EIGHT(Y) MATHEMATICAL QUESTIONS ON FLUIDS AND STRUCTURES

DENIS BONHEURE, FILIPPO GAZZOLA, AND GIANMARCO SPERONE

Abstract. Turbulence is a long-standing mystery. We survey some of the existing (and sometimes contradictory) results and suggest eight natural questions whose answers would increase the mathematical understanding of this phenomenon; each of these questions, yet, gives rise to ten sub-questions.

Dedicated to Vladimir Maz'ya in occasion of his eightieth birthday, with great esteem, deep admiration, and eight winks to his musical moments [208, Sect. 4.8].

Contents

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1. Prelude

Nessuna certezza delle scienze è dove non si pò applicare una delle scienze matematiche, ovver che non sono unite con esse matematiche.

Leonardo da Vinci (1452-1519)

In this paper we analyse from a mathematical point of view some problems about fluids and structures arising from physics and engineering, with an emphasis on the understanding of the aeroelasticity of suspension bridges. As we shall see, mathematicians and engineers are quite unsatisfied of the existing models, theories and explanations. Many doubts and natural questions are still waiting for adequate theoretical answers, many phenomena that can be observed cannot be fully explained and rigorously modelled. In the words of Sir Cyril Hinshelwood (1956 Nobel Prize in Chemistry),

> *fluid mechanics was discredited by engineers from the start, which resulted in an unfortunate split - between the field of hydraulics, observing phenomena which could not be explained, and theoretical fluid mechanics explaining phenomena which could not be observed.*

About mathematics and reality, Albert Einstein (1921 Nobel Prize in Physics), in his *Geometry and Experience* talk at the Prussian Academy of Sciences in Berlin on January 27, 1921 said

> *One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of all other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts. [...] How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? [...] In my opinion the answer to this question is, briefly, this: As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

Our trio is organised as follows. In Section 2, we briefly overview some general historical facts without intending to give a comprehensive panorama. In Section 3, starting from the d'Alembert paradox, we look at the drag and lift forces acting on a body immersed in a fluid computed through Euler, Stokes and Navier-Stokes equations. In Section 4, we describe the various flow behaviours arising past a cylinder and we discuss some theoretical results when the obstacle is either fixed or moving (with a prescribed movement). Section 5 is devoted to the description of several unexplained aeroelastic phenomena observed in suspension bridges. In Section 6, we emphasise how much work is still needed for a full understanding of fluid-structure interactions. Finally, Section 7 deals with the role of the shape of the obstacle in the analysis of turbulence. We also provide a rich (yet incomplete) interdisciplinary bibliography.

2. Blowing in the wind

Strong blowing winds, especially when they hit an obstacle, generate air turbulence with subsequent appearance of vortices behind the obstacle. The first documented and surviving realization of vortices is usually attributed to some sketches by Leonardo da Vinci, see Figure 1. Nowadays, wind tunnel experiments artificially blow air flows and give precise pictures of turbulence and of the dependence of the vortex shedding on the parameters of the flow [87, 133, 233, 234], see e.g. the left picture in Figure 2. Vortex shedding is the cause of so-called vortex-induced vibrations [6, 82, 83, 305–307], namely oscillatory motions of the obstacle. Thanks to the huge progresses of the numerical analysis of fluid flows and the increasing computer capacities, turbulence may also be detected by refined numerics using Computational Fluid Dynamics (CFD), see e.g. [88, 96, 99, 105, 240]. However, the current knowledge of turbulence is still foggy with frequent updates. We refer to [103] for a general introduction and to [16] for

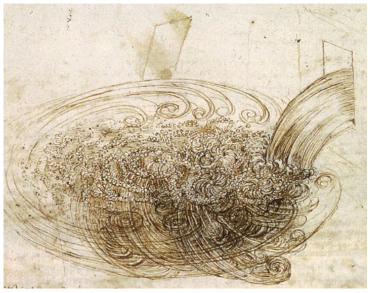


FIGURE 1. Drawing of water vortex by Leonardo Da Vinci, ca. 1510-1513.

> *the most recent advances in attacking these questions* [the fundamental questions in turbulence] *using rigorous mathematical tools.*

Helmholtz [135] published the foundation of the theory 160 years ago, followed by Stokes (1845), Strouhal (1878), Prandtl (1904), Bénard (1908), von Kármán (1912) and, nowadays, according to [234, Section 1.1],

> *it is not only that the accumulated knowledge is vast, but also that the accretion of knowledge and experience on the topic continues to grow unabated, perhaps exponentially.*

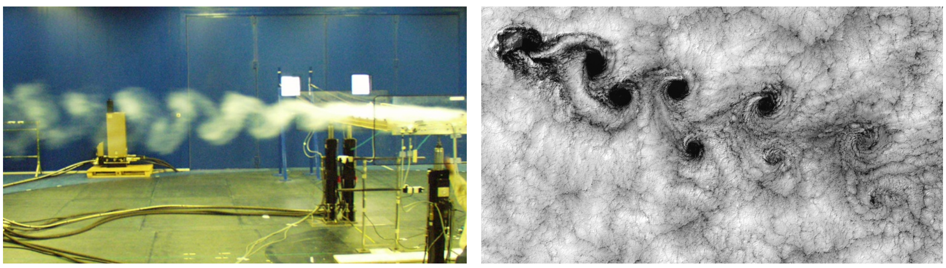


FIGURE 2. Left: vortices around the deck of a scaled bridge obtained experimentally in the wind tunnel of the Politecnico di Milano. Right: clouds off the Chilean coast showing Kármán vortex streets (Landsat 7 image-NASA).

The vortex formation within a flow surrounding an object is the basic observation that laid the foundations of aerodynamics. Complicated phenomena were quickly observed, and the important parameters were identified. A general understanding of viscosity effects began to emerge during the mid-nineteenth century, particularly in the works of Stokes [275, 276], followed later by Prandtl [241] who introduced his *boundary layer theory*. Prandtl claims that the no-slip condition holds even for very small viscosity, but its influence is confined to a small region along the body, the so-called boundary layer. Within this layer the velocity of the fluid rapidly changes from zero on the surface of the body to the free-stream velocity of the flow. In presence of high curvature of the obstacle surface, the flow can be interrupted entirely and the boundary layer may detach from the surface: this phenomenon is called *separation*.

The separation process depends on viscosity and stream velocity whose important influence is collected in the Reynolds number Re that expresses the ratio between inertial forces and viscous forces of the flow. In the year 1883, Reynolds [247] investigated which factors determine whether the motion of water in a pipe is direct or sinuous, thereby introducing the dimensionless parameter

$$\mathrm{Re} = \frac{\rho u L}{\mu} = \frac{uL}{\nu},$$

where ρ is the density of the fluid, u is its velocity, μ is its dynamic viscosity, ν is the kinematic viscosity and L is the diameter of the pipe. Reynolds was interested in the transition from laminar to turbulent regime: a flow is called *laminar* or *streamlined* if it follows parallel layers, with no disruption between the layers, whereas it is called *turbulent* if it undergoes irregular fluctuations or mixing, see Figure 3. In a turbulent flow, the speed of the fluid is widely changing both in

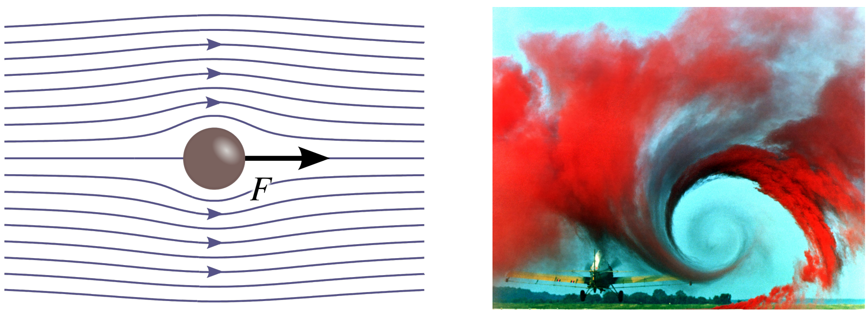


FIGURE 3. Left: laminar flow around a bluff body. Right: turbulent flow from an airplane wing (NASA-Photo ID: EL-1996-00130).

magnitude and direction. Experiments and numerics show that for $\mathrm{Re} \ll 1$, the flow is laminar. For a Reynolds number in the range between 1 and 100, the flow exhibits a complicated (chaotic) structure, while for $\mathrm{Re} \gg 100$, the flow is turbulent, displaying a complex pattern formed by the velocity field. Quoting [93]:

> *While much of the hemodynamics in a healthy human body has low Reynolds number, resulting in laminar flow, relatively high Reynolds number flow is observed at some specific locations [...] For instance, the peak Reynolds number in the human aorta has been measured to be approximately* 4000 [172].

Besides a blood flow in arteries, other turbulent flows include most natural rivers which have Reynolds numbers well above 2000, lava flow, atmosphere and ocean currents, wind-turbines wake, boat and building wakes or aircraft-wing tips. The Reynolds number for the air surrounding an aircraft during flight varies from about 2×10^6 for small slow-speed airplanes to 2×10^7 for large high-speed airplanes.

According to Batchelor [18, Section 5.11], in practice, the most significant feature of a flow past a fixed body (fully immersed in a steady stream that is constant at infinity), is the force exerted on the body by the fluid, which is usually decomposed into two components: the drag force F_D parallel to the flow direction and the lift force F_L perpendicular to the flow. In practice, these forces are computed through the formulas

$$F_D = \frac{C_D}{2}\,\rho A_f W^2\,, \qquad F_L = \frac{C_L}{2}\,\rho A_p W^2\,, \tag{1}$$

where ρ is the fluid density, W is the upstream velocity, A_f is the frontal area (the projected area seen by an observer looking towards the object from a direction parallel to the upstream velocity), and A_p is the planform area (the projected area seen by an observer looking towards the object from a direction normal to the upstream velocity). In (1), C_D and C_L denote, respectively, the drag and lift coefficients, giving dimensionless forms of the drag and lift forces. They are usually determined by help of a simplified analysis, some numerical procedures or empirical rules based on (e.g. wind tunnel) experiments. We refer to [151, Chapter 9] for more details and to [5, 133] in the particular case of suspension bridges. The lift force is intimately related to the vortex shedding process: when asymmetric vortices appear behind the bluff body, the asymmetry generates a forcing lift which starts the vortex-induced vibrations. The vortex shedding in the wake of a structure may also achieve one of its natural frequencies, resulting in a vortex-induced resonance, with subsequent vibrations of the structure. A large variety of models were used to phenomenologically study vortex shedding and vortex-induced vibrations but a unified theory seems lacking: from [234] we quote

> *literature on vortex-induced vibrations is vast and continuously growing, both on fundamental issues and on methods for their prediction in engineering, where applications are numerous. [...] In fact, because of the practical and theoretical importance of vortex-induced vibrations, models have been developed and used since the 1960s. Reviews show not only a large number of them, but also significant differences in the fundamental aspects of their formulations.*

For instance, the aerodynamic forces acting on the deck of a suspension bridge vary with respect to many parameters. It is therefore important to study the aerodynamic derivatives which measure how those forces and moments change as other parameters (such as airspeed, angle of attack, etc.) related to stability are perturbed. The aerodynamic derivatives have been so far determined experimentally, and given the complexity of the vortex shedding phenomena and vortex-induced vibrations, one needs a huge amount of experimental data before attempting a theoretical analysis. Still concerning suspension bridges, we quote [234]:

some recent effort has gone into obtaining the aerodynamic derivatives using numerical methods. For example, Larsen [182] *uses a discrete vortex method to obtain the aerodynamic derivatives for two different cross-sections. A comparison between his results and the experimental data of Scanlan-Tomko* [259] *shows the numerical data to be reasonably good, but probably not good enough to obtain accurate stability predictions.*

The study of vortex shedding is intimately related to vortex dynamics for which a huge literature is available from the physical, engineering and mathematical communities, see for instance [3, 17, 98, 169, 192, 193, 195, 202, 224, 242, 243, 251, 254, 260] and the numerous citations therein. Vortices appear in a great variety of Ginzburg-Landau theories, models in fluid-mechanics, superconductivity and superfluidity [4, 30, 97, 166, 167, 232, 255, 265, 282, 303].

The unforced incompressible Euler equations

$$u_t + (u \cdot \nabla)u + \nabla p = 0\,, \qquad \nabla \cdot u = 0 \qquad (x, y, z) \in \Omega\,, \quad t > 0 \tag{2}$$

play a central role in theoretical fluid mechanics and even in mathematical physics, not only because they model adiabatic and inviscid flows, but also because they can be seen, in some particular situations, as the inviscid limit of the Navier-Stokes system [195, 200] or as the limit of other model equations in some asymptotical regime, see for instance [47, 266]. Nevertheless, if one wishes to model turbulence, there are several reasons not to consider (2). One is that vortices do not only appear in high Reynolds regimes (e.g. for small viscosity), for which (2) would be a good approximation; indeed, vortices can also be generated at low Reynolds, for instance by singularities in the domain and, in particular, by possible obstacles in the flow. Another one is the celebrated d'Alembert paradox [71–74], see next section, which shows that the Euler equations (2) are not appropriate to directly describe the lift and drag exerted from fluids on bluff bodies.

3. I believe I can fly

Why do airplanes fly? On the authority NASA website [219] one may read:

There are many explanations for the generation of lift found in encyclopedias, in basic physics textbooks, and on Web sites. Unfortunately, many of the explanations are misleading and incorrect. Theories on the generation of lift have become a source of great controversy and a topic for heated arguments

> *for many years. [...] To truly understand the details of the generation of lift, one has to have a good working knowledge of the Euler equations.*

The conclusion is a quite strong mathematical statement. So, let us start modelling an incompressible non-viscous fluid in $\mathbb{R}^3 \setminus B$, where B is a solid ball, with the Euler equations (2). We suppose that the stream velocity is constant at infinity, i.e. there exists $u_\infty \in \mathbb{R}^3$ such that $u(x) \to u_\infty$ as $|x| \to \infty$. Denoting by R the radius of the ball and assuming that it is centered at the origin, it is readily seen that the potential

$$U(x,y,z) = \left(1 + \frac{R^3}{2(x^2+y^2+z^2)^{3/2}}\right)x \tag{3}$$

yields a steady state solution $u = \nabla U$ of (2) in $\mathbb{R}^3 \setminus B$ with constant velocity $u_\infty = (1,0,0)$ at infinity and such that u is tangent to ∂B, by which we mean that $u \cdot n = 0$ on ∂B. Due to the symmetry of the field $u = \nabla U$, one easily checks that the flow pressure on the boundary of the ball is zero, i.e.

$$\int_{\partial B} pn\, d\sigma = -\frac{1}{2}\int_{\partial B} |u|^2 n\, d\sigma = 0,$$

which means that the fluid neither produces a drag, nor a lift. This obviously contradicts everyday experience. Moreover, this theoretical paradox is not a consequence of the symmetry of the obstacle B (that induces the symmetry of u). Indeed, in the 18th century, d'Alembert [71–74] proved a surprising result about stationary solutions of the Euler equations:

> *Après avoir ainsi développé mes principes, j'examine une hypothèse dont plusieurs auteurs d'hydrodynamique se sont servis jusqu'ici, & je fais voir que si on suivait une telle hypothèse pour déterminer la résistance d'un fluide, cette résistance se trouverait nulle, ce qui est contraire à toutes les expériences.*

This result, nowadays known as the *d'Alembert paradox* has been and still is a source of debate. In modern terminology, the d'Alembert paradox may be stated as follows.

Theorem 1 ([71–74])**.** *Let $D \subset \mathbb{R}^3$ be a compact smooth set and let n be the inward unit normal vector to ∂D. Let $u = u(x)$ $(x \in \Omega = \mathbb{R}^3 \setminus D)$ be a smooth field over the closure of Ω, divergent-free, tangent to ∂D, and constant at infinity. If u is irrotational, then u is a stationary solution of* (2) *in Ω and the fluid force on the obstacle is zero, that is,*

$$F = \int_{\partial D} pn\, d\sigma = 0\,.$$

The proof of Theorem 1 is based on classical tools from potential theory and on the Divergence Theorem, see e.g. [126, Theorem 2.1], [201, Theorem 4.3] or [277, Section 8.2]. Some comments about the irrotational assumption on the flow are in order. A physical justification of this assumption is based on the fact that, at very large distances from the obstacle, the flow may be seen as uniform ($u \equiv$

constant) so that it is indeed irrotational. But whether this condition remains true all over $\mathbb{R}^3 \setminus D$ is a delicate matter. In fact, by the vorticity-transport formula [195, Proposition 1.8, p.20], the behavior of the vorticity at infinity is transported in all the domain, provided the particle trajectories are smooth and invertible, which would justify the assumption of irrotational flows in Theorem 1, see also [117]. Even though this was already a concern of Birkhoff [32] (see below), it is still an open problem whether (2) admits steady rotational solutions.

Among other things, a consequence of Theorem 1 is that birds and airplanes cannot fly in an ideal incompressible fluid: probably, they fly just because... they believe they can, see Figure 4 which is displayed in order to celebrate the artistic contribution of V.G. Maz'ya in his amazing *Tales for children and grown-ups* [209]; the left picture is taken from [210].

FIGURE 4. They believe they can fly! Left picture: cover of V.G. Maz'ya's book [210]. Right picture: Dumbo (1941) by Disney Enterprises, Inc.

We refer to [131, 197, 273, 284] and the numerous references therein for further discussions on the paradox. As shown by Theorem 1, although the Euler equations (2) provide a good model of reality for many problems of fluid dynamics, they cannot directly account for the lift force. Since only a viscous fluid satisfies the no-slip condition of its particles on the surface of the body immersed in the flow, it is nowadays commonly accepted that viscosity is needed to generate a lift, as first suggested by Saint-Venant [253]. However, any rigorous physical justification or mathematical proof remains far out of reach [126, 277]. Birkhoff [32, p.21] conjectured the drag could be the result of an instability of potential flows:

> *the paradoxes of ideal fluid theory may be, in part, paradoxes of topological oversimplification* [by which he meant that there is no valid mathematical reason to consider potential flows only]. *[...] Though Dirichlet flows and other steady flows are mathematically possible, there is no reason to suppose that any steady flow is stable. It is perfectly conceivable that, in an "ideal"*

> *fluid, initially departing slightly from Dirichlet flow, irregularly varying turbulent "eddies" are built up mathematically in the "wake" of an obstacle-reproducing mathematically what is observed physically at large Reynolds' numbers R. [...] To admit this possibility, we must reject the idea that there is a necessary tendency towards symmetry in natural phenomena, and admit the possibility that a symmetrically stated problem may not have any stable symmetric solution.*

Birkhoff was violently criticised by Stoker [274], especially for invoking instability, and he did not insist more on this idea. Even more, in the second version of Birkhoff's book [33], these thoughts disappeared. More recently, Hoffman and Johnson [143] reconsidered Birkhoff's attempt to explain the paradox. Part of the conclusion in [143] says:

> *We have presented a resolution of d'Alembert's paradox based on analytical and computational evidence that a potential solution with zero drag is ill-posed as a solution of the Euler equations, and under perturbations develops into a wellposed turbulent solution with substantial drag in accordance with observations.*

In a followup paper based on this explanation, Hoffman-Jansson-Johnson [141] presented a new mathematical theory of flight, see also [142], which is fundamentally different from the theory by Prandtl-Kutta-Zhukovsky [35, 298, 300]. Quoting the authors:

> *The new theory shows that the miracle of flight is made possible by the combined effects of (i) incompressibility, (ii) slip boundary condition and (iii) 3d rotational slip separation, creating a flow around a wing which can be described as (iv) potential flow modified by 3d rotational separation. The basic novelty of the theory is expressed in (iii) as a fundamental 3d flow phenomenon only recently discovered by advanced computation and analysed mathematically, and thus is not present in the classical theory. Finally, (iv) can be viewed as a realization in our computer age of Euler's original dream to in his equations capture an unified theory of fluid flow.*

The paper curiously starts with an *Editorial Foreword* which states:

> *The special character of this article requires some comments by the editors on the purpose of its publication. Though, its mathematical content does not meet the degree of mathematical rigor usually expected by articles in this journal, the implications of the argument and the accompanying novel numerical computations are of such far reaching importance for technical fluid dynamics, particularly for the computation of certain features in turbulent flow, that it deserves serious considerations. The main purpose of this publication is therefore to stimulate critical discussion among the experts in this area about the relevance and justification of the view taken in this article and its possible consequences for modelling and computation of turbulent flow.*

It is surprising that this paper has not received much attention and did not stimulate neither public criticism nor interest so far, see however the (publicly revealed) private debate on Johnson's blog [161]. Birkhoff's doubt on the instability is, at least mathematically, quite natural: it is well-known that symmetric problems can simultaneously have unstable symmetric solutions and non symmetric stable solutions. Among others, Tang and Aubry [281] have numerically studied Föppl's vortex model [98, 178, 254] which aims to describe an incompressible fluid past a cylinder (this experiment is described with more details in Section 4). Tang and Aubry analysed the symmetry breaking instability leading to vortex shedding:

> *It is well known that if a circular cylinder starts moving from rest in an incompressible fluid, twin vortices spinning in opposite directions form behind the cylinder soon after motion begins. These vortices grow and become more and more elongated as time increases until they reach their maximal size. After that time, the bubble of vortices remains steady at low Reynolds numbers, develops into a time-dependent oscillating wake regime in which the bubble remains attached to the body at about Re* 48 − 50 *or breaks down into a Kármán vortex street at higher Reynolds numbers. It is interesting to notice that if the initial condition is symmetric, the solution formally remains symmetric at all later times. In other words, the subspace of symmetric solutions is an invariant subspace of the Navier-Stokes equations* [see Proposition 2 below] *subject to the boundary conditions considered here. The fact that the flow goes away from this subspace beyond the critical Reynolds number in both physical and numerical experiments means that the symmetric bubble becomes unstable beyond the critical Reynolds number. It remains, nevertheless, a solution at all Reynolds numbers. This observation led Föppl* [98] *to investigate whether one can find steady solutions in the form of twin vortices and study their stability property. Föppl represented the system by building a two-dimensional, incompressible potential flow consisting of a uniform oncoming flow, a pair of point vortices symmetrically located with respect to the centerline behind the cylinder, and inner vortices placed to satisfy the boundary condition on the body* [see e.g. [178, 254]]. *He found fixed points i.e., steady flows for which the twin vortices can indeed maintain their locations relative to the cylinder. Such equilibrium positions are located on two symmetric curves starting from the rear stagnation point of the bubble. Föppl, who also studied the stability of the equilibrium, showed that the vortices are stable to all symmetric perturbations and unstable to some asymmetric perturbations. However, there was a mistake in Föppl's analytical results which was later detected and corrected by Smith* [270] *who showed that the equilibrium is only marginally stable to all symmetric perturbations instead of being stable as originally found by Föppl.*

The symmetry breaking is well documented by experimental works, see e.g. [65–67, 235]. Jackson [160] and Zebib [310] computationally tackled the symmetry breaking instability from the Navier-Stokes equations

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \qquad \nabla\cdot u = 0, \tag{4}$$

where, as usual, $\nu > 0$ is the kinematic viscosity, in the neighbourhood of the critical Reynolds number. The transition is marked by a Hopf bifurcation which is not fully understood as the Navier-Stokes equations yield an infinite-dimensional dynamical system.

