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A Sharp Upper Bound for the Torsional Rigidity of Rods by Means of Web Functions

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Abstract

Using web functions, we approximate the Dirichlet integral which represents the torsional rigidity of a cylindrical rod with planar convex cross-section Ω. To this end, we use a suitably defined piercing function, which enables us to obtain bounds for both the approximate and the exact value of the torsional rigidity. When Ω varies, we show that the ratio between these two values is always larger than $\frac{3}{4}$; we prove that this is a sharp lower bound and that it is not attained. Several extremal cases are also analyzed and studied by numerical methods.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex domain. We consider the torsion problem for a long cylindrical rod in the space $\mathbb{R}^3$ of uniform planar cross-section Ω in the (x_1, x_2)-plane and whose axis is the x_3 axis. The state of stress in the interior of the rod does not depend on x_3 and is determined by a warping function $u = u(x)$, $x \in \Omega$, which solves the boundary value problem

$$\begin{aligned} -\Delta u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The torsional rigidity (or simply torsion) of the rod $\Omega \times \mathbb{R}$ is the torque required for unit angle of twist per unit length and, up to a multiplicative constant, can be expressed by the Dirichlet integral

$$\int_\Omega |\nabla \overline{u}|^2, \tag{2}$$

where $\overline{u}$ is the unique solution of (1). From a mathematical point of view, (2) is the best (smallest) constant $C = C(\Omega)$ for the Sobolev inequality $\|v\|_1^2 \leqq C\|\nabla v\|_2^2$ which holds for all $v \in H_0^1(\Omega)$. For a brief story of the torsion problem we refer to

Section 5. We also point out that (1) is related to other mechanical problems, such as the bending of a uniformly loaded plane membrane, or the motion of a viscous fluid in a pipe with fixed walls, see for instance [17].

Except for some particular cases (e.g., when Ω is a disk, a rectangle) the explicit form of the unique solution $\overline{u}$ of (1) is not known and therefore the corresponding torsion (2) may not be computed. Hence the problem of finding some estimates, as accurate as possible, for the Dirichlet integral (2), arises. The approximation considered in this paper is based on the idea of restricting the variational formulation of (1) to the class of functions u which depend only on the distance from the boundary $\partial\Omega$. More precisely, note first that (1) is the Euler-Lagrange equation of the convex functional

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - u \right), \qquad u \in H_0^1(\Omega).$$

Therefore, the unique solution $\overline{u}$ of (1) coincides with the unique minimum of J; in particular, the torsion (2) can be recovered by solving the infimum problem for J, since integration by parts gives

$$\min_{u \in H_0^1(\Omega)} J(u) = J(\overline{u}) = -\frac{1}{2} \int_\Omega |\nabla \overline{u}|^2. \tag{3}$$

Consider now the following minimization problem:

$$\min_{u \in \mathcal{K}(\Omega)} J(u), \tag{4}$$

where $\mathcal{K}(\Omega)$ is the subset of $H_0^1(\Omega)$ of functions depending only on the distance from the boundary $\partial\Omega$. We call the functions in $\mathcal{K}(\Omega)$ *web functions*. We observe that $\mathcal{K}(\Omega)$ is a linear closed subspace of $H_0^1(\Omega)$ and that the unique solution $\overline{v}$ of (4) satisfies the corresponding weak formulation of (1) when the class of test functions is restricted to $\mathcal{K}(\Omega)$, namely

$$\int_\Omega \nabla \overline{v} \cdot \nabla \varphi = \int_\Omega \varphi \qquad \forall \varphi \in \mathcal{K}(\Omega). \tag{5}$$

Since $\overline{u}$ satisfies the same relation for all $\varphi \in H_0^1(\Omega)$, we deduce by subtracting that $\overline{v}$ is the orthogonal projection of $\overline{u}$ onto $\mathcal{K}(\Omega)$ in the Hilbert space $H_0^1(\Omega)$; moreover, taking $\varphi = \overline{v}$ in (5), we infer

$$\min_{u \in \mathcal{K}(\Omega)} J(u) = J(\overline{v}) = -\frac{1}{2} \int_\Omega |\nabla \overline{v}|^2. \tag{6}$$

Web functions were introduced for planar regular polygons in [12] in order to approximate the infimum over $W_0^{1,1}(\Omega)$ of a more general class of functionals J with their minimum over $\mathcal{K}(\Omega)$. A full generalization to any convex domain in $\mathbb{R}^n$ ($n \geqq 2$) was given in [8]. It is shown in [8, 12] that under very mild assumptions on J (not including convexity), the minimum of J over $\mathcal{K}(\Omega)$ always exists and is unique. The explicit form of the unique minimizing web function and of (4) is also given. While preparing this manuscript, we discovered that functions depending on the distance from the boundary have been previously considered by Makai [18] and Pólya [25] precisely in connection with the torsion problem.

