An optimal control problem for virus propagation and economic loss

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Abstract

We introduce a new optimal control problem aiming to suggest possible lockdown strategies in case of pandemic virus proliferation. Starting from the (nonlinear) logistic equation, the model takes into account the drawback of the lockdown strategy by coupling it with a second logistic equation evaluating the health effects deriving from economic losses. The control parameter is the lockdown strength and the optimal control is sought in order to minimize the percentage of the overall affected population, counting both the directly affected humans and the humans affected by the economic loss. Our results show that, in some cases, the optimal control is a total lockdown strategy while, in other cases, the lockdown should be milder.
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1 Introduction

In 1798, in his masterpiece study of the "future improvements of Society", the English economist Thomas Robert Malthus [10] suggested a model for the dynamics of populations. His monograph was published under the pseudonym of J. Johnson, only much later discovered to be Malthus himself. The Malthus model, which assumes the existence of infinite quantities of both space and food, may be briefly described as follows. The variation of a population of individuals (e.g. viruses) merely depends on the natality and mortality rates, assumed to be constants. If the population is initially (at time $t = 0$) made by $y_0 > 0$ individuals and if we denote by $y(t)$ the population at time t, we expect that, in average, in any interval of time Δt there is a quantity of individuals born which is proportional to the population and to the interval of time, that is, equal to $ny(t)\Delta t$ where $n > 0$ is the natality rate. Similarly, we expect a number $my(t)\Delta t$ of deaths in the same interval of time, where $m > 0$ is the mortality rate. The population at time $t + \Delta t$ is given by the population at time t plus the born population and minus the dead population, namely

$$y(t+\Delta t) = y(t) + (n-m)y(t)\Delta t \iff \frac{y(t+\Delta t) - y(t)}{\Delta t} = (n-m)y(t)$$

and, taking the limit as $\Delta t \to 0$, we obtain the differential equation

$$\dot{y}(t) = \rho y(t) \qquad (\rho = n - m). \tag{1.1}$$

Its unique solution under the condition $y(0) = 1$ is given by $y(t) = e^{\rho t}$, which is an increasing exponential if $\rho > 0$ (natality larger than mortality) and decreasing if $\rho < 0$; if $n = m$ then

$y(t)$ remains constant. Clearly, this model is fairly simplified and gives inaccurate responses. The weakest point is the initial assumption of infinite quantities of space and food, an assumption allowing for an unbounded increment of the population. In fact, the lack of food and space decreases the natality rate and increases the mortality rate. It is then quite natural to expect ρ to be decreasing with respect to y, which leads to a nonlinear equation.

It was the Belgian mathematician Pierre François Verhulst [11, 12] who introduced the so-called *logistic equation* in 1838, an equation able to take into account the decrement of food and space as the population increases. Before deriving it, let us observe that, in case of pandemic, $y = y(t)$ is also proportional to the number of humans affected by the virus. Then, the probability to have an increment of affected people is proportional to the remaining population with risk of being affected. Denoting by p the overall human population, we then obtain the following variant of (1.1):

$$\dot{y}(t) = \rho \left(1 - \frac{y(t)}{p}\right) y(t). \tag{1.2}$$

This is the logistic equation and it belongs to the class of Bernoulli equations, taking their name from the work of Jacob Bernoulli [3] in 1695. Also (1.2) can be solved explicitly, as we recall in Section 2. Assuming that only one individual is affected at time $t = 0$, we obtain the Cauchy problem $y(0) = 1$ and the resulting solution is

$$y(t) = \frac{p}{1 + (p-1)e^{-\rho t}} \tag{1.3}$$

whose graph is displayed in Figure 1 for $\rho = 0.1$, $p = 10^5$. Figure 1 shows a good agreement with the curves of the Coronavirus affections during year 2020, all over the world, see [13]. Therefore, (1.2) is nowadays considered a good model to describe the dynamics of populations.

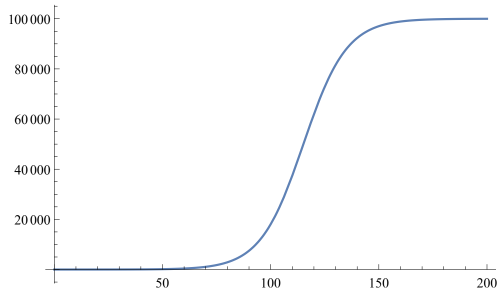

Figure 1: Graph of the solution (1.3) of the logistic equation (1.2) for $\rho = 0.1$, $p = 10^5$.

In order to build a model able to take into account the problems that arose during the Coronavirus propagation, we make a couple of preliminary remarks. First, one may wonder whether a continuous function (in fact, C^1) is well-suited to measure the number of affected humans which is an integer number. This may be explained by taking into account the *percentage of affection* of each human. More precisely, the affection itself cannot be considered just on or off, different degrees of affection are present in humans, starting with weak symptoms until a full infection is reached. Therefore, the assumption that $y \in C^0$ is fully justified.

The second remark concerns the *lockdown*, namely a control imposed by the Governments during the Coronavirus diffusion in order to reduce the speed of propagation of the virus. The lockdown consists in prohibiting to a certain amount of the population to circulate freely. As shown by a statistic published in the Financial Times [2], each Country/Region decided its own lockdown strategy in order to minimize the number of affected people and the economic impact, see also

[7]. Clearly, these two targets are competing with each other: a strong lockdown decreases the number of affected people but increases the economic losses and viceversa. Since also the economic losses themselves represent an affection of the population in terms of welfare, these two numbers may somehow be put on the same level. In particular, economic losses induce both decrements of salaries and of the quality of food and, hence, increment of diseases due to the bad quality of food and decrement of the possibility of treating health problems due to lower incomes. Moreover, an iterated quarantine prevents primary medical treatments and has physical and psychological consequences such as post-traumatic stress symptoms, confusion, and anger [6].

In the sequel, we call

directly affected the humans attained by the virus,
indirectly affected the humans losing health as a consequence of the lockdown.

The data from [2] show the results, in terms of the (directly and indirectly) affected humans, depending on the adopted lockdown strategies. But, among so many data, it appears impossible to derive an *optimal lockdown strategy* able to minimize the overall affected humans.

The purpose of this paper is to suggest a new optimal control problem aiming to determine the best lockdown strategy. Denoting by $y_1(t)$ and by $y_2(t)$, respectively, the directly affected and indirectly affected humans at time t, the suggested control problem reads

$$\begin{cases} \dot{y}_1(t) = \rho\left(1 - \frac{y_1(t)}{p-\alpha(t)}\right) y_1(t)\,, \\ \dot{y}_2(t) = k\alpha(t)\big(\alpha(t) + 1 - y_2(t)\big)y_2(t)\,, \\ y_1(0) = 1\,, \quad y_2(0) = 1\,, \end{cases} \qquad \alpha(t) \in [0, p-1] \quad \forall t \geq 0\,, \tag{1.4}$$

where p is the population, $\alpha(t)$ is the control function measuring the lockdown restrictions. The unknowns y_1 and y_2 are governed by different equations and are independent quantities. Some individual may be affected *both* directly and indirectly so that the constraints are that $y_1 \leq p$ and $y_2 \leq p$, but there is no such constraint on their sum. The derivation and a detailed interpretation of (1.4) is given in Section 2. According to the two equations in (1.4), *larger α yield both smaller y_1 and larger y_2* and viceversa, which emphasizes the opposite impact that α has on y_1 and y_2. The optimization problem consists in minimizing the sum $y_1(T) + y_2(T)$ at some time $T > 0$.

We first analyze the case where the admissible controls are constant, namely the Government decides once forever the lockdown strategy. In Theorem 1 we prove that the optimal (constant) control is never a null lockdown while, under suitable assumptions on k and ρ, it can be a total lockdown. We recall that the total lockdown strategy was adopted by some African Countries [1].

Then we analyze the case of general bounded controls α. In 1956, the Russian mathematician Lev Pontryagin, together with his students [4], formulated what is nowadays called the Pontryagin Minimum Principle (PMP in the sequel); we refer to [8] for a bibliography list of 54 items which appeared before year 1961 on the PMP. Thanks to the PMP, in Theorem 3 we prove that, also among bounded controls, the optimal control is bounded away from zero and can be a total lockdown. Necessary and sufficient conditions are provided, as well as some information on the optimal lockdown strategy. Our proofs take great advantage of the structure of the problem which allows for an almost explicit characterization of the costate, see (5.6).