Even if a direct connection cannot be established with a symmetry breaking instability of a steady state of the Euler equations, it is certainly worth mentioning the following striking theoretical result due to Bardos et. al [15] that somehow suggests that Birkhoff's feeling is maybe not unreasonable:

Proposition 2 ([15]). *Let u_0 be a function of (x,y) only, then the weak solution of the 3D Euler equations (2) might become spontaneously a function of (x,y,z). If the initial data is axi-symmetric or helical symmetric, the weak solutions of the Euler equations might spontaneously break the symmetry. On the contrary, if u_0 is a function of (x,y), then the Leray-Hopf weak solution of the 3D Navier-Stokes equations (4) remains a function of (x,y) only. For axi-symmetric initial data, or helical initial data, the symmetry is also preserved.*

In fact, the wild weak solutions of the Euler equations that do not obey the two-dimensional symmetry of the initial data should be ruled out because they cannot be obtained as vanishing viscosity limit solutions of the Navier-Stokes equations (4). The existence of weak solutions of the Navier-Stokes equations has been treated in pioneering works [148, 177, 188, 191] in cylindrical domains. In the case of a non-cylindrical, but a priori known domain, weak solutions were first studied in [104] for the case of homogeneous Dirichlet boundary conditions. For details, we refer to some classics [102, 109, 110, 112, 187, 285] and to [248, 289] for two more recent additions to the literature.

Having in mind obstacles modelling suspension bridges, we consider the case where the fluid is enclosed in a *bounded box* of $\mathbb{R}^3$ and we assume that the obstacle is a cylinder, namely a 2D object times an interval. More precisely, we consider

$$\Omega = \{(-L,L)^2 \times (0,\Lambda)\} \ \setminus \ \{\overline{K}\times(0,\Lambda)\}$$

for some $L,\Lambda > 0$ and some 2D obstacle K with $D = K\times(0,\Lambda)$. Since our purpose is to analyse the drag and lift forces acting on the obstacle D, it is sometimes convenient (especially for the lift) to restrict the attention to a 2D section of the box, for instance at the midpoint. The domain Ω and its intersection Σ with the plane $z = \frac{\Lambda}{2}$ are represented in Figure 5 (not in scale!), together with a sketch of the flow and the appearance of vortices. The rectangular shape of the cross section K of the obstacle D has been chosen here for simplicity of the picture; this model was first suggested in [38, 121] and subsequently applied in [122] for a study of non-standard boundary conditions for the planar Stokes equations inducing vortices around concave corners.

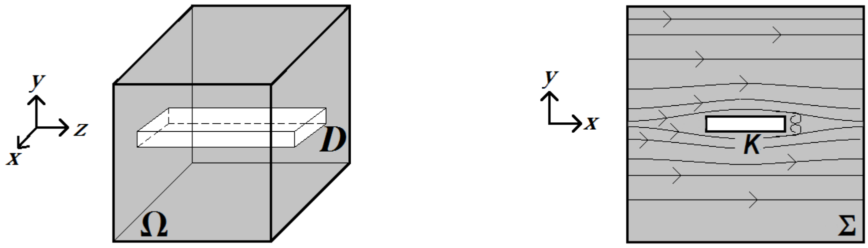


FIGURE 5. The domain Ω and its intersection Σ with the plane $z = \frac{\Lambda}{2}$.

We next discuss the computation of the drag and lift forces exerted on an obstacle by the flow of a viscous fluid. The rate of strain tensor σ and the stress tensor $\mathbb{T}$ of any viscous incompressible fluid are given by (see [180, Chapter 2]):

$$\sigma(u) = \nabla u + \nabla^T u\,, \quad \mathbb{T}(u,p) = -p\mathbb{I} + \nu\sigma(u)\,, \tag{5}$$

where $\mathbb{I}$ is the identity matrix (either 2×2 or 3×3, according to the space dimension). As expressed by (5), in a viscous fluid, in addition to the pressure drag, one needs to take into account the skin friction so that the total force exerted by the fluid over the obstacle D is given by the vector field

$$F_D = -\int_{\partial D} \mathbb{T}(u,p) \cdot n\,, \tag{6}$$

where the minus sign is due to the fact that the outward unit normal n to Ω is directed towards the interior of D. Assuming that the inflow is horizontal, namely the only nonzero component of the boundary velocity is the x-component on the boundary of the box $(-L,L)^2 \times (0,\Lambda)$, the horizontal component in (6) is the drag force, while the orthogonal component is the lift force. For smooth obstacles $D \subset \mathbb{R}^3$ the drag force may also be written as

$$\frac{\nu}{2}\int_\Omega |\sigma(u)|^2\,. \tag{7}$$

see e.g. [21] for the details. It is clear that while the drag force is always acting in the direction of the flow and hence in a one-dimensional direction, the lift force is orthogonal to the drag and has two degrees of freedom in a 3D setting; this is the precise reason why it may be convenient to focus on 2D cross sections of the obstacle, especially when the obstacle is a cylinder aiming to model the deck of a bridge as in Figure 5. In this case, for the drag force in (7), the integral must be computed over the cross-section Σ.

It is possible to derive exact formulas for the drag exerted by a creeping flow over bodies displaying special symmetries like spheres, ellipsoids and cylinders. In 1851, Stokes [276] addressed the problem of the steady flow of a viscous fluid (having constant density ρ and a constant free-stream velocity equal to $u_0 \in \mathbb{R}^3$) surrounding a rigid sphere of radius R. By neglecting, with respect to viscosity, the convective term $(u\cdot\nabla)u$ appearing in the Navier-Stokes equations, he explicitly computed the velocity field of the flow and provided the following formula for the drag over the sphere:

$$F_D = 6\pi\rho\nu R|u_0|, \tag{8}$$

a result that remained in history as the *Stokes law*, see [181, Chapter 6]. Similar expressions for an ellipsoid, a circular disk moving broadside-on, or a circular disk moving edge-ways can be found in the book of Lamb [178, Article 339] (the first edition of this work was published in 1879), from where we quote:

> *The formula of Stokes* (8) *for the resistance experienced by a slowly moving sphere has been employed in physical researches of fundamental importance, as a means of estimating the size of minute globules of water, and thence the number of globules contained in a cloud of given mass. Consequently the conditions of its validity have been much discussed both from the experimental and from the theoretical side.*

A rigorous refutation of the validity of Stokes law was performed by Oseen in 1910, see [231], where it was proven that the convective term may be neglected only at a sufficiently short distance from the sphere, precisely when $|x| \ll \nu/|u_0|$. Far away from the body one may approximate u with u_0, and subsequently $(u\cdot\nabla)u$ with $(u\cdot\nabla)u_0$, by means of which Oseen presented the following linear model for the far-field velocity:

$$-\nu\Delta u + (u\cdot\nabla)u_0 + \frac{1}{\rho}\nabla p = 0 \qquad \nabla\cdot u = 0, \tag{9}$$

usually known as the *Oseen equations*, which constitute an intermediate step between the linear Stokes system and the fully non-linear Navier-Stokes system. An exact resolution of (9) yields an improvement of Stokes law given by:

$$F_D = 6\pi\rho\nu R|u_0|\left(1 + \frac{3R|u_0|}{8\nu}\right),$$

as well as the following expression for the drag, by unit length, applied over an infinite-length cylinder of radius R that is being held orthogonally to the stream, see [180, Chapter II]:

$$F_D = \frac{4\pi\rho\nu|u_0|}{\frac{1}{2} - \gamma - \log\left(\frac{R|u_0|}{4\nu}\right)},$$

where $\gamma = 0.57721...$ is the Euler-Mascheroni constant.

In order to highlight the role of the nonlinear term in the stationary Navier-Stokes equations

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \qquad \nabla\cdot u = 0, \tag{10}$$

following [123], we set up the problem in a perfectly symmetric 2D situation, for instance $\Sigma = (-L, L)^2 \setminus \overline{K}$ with $L \gg \mathrm{diam}(K)$; then Σ approximates the unbounded region outside $\overline{K}$, see again the right picture in Figure 5. We decompose the boundary of Σ as $\partial\Sigma = \partial K \cup \Gamma$ with

$$\Gamma = \{(x,y) \in \mathbb{R}^2 \mid x = -L,\, y \in (-L,L)\ \} \cup \{(x,y) \in \mathbb{R}^2 \mid x \in (-L,L),\, y = -L\,\}$$
$$\cup\ \{(x,y) \in \mathbb{R}^2 \mid x = L,\, y \in (-L,L)\ \} \cup \{(x,y) \in \mathbb{R}^2 \mid x \in (-L,L),\, y = L\,\}.$$

We consider the boundary conditions

$$u = (U,V) \text{ on } \Gamma, \qquad u = (0,0) \text{ on } \partial K, \tag{11}$$

for some given functions $U, V \in H^{1/2}(\Gamma)$ satisfying the compatibility condition (zero total flux across Γ):

$$\int_{-L}^{L} [U(L,y) - U(-L,y)]\, dy + \int_{-L}^{L} [V(x,L) - V(x,-L)]\, dx = 0. \tag{12}$$

Observe that a constant couple (U,V) is an admissible data on Γ. The couple (U,V) models the fluid flow entering Σ with the usual no-slip condition on the obstacle. We suppose first that the fluid is governed by the (linear) stationary $2D$ Stokes equations

$$-\nu\Delta u + \nabla p = 0 \qquad \nabla\cdot u = 0, \qquad \text{in } \Sigma \subset \mathbb{R}^2. \tag{13}$$

It is well known (see for example [285]) that for any $(U,V) \in H^{1/2}(\Gamma)$ satisfying (12) there exists a unique weak solution $(u,p) \in H^1(\Sigma) \times L^2_0(\Sigma)$ of (13)-(11) (here $L^2_0(\Sigma)$ denotes the space of zero mean value functions in L^2).

In a symmetric setting, the following result holds.

Theorem 3 ([123])**.** *Suppose that Σ and K are symmetric with respect to the x-axis. Assume also that the boundary data in* (11) *satisfy* (12) *and*

$$U(x,-y) = U(x,y) \quad \textit{and} \quad V(x,-y) = -V(x,y) \quad \textit{a.e. on } \Gamma. \tag{14}$$

Then the solution $(u,p) = (u_1,u_2,p)$ of (13)-(11) *satisfies the following symmetry property for a.e. $(x,y) \in \Sigma$:*

$$u_1(x,-y) = u_1(x,y),\ \ u_2(x,-y) = -u_2(x,y),\ \ p(x,-y) = p(x,y). \tag{15}$$

Theorem 3 is in clear contradiction with reality, see for instance Figure 2. This discrepancy between Theorem 3 and Figure 2 shows the mere viscosity within the linear Stokes equations (13) does not help to model turbulence and the subsequent drag and lift forces on an obstacle. Furthermore, it should be pointed out that, in the case of an incompressible fluid governed by (13) past a finite obstacle (with a locally Lipschitz boundary), Stokes discovered in 1851 that there is no bounded solution of (13) which vanishes on the surface of the obstacle and that tends to a non-zero limit at infinity: this constitutes the so-called Stokes paradox, see [276]. Indeed, the velocity grows logarithmically with the distance from the body, see [137, Section 3].

For the Navier-Stokes equations, we have the following result which supports the instability of the symmetric steady state at a critical value of the stream velocity.

Theorem 4 ([123]). *For any $(U,V) \in H^{1/2}(\Gamma)$ satisfying* (12) *there exists a weak solution $(u_1, u_2, p) \in H^1(\Sigma) \times L^2_0(\Sigma)$ of* (10)-(11). *Moreover:*

- *there exists $\delta = \delta(\nu) > 0$ such that if $\|(U,V)\|_{H^{1/2}(\Gamma)} < \delta$, then the weak solution of* (10)-(11) *is unique;*
- *if K (and Σ) are symmetric with respect to the x-axis and if (U,V) verifies* (14), *then also (v_1, v_2, q) with*

$$v_1(x,y) = u_1(x,-y), \;\; v_2(x,y) = -u_2(x,-y), \;\; q(x,y) = p(x,-y)$$

for a.e. $(x,y) \in \Sigma$ solves the same problem;

- *if K (and Σ) are symmetric with respect to the x-axis and if (U,V) verifies* (14) *and $\|(U,V)\|_{H^{1/2}(\Gamma)} < \delta$, then the unique weak solution of* (10)-(11) *satisfies* (15).

Existence and uniqueness for small boundary data (U,V) are well-known, see e.g. [285, Theorems 1.5 and 1.6, Chapter II]. Since δ depends increasingly on ν and, therefore decreasingly on Re, Theorem 4 is compatible with Figure 6: as long as Re is small the flow is symmetric, while if Re is large uniqueness may be lost and asymmetric solutions may arise. If (14) holds, for large values of Re the

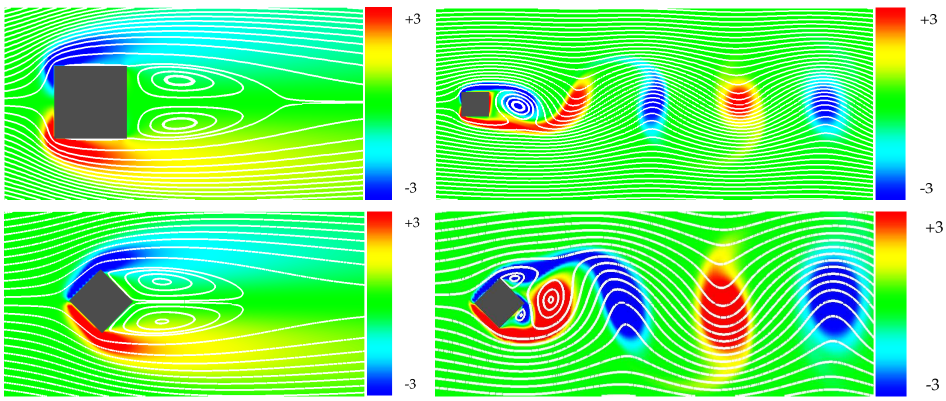


FIGURE 6. CFD simulation of a flow around a square cylinder (on the left, Re= 30; on the right Re= 200) by Fuka-Brechler [105], reproduced with courtesy of the authors. The scale indicates the vorticity.

solutions of (10)-(11) exist by couples: if there is an asymmetric solution, there is also its “reflected” solution. This discussion shows that only the combination of viscosity and nonlinearity gives solutions in line with experiments even if this *does not guarantee that* (10) *are perfectly suited to describe turbulence and vortex formation.* As long as (10) is in a uniqueness regime and that the flow and the obstacle are symmetric or “almost symmetric” one deduces that the lift is zero or small, according to the following statement.

Theorem 5 ([123]). *Assume that K (and Σ) are symmetric with respect to the x-axis, assume that $(U,V) \in H^{3/2}(\Gamma)$ satisfy* (12) *and $\|(U,V)\|_{H^{1/2}(\Gamma)} < \delta$ with δ*

as in Theorem 4. Let F_K be as in (6) *and let L_K be the corresponding intensity of the (vertical) lift force.*
• *If $V = 0$ and U verifies* (14)*, then $L_K = 0$.*
• *For all $\varepsilon > 0$ there exists $\eta = \eta(U,V) > 0$ such that if $\|(\widetilde{U},\widetilde{V})\|_{H^{3/2}(\Gamma)} < \eta$, then $L_K < \varepsilon$; here,*

$$\widetilde{U}(x,y) = \frac{U(x,y) - U(x,-y)}{2}\,,\ \widetilde{V}(x,y) = \frac{V(x,y) + V(x,-y)}{2} \quad a.e.\ on\ \Gamma$$

Although perfect symmetry does not exist in nature, almost symmetric flows do exist, for instance when the wind is laminar and only in the x-direction. Theorem 5 impliess that large lifts may occur only when uniqueness of the solution is no longer true. If the smallness assumption of (U,V) is violated one may obtain multiplicity results for (10)-(11), see [158, 244, 295, 296]. In particular, Velte [296] showed that at a certain Reynolds number there is more than one solution of (10) in the domain between two concentric rotating cylinders, see Section 4 and [285, Section 4 in Chapter II]. In fact, with slightly more regularity of the data, one also has the following generic property of a finite number of solutions.

Theorem 6 ([100])**.** *Assume that the obstacle K has a C^2-boundary ∂K. There exists a dense open set $\mathcal{O} \subset \{(U,V) \in H^{3/2}(\Gamma);\ (U,V)$ satisfies* (12)$\}$ *such that if $(U,V) \in \mathcal{O}$ then the number of solutions of* (10)-(11) *is finite. Moreover, on any connected component of $\mathcal{O}$ the number of solutions of* (10)-(11) *is constant.*

We point out that, contrary to [100], we do not need the whole boundary of Σ to be smooth since the equations in convex polygons (in our case, a square) can be handled with ad-hoc techniques [212]. We expect that by crossing the separation between two distinct connected components a bifurcation occurs. We also expect the bifurcation to generate changes in the dynamics of the flow around the obstacle and of the subsequent fluid-obstacle interaction. We are rather convinced that the appearance of a lift force acting on an obstacle is due to an instability which arises when the stream flow velocity overcomes a critical threshold. In fact, the physical word *instability* mathematically translates into *bifurcation* which, in turn, may appear only in presence of multiple solutions. A natural question then arises from Figure 6, Theorem 6 and the non-uniqueness result in [296].

Problem 1. *Prove that we always have multiple steady flows past any symmetric body for large* Re*. Can we have multiple symmetric steady flows?*

To our knowledge, multiple symmetric solutions of (10)-(11) have not even been detected numerically.