Our purpose is to determine an optimal estimate for the "error" made when (6) is used as an approximation for (3). The *relative error* of such approximation can be expressed by the ratio

$$\mathcal{E}(\Omega) = \frac{\min_{u\in\mathcal{K}(\Omega)} J(u)}{\min_{u\in H_0^1(\Omega)} J(u)}. \tag{7}$$

Before stating the main result of the paper, which is a sharp lower bound for $\mathcal{E}$, it is worth adding a few comments about definition (7). First note that $\mathcal{E}$ is well defined, because $u \equiv 0$ does not minimize J over $H_0^1(\Omega)$. Moreover, since $\mathcal{K}(\Omega) \subset H_0^1(\Omega)$ we have $\mathcal{E} \in [0, 1]$ and the closer $\mathcal{E}$ is to 1, the better the approximation is. This becomes more evident if, recalling (3) and (6), we see $\mathcal{E}(\Omega)$ as the square of the ratio between the $H_0^1(\Omega)$ norm of the component of $\overline{u}$ along $\mathcal{K}(\Omega)$ and the $H_0^1(\Omega)$ norm of $\overline{u}$. It is convenient to set

$$\mathcal{N}(\Omega) = -2 \min_{u\in\mathcal{K}(\Omega)} J(u), \qquad \mathcal{D}(\Omega) = -2 \min_{u\in H_0^1(\Omega)} J(u),$$

so that both the numerator $\mathcal{N}$ and the denominator $\mathcal{D}$ of $\mathcal{E}$ are nonnegative and homogeneous of degree 4, namely

$$\mathcal{N}(k\Omega) = k^4\mathcal{N}(\Omega), \qquad \mathcal{D}(k\Omega) = k^4\mathcal{D}(\Omega) \qquad\qquad \forall k > 0.$$

Therefore, the functional $\mathcal{E}$ is *invariant under dilations* and we may restrict our attention to convex sets in the plane having the same measure as the unit disk:

$$\mathcal{C} = \left\{\Omega \subset \mathbb{R}^2;\ \Omega \text{ is convex, } |\Omega| = \pi\right\}.$$

Clearly, an upper bound (or lower bound) for $\mathcal{E}$ gives a lower bound (resp., upper bound) for the torsion $\mathcal{D}$ in terms of $\mathcal{N}$. The upper bound $\mathcal{E} \leqq 1$ is straightforward and has been already pointed out by Pólya [25, (3.3)]. Much less is known about lower bounds for $\mathcal{E}$: some of them and some numerical results for more general problems are available in the previous works [9, 10]. Therefore, as pointed out by Buttazzo [6], it is of some interest to study the minimization problem of $\mathcal{E}$ over $\mathcal{C}$ and to find out if there exists an optimal design. Our main result gives a complete answer to these questions. It states that the infimum of $\mathcal{E}$ over $\mathcal{C}$ is $\frac{3}{4}$ and that it is not attained:

Theorem 1. *For all* $\Omega \in \mathcal{C}$,

$$\mathcal{E}(\Omega) > \inf_{D\in\mathcal{C}} \mathcal{E}(D) = \tfrac{3}{4}.$$

We stress that most of the usual techniques fail when we try to use them to prove the above result. In fact, these methods may give some information on $\mathcal{N}$ and $\mathcal{D}$ separately, but either they do not work simultaneously for both, or they are not fine enough to emphasize different behaviors and allow us to prove Theorem 1. For instance, the use of maximum principles is ruled out since $\mathcal{E}$ is a quotient whose numerator $\mathcal{N}$ and denominator $\mathcal{D}$ have essentially the same behavior under small perturbations. On the other hand, the derivative with respect to the domain, often employed in shape optimization, can be performed only for smooth sets and turns out to have a complicated form. Therefore, we are led to set up a different, specific approach. It is based on the definition of a *piercing function* λ measuring roughly how far we can enter into Ω starting from a boundary point and following the inner normal. This piercing function, inspired by a work of CELLINA [7], must be handled very carefully because it does not "vary with continuity" with respect to the Hausdorff distance d_H of domains, see Remark 4. When Ω is a convex polygon, we obtain both an upper bound for $\mathcal{D}(\Omega)$ (see Theorem 3) and a lower bound for $\mathcal{N}(\Omega)$ (see Theorem 4) in terms of the piercing function λ. In connection with this, we heavily exploit the explicit expression of $\mathcal{N}(\Omega)$ given in [9] in terms of the *parallel sets* $\Omega_t := \{x \in \Omega;\ d(x, \partial\Omega) > t\}$, see formula (9), and an improved isoperimetric inequality for convex polygons, see Theorem 2. These tools enable us to prove the strict inequality $\mathcal{E} > \frac{3}{4}$ in the class of convex polygons. By density we extend such inequality to the whole class $\mathcal{C}$, and it remains strict by a suitable contradiction argument. Finally, to show that $\frac{3}{4}$ is the sharp lower bound, we exhibit a minimizing sequence; it is suggested by numerical computations, and it is given by isosceles triangles $\{T^h\}_h$ having infinitesimal height as $h \to +\infty$.