We believe that these results validate our model and we hope that the suggested model might be considered as a good starting point towards a "perfect model" able to take into account also other factors, such as memory effects or different payoff functionals. In Section 6 we complement our results with some remarks and some possible future developments.

2 Derivation of the model and statement of the main results

Assume that a pandemic virus has affected one human, among a given population formed by $p \gg 1$ humans, and we denote by $\alpha = \alpha(t)$ the lockdown control parameter, namely the measure of the restrictions imposed by the Government at time t in order to reduce the directly affected humans. We consider (e.g.) the case where

$$p \geq 5000\,, \qquad \alpha \in \mathcal{A} := L^\infty(\mathbb{R}_+; [0, p-1]) \tag{2.1}$$

although much larger p are expected to better describe a population. The control α may also be discontinuous but, since it is bounded, both equations in (1.4) admit a unique Lipschitz-continuous solution, see e.g. [5]. The two unknown functions $y_1 = y_1(t)$ and $y_2 = y_2(t)$ represent, respectively, the directly and indirectly affected humans at time t.

We assume that $\rho > 0$ (natality larger than mortality among viruses), that $k > 0$ (positive speed for indirect affections), these parameters having obvious meanings: ρ and k represent, respectively, the speed of propagation of direct and indirect affections. If a Region has good quality of health assistance then ρ is small, while a bad health assistance means that ρ is large. Moreover, if a Region has wealth (in any sense) then k is large, while if it is poor then k is small (e.g., low wealth means low risk for economic losses). Clearly, the optimal choice of the Government strongly depends on the values of these parameters: large ρ and small k suggest a strategy with less direct affections and viceversa. The optimal lockdown strategy should follow the already mentioned principle that *larger α yield both smaller y_1 and larger y_2*, and viceversa.

We set up a *fixed-time-free-endpoint control problem* by introducing the payoff functional

$$J_T(\alpha) = y_1(T) + y_2(T)\,, \tag{2.2}$$

measuring the overall affected humans at time $T > 0$. The purpose of the Government is to choose the optimal control α minimizing the functional J_T, that is,

$$\text{find } \alpha^* \in \mathcal{A} \text{ such that } J_T(\alpha^*) = \inf_{\alpha \in \mathcal{A}} J_T(\alpha)\,. \tag{2.3}$$

Let us first restrict our attention to the case of a “lazy Government” that aims to take a decision once forever and never change strategy. In this case, the admissible controls are constants:

$$\mathcal{A}_0 = \{\alpha \in \mathcal{A};\ \exists \gamma \in [0, p-1],\ \alpha(t) \equiv \gamma\}\,.$$

By maintaining the very same payoff functional (2.2), in Section 3 we prove the following result.

Theorem 1. *Assume* (2.1) *and let $T > 0$. There exists an optimal control $\alpha^0 \in \mathcal{A}_0$, associated with a C^∞ solution (y_1^0, y_2^0) of* (1.4)*, minimizing J_T in* (2.3) *over $\mathcal{A}_0$, that is,*

$$J_T(\alpha^0) = y_1^0(T) + y_2^0(T) = \min_{\alpha \in \mathcal{A}_0} J_T(\alpha)\,.$$

Moreover, denoting by $\gamma_0 \in [0, p-1]$ the constant value of $\alpha^0(t)$, we have:

- *for all $T, \rho, k > 0$, $\gamma_0 > 0$;*
- *for all $T, \rho > 0$, if*

$$k \geq \frac{p}{(p-1)(2p-1)T}\,, \tag{2.4}$$

then $\gamma_0 < p - 1$;

- *for all $T, \rho > 0$, if*

$$k \leq \overline{k}(T, \rho) := \frac{1}{T}\,\frac{1 - e^{-\rho T}}{1 + (p-1)e^{-\rho T}}\,\frac{2 + \log(p-1)}{2p(p-1)}\,, \tag{2.5}$$

then $\gamma_0 = p - 1$.

The interpretation of Theorem 1 is evident and gives strength to our model.

For lazy Governments, the optimal strategy is never to have null lockdown while it can be a total lockdown if the propagation rate k of indirectly affected humans is sufficiently small.

Note that (2.4) is independent of ρ and refined (but less explicit) ρ-dependent bounds for k are available, see (3.7) below. Also (2.5) can be refined at the cost of a more complicated condition, see (3.9) (3.10) (3.12) below. For given $T > 0$, the map $\rho \mapsto \overline{k}(T, \rho)$ is increasing. Hence, the total lockdown strategy is more convenient if the propagation rate ρ of directly affected humans is large.

But the optimal control for (2.3) needs not to be constant. Before tackling the full optimal control problem for $\alpha \in \mathcal{A}$, let us establish some general properties of the solutions of (1.4).

Proposition 2. *For any $T > 0$ and any $\alpha \in \mathcal{A}$ the solution of (y_1, y_2) of (1.4) satisfies:*

$$1 \le y_1(t) \le \frac{p}{1+(p-1)e^{-\rho t}} \le \frac{p}{1+(p-1)e^{-\rho T}} < p \qquad \forall t \in [0,T]\,; \tag{2.6}$$

$$1 \le y_2(t) \le \frac{p}{1+(p-1)e^{-kp(p-1)t}} \le \frac{p}{1+(p-1)e^{-kp(p-1)T}} < p \qquad \forall t \in [0,T]\,. \tag{2.7}$$

Moreover,

- *if $\alpha(t) \equiv 0$ in $[0, T/2]$ and $\alpha(t) \equiv p-1$ in $[T/2, T]$, then $\dot{y}_1(t) < 0$ in $[T/2, T]$;*
- *if $\alpha(t) \equiv p-1$ in $[0, T/2]$ and $\alpha(t) \equiv \frac{(p-1)(1-e^{-kp(p-1)T/2})}{2+(p-1)e^{-kp(p-1)T/2}}$ in $[T/2, T]$, then $\dot{y}_2(t) < 0$ in $[T/2, T]$.*

We believe that (2.6)-(2.7) might be improved with the inequality

$$\forall T > 0\,,\ \forall \alpha \in \mathcal{A}\,, \quad y_1(t) + y_2(t) < p + 1 \quad \forall t \in [0,T]\,. \tag{2.8}$$

This conjecture is based on the validity of (2.8) for $\alpha \in \mathcal{A}_0$ (constant controls), see (3.5) below. Furthermore, concerning the possibility of decreasing y_1 and y_2, we conjecture that

$$\forall T > 0\,, \quad \forall \alpha \in \mathcal{A}\,, \qquad \dot{y}_1(t) + \dot{y}_2(t) > 0 \quad \forall t \in [0,T]. \tag{2.9}$$

This conjecture is based on some naive numerical experiments, see also Remark 13 below. However, since neither (2.8) nor (2.9) are directly connected with the optimal control problem considered in the present paper, we do not investigate them any further and we leave them as open problems.

We now turn to the general result, for controls $\alpha \in \mathcal{A}$. In this setting, our main result reads:

Theorem 3. *Assume (2.1) and let $T > 0$. There exists an optimal control $\alpha^* \in \mathcal{A}$, associated with a Lipschitzian solution (y_1^*, y_2^*) of (1.4), minimizing J_T in (2.3), that is,*

$$J_T(\alpha^*) = y_1^*(T) + y_2^*(T) = \min_{\alpha \in \mathcal{A}} J_T(\alpha)\,.$$

Moreover, there exists a function $\Phi \in C^0[0,T]$ (depending on the solution) such that

$$\text{if } \Phi(t) > 0, \text{ then } \alpha^*(t) = p-1; \qquad \text{if } \Phi(t) < 0, \text{ then } \alpha^*(t) \in \left(\frac{\rho e^{-\rho T}}{2kp^3}, \frac{3(p-1)}{4}\right)\,.$$

Finally, if

$$k \le \frac{4}{p(2p-1)^2}\ \rho e^{-\rho T}\,, \tag{2.10}$$

then $\alpha^*(t) \equiv p-1$ in $[0,T]$, whereas if

$$k > \frac{4\,\rho}{p^3}\left[1 + (p-1)e^{-kTp(p-1)}\right]\,, \tag{2.11}$$

then $\alpha^*(t) \not\equiv p-1$ in $[0,T]$.