4. Soul meets body

Three main mathematical difficulties arise to model a bluff (elastic or rigid) body surrounded by a fluid: the modelling of the fluid, the modelling of the body and the modelling of the mutual interaction which takes place on the interface where the fluid meets the body. This so-called fluid-solid interaction (also referred to as fluid-structure, liquid-solid, fluid-particle, wind-bridge interaction, depending on the context) became one of the main focuses of theoretical and applied researches in fluid dynamics. Albeit the mathematical theory of the motion of a rigid body in a fluid is a very classical and old problem, mathematicians became interested in a systematic study of those models much later than their colleagues driven by applications. As a consequence, many experiments or numerical observations still require theoretical/mathematical justifications, some of them being currently out of reach. In this section, we focus on models where there is no full interaction, either the body is still or it undergoes a *given* movement. As already discussed in Section 2, turbulence may appear when a flow hits a bluff body. Obvious examples are the motion of a plane and the wind blowing past a structure such as a building or a bridge. Real situations are usually geometrically complicated but even in the case of an obstacle with a very simple geometry, the resulting flow can already display amazing complexities. The most documented experiment in the literature is that of spontaneous oscillations of the wake in the flow of a viscous liquid past a circular cylinder [18, 28, 65, 66, 189, 203, 216, 261, 287, 288, 304]. We reproduce here the setup of [288, Chapter 3]. A cylinder of diameter d is placed with its axis normal to the flow having an upstream constant velocity u_0. The physical experiment is $3D$ but when the cylinder is so long compared with d, the situation can be modelled by an infinite cylinder as the same behaviour likely appears in every plane normal to the axis. Also, the other boundaries to the flow (e.g. the walls of a wind tunnel or the borders of a water channel, in which the cylinder is placed) can be assumed so far away that they have no effect. The pictures and tables in Figure 7 and 8 summarise the observations and the transitions according to the Reynolds number. One can vary the values of the diameter d of the cylinder, the speed u_0 of the flow, the density ρ and the viscosity of the fluid. It happens that the flow pattern depends only on the Reynolds number $(u_0 d/\nu)$. Following [288, Chapter 3], we describe the various changes that occur to the flow pattern as Re varies. The experiment can be done in practice in a very wide range of Re. Indeed, one can make it in the air modifying both u_0 and d to cover the full range; for instance, $\mathrm{Re} = 0.1$ corresponds to a diameter of 10^{-6}m (thin fibre) in an airflow at speed 0.15m/s while $\mathrm{Re} = 10^6$ can be achieved with a diameter of 0.3m and a speed of 50m/s. As emphasised in [288, Chapter 3], the description of the flow patterns is based on experimental observations, some numerics but very few analytical or theoretical clues. For low Reynolds numbers, a 3D flow past a sphere can be approximated by a Stokes flow that can be found explicitly [18, Section 4.9]. When $\mathrm{Re} \ll 1$, the flow is very similar to a laminar flow and qualitatively behaves like the flow arising from the potential (3), it is symmetrical upstream and downstream; the pattern is its mirror image with respect to the diameter of the cylinder which is parallel to the flow. At low Re, the pressure due to the drag force is negligible and the effective drag on the body is entirely due to skin friction.

As Re is increased the upstream-downstream symmetry disappears. When Re exceeds about 4, symmetrical vortices appear behind the cylinder and rotate in opposite directions. These vortices enlarge with increasing Re. For $\mathrm{Re} > 40$, the

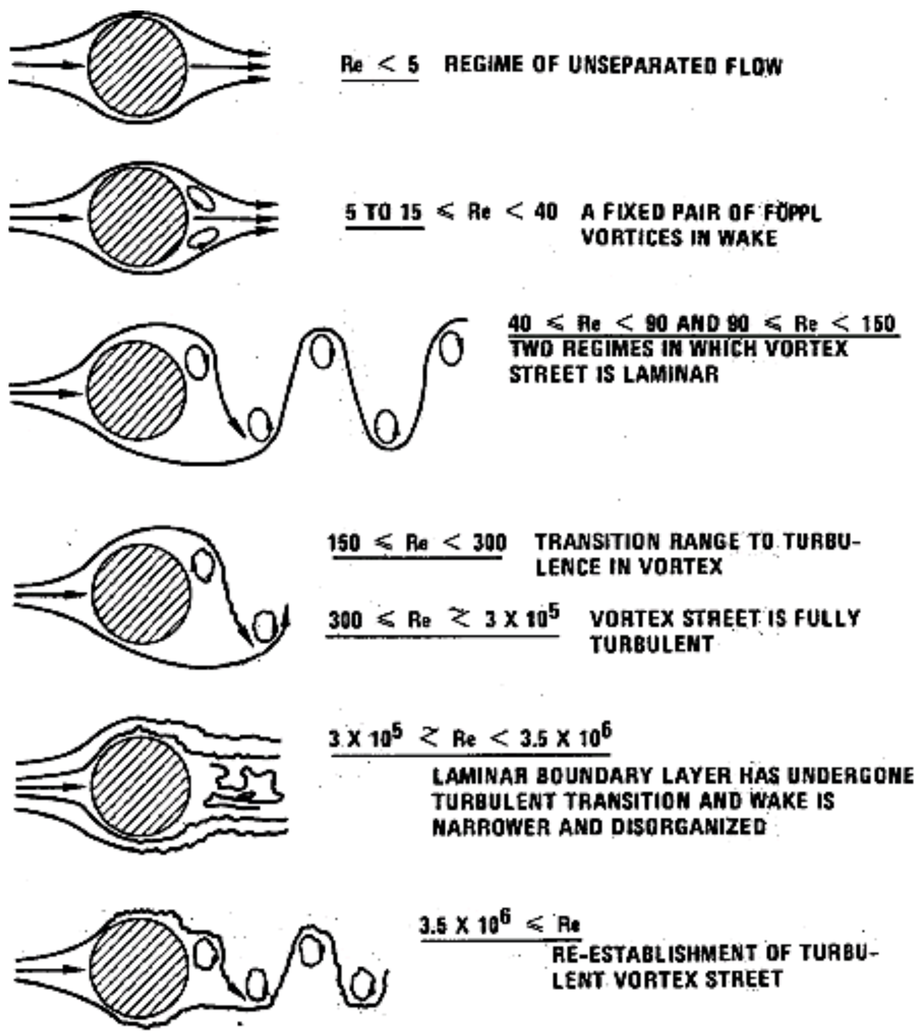


FIGURE 7. Regimes of flow across a smooth cylinder [189].

flow in the wake becomes unsteady. A further increase of the Reynolds number elongates the vortices, which also begin to oscillate until they break away at a Re of approximately 90. The breaking occurs alternatively from one to the other side and the vortices travel downstream. This process is intensified with further increase of Re while the shedding of vortices from alternate sides of the cylinder is regular. This leads to formation of the characteristic wake which is known as Kármán vortex street, see the right picture in Figure 2. The vortex motion is

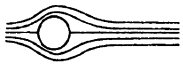
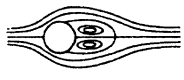
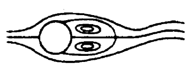
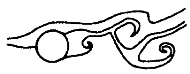
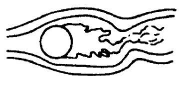
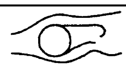
Reynolds number regime	Flow regime	Flow form	Flow characteristic
$\mathrm{Re} \to 0$	Creeping flow		Steady, no wake
$3-4 < \mathrm{Re} < 30-40$	Vortex pairs in wake		Steady, symmetric separation
$\genfrac{}{}{0pt}{}{30}{40} < \mathrm{Re} < \genfrac{}{}{0pt}{}{80}{90}$	Onset of Karman vortex street		Laminar, unstable wake
$\genfrac{}{}{0pt}{}{80}{90} < \mathrm{Re} < \genfrac{}{}{0pt}{}{150}{300}$	Pure Karman vortex street		Karman vortex street
$\genfrac{}{}{0pt}{}{150}{300} < \mathrm{Re} < \genfrac{}{}{0pt}{}{10^5}{1.3 \cdot 10^5}$	Subcritical regime		Laminar, with vortex street instabilities
$\genfrac{}{}{0pt}{}{10^5}{1.3 \cdot 10^5} < \mathrm{Re} < 3.5 \cdot 10^6$	Critical regime		Laminar separation Turbulent reattachment Turbulent separation Turbulent wake
$3.5 \cdot 10^6 < \mathrm{Re}$	Supercritical regime (transcritical)		Turbulent separation

FIGURE 8. Flow regimes at a circular cylinder [261].

periodic both in space and time. The pressure drag at this stage is already larger than the profile drag. Having passed a transition range where the regularity of shedding decreases, above a Re of 300, the vortex shedding becomes irregular. There is still a predominant frequency but the amplitude appears to be random. Notice that the critical regime can be anticipated as the roughness of the body surface increases. At very high Reynolds number from about 3×10^5, the separation point moves rearward on the cylinder, consequently the drag coefficient decreases appreciably. The flow in the wake becomes so turbulent (with highly irregular quick velocity fluctuations), that the vortex street pattern is no longer recognisable. We refer to [136] for an analysis of the drag at the onset of vortex shedding. In the description of this experiment, one sees that critical values of Re appear in the, say, route to turbulence. It is a mathematical challenge to theoretically describe the observed transition from laminar to turbulent flow. It is clearly any nonlinear analyst's dream to explain this experiment with a bifurcation diagram showing successive losses of stability of patterns of less complicated structure giving rise to more complicated ones. Landau and Hopf [147, 149, 179, 180] conjectured that the transition to turbulence indeed occurs through repeated branching of manifolds of quasi-periodic solutions. This conjecture was quickly refuted, see e.g. Iooss [156]:

> *En fait, cette idée a été caduque après l'article de Ruelle et Takens* [252] *remarquant, d'une part qu'une solution quasi-périodique sur un tore invariant, n'est pas structurellement stable : une petite perturbation du système suffit à la faire disparaître, ce qui est très gênant conceptuellement puisqu'on ne sera jamais sûr d'approcher parfaitement, avec les équations choisies, le système réel. D'autre part, ces auteurs ont montré qu'un état turbulent pouvait correspondre à l'apparition d'un "attracteur étrange" dans l'espace des phases, où un nombre fini de dimensions suffirait à le définir (ceci est déjà réalisable sur un tore de dimension 3).*

Despite this objection to the full conjecture, the idea of looking at bifurcations, in particular from steady states to time-periodic solutions, is of great interest and was initiated by Velte [295, 296] followed by Iudovich [158, 159], Iooss [154], Joseph and Sattinger [162] and Rabinowitz [245]. We also refer to [23, 69, 101, 155, 163–165]. The motion of a viscous liquid past a circular cylinder is not the original motivation of those contributions. Many authors were rather motivated by a theoretical explanation of weak turbulence appearing for instance in the celebrated experiment of Rayleigh-Bénard convection. Others by the delicate application of Hopf bifurcation in an infinite dimensional setting such as the PDEs arising from fluids, first in a bounded setting (to model a fluid past a cylinder, one ideally needs a priori to work in an exterior unbounded domain). A good physical model that can fit in that setting [154] is the Taylor-Couette flow which consists of a viscous fluid confined between two rotating cylinders. For low angular velocities, the flow is steady and purely azimuthal: it is nowadays called a circular Couette flow (after Couette, who used this experimental device to measure viscosity). Taylor investigated the stability of a Couette flow [283]. Among other things, he claimed that the no-slip condition, which was in dispute by the scientific community at the time, is the correct boundary condition for viscous flows at a solid boundary, see also [20]. Taylor showed that when the angular velocity of the inner cylinder achieves a critical speed, the Couette flow becomes unstable and a secondary steady state arises. This stationary state, known as the Taylor vortex flow, is characterised by axisymmetric toroidal vortices. Increasing again the angular speed of the cylinder, the system undergoes instabilities which lead to states with greater complexity both in time and space. The onset of turbulence arises beyond a critical Re. If the two cylinders rotate in opposite sense, then spiral vortex flow arises. The theoretical understanding of this experiment yielded many contributions. Velte [296] confirmed that a bifurcation occurs at the same critical Reynolds number as that given by the theory of small perturbations. For the mathematical treatment, we refer to the book of Chossat and Iooss [57] and the citations therein. It is however curious that bifurcation analysis, which is a powerful tool for instance in the theory of dynamical systems or elliptic partial differential equations, was not so much used, at least theoretically, by the community of fluid dynamics after the above mentioned pioneering works. One explanation might come again from the complexity of the phenomena, as emphasised by van Veen [293]:

> *When studying fluid mechanics in terms of instability, bifurcation and invariant solutions one quickly finds out how little can be done by pen and paper. For flows on sufficiently simple domains and under sufficiently simple boundary conditions, one may be able to predict the parameter values at which the base flow becomes unstable and the basic properties of the secondary flow. On more complicated domains and under more realistic boundary conditions, such questions can usually only be addressed by numerical means.*

For some developments on the computational modelling of bifurcations and instabilities in fluid dynamics, we refer to [81, 90, 125, 150, 246, 294] and this list is far from being exhaustive. For some numerical experiments with different shapes of the obstacle, we refer to Section 7. Regarding the behaviour of a fluid in the presence of an obstacle, a special reference needs to be addressed to the works of V.G. Maz'ya and his collaborators in the linear theory of water waves [174, 175, 190, 205, 206, 290–292] starting from the year 1972, precisely when he was at the Department of Applied and Computational Mathematics in the Leningrad Shipbuilding Institute.

Drag and lift are clearly visible in practical experiments [301], although the precise physical mechanism of lift generation is not fully clarified. In our opinion, also in view of Theorems 4 and 5, it could be the consequence of an *instability*: when the drag is too large any tiny perturbation of the equilibrium position may give rise to orthogonal movements of the body. This option and the lack of knowledge suggest the following questions.

Problem 2. *For a given obstacle, is there a critical threshold of the flow velocity which initiates the lift? Is there a deterministic law (at least, an approximate law) for the dependence of the lift on the flow?*

We next discuss the behaviour (in an exterior domain $\Omega = \mathbb{R}^3 \setminus D$) of a fluid around a body D moving with a given law. The most studied case consists of a solid rotating with a prescribed constant angular velocity and translating with a constant velocity. The fluid is governed by the evolution Navier-Stokes equations and it satisfies the no-slip condition both at infinity and on the obstacle. The system is then written in a reference frame attached to the body, so that the fluid domain becomes fixed, see [111]. Moreover, the no-slip condition on the body becomes a dynamic Dirichlet condition. More precisely, we consider the following evolution Navier-Stokes equations

$$
(16) \qquad \begin{cases} u_t - \nu\Delta u + \big((u - V(t)) \cdot \nabla\big)u + \nabla p = 0, \qquad \nabla \cdot u = 0, \qquad \text{in } \Omega \times \mathbb{R} \\ u = V(t) \text{ on } \partial D, \quad \lim_{|x|\to\infty} u = 0 \ \text{ for } t \in \mathbb{R}, \end{cases}
$$

where $V = V(t)$ is the velocity field associated with the rigid motion of D_t, that is, the velocity of the center of mass of D: $D_t = D + V(t)$, which is assumed to be periodic.

Theorem 7 ([115])**.** *Let* $\Omega = \mathbb{R}^3 \setminus D$ *be an exterior domain of* $\mathbb{R}^3$*, let* $V \in H^1(0,T)$ *be a* T*-periodic function. Then there exists at least one* T*-periodic weak solution of* (16)*.*

The extension to a rotating body, also contained in [115], brings additional difficulties because, in the reference frame of the body, the equations contain a linear unbounded coefficient, which excludes the use of a perturbation of the model without rotation. Under additional assumptions, a uniqueness and stability result has been obtained in [225], see also [124]. We refer to the handbook chapter [111] and to the recent book [223] for an account of the recent progresses.

We conclude this section with one of the few recent theoretical bifurcation results concerning viscous fluids. Namely, we state Galdi's existence and uniqueness result [114] of a branching out time-periodic family of solutions arising from a non degenerate steady-state. The framework is the two-dimensional Navier-Stokes equations in the exterior of a cylinder so that the result provides a rigorous analysis of part of the experiment described above concerning the motion of a fluid past a cylinder. Indeed, it is experimentally observed that until a critical Reynolds number is reached, the motion of the liquid in a region sufficiently far from the ends of the cylinder (including it), is planar, steady and stable, whereas as soon as we pass the critical value, the motion is still planar, but its regime becomes oscillatory.

More precisely, we recall the setup: a cylinder D, of diameter d, is placed with its axis orthogonal to the flow of a viscous liquid having an upstream constant velocity u_0. Let Σ be the relevant two-dimensional unbounded region of flow (the entire portion of the plane outside the normal cross-section K of D), and let e_1 be a unit vector parallel to u_0. It is known that, under suitable assumptions on λ_0, the equations

$$\begin{cases} V_t + \lambda((V - e_1) \cdot \nabla)V = \Delta V - \nabla P\,, \quad \nabla \cdot V = 0 \quad \text{in } \Sigma \times \mathbb{R}, \\ V = e_1 \text{ on } \partial K \times \mathbb{R}\,, \quad \lim\limits_{|x|\to\infty} V(x,t) = 0 \text{ for all } t \in \mathbb{R}. \end{cases} \tag{17}$$

possess a unique steady-state solution branch $(v_0(\lambda), p_0(\lambda))$, with λ in a neighborhood of λ_0, see for instance [111]. Consider the perturbed fields $V = v(x,t;\lambda) + v_0(x;\lambda)$, $P = p(x,t;\lambda) + p_0(x;\lambda)$ which solve (17) if and only if v and p are solutions of the equations

$$\begin{cases} v_t + \lambda[((v - e_1) \cdot \nabla)v + (v_0(\lambda) \cdot \nabla)v + (v \cdot \nabla)v_0(\lambda)] = \Delta v - \nabla p \ \text{ in } \Sigma \times \mathbb{R}, \\ \nabla \cdot v = 0 \text{ in } \Sigma \times \mathbb{R}, \\ v = 0 \text{ on } \partial K \times \mathbb{R}\,, \quad \lim\limits_{|x|\to\infty} v(x,t) = 0 \text{ for all } t \in \mathbb{R}. \end{cases} \tag{18}$$

Theorem 8 ([114]). *Under nondegeneracy conditions on the steady state* (v_0, p_0) *(for which we refer to the original paper), there exists a non-trivial family of solutions to* (18)*, namely one solution* (v, p) *for each* λ *close to* λ_0*, with (unknown) time-period* $T(\lambda)$*, such that* $(v, \nabla p) \to (0, 0)$ *as* $\lambda \to \lambda_0$*. These solutions exist either for* $\lambda > \lambda_0$ *or* $\lambda < \lambda_0$ *(subcritical or supercritical bifurcation) and provide the unique (up to phase-shift) time-periodic solution branching out from* (v_0, p_0)*.*

It is emphasised in [114] that numerical evidence [90, 246] suggests that the required nondegeneracy conditions hold at some critical Reynolds number. These are spectral conditions on linearised operators around the steady state (v_0, p_0). The pioneering results in [154, 158, 159, 162] give sufficient conditions, basically of the same type as those of Theorem 8, for the existence (and uniqueness) of bifurcating time-periodic solutions from steady-state solutions to the Navier-Stokes equations but the assumption that the flow occupies a bounded domain is fundamental in those previous works.

5. Floating bridge

5.1. **Troubled bridge over water.** The lack of a precise theory for drag and lift discussed in Section 3 is not just a problem of aerospace engineering. Several structures in civil engineering are subject to violent wind attacks and the resulting vortex shedding generates several dangerous oscillations in the structure, in particular tall buildings or long-span bridges. In this respect, Hansen [133] points out that

> *although a great deal of effort has been made during recent decades to improve the analytical models used for predicting vibrations due to vortex shedding, the analytical models available are still rather crude. The cross-wind forcing mechanisms have proved to be so complex that there is no general analytical method available to calculate cross-wind structural response. The main physical parameters involved in the forcing mechanisms have been clarified, but the basic data used in full-scale predictions have not reached a general agreement among researchers. Especially, the methods they use to take aeroelastic effects, i.e. motion-induced wind loads, into account differ considerably.*

Several companies specialised on wind actions on structures invest on experimental and theoretical research on the topic [134]. In this section we discuss the action of winds on suspension bridges; for an entry to the Engineering literature on this subject we refer to [269, 308]. The first suspension bridges were erected about two centuries ago, much earlier than the development of essential mathematical tools for their study. Among other missing tools, we mention the knowledge of elasticity, of nonlinear analysis, of higher order PDEs, of numerical analysis. The need of a purely theoretical approach is made clear by Claude-Louis Navier (1785-1836) who wrote that [221, p.xxj]

> *L'étude des ponts suspendus n'aurait pas été possible sans les progrès que l'analyse mathématique a faits dans ces derniers temps, et sans les institutions au moyen desquelles les personnes chargées de la direction des travaux publiques se trouvent initiées aux connaissances mathématiques les plus élevées.*

Until some 70 years ago, the only mathematical treatises on suspension bridges were the celebrated report by Navier [221] and the monograph by Melan [215]. In November 1940, the Tacoma Narrows Bridge (TNB) collapsed and several videos [278] witnessed its troubled behaviour over the water and its inevitable eventual failure. The emotional reaction to the TNB collapse led different communities to seek reliable models [31, 36, 238, 239, 249, 257, 258, 309]. In spite of these attempts, no real connection with the aerodynamics of actual suspension bridges was ever attained, and the suggested equations yield conclusions which all differ from each other. A typical example illustrating this discrepancy between theory and practice is the formula used to compute the flutter velocity of the wind [249, p.163], [264], [204, Formula (20)], [157, Formula (4.91)], [60, § 8] and many others. Holmes [146, p.293] shows that none of them perfectly agrees with experimental measurements whereas, from a theoretical point of view, in [24] one finds a proof that there exist no "magic formula" able to compute flutter and satisfying all the rules expected by engineers. This is a further defeat of the current models for turbulence: there exists no reliable way to predict flutter within structures.