The paper is organized as follows. In Section 2, we study in some detail the functional $\mathcal{E}$ on two subclasses $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{C}$ which are extremal in the sense that they achieve the equality in suitable inequalities for convex planar sets. In particular, we analyze the behavior of $\mathcal{E}$ on the class of triangles. In Section 3, we deal with minimizing sequences and we prove the inequality $\inf_{\mathcal{C}} \mathcal{E} \leqq \frac{3}{4}$. Section 4 is devoted to the proof of the strict inequality $\mathcal{E} > \frac{3}{4}$. Finally, in Section 5, we conclude by giving some historical notes and open problems.

2. Two extremal cases

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set and let W_Ω denote its inradius, namely the supremum of the radii of the open disks contained in Ω. The Lebesgue measure of Ω and the 1-dimensional Hausdorff measure of its boundary $\partial\Omega$ will be denoted respectively by $|\Omega|$ and $|\partial\Omega|$. For every convex set $\Omega \subset \mathbb{R}^2$, the geometrical quantities $|\Omega|$, $|\partial\Omega|$, and W_Ω are related by the following inequalities, which can be found for instance in the book by BONNESEN & FENCHEL [5]:

$$\pi W_\Omega + \frac{|\Omega|}{W_\Omega} \leqq |\partial\Omega| \leqq \frac{2|\Omega|}{W_\Omega}. \tag{8}$$

Moreover, it is known (see [30, Section 8]), that (8) represents a *complete system of inequalities* for $(|\Omega|, |\partial\Omega|, W_\Omega)$, i.e., for every triplet of positive real numbers $(\alpha_1, \alpha_2, \alpha_3)$ satisfying $\pi\alpha_3 + \alpha_3^{-1}\alpha_1 \leqq \alpha_2 \leqq 2\alpha_3^{-1}\alpha_1$, there exist a convex planar set Ω such that $|\Omega| = \alpha_1$, $|\partial\Omega| = \alpha_2$, and $W_\Omega = \alpha_3$. In particular, setting $|\Omega| = \pi$, and representing on the coordinate axes

$$x = \frac{2\pi W_\Omega}{|\partial\Omega|} \qquad \text{and} \qquad y = \frac{4\pi^2}{|\partial\Omega|^2},$$

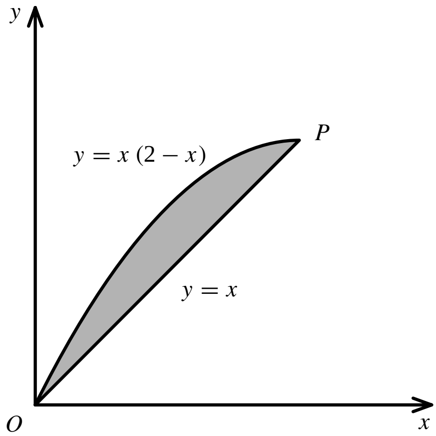

Fig. 1. The class $\mathcal{C}$ represented in the (x, y)-plane

the class $\mathcal{C}$ can be identified as the shaded set in the BLASCHKE-type diagram [3] represented in Fig. 1.

In this section we restrict the study of the functional $\mathcal{E}$ to the class of convex domains Ω which lie on the boundary of the set represented in Fig. 1. The upper parabola and the lower segment delimiting such a region correspond respectively to the following subclasses of $\mathcal{C}$:

$$\mathcal{C}_1 = \left\{ \Omega \in \mathcal{C};\ \pi \left(W_\Omega + \frac{1}{W_\Omega} \right) = |\partial\Omega| \right\}, \quad \mathcal{C}_2 = \left\{ \Omega \in \mathcal{C};\ |\partial\Omega| = \frac{2\pi}{W_\Omega} \right\}.$$

It is known that a set belongs to $\mathcal{C}_1$ if and only if it is in the form of a rectangle with to opposite sides rounded into two half circles [1, p. 8], whereas it belongs to $\mathcal{C}_2$ if and only if it circumscribes a disk [31, p. 321]. In particular, among the two extremal points P and O in Fig. 1, the former corresponds to the disk, which is the only element of $\mathcal{C}_1 \cap \mathcal{C}_2$, while the latter corresponds to the degenerate case of a straight line.