Translating the results into the lockdown language, Theorem 3 yields the conclusions

the possible transition from a total lockdown to a weaker strategy is not continuous;
a total lockdown is the best strategy if the indirect diffusion coefficient k is small;
a total lockdown is not the best strategy if the indirect diffusion coefficient k is large;
a null lockdown is <u>*never*</u> *the best strategy.*

If a Government is ready to modify the lockdown strategy in the considered interval of time $[0, T]$, then the optimal control may be variable in time. As expected in view of the inclusion $\mathcal{A}_0 \subset \mathcal{A}$, the condition (2.10) is more restrictive than (2.5), see Proposition 11 below. In Theorem 3, (2.10) (resp. (2.11)) gives a sufficient condition (resp. necessary condition) for the optimal control to be a total lockdown, both depending on T. What happens in the range

$$\frac{4}{p(2p-1)^2}\,\rho e^{-\rho T} < k < \frac{4\,\rho}{p^3}\left[1+(p-1)e^{-kp(p-1)T}\right],$$

is an open problem: which is the critical value of $k = k(\rho, T)$ for which we have a full lockdown optimal control? As a byproduct of our proof, we obtain a slightly more precise characterization of α^* (in a non total lockdown regime), see Remark 12. A further open problem is to give a better description of the optimal control in the cases when it is not constant.

It is well-known that for nonlinear optimal control problems some physical constraints may appear *a posteriori*. This is the case of the moon lander problem, see e.g. [9], in which constrains on the position, the velocity and the mass of the spacecraft are derived *after* having characterized the optimal control. As we shall see, the same happens for problem (2.3) related to (1.4).

3 Proof of Theorem 1

The payoff functional in (2.2) may be written in a more explicit (but complicated) form. To see this, let us start with the Cauchy problem for the autonomous Bernoulli equation considered in the introduction:

$$\dot{y}(t) = \rho\left(1 - \frac{y(t)}{p}\right)y(t)\,, \qquad y(0) = 1. \tag{3.1}$$

With the change of unknown $z(t) = 1/y(t)$, we obtain the linear equation $\dot{z} + \rho z = \rho/p$. By solving this equation with $z(0) = 1$ and by taking $y = 1/z$, we find the solution (1.3). In fact, (3.1) may also be solved by separating variables but this is no longer possible if a non-autonomous control is inserted in (3.1). We may then apply the same strategy to determine explicitly y_1 solving (1.4) for a given α. We obtain that, for given $\alpha \in \mathcal{A}$, the unique solution of $(1.4)_1$ satisfying $y_1(0) = 1$ is

$$y_1(t) = \frac{e^{\rho t}}{1 + \rho\int_0^t \frac{e^{\rho s}}{p-\alpha(s)}\,ds}\,, \tag{3.2}$$

while the unique solution of $(1.4)_2$ satisfying $y_2(0) = 1$ is

$$y_2(t) = \frac{e^{A_2(t)}}{1 + k\int_0^t \alpha(s)e^{A_2(s)}\,ds}\,, \qquad \text{where} \quad A_2(t) = k\int_0^t [\alpha(\tau)^2 + \alpha(\tau)]\,d\tau\,. \tag{3.3}$$

The expressions (3.2)-(3.3) enable us to write the payoff functional as

$$J_T(\alpha) = \frac{e^{\rho T}}{1 + \rho\int_0^T \frac{e^{\rho s}}{p-\alpha(s)}\,ds} + \frac{e^{A_2(T)}}{1 + k\int_0^T \alpha(s)e^{A_2(s)}\,ds}\,. \tag{3.4}$$

This form is fairly complicated but, at least for constant controls, it may be fruitfully exploited to derive some important information. Let us compute the payoff in the particular case of *constant controls* $\alpha(t) \equiv \gamma \in [0, p-1]$. From (3.2)-(3.3) we infer that

$$y_1(t) = \frac{p-\gamma}{1+(p-\gamma-1)e^{-\rho t}}\,, \qquad y_2(t) = \frac{\gamma+1}{1+\gamma e^{-k(\gamma^2+\gamma)t}} \tag{3.5}$$

and, in turn,

$$J_T(\gamma) = \frac{p-\gamma}{1+(p-\gamma-1)e^{-\rho T}} + \frac{\gamma+1}{1+\gamma e^{-kT(\gamma^2+\gamma)}}\,. \tag{3.6}$$

By differentiating with respect to γ, we find

$$\frac{d}{d\gamma}J_T(\gamma) = \frac{e^{-\rho T}-1}{[1+(p-\gamma-1)e^{-\rho T}]^2} + \frac{1+[kT\gamma(\gamma+1)(2\gamma+1)-1]e^{-kT(\gamma^2+\gamma)}}{[1+\gamma e^{-kT(\gamma^2+\gamma)}]^2}\,.$$

As expected, the first term is negative while the second is positive. We then have

$$\frac{d}{d\gamma}J_T(\gamma)\Big|_{\gamma=0} = \frac{e^{-\rho T}-1}{[1+(p-1)e^{-\rho T}]^2} < 0 \qquad \forall T, \rho, k > 0\,,$$

which shows that $J_T(\gamma) < J_T(0)$ for $\gamma > 0$ sufficiently small; therefore $\alpha(t) \equiv 0$ is not the optimal control among constant controls. This proves the first item in Theorem 1.

Moreover, we have that

$$\frac{d}{d\gamma}J_T(\gamma)\Big|_{\gamma=p-1} = e^{-\rho T} - 1 + \frac{1+[kTp(p-1)(2p-1)-1]e^{-kTp(p-1)}}{[1+(p-1)e^{-kTp(p-1)}]^2}\,.$$

The limit, as $k \to \infty$, of the right hand side of this identity is $e^{-\rho T} > 0$ for all given $T > 0$. Hence, if k is sufficiently large (depending on ρ), then

$$\frac{d}{d\gamma}J_T(\gamma)\Big|_{\gamma=p-1} > 0\,, \tag{3.7}$$

which shows that $J_T(\gamma) < J_T(p-1)$ for γ in a left neighborhood of $p-1$; hence, if (3.7) holds, then $\alpha(t) \equiv p-1$ is not the optimal control among constant controls.

Let us then seek a sufficient condition, independent of ρ, for (3.7) to hold. By dropping $e^{-\rho T}$, we see that (3.7) is certainly satisfied if

$$\frac{1+[kTp(p-1)(2p-1)-1]e^{-kTp(p-1)}}{[1+(p-1)e^{-kTp(p-1)}]^2} \geq 1\,,$$

that is, after computing the squared term and simplifying, if

$$kTp(p-1)(2p-1) - 1 - 2(p-1) \geq (p-1)^2 e^{-kTp(p-1)}\,.$$

In turn, since $e^{-kTp(p-1)} < 1$, the latter is certainly satisfied if (2.4) holds. This proves the second item in Theorem 1.

For the third item, we first notice that (3.6) yields

$$
\begin{aligned}
J_T(\gamma) - J_T(p-1) &= \frac{p-\gamma}{1+(p-\gamma-1)e^{-\rho T}} - 1 + \frac{\gamma+1}{1+\gamma e^{-kT(\gamma^2+\gamma)}} - \frac{p}{1+(p-1)e^{-kTp(p-1)}} \\
&= \frac{(p-\gamma-1)(1-e^{-\rho T})}{1+(p-\gamma-1)e^{-\rho T}} + \frac{\gamma+1-p+(\gamma+1)(p-1)e^{-kTp(p-1)} - p\gamma e^{-kT(\gamma^2+\gamma)}}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]} \\
&= \frac{(p-\gamma-1)(1-e^{-\rho T})}{1+(p-\gamma-1)e^{-\rho T}} + \frac{(p-\gamma-1)[e^{-kTp(p-1)}-1] + p\gamma[e^{-kTp(p-1)} - e^{-kT(\gamma^2+\gamma)}]}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]} \\
&= (p-\gamma-1)\left[\frac{1-e^{-\rho T}}{1+(p-\gamma-1)e^{-\rho T}} - \frac{1-e^{-kTp(p-1)}}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\right] \\
&\quad - \frac{p\gamma(p-\gamma-1)}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\frac{e^{-kT(\gamma^2+\gamma)} - e^{-kTp(p-1)}}{p-\gamma-1} \qquad \forall \gamma \in [0, p-1)\,.
\end{aligned}
\tag{3.8}
$$

Therefore, $J_T(\gamma) > J_T(p-1)$ for all $\gamma \in [0, p-1)$ if and only if

$$
\begin{aligned}
&\frac{1-e^{-kTp(p-1)}}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]} + \frac{p\gamma[e^{-kT(\gamma^2+\gamma)} - e^{-kTp(p-1)}]}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\frac{1}{p-\gamma-1} \\
&\qquad < \frac{e^{\rho T}-1}{e^{\rho T}+p-\gamma-1} \qquad \forall \gamma \in [0, p-1)\,.
\end{aligned}
\tag{3.9}
$$

Let us now derive explicit (simpler) sufficient conditions for (3.9) to hold. To this end, we need a couple of calculus inequalities.