Furthermore, due to turbulence and to the subsequent vortex shedding, vertical oscillations of the deck are to be expected, but the reason of the sudden transition from vertical to torsional oscillations is not clear. These oscillations were seen prior to the collapses of several suspension bridges, see e.g. [120, Chapter 1]. Early attempts [36, 249, 271] to explain this phenomenon were made by von Kármán, a member of the Board appointed for the Federal Report [2]: he was convinced that the torsional motion seen on the day of the TNB collapse was due to the vortex shedding that amplified the already present torsional oscillations and caused the center span to violent twist until the collapse [78, p.31]. Since then, many different theories were suggested, each one claiming to have the full aerodynamic explanation but, in fact, all being denied in subsequent investigations. Scanlan [258, p.841] denied the von Kármán explanation due to a mismatch between frequencies. Green-Unruh [130, Section III] believe that vortices form independently of the motion and are not responsible for the catastrophic oscillations of the TNB. Larsen [183, p.247] stated that

> *vortices may only cause limited torsional oscillations, but cannot be held responsible for divergent large-amplitude torsional oscillations.*

McKenna [214] noticed that the behaviour described by Larsen was never observed at the TNB while Green-Unruh [130] claim that

> *the Larsen model does not adequately explain data or simulations at around 23m/s.*

Bleich [34] suggested a possible connection between the instability in suspension bridges and flutter of aircraft wings but Billah-Scanlan [31, p.122] believe that it is a great mistake to relate these two phenomena and they also claim that their own work proves that the failure of the TNB was in fact related to an aerodynamically induced condition of self-excitation in a torsional degree of freedom: from [234, Section 1.2] we quote

the truth about self-excited oscillations is that they are not truly self-excited.

Moreover, Larsen [183, p.244] believes that the work in [31] fails to connect the vortex pattern to the switch of damping from positive to negative whereas McKenna [214] states that [31]

is a perfectly good explanation of something that was never observed, namely small torsional oscillations, and no explanation of what really occurred, namely large vertical oscillations followed by torsional oscillations.

A linear suspension bridge model introduced by Pittel-Yakubovich [238, 239] (see also [309, Chapter VI]) was used in [309, p.457] to conclude that

the most dangerous phenomenon for the stability of suspension bridges is a combination of parametric resonance.

But Scanlan [258, p.841] comments this explanation by writing that

Others have added to the confusion. A recent mathematics text [309]*, for example, seeking an application for a developed theory of parametric resonance, attempts to explain the Tacoma Narrows failure through this phenomenon.*

To conclude this short overview, we mention that Scanlan [257, p.209] writes that

the original TNB withstood random buffeting for some hours with relatively little harm until some fortuitous condition broke the bridge action over into its low antisymmetrical torsion flutter mode.

The words *fortuitous condition* show that he also had no full answer. Finally, McKenna [213, Section 2.3] writes that

there is no consensus on what caused the sudden change to torsional motion,

whereas Scott [262] says

opinion on the exact cause of the TNB collapse is even today not unanimously shared.

Moreover, Paidoussis-Price-de Langre [234] raise several doubts on many existing theories (such as the "vortex shedding hypothesis" [234, Chapter 6]) and the overall conclusion is that the involved phenomena are too many and too complicated to be modelled through simple equations. Summarising, all the attempts to find a purely aeroelastic answer fail either because the quantitative parameters do not fit the theoretical explanations or because the experimental results do not confirm the underlying theory. This raises further fog on turbulence, the lack of knowledge prevents the scientific community to explain a catastrophe such as the TNB collapse.

In order to thin the fog, some attention should be devoted to the nonlinear behaviour of structures. Recently, the transition from vertical and torsional oscillations was shown to be also the consequence of the nonlinear behaviour of structures [7–9, 19, 25–27, 70, 94, 118, 119]; see also the monograph [120]. The tools used to reach this conclusion take their roots in extensions of the Floquet theory to infinite dimensional dynamical systems. Several different models were considered, all leading to the same response: if vertical oscillation reach a critical amplitude, then they becomes unstable and may suddenly transfer energy towards torsional oscillations. Since the purpose of these papers was to emphasise the contribution of nonlinear elasticity for the appearance of instabilities in a suspension bridge, most of the considered models were isolated, with no damping and no aerodynamic effects. Therefore, although they bring a further crucial ingredient for the description of instability, they fail to reproduce the fundamental aerodynamic phenomena acting on a suspension bridge, from the vortex shedding to the appearance of turbulence. It is unreasonable to view a bridge as an isolated system but, even in presence of damping and forcing the same instability issue has been shown in [37, 52], as we outline in the next subsection.

5.2. **Twist and shout.** The TNB collapse is not an isolated event, several other bridges failed in similar circumstances, see [120, Chapter 1]. Particularly impressive appears the description for the Wheeling Suspension Bridge in West Virginia, erected in 1849 and collapsed in 1854 during a violent storm. From [286] we quote:

> *...for a few moments we watched it with breathless anxiety, lunging like a ship in the storm; at one time it rose to nearly the height of the towers then fell, and twisted and writhed, and was dashed almost bottom upward. At last there seemed to be a determined twist along the entire span, about one half of the flooring being nearly reversed, and down went the immense structure from its dizzy height to the stream below, with an appalling crash and roar...*

Therefore, as for the TNB, the "twisted and writhed" (or "torsional") movements of the deck were the sign for an imminent collapse: when one sees such movement on a bridge he should... shout and escape.

In a first simplified approach, the deck of a suspension bridge may be seen as a thin fixed plate defined by $D = (-\ell, \ell) \times (-d, d) \times (0, \Lambda) \subset \mathbb{R}^3$, where $d \ll \ell \ll \Lambda$, while the 3D region where the air surrounds the deck (either a wind tunnel or a large region of the space) can be taken to be $\Omega = (-L, L)^2 \times (0, \Lambda) \setminus \overline{D}$, where $L \gg \Lambda$. The domain Ω and its intersection with the plane $z = \frac{\Lambda}{2}$ are represented in Figure 5 (not in scale): the rectangular shape of the cross section K has been chosen for simplicity of the picture, other shapes closer to real designs will be discussed in Section 7.

As we have seen in Section 3 for general cylindrical obstacles, the deck D of the suspension bridge in Figure 5 is subject to the lift force $g(x, z, t)$, generated by the vortices, which moves the deck in the y-direction. In many instances of fluid-structure interaction the deck is modelled as a Kirchhoff-Love plate [168, 194] so that a 3D cylinder is reduced to a 2D plate in the (x, z)-plane. Indeed, since the thickness $2d$ is constant, it may be considered as a rigidity parameter and one focuses the attention on the middle horizontal section D_* (the intersection of D with the plane $y = 0$), that is $D_* = (-\ell, \ell) \times (0, \Lambda) \subset \mathbb{R}^2$. This is physically justifiable as long as the vertical displacements remain in a certain range that usually covers the displacements of the deck. If the cross section of the deck is not a perfect rectangle, as in most suspension bridges, then D_* represents the projection of D on the plane $y=0$. The deflections of this plate are described by a function $u=u(\xi,t)$ with $\xi=(x,z)\in D_*$. Starting for the Kirchhoff-Love plate model, Berger [29] introduced a nonlocal nonlinear plate equation which appears suitable to describe the deck of a bridge. Moreover, for a partially hinged plate such as D_*, the buckling load only acts in the z-direction as for a one-dimensional beam, see [170]. We are so led to consider the PDE

$$\begin{cases} u_{tt}+\delta u_t+\Delta^2 u - S\left[\int_{D_*} u_z^2\right] u_{zz} = g(\xi,t) & \text{in } D_*\times(0,T)\\ u=u_{zz}=0 & \text{on } [-\ell,\ell]\times\{0,\Lambda\}\\ u_{xx}+\sigma u_{zz}=u_{xxx}+(2-\sigma)u_{xzz}=0 & \text{on } \{-\ell,\ell\}\times[0,\Lambda]\,, \end{cases} \tag{19}$$

possibly complemented with some initial conditions

$$u(\xi,0)=u_0(\xi),\qquad u_t(\xi,0)=v_0(\xi)\qquad \text{in } D_*\,. \tag{20}$$

In absence of forces, the plate returns in its horizontal position $D_*\times\{y=0\}$ in the 3D space, the functions u_0 and v_0 are, respectively, the initial position and velocity of the deck. The boundary conditions in (19) on the short edges are named after Navier [220] and model the fact that the plate is hinged in connection with the ground while the boundary conditions on the long edges model the fact that the plate is free [199, 297]. The constant σ is the Poisson ratio and one has $\sigma\approx 0.2$. The parameter δ models damping while both the surface density of mass M of the plate and its flexural rigidity EI are written in adimensional form and are then set to 1. Finally, $S>0$ depends on the elasticity of the material composing the plate and $S\int_\Omega u_z^2$ measures the geometric nonlinearity of the plate due to its stretching.

The energy space for the study of (19) is

$$H^2_*(D_*)=\{v\in H^2(D_*);\ v=0 \text{ on } [-\ell,\ell]\times\{0,\Lambda\}\}\,.$$

Also consider its dual space $(H^2_*(D_*))'$. We use the angle brackets $\langle\cdot,\cdot\rangle$ to denote the duality of $(H^2_*(D_*))'\times H^2_*(D_*)$, $(\cdot,\cdot)_{L^2}$ for the inner product in $L^2(D_*)$ with the corresponding norm $\|\cdot\|_{L^2}$, $(\cdot\,,\cdot)_{H^2_*}$ for the inner product in $H^2_*(D_*)$ defined by

$$(v,w)_{H^2_*}=\int_{D_*}\big(\Delta v\Delta w-(1-\sigma)\big(v_{xx}w_{zz}+v_{zz}w_{xx}-2v_{xz}w_{xz}\big)\big)\,,\quad v,w\in H^2_*(D_*)\,.$$

Since $\sigma\in(0,1)$, this defines a norm which makes $H^2_*(D_*)$ a Hilbert space; see [95, Lemma 4.1]. Assuming that $g\in C^0(\mathbb{R}_+,L^2(D_*))$, by solution of (19) we mean a function

$$u\in C^0(\mathbb{R}_+,H^2_*(D_*))\cap C^1(\mathbb{R}_+,L^2(D_*))\cap C^2(\mathbb{R}_+,(H^2_*(D_*))')$$

such that

$$\langle u_{tt},v\rangle+\delta(u_t,v)_{L^2}+(u,v)_{H^2_*}+S\|u_z\|^2_{L^2}(u_z,v_z)_{L^2}=(g,v)_{L^2}\,,$$

for all $t \in \mathbb{R}_+$ and all $v \in H^2_*(D_*)$. Existence, uniqueness, and regularity results for (19) may be found in [37]. Here we focus on the torsional stability issue, in order to show that a simplified lift force within (19) may qualitatively reproduce what occurred at the TNB. We then introduce the subspaces of even and odd functions with respect to x:

$$H^2_{\mathcal{E}}(D_*) = \{u \in H^2_*(D_*) :\ u(-x,z) = u(x,z)\ \forall (x,z) \in D_*\},$$
$$H^2_{\mathcal{O}}(D_*) = \{u \in H^2_*(D_*) :\ u(-x,z) = -u(x,z)\ \forall (x,z) \in D_*\}.$$

Then $H^2_{\mathcal{E}}(D_*) \perp H^2_{\mathcal{O}}(D_*)$, $H^2_*(D_*) = H^2_{\mathcal{E}}(D_*) \oplus H^2_{\mathcal{O}}(D_*)$ and, for all $u \in H^2_*(D_*)$, we denote by $u^V \in H^2_{\mathcal{E}}(D_*)$ and $u^T \in H^2_{\mathcal{O}}(D_*)$ its components according to this decomposition. The space $H^2_{\mathcal{E}}(D_*)$ contains the *vertical displacements* of the plate whereas the space $H^2_{\mathcal{O}}(D_*)$ contains the *torsional displacements*. We use this decomposition in order to write any solution of (19) as

$$u(\xi,t) = u^V(\xi,t) + u^T(\xi,t)\,, \tag{21}$$

that is, by emphasising its vertical and torsional parts whose sketches are given in Figure 9 where the grey pictures refer to the equilibrium position whereas the dots (both black and grey) represent the position of the section of the sustaining cables that are linked to the deck through hangers.

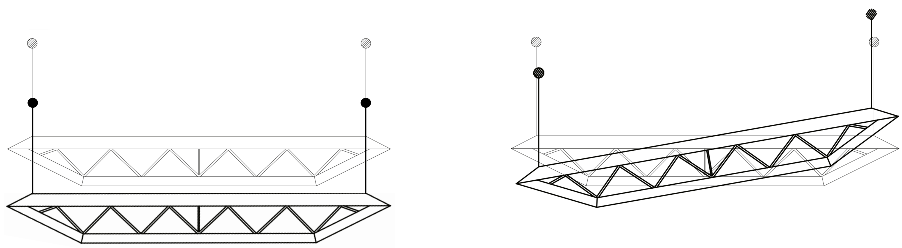


FIGURE 9. On the left (resp. right), vertical (resp. torsional) displacement of a deck (cross-section).

Definition 1 (Torsional stability/instability)**.** *We say that $g = g(\xi,t)$ makes the system* (19) *torsionally stable if every solution of* (19)*, written in the form* (21)*, satisfies $\|u^T_t(t)\|_{L^2} + \|u^T(t)\|_{H^2_*} \to 0$ as $t \to \infty$. We say that $g = g(\xi,t)$ makes the system* (19) *torsionally unstable if there exists a solution of* (19) *such that* $\limsup_{t\to\infty}(\|u^T_t(t)\|_{L^2} + \|u^T(t)\|_{H^2_*}) > 0$.

In some cases of winds acting on the deck of a bridge, the lift force g does not depend on the space variable ξ, that is $g = g(t)$, even when computed pointwise. In some other cases, in particular if the towers interact with the flow, g may depend on the longitudinal position z. In most cases, it does not depend on x: therefore, it is reasonable to assume that it is even with respect to x.

We now give sufficient conditions for the torsional stability.

Theorem 9. [37] *Let $S > 0$, assume that g is even with respect to x and*

$$g \in C^0(\mathbb{R}_+, L^2(D_*)) \ , \qquad g_\infty = \limsup_{t\to\infty} \|g(t)\|_{L^2} < +\infty.$$

• *For any $\delta > 0$ there exists $g_0 = g_0(\delta, S) > 0$ such that if $g_\infty < g_0$, then g makes the system* (19) *torsionally stable.*
• *For any $g_\infty > 0$ there exists $\delta_0 = \delta_0(g_\infty, S) > 0$ such that if $\delta > \delta_0$, then g makes the system* (19) *torsionally stable.*

Several comments are in order. Theorem 9 ensures torsional stability provided that the lift force g is sufficiently small or the damping parameter δ is sufficiently large. In real life, the latter statement appears more interesting: given a maximal intensity of the wind in the region where the bridge will be built, one can design a structure that remains torsionally stable under that wind, provided one inserts strong enough dampers. Theorem 9 is not a perturbation statement, both the constants g_0 and δ_0 can be explicitly computed, see [37]. In fact, the nonlinearity plays against the torsional stability: for large S, g_0 needs to be small whereas δ_0 needs to be large. Theorem 9 is somehow sharp; numerical results in [37] show that if g_∞ is large or δ is small, then torsional instability appears. We believe that this is what happened on the day of the TNB collapse: the wind was too strong compared with the structural damping and this lead not only to large vertical oscillations but also to their instability, that triggered the destructive torsional oscillation.

6. (I can get now) interaction

After analysing the behaviour of the fluid around a fixed or moving body (Section 4) and the behaviour of a bridge excited by a given lift force (Section 5.2), we show here that we can get no satisfaction from a full wind-structure interaction model, where not only the fluid generates movements of structures but also the movements of the structure modify the fluid flow.

Most of the current fluid-structure interaction models that are used in practical applications rely on experimental and numerical tools. In the case of wind-bridge interaction, these tools, that are nowadays consolidated, are fairly simple and are based on the following assumptions: the wind is considered ergodic and stationary, the bridge behaviour is considered linear, the aerodynamic loads are governed by linear laws. As we have seen in Section 5, see also [176], the assumption of linear behaviour of bridges is unreasonable. In the Engineering literature, the studies started from the approaches used in the aeronautical field almost one century ago since the works of Küssner [173], Sears [263, 299], Wagner [302] and Theodorsen [302], and later applied to wind engineering by Davenport [302], Scanlan [75, 259] and others. The aeroelastic problem was initially studied on simple geometries like flat plates, where simplified analytical solutions are achievable, and then extended to more complex shapes like airfoils or deck bridges through semi-empirical methodologies. In the Mathematical literature, most of the contributions to fluid-structure interactions are numerical. The reason is that even simple models give rise to extremely difficult problems: already well-posedness turns out to be quite challenging. Let us survey some of the existing models and results.

After the seminal paper of Serre [267], the breakthrough theoretical results on fluid-structure interaction appeared around 2000, see [61, 62, 79, 80, 129, 144, 145]. For a finite number of rigid bodies and incompressible as well as compressible fluid models, we refer to Desjardins and Esteban [80]. We recall here the simpler case of one spherical body following Conca, San Martín and Tucsnak [62]. Let $A \subset \mathbb{R}^3$ be an open bounded set representing the domain occupied by both the fluid and the body, assumed to be a moving ball of radius 1. Denote, respectively, by $\Omega_t \subset A$ and $B_t = A \setminus \Omega_t$ the parts of A occupied by the fluid and the body at a given instant t. Then the system of equations modelling this fluid-structure interaction reads

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0\,, \quad \nabla\cdot u = 0 \quad \text{in } \Omega_t\,,\ t>0\,,\\ u = 0 \quad \text{on } \partial A\,,\ t>0\,, \quad u = h'(t) - \omega(t)\wedge n \quad \text{on } \partial B_t\,,\ t>0\,,\\ Mh''(t) = -\int_{\partial B_t} \sigma n\,,\ t>0\,, \quad J\omega'(t) = \int_{\partial B_t} n\wedge\sigma n\,,\ t>0\,,\\ u(x,0) = u_0(x) \quad \text{in } \Omega_0\,, \quad h'(0) = h_1 \in \mathbb{R}^3\,, \quad \omega(0) = \omega_0 \in \mathbb{R}^3\,. \end{cases} \tag{22}$$

In the above system, the unknowns are $u(x,t)$, $h(t)$ and $\omega(t)$, namely the velocity field of the fluid, the position of the center of the ball and the angular velocity of the ball, respectively. Therefore, the second identity in $(22)_2$ imposes the no-slip condition at the fluid-solid interface whereas $(22)_3$ expresses the conservation of linear and angular momentum for the body (as in (5), σ denotes the rate of strain tensor of the fluid). The existence of weak solutions, up to collision, for problem (22) is established in the following theorem.

Theorem 10. [62] *Assume that the open set $\widetilde{A} = \{x-y\ |x,y\in A\}$ has smooth boundary. Given $h_0 \in A$ such that $dist(h_0,\partial A) > 1$, suppose that (u_0,h_1,ω_0) is an element of the following space:*

$$\mathbb{H}_{h_0} = \{(v,\ell,k) \in L^2(\widetilde{A})\times\mathbb{R}^3\times\mathbb{R}^3\ |\nabla\cdot v = 0 \ \text{in}\ \widetilde{A},\\ v\cdot n = 0 \ \text{on}\ \partial\widetilde{A},\ v|_{B_1}(y) = \ell + k\times y,\ v|_{E_{h_0}} = 0\},$$

where B_1 is the unit ball of $\mathbb{R}^3$ and $E_{h_0} = \widetilde{A}\setminus(A-h_0)$. Then, there exists $T_0>0$ such that the problem (22) has a weak solution (U,h,w) for any $T<T_0$. Moreover, one of the following alternatives holds true:

$$T_0 = +\infty \qquad or \qquad \lim_{t\to T_0} dist(B(t),\partial A) = 0. \tag{23}$$

The "no-contact" assumption is crucial. Starovoitov [272] proved that there exist at least two generalised solutions to the problem if collisions of the body with the boundary of the flow region are allowed. These solutions are distinguished by the behaviour of the body after collision with the boundary: in the first solution, the body moves away from the boundary after the collision while in the second solution, the body and the boundary remain in contact. Also, in the case of a compressible fluid, Feireisl [91] constructed a solution in which a ball remains attached to the top surface of the cavity A regardless of the intensity of the gravity force, thus showing that collisions may lead to non-physical situations in a standard mathematical framework. The problem discussed in Theorem 10 was also tackled for the Euler equations [230]. For further developments, we refer to [45, 50, 68, 86, 92, 113, 116, 132, 153, 222, 256, 279] and the references therein. A uniqueness result has been obtained by Glass and Sueur [128] (both when the fluid is governed by the Euler equations or the Navier-Stokes equations). It is also worth mentioning that fluid-structure interaction problems have been considered for compressible fluids in [41–43, 46] and stabilisation or control issues have been tackled e.g. in [11, 12, 44, 280].