Concerning the disk B, not only it is the unique set in $\mathcal{C}_1 \cap \mathcal{C}_2$, but it also has the following extremality property:

Proposition 1. *Let* $\Omega \in \mathcal{C}$. *Then*

$$\mathcal{E}(\Omega) = 1 \quad \Longleftrightarrow \quad \Omega = B.$$

Proof. The implication $\Longleftarrow$ is an immediate consequence of the fact that the unique solution $\overline{u}$ of (1) on B is given by $\overline{u}(x) = (1 - |x|^2)/4$. Vice versa, let us assume that $\mathcal{E}(\Omega) = 1$, with $\partial\Omega \in C^2$. This means that the unique solution $\overline{u}$ of (1) is a web function, say $\overline{u}(x) = g(d(x))$, being $d(x) = d(x, \partial\Omega)$, and g a real function on $[0, W_\Omega]$. Then $\nabla\overline{u} = g'(d)\nabla d$. In particular, on $\partial\Omega$ we have

$$\frac{\partial\overline{u}}{\partial n} = g'(d)\nabla d \cdot n = -g'(0) n \cdot n = -g'(0) = \text{constant},$$

where n is the unit outer normal. It then follows by a theorem of SERRIN [32] that Ω must be a disk. In order to drop the restriction $\partial\Omega \in C^2$ we refer to the different proof of Serrin's result given by WEINBERGER in the subsequent paper [34]. $\square$

For every $\Omega \in \mathcal{C}_1 \cup \mathcal{C}_2$, we are able to compute the numerator $\mathcal{N}(\Omega)$ of the ratio $\mathcal{E}(\Omega)$ (see Propositions 2 and 3 below). This can be obtained as a particular case of the representation formula (25) in [9] (see also [25, (3.30)]), which states that

$$\mathcal{N}(\Omega) = \int_0^{W_\Omega} \frac{|\Omega_t|^2}{|\partial\Omega_t|}\, dt, \tag{9}$$

where

$$\Omega_t = \{x \in \Omega;\ d(x, \partial\Omega) > t\}. \tag{10}$$

The aim of computing explicitly $\mathcal{N}(\Omega)$ on $\mathcal{C}_1 \cup \mathcal{C}_2$ is twofold. First, the knowledge of the exact value of $\mathcal{N}(\Omega)$, combined with Theorem 1, allows us to deduce a quite simple way to estimate the torsion (see Corollaries 1 and 2). Second, once $\mathcal{N}(\Omega)$ is known, the numerical determination of $\mathcal{D}(\Omega)$ enables us to evaluate $\mathcal{E}(\Omega)$ on $\mathcal{C}_1 \cup \mathcal{C}_2$ (see Figs. 2, 3, 4 below); in particular, this provides an insight on how to construct a minimizing sequence (cf. Section 3).

Propositions 2 and 3 and the lower bounds of Corollaries 1 and 2 are essentially due to PÓLYA, see [25, p. 418] where several details were omitted. We give here a complete proof of these results; we also establish upper bounds for $\mathcal{D}$ and we complement these estimates with some numerical experiments. We now proceed separately on $\mathcal{C}_1$ and $\mathcal{C}_2$.

2.1. The case of rectangles ended by semicircles

Up to rigid motions there is a one-to-one correspondence between the interval $(0, 1]$ and the class $\mathcal{C}_1$. More precisely, for every $W \in (0, 1]$ (up to translations and rotations) the unique set R_W belonging to $\mathcal{C}_1$ and having W as inradius is given by

$$\begin{aligned} R_W = &\left[\left(\frac{\pi(W^2-1)}{4W}, \frac{\pi(1-W^2)}{4W}\right) \times (-W, W)\right] \\ &\bigcup B_W\left(\frac{\pi(W^2-1)}{4W}\right) \bigcup B_W\left(\frac{\pi(1-W^2)}{4W}\right) \end{aligned} \tag{11}$$

where $B_W(X)$ denotes the disk centered at $(X, 0)$ with radius W. Then, we have the following explicit characterization of $\mathcal{N}(R_W)$ in terms of W.