Lemma 4. *Assume* (2.1). *For all* $k, T > 0$ *one has*

$$
\frac{1-e^{-kTp(p-1)}}{1+(p-1)e^{-kTp(p-1)}} < \frac{p(p-1)}{2+\log(p-1)}\, kT\,. \tag{3.10}
$$

Proof. Putting $s = p(p-1)kT$, we see that (3.10) follows if we show that

$$
\frac{e^s-1}{e^s+p-1} < \frac{s}{2+\log(p-1)} \qquad \forall s > 0\,. \tag{3.11}
$$

To this end, we define $f(s) := \big(e^s+p-1\big)s - \big[2+\log(p-1)\big](e^s-1)$. Then we notice that $f'(s) = \big[s-1-\log(p-1)\big]e^s + p - 1$ and $f''(s) = \big[s - \log(p-1)\big]e^s$. Therefore, $s \mapsto f'(s)$ attains its absolute minimum at $s = \log(p-1)$ and $f'\big(\log(p-1)\big) = 0$. Hence, $s \mapsto f(s)$ is strictly increasing in $[0, \infty)$ and, since $f(0) = 0$, this finally means that $f(s) > 0$ for all s. This proves (3.11) and, consequently, (3.10). □

Lemma 5. *Assume* (2.1) *and* (2.5), *then the following inequality holds:*

$$
\frac{p\gamma[e^{-kT(\gamma^2+\gamma)} - e^{-kTp(p-1)}]}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\frac{1}{p-1-\gamma} < \frac{p(p-1)}{2+\log(p-1)}\, kT\,. \tag{3.12}
$$

Proof. We rewrite the fraction on the left hand side of (3.12) and we split it as

$$
\frac{\gamma\, e^{-kT(\gamma^2+\gamma)}}{1+\gamma e^{-kT(\gamma^2+\gamma)}}\,\frac{p}{1+(p-1)e^{-kTp(p-1)}}\,\frac{1-e^{-kT(p-1-\gamma)(p+\gamma)}}{p-1-\gamma}\,. \tag{3.13}
$$

Let us estimate the three terms in (3.13). For the first term, we observe that

$$\frac{x}{1+x} < \frac{p-1}{p} \qquad \text{if } 0 \le x < p-1\,.$$

Since $\gamma\, e^{-kT(\gamma^2+\gamma)} \le \gamma < p-1$, we then deduce

$$\frac{\gamma\, e^{-kT(\gamma^2+\gamma)}}{1+\gamma e^{-kT(\gamma^2+\gamma)}} < \frac{p-1}{p}\,. \tag{3.14}$$

For the second term in (3.13), we notice that (2.5) implies

$$kT < \frac{2+\log(p-1)}{2p(p-1)}$$

and, therefore,

$$p(p-1)\,kT < \frac{2+\log(p-1)}{2} \implies e^{kTp(p-1)} < e\sqrt{p-1} \implies (p-1)e^{-kTp(p-1)} > \frac{\sqrt{p-1}}{e}\,,$$

which implies that

$$\frac{p}{1+(p-1)e^{-kTp(p-1)}} < \frac{e\,p}{\sqrt{p-1}}\,. \tag{3.15}$$

For the third term in (3.13) we use the inequality $1-e^{-x} < x$, valid for all $x > 0$, and we find

$$\frac{1-e^{-kT(p-1-\gamma)(p+\gamma)}}{p-1-\gamma} = \frac{1-e^{-kT(p-1-\gamma)(p+\gamma)}}{kT(p-1-\gamma)(p+\gamma)}\,kT(p+\gamma) < kT(2p-1)\,, \tag{3.16}$$

since $\gamma < p-1$. By inserting (3.14)-(3.15)-(3.16) into (3.13) we obtain

$$\frac{p\gamma[e^{-kT(\gamma^2+\gamma)} - e^{-kTp(p-1)}]}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\,\frac{1}{p-1-\gamma} < kT\,e\,(2p-1)\sqrt{p-1}\,.$$

Then (3.12) follows by recalling (2.1). $\square$

Finally, let us go back to (3.8) and estimate

$$\begin{aligned}
\frac{J_T(\gamma)-J_T(p-1)}{p-\gamma-1} \quad &= \quad \frac{1-e^{-\rho T}}{1+(p-\gamma-1)e^{-\rho T}} - \frac{1-e^{-kTp(p-1)}}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]} \\
&\quad - \frac{p\gamma[e^{-kT(\gamma^2+\gamma)} - e^{-kTp(p-1)}]}{[1+\gamma e^{-kT(\gamma^2+\gamma)}][1+(p-1)e^{-kTp(p-1)}]}\,\frac{1}{p-\gamma-1} \\
\text{by Lemmas 4-5} \quad &> \quad \frac{1-e^{-\rho T}}{1+(p-1)e^{-\rho T}} - \frac{2p(p-1)}{2+\log(p-1)}\,kT\,.
\end{aligned}$$

Therefore, if (2.5) holds, then $J_T(\gamma) > J_T(p-1)$ for all $\gamma \in [0,p-1)$, which proves the third item in Theorem 1.

4 Proof of Proposition 2

Recalling that

$$1-\frac{y_1(t)}{p-\alpha(t)}\geq 1-y_1(t)\qquad\text{and}\qquad \alpha(t)+1-y_2(t)\geq 1-y_2(t)\,,$$

the inequalities $y_1(t), y_2(t)\geq 1$ for all $t\in[0,T]$ are straightforward.

Since $\alpha(t)\geq 0$, we have that

$$\dot y_1(t)=\rho\left(1-\frac{y_1(t)}{p-\alpha(t)}\right)y_1(t)\leq\rho\left(1-\frac{y_1(t)}{p}\right)y_1(t)\,.$$

Proceeding exactly as for (3.1) (but with the inequality $\leq$) we reach (1.3) with the same inequality, that is, (2.6).

Similarly, since $\alpha(t)\leq p-1$, we have that

$$\dot y_2(t)=k\alpha(t)\big(\alpha(t)+1-y_2(t)\big)y_2(t)\leq k(p-1)\big(p-y_2(t)\big)y_2(t)$$

at least in a right neighborhood of $t=0$; in fact, this inequality holds as long as $y_2(t)<p$ and we now show that this is always the case. Indeed, by proceeding as for (3.1) (but with the inequality $\leq$) we reach the inequality (2.7), which confirms that $y_2(t)<p$ for all $t\geq 0$.

For the second statement, note that if $\alpha(t)\equiv 0$ in $[0,T/2]$, then by (3.5) we have

$$y_1(t)=\frac{p}{1+(p-1)e^{-\rho t}}\quad\forall t\in\left[0,\tfrac{T}{2}\right]\implies y_1\left(\tfrac{T}{2}\right)=\frac{p}{1+(p-1)e^{-\rho T/2}}>1\,.$$

Furthermore, if $\alpha(t)\equiv p-1$ in $[T/2,T]$, then y_1 is governed by the Bernoulli equation $\dot y_1(t)=\rho\big(1-y_1(t)\big)y_1(t)$ and, therefore, $\dot y_1(t)<0$.

Finally, if $\alpha(t)\equiv p-1$ in $[0,T/2]$, then by (3.5) we have

$$y_2(t)=\frac{p}{1+(p-1)e^{-kp(p-1)t}}\quad\forall t\in\left[0,\tfrac{T}{2}\right]\implies y_2(T)=\frac{p}{1+(p-1)e^{-kp(p-1)T/2}}$$

$$\implies y_2(T)>\frac{(p-1)(1-e^{-kp(p-1)T/2})}{2+(p-1)e^{-kp(p-1)T/2}}+1\,.$$

Then the same argument as for y_1 shows that $\dot y_2(t)<0$ in $[T/2,T]$. This completes the proof of Proposition 2.