Related to the unrealistic situation discovered in [91] lies the *no-collision paradox*, firstly encountered by O'Neill et al. [63, 64, 77, 229] during the 1960s. By considering a rigid sphere, immersed in a stationary Stokes flow and falling over a flat wall, they showed that the drag over the body diverges rapidly as it approaches the ramp, thus impeding the sphere from touching the wall in finite time. The paradox was later extended to the case of a Navier-Stokes flow, first in $2D$ and subsequently in $3D$ [138, 139]. Only frontal collisions are taken into account in those papers. In the $3D$ setting, as shown in [140], grazing collisions between smooth bodies can occur. Here we just recall a result by Gérard-Varet and Hillairet [127] who, in an attempt to explain the no-collision paradox, consider a general solid body $S_t \subset A$ and take into account that if the distance between ∂A and ∂S_t becomes very small (less than 10^{-6}m), the no-slip condition is no longer accurate and must be replaced by the following Navier condition:

$$\begin{cases} (u - u_S) \cdot n = 0, \quad (u - u_S) \wedge n = -2\alpha(\sigma \cdot n) \wedge n \ \text{ on } \ \partial S_t \\ u \cdot n = 0, \quad u \wedge n = -2\beta(\sigma \cdot n) \wedge n \ \text{ on } \ \partial A, \end{cases} \tag{24}$$

where $u_S(x,t) = h'_S(t) + \omega(t) \wedge (x - h(t))$ is the velocity, at every point x of the solid body S_t, whose center of mass is in position $h(t) \in \mathbb{R}^3$ at time $t > 0$. In (24), impermeability is ensured by imposing that the normal component of the relative velocity of the fluid is zero, whereas the coefficients $\alpha, \beta > 0$ are the so-called slip lengths (note that the tangential component of the relative velocity may exhibit discontinuities). The existence of weak solutions, up to collision, for problem (22)-(24) (exchanging B_t by S_t) is established in the following:

Theorem 11. [127] *Let $S \subset A$ be two $C^{1,1}$ bounded domains of $\mathbb{R}^3$. Let $u_0 \in \overline{\mathcal{D}(A)}^{L^2(A)}$, with $\mathcal{D}(A)$ being the subspace of solenoidal vector fields belonging to $C_0^\infty(A)$, and assume that there exist $V, W \in \mathbb{R}^3$ such that $u_0^S(x) = V + W \wedge (x - h(0))$, for every $x \in \partial S$. Furthermore, suppose that $(u_0 - u_0^S) \cdot n = 0$ on ∂S. Then, there exists $T_0 \in (0, +\infty]$ and a weak solution of (22)-(24) over $[0, T)$ associated to the initial data u_0 and u_0^S. Moreover, such a weak solution exists up to collision, that is, the alternative (23) holds.*

Further theoretical results are related to models with a linear elastic hyperbolic-type equation describing the dynamics of the solid, by the Euler equations [53] or the Navier-Stokes equations [13, 14] or the Stokes equations [184] for the dynamic of the fluid, and by suitable Neumann-type transmission boundary conditions (see also [152] for the case of a non-Newtonian fluid). A major difficulty is then to deal with the mismatch between parabolic and hyperbolic regularity and, so far, only very few satisfactory regularity results have been obtained [10, 14, 184], thereby proving that the setting is correct. Nonlinear plates interacting with fluids have also been studied [54, 79, 218]. In fact, there are further models, with nonstandard interface conditions [185], with mechanical damping [186] or stochastic forcing [56]. Finally, let us mention the survey [55] where a variety of models mathematically describing the interaction between flows and oscillating structures are discussed.

7. Shape of things to come

Most of the phenomena discussed in this paper lead to shape optimization problems. In this section we raise some questions on the shapes of bridges to come.

The vortex pattern in dependence of the Reynolds number, as well described for circular cylinders in Figures 7 and 8, depends furthermore strongly on the shape of the bluff body. Davis and Moore [76] were among the first to study numerically the vortex formation around rectangles, see also the incomplete list of references [1, 40, 58, 85, 196, 198, 227, 311] for subsequent developments. Figure

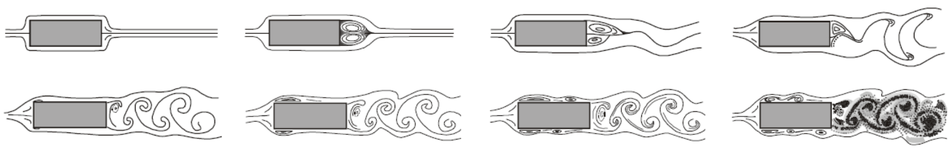


FIGURE 10. From [1], vortices around a rectangle for different Re. From left to right, top to bottom: $\mathrm{Re}<40$, $38<\mathrm{Re}<46$, $42<\mathrm{Re}<56$, $50<\mathrm{Re}<150$, $100<\mathrm{Re}<200$, $150<\mathrm{Re}<250$, $200<\mathrm{Re}<350$, $\mathrm{Re}>350$.

10 displays the pictures taken from [1] sketching the vortex pattern around a rectangle for varying Reynolds numbers; this should be compared with Figure 7. It turns out that vortices on the horizontal sides appear at $\mathrm{Re}\approx100$. As explained for instance in [228], the flow patterns are strongly dependent of the aspect ratio of the rectangle, as depicted in Figure 11 obtained with the OpenFOAM toolbox [226], through the use of the SIMPLE algorithm for the numerical resolution of the steady-state Navier-Stokes equations in laminar regime, see [48]. For short rectangular cylinders the flow detaches completely and the wake structure is similar to the case of a circular cylinder. For more elongated rectangular cylinders, the flow exhibits a large-scale separation at the leading-edge and also a reattachment before the definitive separation at the trailing-edge. This is in sharp contrast with the case of a circular cylinder. Instabilities in the wake are observed in all bluff bodies but only long bluff bodies present further instabilities due to the separating and

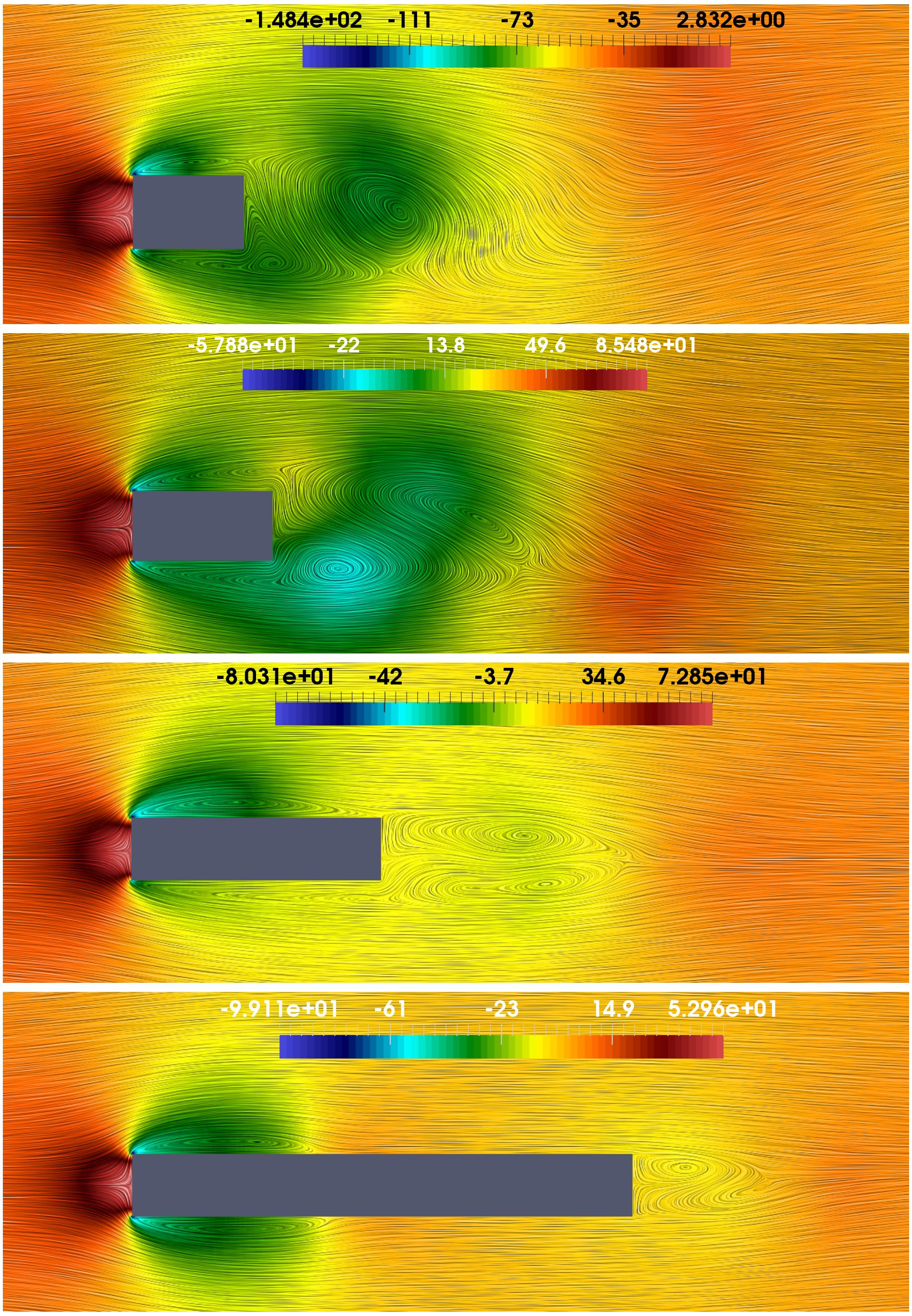


Figure 11. CFD simulation of a 2D flow around rectangles of different aspect ratio: 1.5, 2, 4 and 8 respectively. The scale indicates the pressure.

reattaching leading-edge shear layer. We refer for instance to [51, 59] for detailed studies of this complex mechanism. As highlighted in [58]:

Despite the fact that these kind of flows have been the subject of several numerical and experimental studies, the topic is still attractive [...] the interest is given by the fact that both experimental and numerical techniques appear to be unable to tackle the problem in an unequivocal way. Indeed, a large variability of results is found in the literature [...] The reason of these discrepancies is the high sensitivity of the flow on the test boundary conditions and measurement accuracy in experiments and on the turbulence model, numerical schemes and mesh properties in CFD analysis.

The Strouhal number (St) is a dimensionless parameter introduced to describe the frequency of the vortex-induced vibrations arising from the vortex shedding. It is usually defined by

$$\mathrm{St} = \frac{\omega L}{W},$$

where ω is the frequency of the vortex shedding, L is the characteristic length (such as the hydraulic diameter or the airfoil/deck thickness) and W is the flow velocity. For rectangular cross sections of decks of bridges, having thickness b and width d, the Eurocode1 [89, Figure E.1] suggests the dependence of the Strouhal number St on the aspect ratio d/b as plotted in the left picture of Figure 12. But this characterization is not unanimously accepted by the entire engineering community: in the (fairly discrepant) right picture of Figure 12 we reproduce several different values computed by Japanese engineers [227, 268].

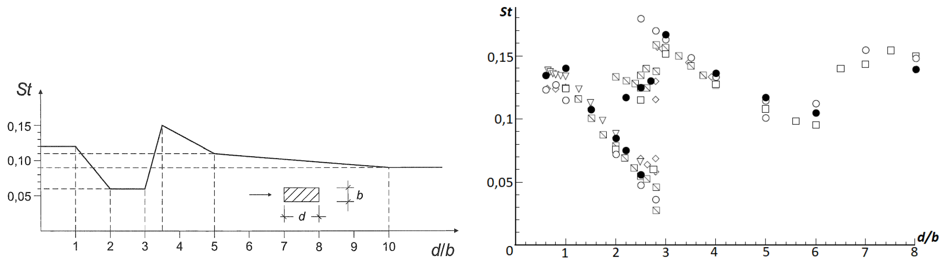


FIGURE 12. Strouhal number in dependence of the thickness/width ratio. On the left: Eurocode1 [89, Figure E.1]. On the right: data taken from [268]; each symbol corresponds to a different source.

To simplify the description of the phenomenon, the main vortex shedding regimes have been defined and classified on the basis of the characteristics of the main vortices involved. Each vortex type allows a specific dynamical state. Under the flow conditions above mentioned and for a wide range of Reynolds numbers, the aspect ratio d/b affects the vortex shedding and the loading on the structure significantly.

The wind force acting in the transversal direction depends on several structural parameters and on the air density [89, (8.2); 269; 308], whereas the frequency of the vortices is computed in terms of the Strouhal number, of the scalar wind velocity and of the cross-wind direction of the structure considered [133, (1)]. In turn, the Strouhal number is usually computed experimentally and strongly depends on the shape of the structure [89, E.1.3.2]. But the suggested values appear more like "reasonable values" rather than "exact values", see [99, Figure 7]: in particular, it appears unlikely that the Strouhal number varies as a piecewise affine function in terms of the ratio of a rectangular cross section, as suggested in [89, Figure E.1]. Needless to say, this picture appears unrealistic from a purely mathematical point of view. Determining how the Strouhal number affects the drag/lift forces in dependence of the wind flow is of crucial importance for the stability of bridges. This raises the following question.

Problem 3. *Find a more reliable rule for the dependence* $\mathrm{St}=\mathrm{St}(d/b)$*, possibly by smooth interpolation of experimental results. Does there exist a minimum? What are its physical and geometrical characterizations?*

One of the great challenges in engineering consists in deducing the drag and lift forces on the structure depending on the parameters of the flow (wind speed, angle of attack, shape of the obstacle, etc.). Assuming that the fluid is governed by the Navier-Stokes equations (10) in $\Omega = \{(-L,L)^2 \times (0,\Lambda)\} \setminus D$, where D *does not* intersect the boundary of the box $(-L,L)^2 \times (0,\Lambda)$, there exists an overwhelming literature concerning the drag-minimisation problem in terms of a boundary control of the velocity field, specially in the works of Fursikov et al. [106–108]. But one can also exploit the explicit form (7) in order to seek the optimal shape of the body D (with given volume) that minimises the drag. This optimization problem was tackled in two seminal papers by Pironneau [236, 237] who associates to (10) the following linear problem:

$$-\nu\Delta w+(w\cdot\nabla)u-(u\cdot\nabla)w=(u\cdot\nabla)u+\nabla q,\quad \nabla\cdot w=0 \quad \text{in } \Omega, \qquad w=0 \text{ on } \partial\Omega,$$

where u solves (10) under the boundary conditions

$$u=v \text{ on } \partial\{(-L,L)^2\times(0,\Lambda)\}\,, \qquad u=0 \text{ on } \partial D\,. \tag{25}$$

In optimal control theory, w is called the co-state vector of u although it has no simple mechanical interpretation. The following result holds.

Theorem 12. [237] *Assume that the boundary datum* v *in* (25) *is sufficiently small, so that the solution* u *of* (10)-(25) *is unique. Then the body* D_0 *of given volume minimising the drag energy dissipation* (7)*, if any, must be such that*

$$\left|\frac{\partial u}{\partial n}\right|^2+2\frac{\partial w}{\partial n}\cdot\frac{\partial u}{\partial n} \quad \textit{is constant a.e. on } \partial D_0\,.$$

As far as we are aware, this optimization problem has not been fully solved from a theoretical point of view: in particular it is not clear in which class of obstacles the minimiser has to be sought, see [22]. Some progress has been done numerically [217] and for the Stokes case [49]. Moreover, much less is known for the lift and, therefore, a further natural shape optimization problem arises.

Problem 4. *Find a reliable lift force formula. For a given cross-sectional area of the deck of a suspension bridge, find the optimal shape minimising the (effects of the) lift force. Same problem for the full* $3D$ *deck (here the extreme parts of the obstacle are fixed on the boundary of the box).*

This problem was recently addressed numerically in [123] but its full solution still appears very far away. In fact, there exists a guideline for European engineers, the Eurocode1 [89], but it mainly refers to short-span bridges (less than 200m) and to particular situations that exclude cable-suspended bridges [89, p.9]. The Annex E [89, p.114] provides criteria for the study of the effect of vortex shedding and aeroelastic instabilities but, as we shall see below, the rules and parameters for the computation of the lift are not unanimously shared. The deck hit transversally by the flow creates vortices that generate the forcing lift which starts the vertical oscillations of the bridge, see the sketch of a 2D cross section in the left picture in Figure 13, yielding the vertical displacement of the whole 3D deck, see the right picture in Figure 13. This qualitative explanation is shared by the entire engineering community [183, 258, 262] and it has been experimented in wind tunnels. As we have seen, it is not supported by a theoretical approach or a sound modelling. The air and the deck occupy *time-dependent domains*, denoted respectively

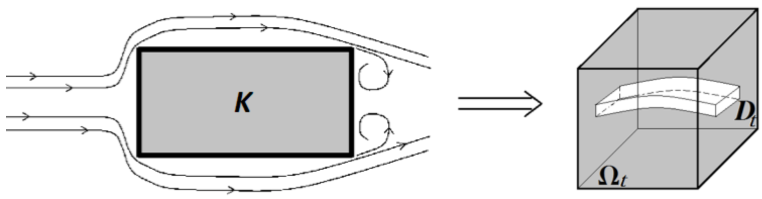


FIGURE 13. Vortex shedding around a cross section K (left) and its effect on the movement of the deck D (right).

by Ω_t and D_t, giving a full fluid-structure interaction flavor to the just described phenomenon. Moreover the cross section of a suspension bridge can have very unpleasant shapes (quite different from circles and rectangles!) due to side-walks, lamps, guard-rails, see the sketch in the right picture in Figure 15. More general shapes are used in engineering plans; some of them are analysed in detail by the EU regulatory of Eurocode1, see Figure 14 where we reproduce Figure 8.1 p.83 [89] (whose caption is "Cross-sections of normal construction decks").

One should compare all these figures with those in [207, Chapter 1]. From a mathematical point of view, domains with singularities require a delicate functional setting since it is known [212] that existence and regularity results for PDEs may fail even if the data are smooth. The singularities of solutions in a neighborhood of a concave corner are described through functional spaces with weighted norms [207]; see also [39, Chapter 5]. Refined regularity results for the Navier-Stokes

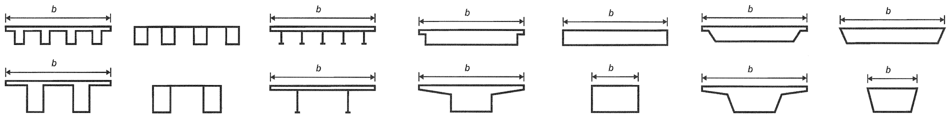


FIGURE 14. Typical cross sections of suspension bridges from the Eurocode1 [89].

equations may be obtained in convex domains [171]: the obstacle problem considered here is clearly set in a nonconvex domain but, still, the following question arises:

Problem 5. *Can the regularity results by Kozlov-Maz'ya* [171] *be extended to planar domains being the difference between two convex domains?*

Some hints and updates on this problem may be found in [211]. This combination of regularity results in PDEs, shapes of structures, and resulting turbulence gives rise to the following interdisciplinary question.

Problem 6. *Can elliptic regularity in (theoretical) PDEs help to understand the (physical) behaviour of turbulence in reality?*

In this respect, we quote from [151, p.380]

> *the dimples on a golf ball are used to reduce the drag over that which would occur for a smooth golf ball. Although this is undoubtedly of great interest to the avid golfer, it is also important to engineers responsible for fluid-flow models, since it does emphasise the potential importance of the surface roughness. However, for bodies that are sufficiently angular with sharp corners, the actual surface roughness is likely to play a secondary role compared with the main geometric features of the body.*

This seems to lower the interest of Problem 6 but one should then quantify what is a “secondary role”.

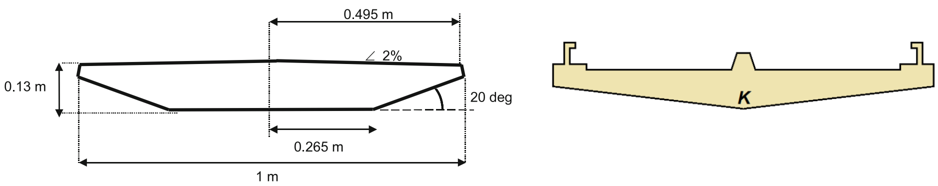


FIGURE 15. Possible cross section of a suspension bridge. Left: scaled model proposed in [84].

We tested numerically four slight variants of deck shapes, each one having a characteristic length of 1m and a characteristic height of 0.13m, based on the design proposed by Diana et al. [84, Figure 1] reproduced on the left of Figure 15, also obtained with the OpenFOAM toolbox. The main purpose being a qualitative comparison of the aerodynamic response of those shapes (in terms on the intensity of the developed vortices in the wake of each obstacle), each numerical experiment was carried out with a Reynolds number of 10^4. The pressure distribution as well as the streamlines of the velocity field are displayed in Figure 16. Observe that the singularities of the cross-section create more vortices and more separation/reattachment points; in particular, the shape in the bottom figure influences the fluid at larger distance.