Proposition 2. *For all* $R_W \in \mathcal{C}_1$ (*with* $0 < W < 1$),

$$\mathcal{N}(R_W) = \frac{\pi}{32}\left(\frac{W^2-1}{W}\right)^4 \log\left(\frac{1+W^2}{1-W^2}\right) - \frac{\pi}{16}\,\frac{1-4W^2+W^4}{W^2}. \tag{12}$$

Proof. For all $t \in (0, W)$ we have

$$|\partial(R_W)_t| = \pi \left(W + \frac{1}{W} - 2t \right).$$

Since also $(R_W)_t$ belongs to $\mathcal{C}_1$, it still satisfies the equality in the (rescaled) first inequality of (8); hence

$$|(R_W)_t| = (W - t)|\partial(R_W)_t| - \pi(W - t)^2.$$

Therefore,

$$\frac{|(R_W)_t|^2}{|\partial(R_W)_t|} = \frac{\pi}{W} \frac{(W-t)^2(1-tW)^2}{W^2 - 2tW + 1}, \qquad 0 < t < W,$$

and the statement follows from (9) after integration over $[0, W]$. $\square$

By combining Theorem 1 with Proposition 2 we obtain an explicit way to estimate the torsion when $\Omega \in \mathcal{C}_1$:

Corollary 1. *For all* $R_W \in \mathcal{C}_1$ (*with* $0 < W < 1$),

$$\frac{\pi}{32} \left(\frac{W^2 - 1}{W} \right)^4 \log\left(\frac{1 + W^2}{1 - W^2} \right) - \frac{\pi}{16} \frac{1 - 4W^2 + W^4}{W^2} \leqq \mathcal{D}(R_W),$$

$$\mathcal{D}(R_W) < \frac{\pi}{24} \left(\frac{W^2 - 1}{W} \right)^4 \log\left(\frac{1 + W^2}{1 - W^2} \right) - \frac{\pi}{12} \frac{1 - 4W^2 + W^4}{W^2}.$$

Using the toolbox PDE of the program Matlab, we determined numerical values for $\mathcal{D}(R_W)$ for all $0 < W \leqq 1$. Thanks to Proposition 2, we then obtained the graph in Fig. 2, which represents the function $\Phi(W) = \mathcal{E}(R_W)$ for $0 < W \leqq 1$.

Note that the function Φ admits a global minimum for $W \approx 0.585$. This fact appears somehow natural since, for $W = 1$, R_W is the disk, so that $\mathcal{E}(R_1) = 1$

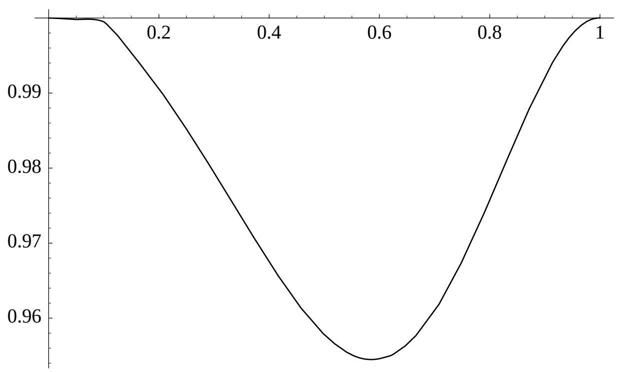

Fig. 2. The plot of $\Phi(W) = \mathcal{E}(R_W)$ for $W \in (0, 1]$

(see Proposition 1), whereas, for $W \to 0$, $\mathcal{E}(R_W)$ tends to 1 (see Proposition 5 below). As $W \to 0$, R_W approaches the sequence of thinning rectangles contained in the class $\mathcal{C}$, which are obtained by deformation of the square when stretching two opposite sides. Along such a sequence $\mathcal{E}$ tends to 1 and turns out, surprisingly, not to be monotonic, see [9, Proposition 4]. This unexpected behavior of thinning rectangles may find an explanation in the existence of a global minimum for the function Φ.

2.2. The case of circumscribed domains

Let $\Omega \in \mathcal{C}_2$. Then the following simple characterization of $\mathcal{N}(\Omega)$ in terms of W_Ω holds.

Proposition 3. *For all* $\Omega \in \mathcal{C}_2$,

$$\mathcal{N}(\Omega) = \frac{\pi}{8}\, {W_\Omega}^2 \;=\; \frac{\pi^3}{2|\partial\Omega|^2}.$$

Proof. By a density argument, it suffices to prove the statement when $\Omega \in \mathcal{C}_2$ is a polygon. Indeed, every $\Omega \in \mathcal{C}_2$ (circumscribing some disk D of radius W_Ω) can be approximated in the Hausdorff topology by a sequence of polygons $\{P_h\} \subset \mathcal{C}_2$ circumscribing the same disk D; then, we can pass to the limit in the equality $\mathcal{N}(P_h) = \frac{\pi}{8}\, {W_\Omega}^2$ thanks to the continuity of the mapping $\Omega \mapsto \mathcal{N}(\Omega)$ with respect to the Hausdorff convergence of domains, see [8, Section 6]. So, assume that Ω is a polygon, and let

$$C = \sum_{\vartheta} \operatorname{cotan} \frac{\vartheta}{2}$$

where the sum is extended over all inner angles ϑ of the polygon. By a straightforward computation, for all $t \in (0, W_\Omega)$, we have

$$|\Omega_t| = \pi - \frac{2\pi}{W_\Omega} t + Ct^2, \qquad |\partial\Omega_t| = \frac{2\pi}{W_\Omega} - 2Ct. \tag{13}$$