5 Proof of Theorem 3

We first prove a fundamental technical statement.

Lemma 6. *Let $p>1$, $A>0$, $0\leq B<(p-1)A$, and consider the function*

$$\phi(s)=As^2-Bs-\frac{1}{p-s}\quad\forall s\in[0,p)\,.$$

(i) If $B>(2p-1)A-2\sqrt{A}$, then $\phi(p-1)=\min_{s\in[0,p-1]}\phi(s)$.
(ii) If $B<(2p-1)A-2\sqrt{A}$, then there exists $\sigma\in(\frac{B}{2A},p-1)$ such that $\phi(\sigma)=\min_{s\in[0,p-1]}\phi(s)<\phi(p-1)$.

Proof. Although based on simple arguments, the proof is quite delicate and, for the sake of clarity, we prefer to start *without assuming* the inequality $B < (p-1)A$.

We first observe that

$$\begin{aligned}\phi(s) > \phi(p-1) \quad &\iff \quad As^2 - Bs - \frac{1}{p-s} > A(p-1)^2 - B(p-1) - 1 \\ &\iff \quad A(s-p+1)(s+p-1) - B(s-p+1) - \frac{s-p+1}{p-s} > 0 \,.\end{aligned}$$

Hence, after simplification by $(s-p+1) < 0$ and multiplication by $(s-p) < 0$, we have

$$\phi(s) > \phi(p-1) \quad \forall s \in [0, p-1) \iff A(s+p-1) - B - \frac{1}{p-s} < 0 \quad \forall s \in [0, p-1)$$

$$\iff \psi(s) := As^2 - (A+B)s + Ap + Bp + 1 - Ap^2 > 0 \quad \forall s \in [0, p-1)\,. \tag{5.1}$$

The second order polynomial $\psi(s)$ maintains positive sign over $\mathbb{R}$ provided that its discriminant is negative, which means $[B-(2p-1)A]^2 < 4A$ and, hence,

$$(2p-1)A - 2\sqrt{A} < B < (2p-1)A + 2\sqrt{A} \implies \psi(s) > 0 \quad \forall s \in [0, p-1)\,. \tag{5.2}$$

On the other hand, if the discriminant is positive, namely

$$\text{either} \quad B < (2p-1)A - 2\sqrt{A} \quad \text{or} \quad B > (2p-1)A + 2\sqrt{A}\,,$$

then a necessary and sufficient condition for (5.1) to hold is that the least zero of $\psi(s)$ is greater than or equal to $p-1$ (recall $A + B > 0$), that is,

$$\frac{A + B - \sqrt{[B-(2p-1)A]^2 - 4A}}{2A} \geq p - 1 \iff B - (2p-3)A \geq \sqrt{[B-(2p-1)A]^2 - 4A}\,.$$

Before squaring, we need to impose $B \geq (2p-3)A$; then the above condition is equivalent to

$$B \geq \max\big\{2(p-1)A - 1, (2p-3)A\big\}\,. \tag{5.3}$$

Note that the lower bound for B in (5.2) is tangent to the first lower bound in (5.3) at the point $(A, B) = (1, 2p-3)$ and, for $A \neq 1$, the corresponding graph of the former is above the line defined by the latter; see Figure 2, where the thick straight line has equation $B = 2(p-1)A - 1$ and represents the first equality case in (5.3), the dotted straight line has equation $B = (2p-3)A$ and represents the second equality case in (5.3), while the two thin curved lines represent the equality cases in (5.2).

Figure 2 clarifies the general behavior of ϕ for varying A and B, with no constraint. Let us now impose the constraint that $B < (p-1)A$. The region defined by this inequality should be intersected with the former shaded region, giving the shaded region in Figure 3, thereby simplifying the above analytic description.

Together with (5.1), this proves item (i) in the statement and also item (ii) but with $\sigma \in (0, p-1)$. Therefore, we still need to improve the lower bound for σ when $B > 0$. To this end, we go back to the function ϕ and we notice that

$$\phi'(s) = 2As - B - \frac{1}{(p-s)^2} \leq -\frac{1}{(p-s)^2} < 0 \qquad \forall s \in \left(0, \frac{B}{2A}\right]$$

so that the minimum of $\phi(s)$ over $[0, p-1]$ cannot be attained in the above interval. $\square$

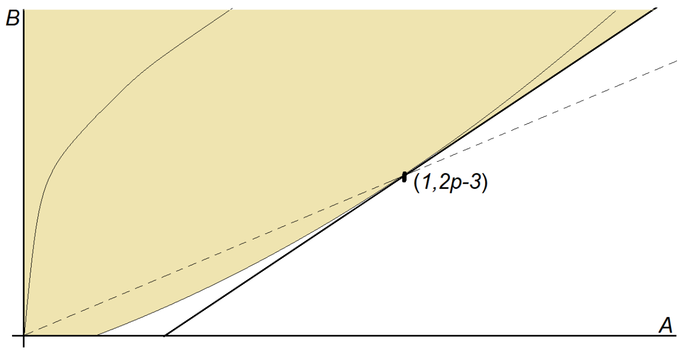

Figure 2: Shaded (resp. white) region representing item (i) (resp. (ii)), without assuming $B < (p-1)A$.

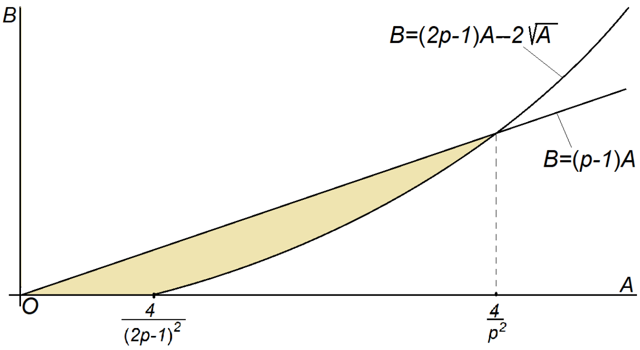

Figure 3: Shaded (resp. white) region representing item (i) (resp. (ii)), assuming $B < (p-1)A$.

A full comprehension of Lemma 6 will be possible only after introducing *dynamic* coefficients A and B, see Proposition 10 below. Here, we just emphasize that Lemma 6 has the following consequence and refinement.

Lemma 7. *Let $p > 1$, $0 < A < \frac{kpe^{\rho T}}{\rho}$, $0 \le B < (p-1)A$, and consider the function ϕ in Lemma 6.*
(i) If either $4/(2p-1)^2 < A < 4/p^2$ and $B > (2p-1)A - 2\sqrt{A}$ or $A \le 4/(2p-1)^2$, then $\phi(p-1) = \min_{s\in[0,p-1]} \phi(s)$.
(ii) If either $4/(2p-1)^2 < A < 4/p^2$ and $B < (2p-1)A - 2\sqrt{A}$ or $A \ge 4/p^2$, then there exists $\sigma \in (\frac{\rho e^{-\rho T}}{2kp^3}, \frac{3(p-1)}{4})$ such that $\phi(\sigma) = \min_{s\in[0,p-1]} \phi(s) < \phi(p-1)$.

Proof. (i) This is a restatement of item (i) in Lemma 6, see also Figure 3.

(ii) The three least derivatives of ϕ (as in Lemma 6) are

$$\phi'(s) = 2As - B - \frac{1}{(p-s)^2}\,, \qquad \phi''(s) = 2A - \frac{2}{(p-s)^3}\,, \qquad \phi'''(s) = -\frac{6}{(p-s)^4}\,. \tag{5.4}$$

A first consequence of (5.4) is that ϕ'' vanishes at most once, whatever A is. But since we are assuming here that $A > 1/p^3$ (recall (2.1)), we infer that ϕ'' vanishes exactly once. More precisely, $\phi''(s) = 0$ for $s = p - 1/\sqrt[3]{A}$.