The Eurocode1 [89] suggests to consider the force due to the vortex shedding as a periodic function of time, whose frequency can be determined in terms of the wind velocity, of the Strouhal number and other parameters. If g in (19) is taken periodic, then one expects the existence of periodic solutions whose stability plays a crucial role in the overall stability of (19): this was studied in the recent work [37] as we now recall. The wind flow creates vortices around the deck of a suspension bridge and the vortices increase the internal energy of the structure, generating wide vertical oscillations which look somehow periodic in time. In fact, the oscillations in suspension bridges display a prevailing mode of oscillation: from [2, p.20] we learn that, in the months prior to the Tacoma collapse, *one principal mode of oscillation prevailed and the modes of oscillation frequently changed.* Hence, even if the oscillations are governed by (disordered) forcing and damping, the resulting oscillation is simple to describe. In fact, after some transient time (depending on the size of the damping parameter δ), it can be shown that the motions asymptotically approach periodic movements, at least in some particular situations.

Theorem 13. [37] *Let $S > 0$, assume that $g \in C^0(\mathbb{R}_+, L^2(D_*))$, that g is τ-periodic for some $\tau > 0$, and let $g_\infty = \max_{t\in[0,\tau]} \|g(t)\|_{L^2}$. There exists $g_0 = g_0(\delta, S) > 0$ such that if $g_\infty < g_0$, then:*
- *the equation* (19) *admits a unique periodic solution U^p;*
- *there exists $\eta > 0$ such that*

$$\lim_{t\to\infty} \mathrm{e}^{\eta t}\Big(\|u_t(t) - U_t^p(t)\|_{L^2}^2 + \|u(t) - U^p(t)\|_{H_*^2}^2\Big) = 0$$

for any other solution u of (19).

Theorem 13 states that small periodic forcing terms (such as small vortex shedding) generate a *stable* prevailing mode of oscillation to which all the solutions approach as t increases. But since each periodic movement (or mode of oscillation) has its own threshold of instability, see [9, 19, 25, 27], it is more realistic to address the following fluid-structure interaction problem.

Problem 7. *Find a precise rule giving the prevailing mode of oscillation of a plate in dependence of the features (velocity, angle of attack, Reynolds number, turbulence) of the fluid hitting the plate.*

An answer to this problem would be very useful for an unimodal analysis, as suggested by the Eurocode1 [89, p.9 and p.118].

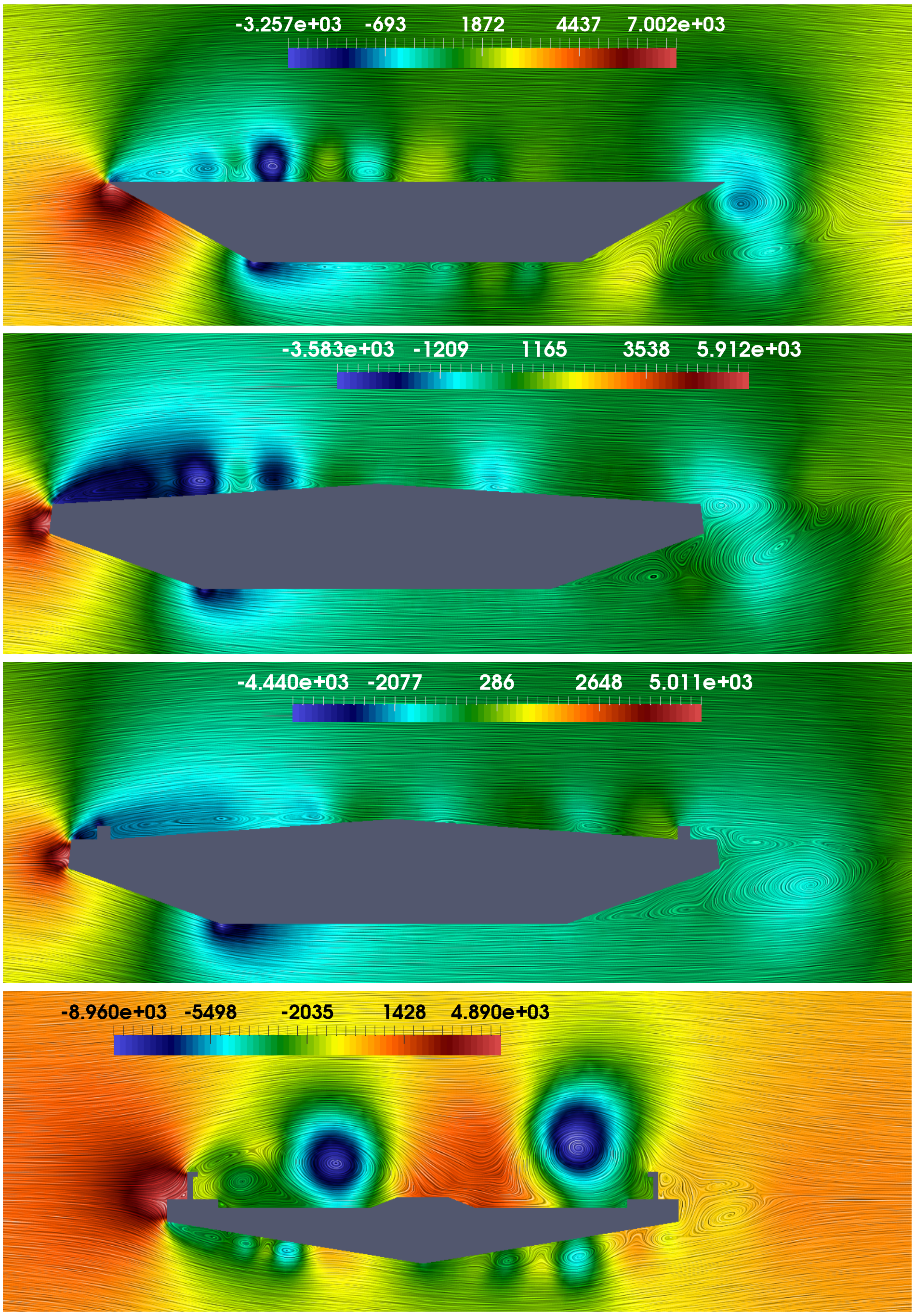


FIGURE 16. CFD simulation of a 2D flow with Reynolds number 10^4 around different obstacles. The scale indicates the pressure.

It is clear that (19) does not take into account that the deck may be composed by several spans, in general three, or even many more as it is the case of the San Francisco-Oakland Bay Bridge. The oscillations on each span induce oscillations on the adjacent spans, although a precise rule is unknown. From a purely theoretical point of view, a detailed spectral analysis has been recently performed in [118] and the interaction between spans has been analysed for several prototype nonlinearities. Surprising behaviours of the vortex shedding have been detected experimentally in case of asymmetric spans [250]. All this raises further fog on the turbulent behaviour of the wind flow in case of fluid-structure interactions.

Problem 8. *In a suspension bridge, determine the optimal ratio between spans in order to maximise the stability and to lower the impact of the turbulent action of the wind. Does the optimal configuration have equal side spans?*

Throughout the paper, we have seen how symmetry influences both the vortex formation and the behaviour of drag and lift forces. Clearly, also the symmetry of the structure influences its stability. In Section 3 we also recalled one of Birkhoff's thought [32, p.21]

a symmetrically stated problem may not have any stable symmetric solution.

As a conclusion, we give our belief which is going even one step further: **in general, symmetry plays against stability, both in fluids and structures.**

References

[1] O. Almeida, S.S. Mansur, A. Silveira-Neto, On the flow past rectangular cylinders: physical aspects and numerical simulation, Engenharia Térmica (Thermal Engineering) 7, 55-64 (2008) 32

[2] O.H. Ammann, T. von Kármán, G.B. Woodruff, The failure of the Tacoma Narrows Bridge, Federal Works Agency, Washington D.C., 1941. 24, 38

[3] H. Aref, Point vortex dynamics: A classical mathematical playground, J. Math. Phys. 48, 1-22 (2007) 6

[4] H. Aref, P.K. Newton, M.A. Stremler, T. Tokieda, D. Vainchtein, Vortex crystals, Adv. Appl. Mech. 39, 1-79 (2002) 6

[5] T. Argentini, S. Muggiasca, G. Diana, D. Rocchi, Aerodynamic instability of a bridge deck section model: Linear and nonlinear approach to force modeling. Journal of Wind Engineering and Industrial Aerodynamics 98, 363-374, ISSN: 0167-6105 (2010) 5

[6] T. Argentini, D. Rocchi, A. Zasso, Aerodynamic interference and vortex-induced vibrations on parallel bridges: The Ewijk bridge during different stages of refurbishment. Journal of Wind Engineering and Industrial Aerodynamics 147, 276-282 (2015) 3

[7] G. Arioli, F. Gazzola, A new mathematical explanation of what triggered the catastrophic torsional mode of the Tacoma Narrows Bridge collapse, Appl. Math. Modelling 39, 901-912 (2015) 26

[8] G. Arioli, F. Gazzola, On a nonlinear nonlocal hyperbolic system modeling suspension bridges, Milan J. Math. 83, 211-236 (2015) 26

[9] G. Arioli, F. Gazzola, Torsional instability in suspension bridges: the Tacoma Narrows Bridge case, Communications in Nonlinear Science and Numerical Simulation 42, 342-357 (2017) 26, 38

[10] G. Avalos, I. Lasiecka, R. Triggiani, Higher regularity of a coupled-hyperbolic fluid-structure interactive system, Georgian Mathematical Journal 15, 403-437 (2008) 32

[11] M. Badra, T. Takahashi, Feedback stabilization of a simplified 1d fluid–particle system, Ann. I. H. Poincaré – AN 31, 369–389 (2014) 31

[12] M. Badra, T. Takahashi, Feedback boundary stabilization of 2d fluid-structure interaction systems. Discrete and Continuous Dynamical Systems - Series A, 37 (5), 2315-2373 (2017) 31

[13] V. Barbu, Z. Grujic, I. Lasiecka, A. Tuffaha, Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model, Fluids and waves, Contemporary Mathematics 440, 55-82 (2007) 32

[14] V. Barbu, Z. Grujic, I. Lasiecka, A. Tuffaha, Smoothness of weak solutions to a nonlinear fluid-structure interaction model, Indiana University Mathematics Journal 57, 1173-1207 (2008) 32

[15] C. Bardos, M.C. Lopes Filho, D. Niu, H.J. Nussenzveig Lopes, E.S. Titi, Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking, SIAM J. Math. Anal. 45, 1871-1885 (2013) 11

[16] C. Bardos, E. Titi, Mathematics and turbulence: where do we stand?, Journal of Turbulence 14:3, 42-76 (2013) 3

[17] T. Bartsch, A. Pistoia, Critical points of the N-vortex Hamiltonian in bounded planar domains and steady state solutions of the incompressible Euler equations, SIAM J. Appl. Math. 75, no. 2, 726-744 (2015) 6

[18] G.K. Batchelor, An Introduction to Fluid Dynamics. Cambridge University Press, 2000 5, 17

[19] U. Battisti, E. Berchio, A. Ferrero, F. Gazzola, Energy transfer between modes in a nonlinear beam equation, J. Math. Pures Appl. 108, 885-917 (2017) 26, 38

[20] C. Begue, C. Conca, F. Murat, O. Pironneau, Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression, Nonlinear PDE and their applications - Collège de France seminar - H. Brézis, J.L. Lions Eds. - Vol.IX, 179-264 (1988) 20

[21] J.A. Bello, E. Fernández-Cara, J. Lemoine, J. Simon, The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes flow, SIAM J. Control Optim. 35, 626-640 (1997) 12

[22] J.A. Bello, E. Fernández-Cara, J. Lemoine, J. Simon, On drag differentiability for Lipschitz domains, Control of partial differential equations and applications (Laredo, 1994), 11-22, Lecture Notes in Pure and Appl. Math., 174, Dekker, New York (1996) 35

[23] T.B.Benjamin, Bifurcation phenomena in steady flows of a viscous fluid I. Theory, Proc. R. Soc. Lond. A. 359, 1-26 (1978) 20

[24] E. Berchio, D. Buoso, F. Gazzola, On the variation of longitudinal and torsional frequencies in a partially hinged rectangular plate, ESAIM COCV 24, 63-87 (2018) 24

[25] E. Berchio, A. Ferrero, F. Gazzola, Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions, Nonlin. Anal. Real World Appl. 28, 91-125 (2016) 26, 38

[26] E. Berchio, F. Gazzola, A qualitative explanation of the origin of torsional instability in suspension bridges, Nonlin. Anal. TMA 121, 54-72 (2015) 26

[27] E. Berchio, F. Gazzola, C. Zanini, Which residual mode captures the energy of the dominating mode in second order Hamiltonian systems?, SIAM J. Appl. Dyn. Syst. 15, 338-355 (2016) 26, 38

[28] E. Berger, R. Wille, Periodic Flow Phenomena, Annual Review of Fluid Mechanics 4:1, 313-340 (1972) 17

[29] H.M. Berger, A new approach to the analysis of large deflections of plates, J. Appl. Mech. 22, 465-472 (1955) 27

[30] F. Bethuel, H, Brezis, F. Hélein, Ginzburg-Landau vortices, Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, 1994 6

[31] K.Y. Billah, R.H. Scanlan, Resonance, Tacoma Narrows Bridge failure, and undergraduate physics textbooks, Amer. J. Physics 59, 118-124 (1991) 24, 25

[32] G. Birkhoff, Hydrodynamics: a study in logic, fact and similitude, Princeton University Press, 1950. 8, 40

[33] G. Birkhoff, Hydrodynamics: a study in logic, fact, and similitude. Second edition, revised and enlarged. Princeton University Press, 1960. 9

[34] F. Bleich, Dynamic instability of truss-stiffened suspension bridges under wind action, Proceedings ASCE 74, 1269-1314 (1948) 24

[35] D. Bloor, The Enigma of the Aerofoil: Rival Theories in Aerodynamics, 1909–1930. The University of Chicago Press 2011 9

[36] F. Bleich, C.B. McCullough, R. Rosecrans, G.S. Vincent, The mathematical theory of vibration in suspension bridges, U.S. Dept. of Commerce, Bureau of Public Roads, Washington D.C., 1950 24

[37] D. Bonheure, F. Gazzola, E. Moreira dos Santos, Periodic solutions and torsional instability in a nonlinear nonlocal plate equation, arXiv:1809.09783v1 - `https://arxiv.org/abs/1809.09783` 26, 28, 29, 38

[38] D. Bonheure, F. Gazzola, E. Moreira dos Santos, Mathematical study of the stability of suspended bridges: focus on the fluid-structure interactions, Research proposal FRB 2018-J1150080-210411 BONHEURE, submitted to the Thelam Fund (Fondation Roi Baudouin, Belgium), March 2018, Available online at `http://homepages.ulb.ac.be/~dbonheur/Thelam.pdf` 11

[39] M. Borsuk, V. Kondrat'ev, Elliptic boundary value problems of second order in piecewise smooth domains, Elsevier (2006) 36

[40] A. Bosco, L. Bruno, N. Costez, D. Fransos, 3D flow around a rectangular cylinder: a computational study, Proc. BBAA VI Intl. Colloquium on: Bluff Bodies Aerodynamics & Applications, Milano, Italy (July 20-24, 2008) 32

[41] M. Boulakia, Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid, J. Math. Pures Appl. 84, 1515–1554 (2005) 31

[42] M. Boulakia, S. Guerrero, A regularity result for a solid–fluid system associated to the compressible Navier–Stokes equations, Ann. I. H. Poincaré – AN 26, 777–813 (2009) 31

[43] M. Boulakia, S. Guerrero, Regular solutions of a problem coupling a compressible fluid and an elastic structure, J. Math. Pures Appl. 94, 341–365 (2010) 31

[44] M. Boulakia, S. Guerrero, Local null controllability of a fluid-solid interaction problem in dimension 3, J. Eur. Math. Soc. 15, 825–856 (2013) 31

[45] M. Boulakia, S. Guerrero, On the interaction problem between a compressible fluid and a Saint-Venant Kirchhoff elastic structure, Adv. Differential Equations 22, 1-48 (2017) 31

[46] D. Breit, S. Schwarzacher, Compressible fluids interacting with a linear-elastic shell. Arch. Ration. Mech. Anal. 228, 495-562 (2018) 31

[47] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations. Comm. Partial Differential Equations 25, no. 3-4, 737-754 (2000) 6

[48] L.S. Caretto, A.D. Gosman, S.V. Patankar, D.B. Spalding, Two calculation procedures for steady, three-dimensional flows with recirculation. In Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics (pp. 60-68). Springer, Berlin, Heidelberg (1973) 32

[49] F. Caubet, M. Dambrine, Stability of critical shapes for the drag minimization problem in Stokes flow, J. Math. Pures Appl. 100, 327–346 (2013) 35

[50] A. Chambolle, B. Desjardins, M.J. Esteban, C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, J. Math. Fluid Mech. 7, 368-404 (2005) 31

[51] N. Cherry, R. Hillier, M. Latour, Unsteady measurements in a separated and reattaching flow. Journal of Fluid Mechanics 144, 13-46 (1984). 34

[52] J. Chu, M. Garrione, F. Gazzola, Stability analysis in some strongly prestressed rectangular plates, preprint 26

[53] I. Chueshov, Interaction of an elastic plate with a linearised inviscid incompressible fluid, Comm. Pure Appl. Anal. 13, 1759-1778 (2014) 32

[54] I. Chueshov, Dynamics of a nonlinear elastic plate interacting with a linearised compressible viscous fluid, Nonlinear Analysis TMA 95, 650-665 (2014) 32

[55] I. Chueshov, E.H. Dowell, I. Lasiecka, J.T. Webster, Mathematical aeroelasticity: A survey, Mathematics in Engineering, Science and Aerospace 7, No.1, 5-29 (2016) 32

[56] I. Chueshov, B. Schmalfuss, Stochastic dynamics in a fluid-plate interaction model with the only longitudinal deformations of the plate, Discrete and Continuous Dynamical Systems-B 20, 833-852 (2015) 32

[57] P. Chossat, G. Iooss, The Couette-Taylor problem. Applied Mathematical Sciences, 102. Springer-Verlag, New York, 1994. 20

[58] A. Cimarelli, A. Leonforte, D. Angeli, Direct numerical simulation of the flow around a rectangular cylinder at a moderately high Reynolds number, Journal of Wind Engineering and Industrial Aerodynamics 174, 39-49 (2018) 32, 34

[59] A. Cimarelli, A. Leonforte, D. Angeli, On the structure of the self-sustaining cycle in separating and reattaching flows, Journal of Fluid Mechanics 857, 907-936 (2018) 34

[60] M. Como, S. Del Ferraro, A. Grimaldi, A parametric analysis of the flutter instability for long span suspension bridges, Wind and Structures 8, 1-12 (2005) 24

[61] C. Conca, J. San Martín, M. Tucsnak, Motion of a rigid body in a viscous fluid. C. R. Acad. Sci. Paris Sér. I Math. 328 no. 6, 473-478 (1999) 30

[62] C. Conca, J. San Martín, M. Tucsnak, Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid, Comm. Part. Diff. Eq. 25, 1019-1042 (2000) 30

[63] M.D.A. Cooley, M.E. O'Neill, On the slow rotation of a sphere about a diameter parallel to a nearby plane wall. J. Inst. Math. Appl. 4, 163-173 (1968) 31

[64] M.D.A. Cooley, M.E. O'Neill, On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or stationary sphere. Mathematika 16, 37-49 (1969) 31

[65] M. Coutanceau, R. Bouard, Experimental determination of the viscous flow in the wake of a circular cylinder in uniform translation. Part 1 Steady flow, J. Fluid Mech. 79, 231-256 (1977) 11, 17

[66] M. Coutanceau, R. Bouard, Experimental determination of the viscous flow in the wake of a circular cylinder in uniform translation. Part 2 Unsteady flow, J. Fluid Mech. 79, 257-272 (1977) 11, 17

[67] M. Coutanceau, J. R. Defaye, Circular cylinder wake configurations: A flow visualization survey, Appl. Mech. Rev. 44, 255-305 (1991) 11

[68] D. Coutand, S. Shkoller, Motion of an elastic solid inside an incompressible viscous fluid, Arch. Rational Mech. Anal. 176, 25-102 (2005) 31

[69] M.G. Crandall, P.H. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions. Arch. Rational Mech. Anal. 67 (1977), no. 1, 53-72. 20

[70] G. Crasta, A. Falocchi, F. Gazzola, A new model for suspension bridges involving the convexification of the cables, arXiv:1810.02853v1 - https://arxiv.org/abs/1810.02853 26