Moreover, since Ω_t still circumscribes a disk (of radius $W_\Omega - t$), it satisfies the equality in the (rescaled) second part of inequality (8), hence:

$$|\partial\Omega_t| = \frac{2|\Omega_t|}{(W_\Omega - t)}.$$

Taking $t = W_\Omega$ in the second equation of (13), we obtain $C = \dfrac{\pi}{{W_\Omega}^2}$. Therefore,

$$\frac{|\Omega_t|^2}{|\partial\Omega_t|} = \frac{\pi(W_\Omega - t)}{2}\left(1 - \frac{2}{W_\Omega} t + \frac{1}{{W_\Omega}^2} t^2\right),$$

and the statement follows from (9) after integration over $(0, W_\Omega)$. □

Thanks to Theorem 1 and Proposition 3, we may estimate the torsion of sets in $\mathcal{C}_2$ with the following simple inequalities:

Corollary 2. *For all* $\Omega \in \mathcal{C}_2$,

$$\frac{\pi}{8} W_\Omega{}^2 \leqq \mathcal{D}(\Omega) < \frac{\pi}{6} W_\Omega{}^2.$$

At this point, as in the case of class $\mathcal{C}_1$, we would like to determine numerical values for $\mathcal{D}(\Omega)$ when Ω belongs to $\mathcal{C}_2$. For this purpose, the whole class $\mathcal{C}_2$ is too wide. Therefore, we restrict our attention to the subset of $\mathcal{C}_2$ given by triangles. In order to simplify the numerics, we drop the area constraint, we fix two of the vertices, say $A = (-1, 0)$ and $B = (1, 0)$, and we let the third vertex C vary in the plane sector $\Gamma = [0, \infty) \times (0, \infty)$.

Let us begin by considering the simpler class of isosceles triangles of basis AB: in this case, the third vertex C is free to move along the positive y axis and we have a one-parameter family of triangles. We parametrize it as $\{T_\theta\}_\theta$, where $\theta \in (0, \frac{\pi}{2})$ is the common value for the acute angles adjacent to the base AB. The numerator $\mathcal{N}(T_\theta)$ can be recovered thanks to Proposition 3 and taking into account that

$$W_{T_\theta} = \tan \frac{\theta}{2}.$$

The numerical values for $\mathcal{D}(T_\theta)$ can be determined using the toolbox PDE of the program Matlab. Thus we obtained Fig. 3, representing the function $\Phi(\theta) = \mathcal{E}(T_\theta)$ for $\theta \in (0, \frac{\pi}{2})$.

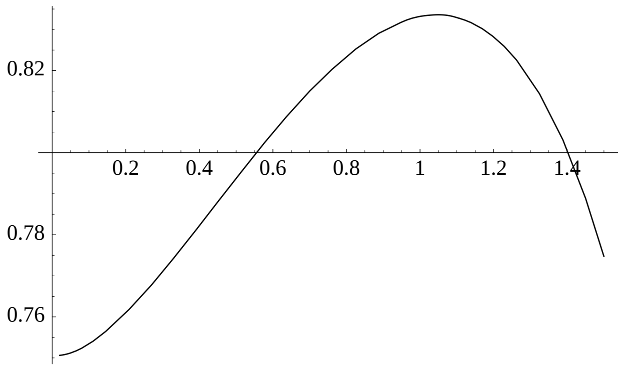

Fig. 3. The plot of $\Phi(\theta) = \mathcal{E}(T_\theta)$ for $\theta \in (0, \frac{\pi}{2})$

The function Φ approaches $\frac{3}{4}$ as $\theta \to 0$ or $\theta \to \frac{\pi}{2}$ (see Proposition 4 and Remark 1 below) and it has a maximum for $\theta = \frac{\pi}{3}$, in correspondence of the equilateral triangle. Actually, the equilateral triangle is the global maximum in the whole class of triangles. In such a class, to obtain a representation for $\mathcal{E}$ is more delicate since the third vertex C has two degrees of freedom in Γ. We parametrize the triangles as $\{T_{x,y}\}_{x,y}$, where $(x, y) \in \Gamma$ are the Cartesian coordinates of C. The numerator $\mathcal{N}(T_{x,y})$ can be recovered thanks to Proposition 3, as some computations give

$$W_{T_{x,y}} = \frac{2}{\operatorname{cotan}\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\alpha+\beta}{2}\right)}$$

where

$$\alpha = \arctan\left(\frac{y}{x+1}\right), \quad \beta = \begin{cases} \pi - \arcsin\left[\dfrac{2\sin\alpha}{\sqrt{(x-1)^2+y^2}}\right] & \text{if } x^2+y^2 \leqq 1, \\ \arcsin\left[\dfrac{2\sin\alpha}{\sqrt{(x-1)^2+y^2}}\right] & \text{otherwise.} \end{cases}$$