Furthermore, under the assumptions of item (ii) of the present lemma, item (ii) in Lemma 6 ensures that there exists $\sigma \in (0, p-1)$ such that $\phi(\sigma) = \min_{s\in[0,p-1]} \phi(s) < \phi(p-1)$. Combined with the uniqueness of the flex point, this means that there exists a unique global minimum $\sigma \in (0, p-1)$ where $\phi'(\sigma) = 0$. In turn, this means that any $s \in (0, p-1)$ such that $\phi'(s) > 0$ gives an upper bound for σ. By taking a convex combination of the assumed inequalities, we obtain

$$B < \frac{p+1}{2p}\Big[(p-1)A\Big] + \frac{p-1}{2p}\Big[(2p-1)A - 2\sqrt{A}\Big] \implies \frac{3(p-1)}{2}\,A > B + \frac{p-1}{p}\,\sqrt{A}\,.$$

By combining this with (5.4), we find that

$$\phi'\left(\frac{3(p-1)}{4}\right) = \frac{3(p-1)}{2}\,A - B - \frac{16}{(p+3)^2} > \frac{p-1}{p}\,\sqrt{A} - \frac{16}{(p+3)^2} > 0\,,$$

where we also used both the assumptions $A > 4/(2p-1)^2$ and (2.1). Hence, $\sigma < 3(p-1)/4$.

Finally, we need to find a lower bound for σ. Since $A < kpe^{\rho T}/\rho$, we know that

$$\phi'(s) = 2As - B - \frac{1}{(p-s)^2} < \frac{2kpe^{\rho T}}{\rho}\,s - \frac{1}{(p-s)^2} < 0 \qquad \forall s \in \Big(0, \tfrac{\rho e^{-\rho T}}{2kp^3}\Big]\,, \qquad \phi'(0) < 0\,.$$

Hence, $\sigma > \frac{\rho e^{-\rho T}}{2kp^3}$. □

As we shall see below, the most difficult situation, possibly leading to discontinuous optimal controls, is the "intermediate range", that is, $4/(2p-1)^2 < A < 4/p^2$. In this case, we have

$$\phi'(s) < \frac{8}{p^2}\,s - \frac{1}{(p-s)^2} < 0 \quad \forall s \in \big(0, \tfrac{1}{8}\big] \implies \sigma > \tfrac{1}{8}\,.$$

With these technical results at hand, we may tackle the optimal control problem.

Lemma 8. *The minimization problem* (2.3) *admits a solution* $\alpha^* \in \mathcal{A}$. *Moreover, there exists a function* $\Phi \in C^0[0,T]$ *(depending on the corresponding solution* (y_1^*, y_2^*) *of* (1.4)*) such that*

$$\text{if } \Phi(t) > 0, \text{ then } \alpha^*(t) = p-1; \qquad \text{if } \Phi(t) < 0, \text{ then } \alpha^*(t) \in \left(\frac{y_2^*(t)-1}{2}, \frac{3(p-1)}{4}\right).$$

Proof. By the explicit form of the solutions in (3.2)-(3.3) and by the Alaoglu Theorem, we know that there exists an optimal control: from now on we denote by $\alpha^* \in \mathcal{A}$ the optimal control and by $y^* = (y_1^*, y_2^*)$ the related solution of (1.4). In order to determine α^* we use the PMP.

To this end, we switch to the vector notation and we define

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2\,, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{R}^2\,, \quad a \in [0, p-1]\,.$$

Then, for all $y_1, y_2 \in \mathbb{R}$ and $a \in [0, p-1]$ we put

$$f_1(y_1, a) = \rho\left(y_1 - \frac{y_1^2}{p-a}\right), \qquad f_2(y_2, a) = k\,a\Big((a+1)y_2 - y_2^2\Big)\,,$$

$$f(y,a) = f(y_1, y_2, a) = \begin{pmatrix} f_1(y_1, a) \\ f_2(y_2, a) \end{pmatrix} \in \mathbb{R}^2\,,$$

so that (1.4) can be written as

$$\dot{y}(t) = f\big(y(t), \alpha(t)\big) \qquad \text{with } \alpha \in \mathcal{A}\,.$$

Since the functional J_T, defined in (2.2), is a terminal payoff (with no running payoff), the Hamiltonian relative to (1.4) reads

$$H(y,\lambda,a) = f(y,a)\cdot\lambda = \rho\left(y_1 - \frac{y_1^2}{p-a}\right)\lambda_1 + k\,a\Big((a+1)y_2 - y_2^2\Big)\lambda_2 \quad \forall y,\lambda\in\mathbb{R}^2,\ a\in[0,p-1].$$

According to the PMP, there exists a Lipschitz-continuous costate $\lambda = \big(\lambda_1(t),\lambda_2(t)\big)\in\mathbb{R}^2$ (time-dependent Lagrange multiplier) that satisfies

$$\dot\lambda(t) = -\nabla_y H\big(y^*(t),\lambda(t),\alpha^*(t)\big) \quad \text{for a.e. } t\in[0,T]\,, \qquad \lambda_1(T)=\lambda_2(T)=1\,, \tag{5.5}$$

the final condition being a consequence of the explicit (terminal) payoff functional J_T in (2.2). We claim that the solution $\lambda=(\lambda_1,\lambda_2)$ of (5.5) satisfies

$$\lambda_1(t) = \frac{y_1^*(T)^2}{y_1^*(t)^2}\cdot e^{-\rho(T-t)}\,, \qquad \lambda_2(t) = \frac{y_2^*(T)}{y_2^*(t)}\cdot e^{-k\int_t^T \alpha^*(s)y_2^*(s)ds}\,, \qquad \forall t\in[0,T]. \tag{5.6}$$

Indeed, the system (5.5) may be split in the two independent linear scalar ODEs

$$\dot\lambda_1(t) = \rho\left(\frac{2y_1^*(t)}{p-\alpha^*(t)} - 1\right)\lambda_1(t)\,, \quad \dot\lambda_2(t) = k\alpha^*(t)\big(2y_2^*(t)-\alpha^*(t)-1\big)\lambda_2(t)$$

and, since $\lambda_1(T)=\lambda_2(T)=1$, we find

$$\lambda_1(t) = e^{-\rho\int_t^T\left(\frac{2y_1^*(s)}{p-\alpha^*(s)}-1\right)ds}\,, \qquad \lambda_2(t) = e^{-k\int_t^T \alpha^*(s)\left(2y_2^*(s)-\alpha^*(s)-1\right)ds}\,. \tag{5.7}$$

By using $(1.4)_1$, we see that

$$\frac{2y_1^*(s)}{p-\alpha^*(s)} - 1 = 2\left(\frac{y_1^*(s)}{p-\alpha^*(s)} - 1\right) + 1 = -\frac{2}{\rho}\,\frac{\dot y_1^*(s)}{y_1^*(s)} + 1$$

and, hence,

$$-\rho\int_t^T\left(\frac{2y_1^*(s)}{p-\alpha^*(s)} - 1\right)ds = \int_t^T\left(2\,\frac{\dot y_1^*(s)}{y_1^*(s)} - \rho\right)ds = 2\log\frac{y_1^*(T)}{y_1^*(t)} - \rho(T-t)\,.$$

By plugging this into $(5.7)_1$ we find $(5.6)_1$.

By using $(1.4)_2$, we see that

$$\alpha^*(s)\big(2y_2^*(s)-\alpha^*(s)-1\big) = \alpha^*(s)y_2^*(s) - \alpha^*(s)\big(\alpha^*(s)+1-y_2^*(s)\big) = \alpha^*(s)y_2^*(s) - \frac{1}{k}\,\frac{\dot y_2^*(s)}{y_2^*(s)}$$

and, hence,

$$-k\int_t^T \alpha^*(s)\big(2y_2^*(s)-\alpha^*(s)-1\big)ds = \log\frac{y_2^*(T)}{y_2^*(t)} - k\int_t^T \alpha^*(s)y_2^*(s)ds\,.$$

By plugging this into $(5.7)_2$ we find $(5.6)_2$.