[71] J. le Rond d'Alembert, Theoria resistentiae quam patitur corpus in fluido motum, ex principiis omnino novis et simplissimus deducta, habita ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis compressionis partium fluidi, Berlin-Brandenburgische Akademie der Wissenschaften, Akademie-Archiv call number: I-M478 (1749) 6, 7

[72] J. le Rond d'Alembert, Essai d'une nouvelle théorie de la résistance des fluides, Paris (1752) 6, 7

[73] J. le Rond d'Alembert, Paradoxe proposé aux géomètres sur la résistance des fluides, In: Opuscules mathématiques, Vol. 5, Memoire XXXIV, I, 132-138, 1768 6, 7

[74] J. le Rond d'Alembert, Sur la résistance des fluides, In: Opuscules mathématiques, Vol. 8, Memoire LVII, 13, 210-230, 1780 6, 7
[75] A.G. Davenport, Buffeting of a suspension bridge by storm winds, J. Struct. Division ASCE 883, 233-268 (1962) 29
[76] R.W. Davis, E.F. Moore, A numerical study of vortex shedding from rectangles, J. Fluid Mechanics 116, 475-506 (1982) 32
[77] W.R. Dean, M.E. O'Neill, A slow motion of viscous liquid caused by the rotation of a solid sphere. Mathematika 10, 13-24 (1963) 31
[78] N.J. Delatte, Beyond failure, ASCE Press, 2009 24
[79] B. Desjardins, M.J. Esteban, Existence of weak solutions for the motion of rigid bodies in a viscous fluid, Arch. Rational Mech. Anal. 146, 59-71 (1999) 30, 32
[80] B. Desjardins, M.J. Esteban, On weak solutions for fluid-rigid structure interaction: compressible and incompressible models, Comm. Partial Differential Equations 25, 1399–1413 (2000) 30
[81] N. de Divitiis, Bifurcations analysis of turbulent energy cascade, Annals of Physics 354 604–617 (2015) 21
[82] G. Diana, G. Fiammenghi, Wind tunnel tests and numerical approach for long span bridges: the Messina bridge, The Seventh International Colloquium on Bluff Body Aerodynamics and its Applications (BBAA7) Shanghai, China, September 2-6 (2012) 3
[83] G. Diana, F. Resta, M. Belloli, D. Rocchi, On the vortex shedding forcing on suspension bridge deck, Journal of Wind Engineering and Industrial Aerodynamics 94, 341-363 (2006) 3
[84] G. Diana, D. Rocchi, T. Argentini, S. Muggiasca, Aerodynamic instability of a bridge deck section model: Linear and nonlinear approach to force modeling. Journal of Wind Engineering and Industrial Aerodynamics, 98(6-7), 363-374 (2010) 37
[85] H.N. Dixit, A. Sameen, R. Govindarajan, Flow past a rectangular cylinder in a stratified fluid, Proc. Sixth Intl. Symposium on Stratified Flows (Ed: G.N. Ivey), 189-194. Perth, Australia (December, 2006) 32
[86] T.V. Duoc, Navier-Stokes-Oseen flows in the exterior of a rotating and translating obstacle, Discrete Contin. Dyn. Syst. 38, 3387-3405 (2018) 31
[87] C. Dyrbye, S.O. Hansen, Wind Loads on Structures, Wiley, 1997 3
[88] Encyclopedia of Computational Mechanics. Edited by Erwin Stein, René de Borst and Thomas JR Hughes, John Wiley & Sons (2004) 3
[89] Eurocode1, Actions on structures. Parts 1-4: General actions - Wind actions, The European Union Per Regulation 305/2011, Directive 98/34/EC & 2004/18/EC. `http://www.phd.eng.br/wp-content/uploads/2015/12/en.1991.1.4.2005.pdf` 34, 35, 36, 37, 38
[90] L. Fatone, P. Gervasio, A. Quarteroni, Multimodels for incompressible flows, *J. Math. Fluid Mech.* **2** 126–150 (2000) 21, 23
[91] E. Feireisl, On the motion of rigid bodies in a viscous compressible fluid, Arch. Rational Mech. Anal. 167 (2003) 281-308. 31
[92] E. Feireisl, On the motion of rigid bodies in a viscous incompressible fluid, J. Evol. Equ. 3, no. 3, 419–441 (2003) 31
[93] P. F. Fischer, F. Loth, S.E. Lee, S.-W. Lee, D.S. Smith, H. S. Bassiouny, Simulation of high-Reynolds number vascular flows, Computer Methods in Applied Mechanics and Engineering, Volume 196, Issues 31-32, 3049-3060 (2007) 4
[94] V. Ferreira, F. Gazzola, E. Moreira dos Santos, Instability of modes in a partially hinged rectangular plate, J. Diff. Eq. 261, 6302-6340 (2016) 26
[95] A. Ferrero, F. Gazzola, A partially hinged rectangular plate as a model for suspension bridges, Discr. Cont. Dynam. Syst. A 35, 5879-5908 (2015) 27
[96] J. Ferziger, M. Perić, Computational Methods for Fluid Dynamics. Springer-Verlag, Berlin, 1999 3

[97] M. Flucher, B. Gustafsson, Vortex motion in two-dimensional hydromechanics, Preprint in TRITA-MAT-1997-MA-02, 2015 6
[98] L. Föppl, Wirbelbewegung hinter einem Kreiszylinder, Sitzb. Bayer. Akad. Wiss. 1, 1 (1913) 6, 10
[99] S. Franzetti, M. Greco, S. Malavasi, D. Mirauda, Flow induced excitation on basic shape structures, In: Vorticity and Turbulence Effects in Fluid Structure Interaction (An Application to Hydraulic Structure Design), M. Brocchini & F. Trivellato Eds., WIT Press, Chapter 6, 131-156 (2006) 3, 35
[100] C. Foias, R. Temam, Structure of the set of stationary solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. 30, 149-164 (1977) 16
[101] C. Foias, R. Temam, Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation, Annali Sc. Norm. Sup. Pisa, 5, 29-63 (1978) 20
[102] C. Foias, O. Manley, R. Rosa, R. Temam, Navier-Stokes equations and turbulence. Encyclopedia of Mathematics and its Applications, 83. Cambridge University Press, 2001. 11
[103] U. Frisch, Turbulence, Cambridge University Press, 1995. 3
[104] H. Fujita, N. Sauer, On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo Sect. I 17, 403-420 (1970) 11
[105] V. Fuka, J. Brechler, Large eddy simulation of the stable boundary layer, Finite Volumes for Complex Applications VI - Problems & Perspectives, 485-493 (2011) `http://artax.karlin.mff.cuni.cz/~fukav1am/sqcyl.html` 3, 15
[106] A. Fursikov, Flow of a viscous incompressible fluid around a body: boundary value problems and work minimization for a fluid, J. Math. Sci. 180, 763-816 (2012) 35
[107] A. Fursikov, M. Gunzburger, L. Hou, Optimal boundary control for the evolutionary Navier-Stokes system: the three-dimensional case, SIAM J. Control Optim. 43, no. 6, 2191-2232 (2005) 35
[108] A. Fursikov, R. Rannacher, Optimal Neumann control for the two-dimensional steady-state Navier-Stokes equations. New Directions in Mathematical Fluid Mechanics, 193-221, Adv. Math. Fluid Mech., Birkhäuser Verlag, Basel, 2010. 35
[109] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearised steady problems. Springer Tracts in Natural Philosophy, 38. Springer-Verlag 1994. 11
[110] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II. Nonlinear steady problems. Springer Tracts in Natural Philosophy, 39. Springer-Verlag 1994. 11
[111] G.P. Galdi, On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications, Handbook of mathematical fluid dynamics, Vol. I, 653–791, 2002 21, 22
[112] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Second edition. Springer Monographs in Mathematics. Springer 2011. 11
[113] G.P. Galdi, On time-periodic flow of a viscous liquid past a moving cylinder, Arch. Rational Mech. Anal. 210, 451-498 (2013) 31
[114] G.P. Galdi, On bifurcating time-periodic flow of a Navier-Stokes liquid past a cylinder, Arch. Rational Mech. Anal. 222, 285-315 (2016) 22, 23
[115] G.P. Galdi, A.L. Silvestre, Existence of time-periodic solutions to the Navier-Stokes equations around a moving body, Pacific J. Math. 223, 251-267 (2006) 22
[116] G.P. Galdi, A.L. Silvestre, On the motion of a rigid body in a Navier-Stokes liquid under the action of a time-periodic force, Indiana Univ. Math. J. 58, 2805-2842 (2009) 31
[117] G.P. Galdi, Flow of a viscous liquid past an obstacle at arbitrary Reynolds number, talk given at the Waseda University, November 8 2016 8
[118] M. Garrione, F. Gazzola, Nonlinear equations and stability for beams and degenerate plates with several intermediate piers, Springer Briefs, PoliMi (2019), to appear 26, 40

[119] C. Gasparetto, F. Gazzola, Resonance tongues for the Hill equation with Duffing coefficients and instabilities in a nonlinear beam equation, Comm. Contemp. Math. 20, 1750022, 22 pp. (2018) 26
[120] F. Gazzola, Mathematical models for suspension bridges, MS&A Modeling, Simulation and Applications 15, Springer, Cham, 2015 24, 26
[121] F. Gazzola, G. Sperone, Navier-Stokes equations interacting with plate equations, Annual Report of the Politecnico di Milano PhD School (2017) 11
[122] F. Gazzola, G. Sperone, Boundary conditions for planar Stokes equations inducing vortices around concave corners, Milan J. Math. (2019). DOI 10.1007/s00032-019-00297-0 11
[123] F. Gazzola, G. Sperone, New bounds for the stability of a bluff body immersed in a viscous fluid, in preparation 13, 14, 15, 36
[124] M. Geissert, M. Hieber, T.H. Nguyen, A General Approach to Time Periodic Incompressible Viscous Fluid Flow Problems, Arch. Rational Mech. Anal. 220 (2016) 1095-1118 22
[125] Computational modelling of bifurcations and instabilities in fluid dynamics. Edited by Alexander Gelfgat. Computational Methods in Applied Sciences, 50. Springer, Cham, 2019. 21
[126] D. Gérard-Varet, Some recent mathematical results on fluid-solid interaction. Lectures on the analysis of nonlinear partial differential equations. Part 3, 209-235, Morningside Lect. Math. 3, Int. Press, Somerville, MA, 2013 7, 8
[127] D. Gérard-Varet, M. Hillairet, Existence of weak solutions up to collision for viscous fluid-solid systems with slip, Comm. Pure Appl. Math. 67, 2022-2075 (2014) 31
[128] O. Glass, F. Sueur, Uniqueness results for weak solutions of two-dimensional fluid-solid systems, Arch. Ration. Mech. Anal. 218, no. 2, 907–944 (2015) 31
[129] C. Grandmont, Y. Maday, Existence de solutions d'un problème de couplage fluide-structure bidimensionnel instationnarie, C. R. Acad. Sci. Paris Sér. I Math. 326 no. 4, 525–530 (1998) 30
[130] D. Green, W.G. Unruh, Tacoma Bridge failure, a physical model, Amer. J. Physics 74, 706-716 (2006) 24
[131] G. Grimberg, W. Pauls, U. Frisch, Genesis of d'Alembert's paradox and analytical elaboration of the drag problem, Physica D 237, 1878-1886 (2008) 8
[132] M.D. Gunzburger, H.-C. Lee, G.A. Seregin, Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions J. Math. Fluid Mech. 2, no. 3, 219–266 (2000) 31
[133] S.O. Hansen, Vortex-induced vibrations of structures, Structural Engineers World Congress 2007, November 2-7, 2007, Bangalore, India `http://www.eurocodes.fi/1991/1991-1-4/background/Hansen_2007.pdf` 3, 5, 23, 35
[134] Svend Ole Hansen ApS publications, `http://sohansen.dk/publications/` 23
[135] H. Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, J. Reine Angew. Math. 55, 25-55 (1858) 3
[136] R.D. Henderson, Details of the drag curve near the onset of vortex shedding, Physics of Fluids 7, 2102-2104 (1995) 19
[137] J. Heywood, On some paradoxes concerning two-dimensional Stokes flow past an obstacle. Indiana Univ. Math. J. 24, 443-450 (1975) 14
[138] M. Hillairet, Lack of collision between solid bodies in a 2D incompressible viscous flow, Comm. Partial Differential Equations 32, 1345-1371 (2007) 31
[139] M. Hillairet, T. Takahashi, Collisions in three-dimensional fluid structure interaction problems, SIAM J. Math. Anal. 40, 2451-2477 (2009) 31
[140] M. Hillairet, T. Takahashi, Blow up and grazing collision in viscous fluid solid interaction systems Ann. I. H. Poincaré – AN 27, 291–313 (2010) 31
[141] J. Hoffman, J. Jansson, C. Johnson, New theory of flight, Journal of Mathematical Fluid Mechanics 18, 2, 219-241 (2016) 9
[142] J. Hoffman, C. Johnson, The Mathematical Secret of Flight, Normat 57:4, 1-25 (2009) 9

[143] J. Hoffman, C. Johnson, Resolution of d'Alembert's Paradox, Journal of Mathematical Fluid Mechanics 12, 321-334 (2010) 9
[144] K.-H. Hoffmann, V. N. Starovoitov, On a motion of a solid body in a viscous fluid. Two-dimensional case. Adv. Math. Sci. Appl. 9 no. 2, 633–648 (1999) 30
[145] K.-H. Hoffmann, V. N. Starovoitov, Zur Bewegung einer Kugel in einer zähen Flüssigkeit, Doc. Math. 5, 15–21 (2000) 30
[146] J.D. Holmes, Wind loading of structures, (2nd Ed.) Taylor & Francis, London & New York, 2007 24
[147] E. Hopf, A mathematical example displaying the features of turbulence. Communications on Appl. Math. 1, 303-322 (1948) 20
[148] E. Hopf, Über die anfangwertaufgabe für die hydronamischen grundgleichungen, Math. Nachr. 4, 213-231 (1951) 11
[149] E. Hopf, Repeated branching through loss of stability, an example. Proceedings of the Conference on Differential Equations, 49-56. University of Maryland Book Store, College Park, Md., 1956. 20
[150] C.-H. Hsia, C.-Y. Jung, T.B. Nguyen, M.-C. Shiue, On time periodic solutions, asymptotic stability and bifurcations of Navier-Stokes equations, Numerische Mathematik 135, 607–638 (2017) 21
[151] W.W. Huebsch, B.R. Munson, T.H. Okiishi, A.P. Rothmayer, Fundamentals of Fluid Mechanics, John Wiley & Sons Inc. (7th edition), 2013. 5, 37
[152] A. Hundertmark, M. Lukáčová-Medvidová, Š. Nečasová, On the weak solution of the fluid-structure interaction problem for shear-dependent fluids. Recent Developments of Mathematical Fluid Mechanics, 291-319, Adv. Math. Fluid Mech., Birkhäuser/Springer, Basel (2016) 32
[153] N.T. Huy, Periodic motions of Stokes and Navier-Stokes flows around a rotating obstacle, Arch. Rational Mech. Anal. 213, 689-703 (2014) 31
[154] G. Iooss, Existence et stabilité de la solution périodique secondaire intervenant dans les problèmes d'evolution du type Navier-Stokes. (French) Arch. Rational Mech. Anal. 47 (1972), 301-329 20, 23
[155] G. Iooss, A. Mielke, Bifurcating time-periodic solutions of Navier-Stokes equations in infinite cylinders. J. Nonlinear Sci. 1 (1991), no. 1, 107-14620
[156] G. Iooss, Modélisation de la transition vers la turbulence, Bourbaki seminar, Vol. 1982/83, 179-19720
[157] H.M. Irvine, Cable Structures, MIT Press Series in Structural Mechanics, Massachusetts, 1981 24
[158] V.I. Iudovich, Secondary flows and fluid instability between rotating cylinders, Prikl. Mat. Meh. 30 688-698 (Russian); translated in J. Appl. Math. Mech. 30, 822–833 (1966) 16, 20, 23
[159] V. I. Iudovich, The onset of auto-oscillations in a fluid, J. Appl. Math. Mech. 35 (1971), 587-603 (1972).; translated from Prikl. Mat. Meh. 35 (1971), 638–655 (Russian) 20, 23
[160] C. P. Jackson, A finite-element study of the onset of vortex shedding in flow past variously shaped bodies, J. Fluid Mech. 182, 23-45 (1987) 11
[161] C. Johnson, The Secret of Flight, personal blog, `https://secretofflight.wordpress.com/media/` 10
[162] D.D. Joseph, D.H. Sattinger, Bifurcating time periodic solutions and their stability. Arch. Rational Mech. Anal. 45 (1972), 79-109. 20, 23
[163] D.D. Joseph, Remarks about bifurcation and stability of quasi-periodic solutions which bifurcate from periodic solutions of the Navier-Stokes equations. In Nonlinear Problems in Physical Science and Biology. Springer Lecture Notes in Mathematics, 322, 130, 1973. 20
[164] D.D. Joseph, Repeated supercritical branching of solutions arising in the variational theory of turbulence. Arch. Rational Mech. Anal. 53 (1973/74), no. 2, 101-130. 20

[165] D.D. Joseph, Slow motion and viscometric motion; stability and bifurcation of the rest state of a simple fluid. Arch. Rational Mech. Anal. 56 (1974/75), 99-157. 20
[166] C. Josserand, Y. Pomeau, S. Rica, Vortex shedding in a model of superflow, Physica D 134, 111-125 (1999) 6
[167] P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-González, The defocusing nonlinear Schrödinger equation. From dark solitons to vortices and vortex rings. Society for Industrial and Applied Mathematics, Philadelphia, 2015 6
[168] G.R. Kirchhoff, Über das gleichgewicht und die bewegung einer elastischen scheibe, J. Reine Angew. Math. 40, 51-88 (1850) 26
[169] G.R. Kirchhoff, Vorlesungen über mathematische Physik, Teubner, Leipzig, 1876 6
[170] G.H. Knightly, D. Sather, Nonlinear buckled states of rectangular plates, Arch. Rational Mech. Anal. 54, 356-372, (1974) 27
[171] V. Kozlov, V. Maz'ya, Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary, Ann. Mat. Pura Appl. 184, 185-213 (2005) 37
[172] D.N. Ku, Blood flow in arteries, Annu. Rev. Fluid Mech. 29, 399-434 (1997) 5
[173] H.G. Küssner, Zusammenfassender Bericht über den instationären Auftrieb von Flügeln, Luftfahrtforschung 13, 410-424 (1936) 29
[174] N.G. Kuznetsov, V.G. Maz'ya, Problem concerning steady-state oscillations of a layer of fluid in the presence of an obstacle. Soviet Physics Dokl. 19, 341-343 (1974) 21
[175] N.G. Kuznetsov, V.G. Maz'ya, On unique solvability of the plane Neumann-Kelvin problem. Math. USSR Sborn. 63,425-446 (1989) 21
[176] W. Lacarbonara, Nonlinear structural mechanics, Springer (2013) 29
[177] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows, Gordon and Breach, New York, 1969 11
[178] H. Lamb, Hydrodynamics, Cambridge University Press, 6th Edition, 1975 10, 13
[179] L.D. Landau, On the problem of turbulence. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 311–314 (1944) 20
[180] L.D. Landau, E.M. Lifshitz, Course of Theoretical Physics, Vol. 6: Fluid Mechanics, Pergamon Press, 1987 12, 13, 20
[181] W. Langlois, M. Deville, Slow Viscous Flow, Springer, 2014. 13
[182] A. Larsen, Advances in aeroelastic analyses of suspension and cable-stayed bridges, J. Wind Eng. Industrial Aerodyn. 74, 73-90 (1998) 6
[183] A. Larsen, Aerodynamics of the Tacoma Narrows Bridge - 60 years later, Struct. Eng. Internat. 4, 243-248 (2000) 24, 25, 36
[184] I. Lasiecka, A. Tuffaha, Riccati equations arising in boundary control of fluid structure interactions, Int. Journal of Computer Science and Mathematics 1, 128-147 (2007) 32
[185] I. Lasiecka, J. Webster, Nonlinear plates interacting with a subsonic, inviscid flow via Kutta-Joukowski interface conditions, Nonlinear Analysis: Real World Applications 17, 171-191 (2014) 32
[186] I. Lasiecka, J. Webster, Eliminating flutter for clamped von Kármán plates immersed in subsonic flows, Comm. Pure Applied Analysis 13, 1935-1969 (2014) 32
[187] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC Research Notes in Mathematics 431, 2002. 11
[188] J. Leray, Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodinamique, J. Math. Pures et Appl. 13, 1-82 (1933) 11
[189] J.H. Lienhard, Synopsis of Lift, Drag, and Vortex Frequency Data for Rigid Circular Cylinders. Research Division Bulletin, Washington State University College of Engineering , 32 (1966) 17, 18
[190] E.M. Lifshitz, V.G. Maz'ya, Solvability of the two-dimensional Kelvin-Neumann problem for a submerged circular cylinder. Applicable Analysis 64, 1-5 (1997) 21
[191] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969 11

