exp[(2k+1)\pi y] + \exp[(2k+1)\pi(1-y)] \right\} .$$

Therefore, by (39) we have

$$I_{**} = J(\overline{u}) = -\frac{1}{2}\int_{\Omega} \overline{u}$$
$$= -\frac{1}{24} + \frac{8}{\pi^5}\sum_{k=0}^{\infty} \frac{e^{(2k+1)\pi}-1}{e^{(2k+1)\pi}+1}\frac{1}{(2k+1)^5} \approx -0.0175721 \, . \tag{41}$$

On the other hand, we have $W_{\Omega} = \frac{1}{2}$, $\alpha(t) = 4(1-2t)$, $\nu(t) = \frac{1-2t}{4}$, $f_*(s) = \frac{s^2}{2}$. Therefore, (25) yields

$$I_{\mathcal{K}} = -\frac{1}{8}\int_0^{1/2}(1-2t)^3 dt = -\frac{1}{64} \, .$$

This, together with (41), gives $\mathcal{E} \approx 0.889$. □

Proof of Proposition 4. By arguing as in the proof of Proposition 3 we find that the unique solution $\overline{u}$ of (14) is given by

$$\overline{u}(x,y) = \frac{\ell x - x^2}{2} - \frac{4\ell^2}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)\pi x]}{(2k+1)^3(e^{(2k+1)\pi/\ell}+1)}$$
$$\times \left\{ \exp[(2k+1)\pi y/\ell] + \exp[(2k+1)\pi(1-y)/\ell] \right\} .$$

Therefore, by (39) we have

$$J(\overline{u}) = -\frac{1}{2}\int_{\Omega}\overline{u} = -\frac{\ell^3}{24} + \frac{8\ell^4}{\pi^5}\sum_{k=0}^{\infty}\frac{e^{(2k+1)\pi/\ell}-1}{e^{(2k+1)\pi/\ell}+1}\frac{1}{(2k+1)^5} \, .$$

Since the sum of the series is bounded away from $+\infty$ and 0 as $\ell \to 0$, we infer that

$$J(\overline{u}) = -\frac{\ell^3}{24} + o(\ell^3) \qquad \text{as } \ell \to 0 \, . \tag{42}$$

In order to evaluate the minimum of J over $\mathcal{K}$, note that in this case we have $W_{\Omega} = \frac{\ell}{2}$ and $\alpha(t) = 2(1+\ell-4t)$, $\nu(t) = \frac{(\ell-2t)(1-2t)}{2(1+\ell-4t)}$. Then, from (25) we get

$$I_{\mathcal{K}} = I_{\mathcal{K}}(\ell) = -\frac{1}{4}\int_0^{\ell/2}\frac{(\ell-2t)^2(1-2t)^2}{1+\ell-4t}\,dt$$
$$= \frac{1}{256}(1-\ell)^4\log\frac{1-\ell}{1+\ell} + \frac{\ell}{128}(1+\ell^2-4\ell)$$
$$= -\frac{\ell^3}{24} + o(\ell^3) \quad \text{as } \ell \to 0 \, .$$

This, together with (42) proves that $\mathcal{E} \to 1$ as $\ell \to 0$. □

Proof of Proposition 5. The unique solution $\overline{u}$ of (14) is given by

$$\overline{u}(x,y) = \frac{b^2 - b^2x^2 - y^2}{2(1+b^2)} \, .$$

Then, if B_1 denotes the unit ball in $\mathbb{R}^2$, by (39) we infer

$$\begin{aligned} I_{**} &= J(\overline{u}) = -\frac{1}{2}\int_{\Omega} \overline{u} \\ &= -\frac{b^3}{4(1+b^2)}\int_{B_1}(1-x^2-y^2)dxdy = -\frac{\pi b^3}{8(1+b^2)} \ . \end{aligned} \tag{43}$$

The boundary of the ellipse Ω may be represented parametrically as $x = \cos\theta$, $y = b\sin\theta$, $\theta \in [0, 2\pi]$. The outward normal vector has components

$$\left(\frac{b\cos\theta}{\sqrt{\sin^2\theta + b^2\cos^2\theta}}, \frac{\sin\theta}{\sqrt{\sin^2\theta + b^2\cos^2\theta}} \right) \ ,$$

and therefore a parametric representation of $\partial\Omega_t$ ($0 \le t < b = W_\Omega$) for $x, y \ge 0$ is given by

$$\begin{cases} x = a\cos\theta - tb\dfrac{\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \\ y = b\sin\theta - ta\dfrac{\sin\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \end{cases} \qquad \theta \in \left[\theta_t, \frac{\pi}{2}\right] \ , \tag{44}$$

where

$$\theta_t = \arcsin\sqrt{\max\left[\frac{t^2 - b^4}{b^2(1-b^2)}, 0\right]} \ .$$