The second consequence of the PMP states that

$$H\big(y^*(t),\alpha^*(t),\lambda(t)\big) = \min_{a\in[0,p-1]} H\big(y^*(t),a,\lambda(t)\big) \qquad \text{for a.e. } t\in[0,T]\,.$$

In our setting, this means that

$$\rho\left(y_1^*(t)-\frac{y_1^*(t)^2}{p-\alpha^*(t)}\right)\lambda_1(t)+k\alpha^*(t)\Big((\alpha^*(t)+1)y_2^*(t)-y_2^*(t)^2\Big)\lambda_2(t)$$

$$=\min_{a\in[0,p-1]}\left\{\rho\left(y_1^*(t)-\frac{y_1^*(t)^2}{p-a}\right)\lambda_1(t)+k\,a\Big((a+1)y_2^*(t)-y_2^*(t)^2\Big)\lambda_2(t)\right\}.$$

By dropping the α^*-independent term, this problem also reads

$$k\alpha^*(t)\Big((\alpha^*(t)+1)y_2^*(t)-y_2^*(t)^2\Big)\lambda_2(t)-\frac{\rho y_1^*(t)^2\lambda_1(t)}{p-\alpha^*(t)}$$

$$=\min_{a\in[0,p-1]}\left\{k\,a\Big((a+1)y_2^*(t)-y_2^*(t)^2\Big)\lambda_2(t)-\frac{\rho y_1^*(t)^2\lambda_1(t)}{p-a}\right\}.$$

Furthermore, after dividing by $\rho y_1^*(t)^2\lambda_1(t)>0$ (recall $(5.6)_1$) and introducing the time-dependent functions

$$A(t)=\frac{k\,y_2^*(t)\lambda_2(t)}{\rho\,y_1^*(t)^2\lambda_1(t)},\quad B(t)=\frac{k\left(y_2^*(t)^2-y_2^*(t)\right)\lambda_2(t)}{\rho\,y_1^*(t)^2\lambda_1(t)},\quad \phi_t(a)=A(t)a^2-B(t)a-\frac{1}{p-a},\tag{5.8}$$

the minimization problem can be further simplified to

$$\phi_t\big(\alpha^*(t)\big)=\min_{a\in[0,p-1]}\phi_t(a)\qquad\text{for a.e. } t\in[0,T].\tag{5.9}$$

We then apply Lemma 7 to the minimization problem (5.9) and we infer that, for a.e. $t\in[0,T]$,

$$\begin{cases}\text{if } B(t)>(2p-1)A(t)-2\sqrt{A(t)}, & \text{then } \alpha^*(t)=p-1;\\ \text{if } B(t)<(2p-1)A(t)-2\sqrt{A(t)}, & \text{then } \alpha^*(t)\in\left(\frac{B(t)}{2A(t)},\frac{3(p-1)}{4}\right).\end{cases}\tag{5.10}$$

Since $A,B\in C^0[0,T]$, by taking

$$\Phi(t):=B(t)-(2p-1)A(t)+2\sqrt{A(t)}\tag{5.11}$$

we obtain the characterization of α^*, as in the statement. $\square$

By combining (5.6) with (5.8) we obtain

$$A(t)=\tfrac{k}{\rho}\tfrac{y_2^*(T)}{y_1^*(T)^2}\,e^{\rho(T-t)-k\int_t^T\alpha^*(s)y_2^*(s)ds},\quad B(t)=\tfrac{k}{\rho}\tfrac{y_2^*(T)}{y_1^*(T)^2}\left(y_2^*(t)-1\right)e^{\rho(T-t)-k\int_t^T\alpha^*(s)y_2^*(s)ds},\tag{5.12}$$

so that, by Proposition 2,

$$B(t)\le\frac{(p-1)(1-e^{-kTp(p-1)})}{1+(p-1)e^{-kTp(p-1)}}A(t)<(p-1)A(t),\tag{5.13}$$

which justifies the assumption in Lemma 6. These formulas play a crucial role in the proof of the quantitative properties related to (2.10)-(2.11). We use them to prove the sufficient condition and the necessary condition for the optimal control to be the total lockdown, as stated in Theorem 3.

Lemma 9. *If* (2.10) *holds, then* $\alpha^*(t)\equiv p-1$ *in* $[0,T]$. *If* (2.11) *holds, then* $\alpha^*(t)\not\equiv p-1$ *in* $[0,T]$.

Proof. Assume (2.10). By recalling the initial conditions $y_1^*(0) = y_2^*(0) = 1$ in (1.4), and using the expressions (5.6), we get

$$\lambda_1(0) = y_1^*(T)^2 \cdot e^{-\rho T}, \qquad \lambda_2(0) = y_2^*(T) \cdot e^{-k\int_0^T \alpha^*(s)y_2^*(s)ds} \le y_2^*(T), \qquad \forall t \in [0,T].$$

Note that, whatever α^* is, by Proposition 2 we have $1 \le y_1^*(t), y_2^*(t) < p$ for all $t \ge 0$; hence, from (2.10) and (5.8) we obtain

$$A(0) = \frac{k\,\lambda_2(0)}{\rho\,\lambda_1(0)} \le \frac{k\,y_2^*(T)}{\rho\,y_1^*(T)^2}\,e^{\rho T} < \frac{k\,p}{\rho}\,e^{\rho T} \le \frac{4}{(2p-1)^2}\,. \tag{5.14}$$

Note that (5.12), combined with the facts that $\alpha^*(t) \le p-1$ and $y_2^*(t) < p$ for all t, implies that

$$\dot{A}(t) = \frac{k\,y_2^*(T)}{\rho\,y_1^*(T)^2} \cdot e^{\rho T} \cdot \big(k\alpha^*(t)y_2^*(t) - \rho\big) \cdot e^{-\rho t - k\int_t^T \alpha^*(s)y_2^*(s)ds} < 0,$$

where we used the fact that (2.10) yields

$$k \le \frac{4}{p(2p-1)^2}\,\rho e^{-\rho T} < \frac{\rho}{p(p-1)}\,.$$

Therefore $t \mapsto A(t)$ is strictly decreasing and then, by (5.14),

$$A(t) < A(0) < \frac{4}{(2p-1)^2} \qquad \forall t > 0\,.$$

Therefore, $(2p-1)A(t) - 2\sqrt{A(t)} < 0 \le B(t)$ for all $t \ge 0$, In terms of the function Φ in (5.11), this means that $\Phi(t) > 0$ for all $t \ge 0$. Then we conclude that $\alpha^*(t) \equiv p-1$ in $[0,T]$ by using Lemma 8.

Assume (2.11) and, for contradiction, assume also that $\alpha^*(t) \equiv p-1$ in $[0,T]$. Then (3.5) yields

$$y_1^*(T) = 1\,, \qquad y_2^*(T) = \frac{p}{1 + (p-1)e^{-kTp(p-1)}}\,,$$

which, inserted into (5.12), gives

$$A(T) = \frac{k}{\rho}\,\frac{p}{1 + (p-1)e^{-kTp(p-1)}} > \frac{4}{p^2}$$

since (2.11) holds. By continuity of the function $A(t)$, we infer that $A(t) > \frac{4}{p^2}$ in a right neighborhood of $t = T$. By Lemma 8, we reach a contradiction: we have $\alpha^*(t) < \frac{3(p-1)}{4}$ in the same neighborhood. □

Putting together all the above lemmas completes the proof of Theorem 3.

6 Further results and comments

By going through the proof of Theorem 3, in particular Lemma 7, and after a careful look at Figure 3, we derive the following "dynamic" statement.

Proposition 10. *Let A and B be as in* (5.8)*. If there exists $\bar{t} \in (0,T)$ such that*

$$t \mapsto B(t) - (2p-1)A(t) - 2\sqrt{A(t)} \text{ vanishes and changes sign at } \bar{t}$$

then α^* is discontinuous at $\bar{t}$.

Proof. We give here a mostly geometric proof which, in our opinion, is much more illustrative than a fully rigorous algebraic proof. Moreover, the algebraic proof can be derived from the following geometric arguments.