[192] C.C. Lin, On the motion of vortices in 2D I. Existence of the Kirchhoff-Routh function, Proc. Nat. Acad. Sc. 27, 570-575 (1941) 6
[193] C.C. Lin, On the motion of vortices in 2D II. Some further properties on the Kirchhoff-Routh function, Proc. Nat. Acad. Sc. 27, 575-577 (1941) 6
[194] A.E.H. Love, A treatise on the mathematical theory of elasticity, 4th Ed. Cambridge Univ. Press, 1927 26
[195] A.J. Majda, A.L. Bertozzi, Vorticity and incompressible flow, vol. 27 of Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002 6, 8
[196] S. Malavasi, N. Trabucchi, Numerical investigation of the flow around a rectangular cylinder near a solid wall, Proc. BBAA VI Intl. Colloquium on: Bluff Bodies Aerodynamics & Applications, Milano, Italy (July 20-24, 2008) 32
[197] M. Manarini, Sui paradossi di D'Alembert e di Brillouin nella dinamica dei fluidi, (Italian) Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. 4, 427-433 (1948) 8
[198] C. Mannini, A. Soda, G. Schewe, Unsteady RANS modelling of flow past a rectangular cylinder: Investigation of Reynolds number effects, Computers & Fluids 39, 1609-1624 (2010) 32
[199] E.H. Mansfield, The bending and stretching of plates, Cambridge University Press, Cambridge, second edition, 1989 27
[200] C. Marchioro, On the Inviscid Limit for a Fluid with a Concentrated Vorticity, Comm. Math. Phys. 196, 53-65 (1998) 6
[201] C. Marchioro, M. Pulvirenti, Euler evolution for singular initial data and vortex theory, Comm. Math. Phys. 91, 4, 563-572 (1983) 7
[202] C. Marchioro, M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids, Applied Mathematical Sciences 96, Springer-Verlag, New York, 1994 6
[203] A. W. Marris, A Review on Vortex Streets, Periodic Wakes, and Induced Vibration Phenomena, J. Basic Eng. 86 (2), 185-193 (1964) & Discussion on pages 193–196 17
[204] M. Matsumoto, H. Matsumiya, S. Fujiwara, Y. Ito, New consideration on flutter properties based on step-by-step analysis, J. Wind Engineering Indust. Aerodynamics 98, 429-437 (2010) 24
[205] V.G. Maz'ya, On the steady problem of small oscillations of a fluid in the presence of a submerged body. Proc. Sobolev's Semin. No.2, 57-79. Novosibirsk: Inst. of Maths, Siberian Branch, Acad. Sci. USSR (in Russian), 1977 21
[206] V.G. Maz'ya, Solvability of the problem on the oscillations of a fluid containing a submerged body, J. Soviet Math. 10, 86-89 (1978) 21
[207] V.G. Maz'ya, Sobolev spaces with applications to elliptic partial differential equations, Springer (2011) 36
[208] V.G. Maz'ya, Differential equations of my young years, (Translated from Russian by Arkady Alexeev, with a foreword by V.P. Havin, Birkhäuser/Springer, Cham, 2014 1
[209] V.G. Maz'ya, Tales for children and grown-ups, `http://users.mai.liu.se/vlama82/books-skazki/index.html` 8
[210] V.G. Maz'ya, Novyy blokbaster dedy Vovy, Redaktsiya Kommentarii, Moskva (2016) 8
[211] V.G. Maz'ya, Seventy five (thousand) unsolved problems in analysis and partial differential equations, Integral Equations Operator Theory 90, Art.25, 44 pp. (2018) 37
[212] V.G. Maz'ya, J. Rossmann, Elliptic equations in polyhedral domains, American Mathematical Society 162, 2010 16, 36
[213] P.J. McKenna, Torsional oscillations in suspension bridges revisited: fixing an old approximation, Amer. Math. Monthly 106, 1-18 (1999) 25
[214] P.J. McKenna, Oscillations in suspension bridges, vertical and torsional, Disc. Cont. Dynam. Systems 7, 785-791 (2014) 24, 25
[215] J. Melan, Theory of arches and suspension bridges, Myron Clark Publ. Comp., London (1913). (German original: Handbuch der Ingenieurwissenschaften, Vol. 2, 1906) 24

[216] M.V. Morkovin, Flow around circular cylinder - A kaleidoscope of challenging fluid phenomena. In: Hansen, A.G. (Ed.): Symposium on Fully Separated Flows. American Society of Mechanical Engineers, 102?118. (1964) 17
[217] B. Mohammadi, O. Pironneau, Applied shape optimization for fluids. Second edition. Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2010 35
[218] B. Muha, S. Šanić, Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. Interfaces Free Bound. 17, no. 4, 465-495 (2015) 32
[219] NASA Glenn Research Center, `https://www.grc.nasa.gov/www/k-12/airplane/bernnew.html` 6
[220] C.L. Navier, Extrait des recherches sur la flexion des plans elastiques, Bull. Sci. Soc. Philomathique de Paris 5, 95-102 (1823) 27
[221] C.L. Navier, Mémoire sur les ponts suspendus, Imprimerie Royale, Paris (1823) 23, 24
[222] Š. Nečasová, J. Wolf, On the existence of global strong solutions to the equations modeling a motion of a rigid body around a viscous fluid, Discrete Contin. Dyn. Syst. 36, 1539-1562 (2016) 31
[223] Š. Nečasová, S. Kračmar, Navier-Stokes flow around a rotating obstacle, Atlantis Briefs in Differential Equations, Mathematical analysis of its asymptotic behaviour, 2016 22
[224] P.K. Newton, The N-vortex problem, Springer-Verlag, Berlin, 2001 6
[225] T.H. Nguyen, Periodic Motions of Stokes and Navier-Stokes Flows Around a Rotating Obstacle, Arch. Rational Mech. Anal. 213 (2014) 689-703 22
[226] OpenFOAM online documentation: `https://www.openfoam.com/` 32
[227] A. Okajima, Strouhal number of rectangular cylinder, J. Fluid Mech., vol 123, 379-398 (1982) 32, 34
[228] A. Okajima, H. Ueno, H. Sakay, Numerical Simulation of Laminar and Turbulent Flows around Rectangular Cylinders, Int. J. Num. Methods in Fluids, Vol. 15, 999-1012 (1992) 32
[229] M.E. O'Neill, K. Stewartson, On the slow motion of a sphere parallel to a nearby plane wall. J. Fluid Mech. 27, 705-724 (1967) 31
[230] J.H. Ortega, L. Rosier, T. Takahashi, Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid, M2AN Math. Model. Numer. Anal. 39, 79-108 (2005) 31
[231] C.W. Oseen, Über die Stokes'sche Formel und über eine verwandte Aufgabe in der Hydrodynamik, Arkiv Mat., Astron. och Fysik, vol.6, p.1 (1910) 13
[232] F. Pacard, T. Rivière, Linear and nonlinear aspects of vortices. The Ginzburg-Landau model. Progress in Nonlinear Differential Equations and their Applications, 39, Birkhäuser, Boston, 2000 6
[233] M. Paidoussis, Fluid-structure interactions. Slender structures and axial flow, Elsevier, 888 pp., 2014 3
[234] M. Paidoussis, S. Price, E. de Langre, Fluid-structure interactions. Cross-flow-induced instabilities, Cambridge University Press, Cambridge, 402 pp., 2011 3, 5, 6, 25
[235] A. E. Perry, M. S. Chong, T. T. Lim, The vortex-shedding process behind two-dimensional bluff bodies, J. Fluid Mech. 116, 77-90 (1982) 11
[236] O. Pironneau, On optimum profiles in Stokes flow, J. Fluid Mech. 59, 117-128 (1973) 35
[237] O. Pironneau, On optimum design in fluid mechanics, J. Fluid Mech. 64, 97-110 (1974) 35
[238] B.G. Pittel, V.A. Yakubovich, Application of the theory of parametric resonance to explain the collapse of the Tacoma Narrows Bridge (Russian), Uspekhi Mat. Nauk. 15, 183-184 (1961) 24, 25
[239] B.G. Pittel, V.A. Yakubovich, A mathematical analysis of the stability of suspension bridges based on the example of the Tacoma Bridge (Russian), Vestnik Leningrad Univ. 24, 80-91 (1969) 24, 25

[240] S. Pope, Turbulent Flows, Cambridge University Press, Cambridge (2000) 3
[241] L. Prandtl, On motion of fluids with very little viscosity, Third International Congress of Mathematics, Heidelberg, `https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19930090813.pdf`, 1904 4
[242] B. Protas, Higher-order Föppl models of steady wake flows, Phys. Fluids 18, 117109 (2006) 6
[243] B. Protas, Vortex dynamics models in flow control problems, Nonlinearity 21, 203-250 (2008) 6
[244] P.H. Rabinowitz, Existence and nonuniqueness of rectangular solutions of the Bénard problem, Arch. Rational Mech. Anal. 29, 32-57 (1968) 16
[245] P.H. Rabinowitz, A priori bounds for some bifurcation problems in fluid dynamics, Arch. Rational Mech. Anal. 49, 270–285 (1973) 20
[246] D. Gerecht, R. Rannacher, W. Wollner, Computational aspects of pseudospectra in hydrodynamic stability analysis, *J. Math. Fluid Mech.*, **14**, 661–692 (2012) 21, 23
[247] O. Reynolds, An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels. Philosophical Transactions of the Royal Society of London Series I, 174, 935-982 (1883). 4
[248] J. Robinson, J.L. Rodrigo, W. Sadowski, The three-dimensional Navier-Stokes equations. Classical theory. Cambridge Studies in Advanced Mathematics, 157. Cambridge University Press 2016. 11
[249] Y. Rocard, Dynamic instability: automobiles, aircraft, suspension bridges, Crosby Lockwood, London, 1957 24
[250] D. Rocchi, private communication 40
[251] E.J. Routh, Some applications of conjugate functions, Proc. London Mat. Soc. 12, 73-89 (1881) 6
[252] D.Ruelle, F.Takens, On the nature of turbulence, Comm. Math. Phys. 20, 167-192 (1971) 20
[253] M. de Saint-Venant, Résistance des fluides: considérations historiques, physiques et pratiques relatives au problème de l'action dynamique mutuelle d'un fluide et d'un solide, spécialement dans l'état de permanence supposé acquis par leurs mouvements, Academie des Science, Mémoires 44, 1-280 (1888) 8
[254] P.G. Saffman, Vortex dynamics, Cambridge University Press, Cambridge, 1992 6, 10
[255] E. Sandier, S. Serfaty, Vortices in the magnetic Ginzburg-Landau model. Progress in Nonlinear Differential Equations and their Applications, 70, Birkhäuser Boston, 2007 6
[256] J.A. San Martín, V. Starovoitov, M. Tucsnak, Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid, Arch. Ration. Mech. Anal. 161 (2002), no. 2, 113-147. 31
[257] R.H. Scanlan, The action of flexible bridges under wind, I: flutter theory, II: buffeting theory, J. Sound and Vibration 60, 187-199 & 201-211 (1978) 24, 25
[258] R.H. Scanlan, Developments in low-speed aeroelasticity in the civil engineering field, AIAA Journal 20, 839-844 (1982) 24, 25, 36
[259] R.H. Scanlan, J.J. Tomko, Airfoil and bridge deck flutter derivatives, Journal of Engineering Mechanics (ASCE) 97, 1717-1737 (1971) 6, 29
[260] B.N. Shashikanth, J.E. Marsden, J.W. Burdick, S.D. Kelly, The Hamiltonian structure of a two-dimensional rigid circular cylinder interacting dynamically with N point vortices, Physics of Fluids 14, 1214-1227 (2002) (Erratum: Phys. Fluids 14, 4099 (2002)) 6
[261] H. Schlichting, K. Gersten, Boundary-Layer Theory, Ninth Edition, Springer, 2017 17, 19
[262] R. Scott, In the wake of Tacoma. Suspension bridges and the quest for aerodynamic stability, ASCE Press, 2001 25, 36
[263] W.R. Sears, Some aspects of non-stationary airfoil theory and its practical application, J. Aero. Sci. 8(3), 104-108 (1941) 29

[264] A. Selberg, Oscillation and aerodynamic instability of suspension bridges, Acta Polytechnica Scandinavica, Civil Engineering and Construction Series 13, Trondheim (1961) 24
[265] S. Serfaty, Coulomb gases and Ginzburg-Landau vortices, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2015 6
[266] S. Serfaty, Mean field limits of the Gross-Pitaevskii and parabolic Ginzburg-Landau equations, J. Amer. Math. Soc. 30, 3, 713-768 (2017) 6
[267] D. Serre, Chute libre d'un solide dans un fluide visqueux incompressible. Existence. Japan J. Appl. Math. 4, 99-110 (1987) 30
[268] K. Shimada, T. Ishihara, Application of a modified $k-\varepsilon$ model to the prediction of aerodynamic characteristics of rectangular cross-section cylinders, J. Fluids Structures 16, 465-485 (2002) 34
[269] E. Simiu, R.H. Scanlan, Wind effects on structures: an introduction to wind engineering, Wiley-Interscience (1986) 23, 34
[270] A.C. Smith, On the stability of Föppl?s Vortices, J. Appl. Mech. 70, 610–612 (1973) 10
[271] F.C. Smith, G.S. Vincent, Aerodynamic stability of suspension bridges: with special reference to the Tacoma Narrows Bridge, Part II: Mathematical Analysis, Investigation conducted by the Structural Research Laboratory, University of Washington Press, Seattle (1950) 24
[272] V.N. Starovoitov, On the nonuniqueness of the solution of the problem of the motion of a rigid body in a viscous incompressible fluid, Zap. Nauchn. Sem. St.Peterburg. Otdel. Mat. Inst. Steklov. 306 (2003), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 34, 199-209, 231-232; translation in J. Math. Sci. (N.Y.) 130 (2005), no. 4, 4893-4898 30
[273] K. Stewartson, D'Alembert's Paradox, SIAM Review 23, 3, 308-343 (1981) 8
[274] J. Stoker, Book Review: Hydrodynamics, a study in logic, fact, and similitude, Bull. Amer. Math. Soc. Am. Math. 57 (6) (1951), 497-499. 9
[275] G.G. Stokes, On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, Trans. Cambridge Phil. Soc., 8, 287-305 (1845). 4
[276] G.G. Stokes, On the effect of the internal friction of fluids on the motion of pendulums, Trans. Cambridge Phil. Soc., 9, 8-106 (1851). 4, 12, 14
[277] V. Sverak, Selected Topics in Fluid Mechanics - `http://www-users.math.umn.edu/~sverak/course-notes2011.pdf` 7, 8
[278] Tacoma Narrows Bridge collapse, `http://www.youtube.com/watch?v=3mclp9QmCGs(Video, 1940)` 24
[279] T. Takahashi, M. Tucsnak, Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid, J. Math. Fluid Mech. 6, 53-77 (2004) 31
[280] T. Takahashi, M. Tucsnak, G. Weiss, Stabilization of a fluid–rigid body system, J. Differential Equations 259, 6459–6493 (2015) 31
[281] S. Tang, N. Aubry, On the symmetry breaking instability leading to vortex shedding, Physics of Fluids 9, 2550-2561 (1997) 10
[282] G. Tarantello, Selfdual gauge field vortices. An analytical approach, Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston, Inc., Boston, MA, 2008 6
[283] G.I. Taylor, Stability of a Viscous Liquid Contained between Two Rotating Cylinders, Philosophical Transactions of the Royal Society of London. Series A Vol. 223 289-343 (1923) 20
[284] R. Tazzioli, D'Alembert's paradox, 1900-1914: Levi-Civita and his Italian and French followers, Comptes Rendus Mécanique, Volume 345, Issue 7, 488-497 (2017) 8
[285] R. Temam, Navier-Stokes equations, theory and numerical analysis, North-Holland, Amsterdam, 1979 11, 14, 15, 16
[286] The Intelligencer, Destruction of the Wheeling Suspension Bridge, Wheeling, Va., vol.2, no.225, p.3 (Thursday, May 18, 1854) 26

[287] D.J. Tritton, Experiments on the flow past a circular cylinder at low Reynolds numbers. Journal of Fluid Mechanics, 6(4), 547-567 (1959) 17
[288] D.J. Tritton, Physical Fluid Dynamics, 2nd edn. Clarendon Press, Oxford, 1988 17
[289] T.-P. Tsai, Lectures on Navier-Stokes equations. Graduate Studies in Mathematics, 192. American Mathematical Society 2018. 11
[290] B.R. Vainberg, V.G. Maz'ya, Certain stationary problems of the linear theory of surface waves (Russian), Dokl. Akad. Nauk SSSR 205, 310-313 (1972) 21
[291] B.R. Vainberg, V.G. Maz'ya, On the plane problem of the motion of a solid immersed in a fluid (Russian), Trudy Moskov. Mat. Obsc. 28, 35-56 (1973) 21
[292] B.R. Vainberg, V.G. Maz'ya, On the problem of the steady oscillations of a layer of fluid of variable depth (Russian), Trudy Moskov. Mat. Obsc. 28, 57-74 (1973) 21
[293] L. van Veen, A Brief History of Simple Invariant Solutions in Turbulence. In: Gelfgat A. (eds) Computational Modelling of Bifurcations and Instabilities in Fluid Dynamics. Computational Methods in Applied Sciences, vol 50, Springer, 2019 21
[294] L. van Veen, The quasi-periodic doubling cascade in the transition to weak turbulence, Physica D 210, 249–261 (2005) 21
[295] W. Velte, Stabilitätsverhalten und verzweigung stationärer lösungen der Navier-Stokesschen gleichungen (German), Arch. Rational Mech. Anal. 16, 97-125 (1964) 16, 20
[296] W. Velte, Stabilität und verzweigung stationärer lösungen der Navier-Stokesschen gleichugen beim Taylorproblem (German), Arch. Rational Mech. Anal. 22, 1-14 (1966) 16, 20
[297] E. Ventsel, T. Krauthammer, Thin plates and shells: theory, analysis, and applications, Marcel Dekker Inc., New York, 2001 27
[298] T. Von Kármán, Aerodynamics. Cornell University press, Mc Graw-Hill 1963 9
[299] T. von Kármán and W. R. Sears, Airfoil Theory for Non-Uniform Motion, Journal of the Aeronautical Sciences, Vol. 5, No. 10, 1938, pp. 379- 390; reprinted in Faeth, G. M. (ed.), Centennial of Powered Flight: A Retrospective of Aerospace Research, AIAA, Reston, VA, 2003, pp. 5-16. 29
[300] R. von Mises, Theory of Flight. Dover Publications 1959 9
[301] Vortex shedding in water, https://www.youtube.com/watch?v=_Hbbkd2d3H8 (Video, 2012) 21
[302] H. Wagner, Über die Entstehung des dynamischen Auftriebes von Tragflügeln, Zeitschrift für Angewandte Mathematik und Mechanik 5, 17-35 (1925) 29
[303] E. Weinan, Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity. Phys. D 77, 4, 383-404 (1994) 6
[304] C.H.K. Williamson, Vortex Dynamics in the Cylinder Wake, Annual Review of Fluid Mechanics 28, 477-539 (1996) 17
[305] C.H.K. Williamson, R. Govardhan, Vortex-induced vibrations Annual Review of Fluid Mechanics 36, 413-455 (2004) 3
[306] C.H.K. Williamson, R. Govardhan, A brief review of recent results in vortex-induced vibrations, Journal of Wind Engineering and Industrial Aerodynamics 96, 6-7, 713-735 (2008) 3
[307] X. Wu, F.Ge, Y. Hong, A review of recent studies on vortex-induced vibrations of long slender cylinders, Journal of Fluids and Structures 28, 292-308 (2012) 3
[308] Y.-L. Xu, Wind effects on cable-supported bridges, John Wiley & Sons Singapore (2013) 23, 34
[309] V.A. Yakubovich, V.M. Starzhinskii, Linear differential equations with periodic coefficients, J. Wiley & Sons, New York, 1975 (Russian original in Izdat. Nauka, Moscow, 1972) 24, 25
[310] A. Zebib, Stability of viscous flow past a circular cylinder, J. Eng. Math. 21, 155-165 (1987) 11

[311] Q. Zhou, S. Cao, Z. Zhou, Shear effects on flow past a rectangular cylinder with side ratio B/D=5, Proc. Seventh Intl. Colloquium on Bluff Body Aerodynamics and Applications (BBAA7). Shanghai, China (September 2-6, 2012) 32

Département de Mathématique - Université Libre de Bruxelles, Belgium
E-mail address: denis.bonheure@ulb.ac.be
URL: homepages.ulb.ac.be/~dbonheur/

Dipartimento di Matematica - Politecnico di Milano, Italy
E-mail address: filippo.gazzola@polimi.it
URL: http://www1.mate.polimi.it/~gazzola/
E-mail address: gianmarcosilvio.sperone@polimi.it
URL: https://www.researchgate.net/profile/Gianmarco_Sperone