Again, the numerical values for $\mathcal{D}(T_{x,y})$ can be determined using the toolbox PDE of the program Matlab. This gives the 3-dimensional plot for the function $\Phi = \Phi(x, y)$ represented in Fig. 4, where (x, y) vary in $[0, 8) \times (0, 10)$.

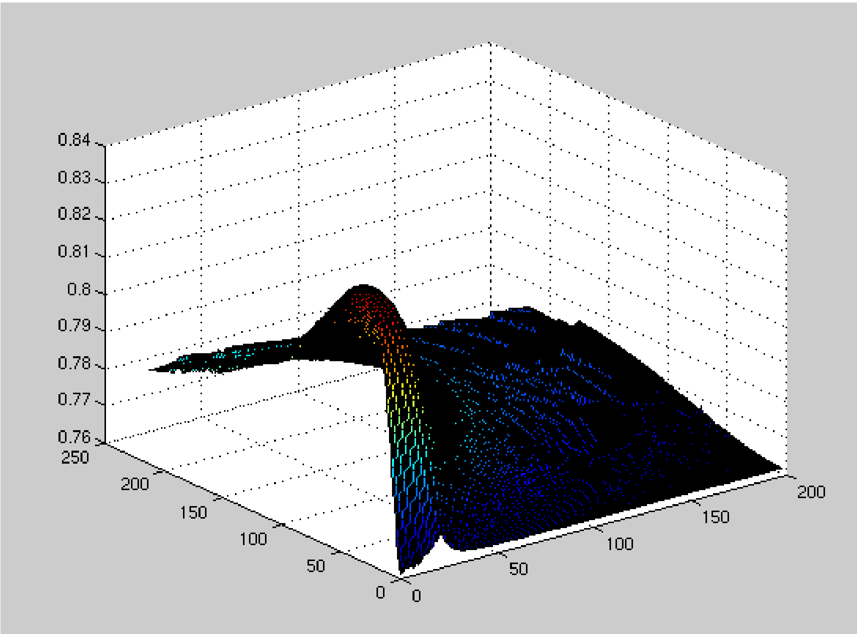

Fig. 4. The plot of $\mathcal{E}(T_{x,y})$ for $(x, y) \in [0, 8) \times (0, 10)$

The absolute maximum corresponds to the equilateral triangle $T_{0,\sqrt{3}}$ and the section of the surface in Fig. 4 with the axis $x = 0$ is precisely the graph for isosceles triangles $T_{0,y}$ $(y > 0)$ in Fig. 3. We also recall that $\mathcal{E}(T_{0,\sqrt{3}}) \approx 0.834$, see [10]. According to Fig. 4, it seems that $\mathcal{E}$ is strictly decreasing on every half line whose origin is $(0, \sqrt{3})$.

3. About minimizing sequences

Note first that, for every $\delta > 0$, the class $\mathcal{C}^\delta = \{\Omega \in \mathcal{C};\ W_\Omega \geqq \delta\}$ is compact. Indeed, every $\Omega \in \mathcal{C}^\delta$ satisfies $|\partial\Omega| \leqq 2\pi/\delta$, so that any sequence $\{\Omega^h\}_h \subset \mathcal{C}^\delta$ is equibounded up to a translation. Hence, from the Blaschke-selection Theorem [31, Theorem 1.8.6] and the continuity of the inradius, there exists a subsequence of $\{\Omega^h\}$ converging to a convex set $\Omega \in \mathcal{C}^\delta$.

We also recall that, if we endow $\mathcal{C}$ with the Hausdorff distance d_{H}, the functional $\mathcal{E}$ is continuous, see [8, Theorem 6.1]. Therefore, $\mathcal{E}$ admits a minimum over $\mathcal{C}^\delta$.