This value of θ_t comes from the fact that if $t > b^2$ then two different inward normal segments to $\partial\Omega$ of length t may intersect. In particular, for these values of t, $\partial\Omega_t$ is not a regular curve and this tells us that any web function which is not constant on Ω_{b^2} is not in $C^1(\Omega)$.

From (44) we infer that for $x, y \ge 0$ we have

$$\begin{cases} \dot{x} = -\sin\theta + tb\dfrac{\sin\theta}{(\sin^2\theta + b^2\cos^2\theta)^{3/2}} \\ \dot{y} = b\cos\theta - tb^2\dfrac{\cos\theta}{(\sin^2\theta + b^2\cos^2\theta)^{3/2}} \end{cases} \qquad \theta \in \left[\theta_t, \frac{\pi}{2}\right]$$

and therefore

$$\begin{aligned} \alpha(t) &= 4\int_{\theta_t}^{\pi/2} \sqrt{|\dot{x}(\theta)|^2 + |\dot{y}(\theta)|^2}\, d\theta \\ &= 4\int_{\theta_t}^{\pi/2} \left[\sqrt{\sin^2\theta + b^2\cos^2\theta} - \frac{tb}{\sin^2\theta + b^2\cos^2\theta}\right] d\theta \ . \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{L}(\Omega_t) &= \frac{1}{2}\int_{\partial\Omega_t}(x\,dy - y\,dx) \\ &= 2\int_{\theta_t}^{\pi/2} \Bigg[b - \frac{tb^2}{(\sin^2\theta + b^2\cos^2\theta)^{3/2}} - t\sqrt{\sin^2\theta + b^2\cos^2\theta} \\ &\qquad + \frac{t^2 b}{\sin^2\theta + b^2\cos^2\theta}\Bigg] d\theta \ . \end{aligned}$$

On the other hand, by (25) we have $I_{\mathcal{K}} = -\frac{1}{2}\int_0^b \frac{\mathcal{L}^2(\Omega_t)}{\alpha(t)}\,dt$ and a numerical computation with Mathematica allows to determine the approximate values of $\mathcal{E}$ given in the statement of Proposition 5.

Finally, if $b \to 1$, by the explicit value of $\alpha(t)$ and $\mathcal{L}(\Omega_t)$ found above and by Lebesgue Theorem, we have $\lim_{b\to 1} I_{\mathcal{K}} = -\pi/16$ which, together with (43), gives $\lim_{b\to 1}\mathcal{E} = 1$. □

Proof of Proposition 6. By (37) we see that f satisfies (7) with $\delta = p/(p-1)$. By a rescaling argument one sees that $\mathcal{E}_1$ in (6) is independent of ℓ. Then, the result follows from Theorem 2. □

Proof of Proposition 7. Consider the modified functional

$$J_+(u) = \int_\Omega \sqrt{1+|\nabla u|^2} - u\ ,$$

so that $J_+(u) = J(u) + \mathcal{L}(\Omega)$ for all $u \in W_0^{1,1}(\Omega)$ and we can argue instead on the functional J_+ (see also Proposition 1) and obtain

$$\mathcal{E} = \frac{\min_{u\in\mathcal{K}} J_+(u) - \ell^n}{\inf_{u\in W_0^{1,1}(\Omega)} J_+(u) - \ell^n}\ . \tag{45}$$

Therefore, we consider the function $g(s) = \sqrt{1+s^2}$. Then, $g_*(s) = -\sqrt{1-s^2}$ which is defined if $s\in[0,1]$ (we only need to consider $s \geq 0$).