Let us maintain $A(t) < 4/p^2$ fixed and discuss the graph of ϕ_t as $B(t)$ varies, see (5.8). Since

$$\phi_t'(p-1) = 2A(t)(p-1) - B(t) - 1 < \frac{8(p-1)}{p^2} - 1 < 0 \qquad \forall p \geq 100\,,$$

the possible (qualitative) graphs are displayed in Figure 4 when $B(t) \lesseqgtr (2p-1)A(t) - 2\sqrt{A(t)}$. If $B(t) > (2p-1)A(t) - 2\sqrt{A(t)}$ (left picture) we have $\phi_t(p-1) = \min_{s\in[0,p-1]} \phi_t(s)$, as stated in

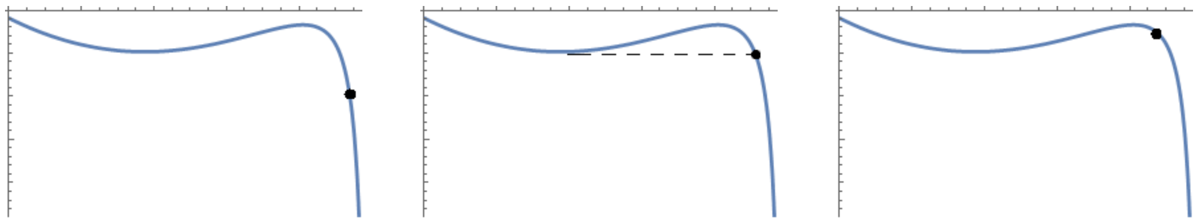

Figure 4: Graph of ϕ_t when $B(t) >$ (left), $=$ (center), $<$ (right) than $(2p-1)A(t) - 2\sqrt{A(t)}$.

Lemma 6: we are here in the situation of negative discriminant for the second order polynomial ψ in (5.2). Then we let $B(t)$ decrease (moving downwards in Figure 2) and we reach the curve $B(t) = (2p-1)A(t) - 2\sqrt{A(t)}$; this corresponds to the middle graph in Figure 4 in which there are two minimum points for ϕ_t in the interval $[0, p-1]$, one in the interior and another one at $p-1$. If $B(t)$ decreases further and enters the region where $B(t) < (2p-1)A(t) - 2\sqrt{A(t)}$ (white region in Figures 2 and 3), the graph of ϕ_t is as on the right in Figure 4: here the minimum of ϕ_t is only in the interior of $[0, p-1]$ creating thereby a discontinuity in the control α^*. This happens every time that the line $B(t) = (2p-1)A(t) - 2\sqrt{A(t)}$ is crossed at some point (A, B) with $4/(2p-1)^2 < A < 4/p^2$, see Figure 3. $\square$

Let us now compare the sufficient conditions in Theorems 1 and 3 for the optimal control to be a total lockdown.

Proposition 11. *Assume* (2.1), $T > 0$. *Then the condition* (2.10) *is more restrictive than* (2.5).

Proof. The statement amounts to prove that

$$\rho T\, e^{-\rho T} \frac{1 + (p-1)e^{-\rho T}}{1 - e^{-\rho T}} < \frac{2 + \log(p-1)}{8(p-1)} (2p-1)^2\,. \tag{6.1}$$

To this end, we first claim that

$$\rho T\, e^{-\rho T} \frac{1 + (p-1)e^{-\rho T}}{1 - e^{-\rho T}} < p \qquad \forall \rho, T > 0\,. \tag{6.2}$$

Put $s = \rho T$ and consider the function $h \in C^0[0, \infty)$ defined by

$$h(s) = \frac{s}{e^s - 1}\left[1 + (p-1)e^{-s}\right] \quad \text{for } s > 0\,, \qquad h(0) = p\,.$$

By differentiating, we find that

$$h'(s) = \frac{(p-1)(s-1)e^{-s} + p - 2 - 2(p-1)s + (1-s)e^s}{(e^s-1)^2} =: \frac{g(s)e^{-s}}{(e^s-1)^2}$$
$$\implies g(s) = (p-1)(s-1) + (p-2)e^s - 2(p-1)se^s + (1-s)e^{2s}\,.$$

Then we compute

$$g'(s) = (1-2s)e^{2s} + (2s-2ps-p)e^s + p - 1\,, \quad g''(s) = (2-3p)e^s - 2(p-1)se^s - 4se^{2s},$$

which shows that $g'(0) = 0$ and $g''(s) < 0$ for all s. This implies that $g'(s) < 0$ for all s and, since $g(0) = 0$, also that $g(s) < 0$ for all s. Back to the function h, this shows that $h'(s) < 0$ for all $s > 0$, so that $h(s) < p$ for all $s > 0$. This proves (6.2).

By (6.2), the inequality (6.1) certainly holds if

$$\frac{2p(p-1)}{2+\log(p-1)} \le \frac{(2p-1)^2}{4} = p(p-1) + \frac{1}{4}\,.$$

Since the left hand side is smaller than $p(p-1)$, this inequality holds. This completes the proof. $\square$

The second part of this section consists in some remarks, used throughout the paper.

Remark 12. *Lemmas 6-7-8 tell us that if $\alpha^*(t) \neq p-1$, then $s = \alpha^*(t)$ is a minimum point for the function ϕ_t in (5.8), see (5.9). Therefore, $\phi_t'\big(\alpha^*(t)\big) = 0$ which, combined with (5.4) and the explicit expressions in (5.12), shows that*

$$\frac{k}{\rho}\frac{y_2^*(T)}{y_1^*(T)^2}\, e^{\rho(T-t)-k\int_t^T \alpha^*(s)y_2^*(s)ds}\,\big(2\alpha^*(t)+1-y_2^*(t)\big)\big(p-\alpha^*(t)\big)^2 = 1 \qquad \text{for a.e. } t \in [0,T]\,.$$

Quite surprisingly, this "explicit" characterization of α^* depends on y_1^* only through its final value while it depends much more directly on y_2^*. Unfortunately, this formula is not usable, precisely because it requires the knowledge of the final state (at time $t = T$). On the other hand, from $(1.4)_1$ we infer that

$$\alpha^*(t) = p - \frac{\rho\, y_1^*(t)^2}{\rho y_1^*(t) - \dot{y}_1^*(t)}\,.$$

This different characterization of α^* merely depends on y_1^* but, again, it is not usable because also its derivative $\dot{y}_1^*$ is involved.

Remark 13. *In connection with conjecture (2.9), notice that the PMP also implies that the Hamiltonian is constant, that is,*

$$\rho\left(y_1^*(t) - \frac{y_1^*(t)^2}{p-\alpha^*(t)}\right)\lambda_1(t) + k\alpha^*(t)\Big((\alpha^*(t)+1)y_2^*(t) - y_2^*(t)^2\Big)\lambda_2(t) \equiv \overline{H}$$

for some $\overline{H} \in \mathbb{R}$. Since at $t = 0$ we have

$$\rho\left(1 - \frac{1}{p-\alpha^*(0)}\right)\lambda_1(0) + k\alpha^*(0)^2\lambda_2(0) > 0$$

whatever $\alpha^*(0) \in [0, p-1]$ is, this proves that $\overline{H} > 0$ and, hence, that at least one between $1 - \frac{y_1^*(t)}{p-\alpha^*(t)}$ and $\alpha^*(t) + 1 - y_2^*(t)$ is strictly positive for any $t \ge 0$. In turn, by (1.4), at least one between $\dot{y}_1^*(t)$ and $\dot{y}_2^*(t)$ is strictly positive. This fact is not as strong as (2.9) but, at least, it gives a hint in its direction.

Remark 14. *We have that*

$$\forall\xi > p \quad \text{the map} \quad t \mapsto \big(y_2^*(t)-\xi\big)e^{-k\int_t^T \alpha^*(s)y_2^*(s)ds} \quad \text{is negative and strictly decreasing.}$$

Indeed, negativity follows directly from Proposition 2, see (2.7)*. For the monotonicity, we differentiate and, by using* $(1.4)_2$*, we get*

$$\frac{d}{dt}\Big(\big(y_2^*(t)-\xi\big)e^{-k\int_t^T \alpha^*(s)y_2^*(s)ds}\Big) = k\alpha^*(t)y_2^*(t)\big(\alpha^*(t)+1-\xi\big)e^{-k\int_t^T \alpha^*(s)y_2^*(s)ds} < 0\,.$$

We conclude this section (and the paper) with some open problems and by mentioning some possible future developments of the model developed here. First, we believe that the optimal control $\alpha^*(t)$ never suggests a mild lockdown at some t and a total lockdown at some later t: is it possible to prove that α^* is non-increasing? Second, is it possible to improve the property in Remark 13 and to prove conjecture (2.9), at least for the optimal state? This means that $\dot y_1^*(t) + \dot y_2^*(t) > 0$ for all t. Finally, instead of the payoff functional (2.2) one could consider a convex combination of y_1 and y_2, that is $J_T(\alpha) = \beta y_1(T) + (1-\beta)y_2(T)$ for some $\beta \in (0,1)$: is there a (sufficiently small) value of β yielding a null lockdown ($\alpha^* \equiv 0$) as optimal control?

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