If $\overline{u}$ minimizes J_+ and u_s minimizes the "symmetrized functional" $\mathcal{S}_J$, by Lemma 2 and (27) we get

$$\begin{aligned} J_+(\overline{u}) &\geq \min_{u\in W_0^{1,1}(\Omega^s)} \int_{\Omega^s} \sqrt{1+|\nabla u|^2} - u = n\omega_n \int_0^R \frac{r^{n-1}}{\sqrt{1+|u_s'(r)|^2}}dr \\ &= \omega_n \int_0^R r^{n-1}\sqrt{n^2-r^2}dr = \omega_n n^{n+1} \int_0^{\ell/n\omega_n^{1/n}} r^{n-1}\sqrt{1-r^2}dr\ . \end{aligned}$$

On the other hand, (25) yields

$$\begin{aligned} I_{\mathcal{K}} &= \int_0^{\ell/2} (\ell-2t)^{n-1}\sqrt{4n^2-(\ell-2t)^2}dt \\ &= 2^n n^{n+1} \int_0^{\ell/2n} r^{n-1}\sqrt{1-r^2}dr\ . \end{aligned}$$

The estimate of $\mathcal{E}$ now follows by applying (28) and (45). □

Proof of Proposition 8. We have $f_{**}(t) = h_1(t)$ if $0 \leq t \leq t_1$, $f_{**}(t) = at+b$ if $t_1 \leq t \leq t_2$, $f_{**}(t) = h_2(t)$ if $t_2 \leq t$, where $t_1 = \sqrt{\frac{\beta\gamma}{\alpha(\alpha-\beta)}}$, $t_2 = \sqrt{\frac{\alpha\gamma}{\beta(\alpha-\beta)}}$, $a = 2\sqrt{\frac{\alpha\beta\gamma}{\alpha-\beta}}$, $b = \frac{\beta\gamma}{\beta-\alpha}$. Moreover, the polar function is given by $f_*(t) = \frac{t^2}{4\alpha}$ if $0 \leq t \leq a$, $f_*(t) = \frac{t^2}{4\beta} - \gamma$ if $t \geq a$, and by Corollary 1 we get

$$\mathcal{E} \geq \frac{\int_0^{\ell/\sqrt{\pi}} t f_*(\frac{\sqrt{\pi}}{4}t)dt}{\int_0^{\ell/\sqrt{\pi}} t f_*(\frac{t}{2})dt}\ .$$

Hence, the result follows after integration. □

Proof of Proposition 9. Let f be as in (20), then $f_{**}(s) = 0$ if $s \in [0,1]$, $f_{**}(s) = s-1$ if $s \in [1,2]$, and $+\infty$ elsewhere. For all $\varepsilon \in (0,1)$ consider the function $f_\varepsilon(s) = 0$ if $s \in [0, 1+\varepsilon]$, $f_\varepsilon(s) = \frac{s-1-\varepsilon}{1-\varepsilon}$ if $s \in [1+\varepsilon, 2]$, and $+\infty$ elsewhere. Denote by J_{**} e J_ε the functionals associated to f_{**} and f_ε

$$J_{**}(u) = \int_\Omega [f_{**}(|\nabla u|) - u] \, , \qquad J_\varepsilon(u) = \int_\Omega [f_\varepsilon(|\nabla u|) - u] \, .$$

Since $f_\varepsilon(s) \geq \max\{0, \frac{1}{1-\varepsilon}(s-1-\varepsilon)\}$ and since $\frac{1}{1-\varepsilon} \geq 1+\varepsilon$, by Theorem 1 in [10] we have that the (unique) minimum u_ε of J_ε over $W_0^{1,1}(\Omega)$ is given by $u_\varepsilon(x) = (1+\varepsilon)d(x, \partial\Omega)$ and hence

$$I_{**} \geq \min_{W_0^{1,1}(\Omega)} J_\varepsilon = J_\varepsilon(u_\varepsilon) = -\frac{4}{3}(1+\varepsilon)^4 \, ,$$

where we have used the fact that $f_\varepsilon \leq f$.

Next, note that in this case we have $W_\Omega = 1+\varepsilon$, $\alpha(t) = 8(1+\varepsilon-t)$, $\nu(t) = (1+\varepsilon-t)/2$ and $f_*(s) = s$ if $s \in [0,1]$, $f_*(s) = 2s-1$ if $s \in [1,\infty)$. Hence, by (25) we infer

$$I_{\mathcal{K}} = -\int_0^{1+\varepsilon} \alpha(t) f_*(\nu(t)) \, dt = -4 \int_0^{1+\varepsilon} (1+\varepsilon-t)^2 \, dt = -\frac{4}{3}(1+\varepsilon)^3 \, .$$

We can now conclude that

$$\mathcal{E} = \frac{I_{\mathcal{K}}}{I_{**}} \geq \frac{1}{1+\varepsilon} \, . \square$$

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