In order to apply such property, suppose for a moment that the strict inequality $\mathcal{E}(\Omega) > \frac{3}{4}$ holds for every $\Omega \in \mathcal{C}$ (this is precisely the first part of Theorem 1 and it will be proved in the next section). Then, either $\inf_{\mathcal{C}} \mathcal{E} > \frac{3}{4}$ or the following implication holds:

$$\forall\{\Omega^h\}_h \subset \mathcal{C}\,,\ \mathcal{E}(\Omega^h) \to \tfrac{3}{4} \quad \Longrightarrow \quad W_{\Omega^h} \to 0. \tag{14}$$

Accordingly, to prove that $\inf_{\mathcal{C}} \mathcal{E} = \frac{3}{4}$, we find a minimizing sequence:

Proposition 4. *There exists a sequence of isosceles triangles $\{T^h\}_h \subset \mathcal{C}$ such that*

$$\limsup_{h\to+\infty} \mathcal{E}(T^h) \leqq \tfrac{3}{4}\ ; \tag{15}$$

in particular, $\inf\limits_{\Omega\in\mathcal{C}} \mathcal{E}(\Omega) \leqq \frac{3}{4}$.

Proof. For all integer $h \geqq 1$ consider the isosceles triangle

$$T^h = \left\{(x, y) \in \mathbb{R}^2;\ 0 < y < \frac{\pi}{h},\ \frac{h^2 y}{\pi} - h < x < h - \frac{h^2 y}{\pi}\right\}.$$

Clearly, $T^h \in \mathcal{C}$ for all h. Moreover, since $W_{T^h} = \pi h(h^2 + \sqrt{h^4 + \pi^2})^{-1}$, by Proposition 3 and by letting $h \to \infty$, we obtain

$$\mathcal{N}(T^h) = \frac{\pi^3 h^2}{8(h^2 + \sqrt{h^4 + \pi^2})^2} \approx \frac{\pi^3}{32h^2} \qquad \text{as } h \to \infty. \tag{16}$$

Now let β_h be the function defining the two equal sides of T^h, namely

$$\beta_h(x) = \min\left\{\frac{\pi}{h^2}(h + x), \frac{\pi}{h^2}(h - x)\right\}, \qquad x \in [-h, h].$$

Set $v_h(x, y) = -\frac{1}{2} y(y - \beta_h(x))$, so that $v_h \in H_0^1(T^h)$. Then, with a simple integration we obtain

$$\mathcal{D}(T^h) = -2 \min_{u\in H_0^1(T^h)} J(u) \geqq -2J(v_h) \approx \frac{\pi^3}{24h^2} \qquad \text{as } h \to \infty.$$

This, together with (16), proves (15) by letting $h \to \infty$. □

Remark 1. An alternative minimizing sequence $\{T^h\}_h$ of isosceles triangles is obtained by letting h tend to zero, with T^h defined as in the above proof.

The next statement shows that the converse implication in (14) is false. Actually, a sequence of domains $\{\Omega^h\}_h \subset \mathcal{C}$ satisfying $W_{\Omega^h} \to 0$ may even be maximizing for $\mathcal{E}$.

Proposition 5. *For all* $W \in (0, 1]$ *let* $R_W \in \mathcal{C}_1$ *be the set defined by* (11). *Then*

$$\lim_{W \to 0} \mathcal{E}(R_W) = 1.$$

Proof. Consider the rectangle

$$Q_W = \left(\frac{\pi(W^2-1)}{4W} - W, \frac{\pi(1-W^2)}{4W} + W \right) \times (-W, W).$$

Since $R_W \subset Q_W$, by the maximum principle we have $\mathcal{D}(R_W) < \mathcal{D}(Q_W)$. For rectangles, explicit computations made by separation of variables allow us to determine $\mathcal{D}$, see (40) in [9]; using such a formula and the homogeneity of degree 4 of $\mathcal{D}$, we deduce that

$$\mathcal{D}(R_W) \leqq \mathcal{D}(Q_W) \approx \frac{\pi}{3} W^2 \text{ as } W \to 0.$$

Hence, we have

$$\liminf_{W \to 0} \mathcal{E}(R_W) \geqq \liminf_{W \to 0} \frac{3}{\pi W^2} \mathcal{N}(R_W) = 1,$$

where the last equality follows using (12). □

Remark 2. In view of Propositions 4 and 5, it is natural to ask which are the sequences $\{\Omega^h\}_h \subset \mathcal{C}$ that fulfill the necessary condition $W_{\Omega^h} \to 0$ and are also minimizing for $\mathcal{E}$. A complete characterization of such sequences seems to be rather difficult. However, we can find minimizing sequences different from the one made by triangles given in Proposition 4. In some sense, it is necessary to consider a sequence of domains for which the "triangular" component dominates the "rectangular" one in the thinning process. As $W_\Omega \to 0$, the rectangular and the triangular components of Ω are in fact its $\mathcal{C}_1$ and $\mathcal{C}_2$ components whose behavior is respectively maximizing and minimizing for $\mathcal{E}$. For instance, let P_b be the parallelogram with base $b > 0$, height $h = \frac{\pi}{b}$ and smallest inner angle of measure $\theta = \theta(b)$. Then the asymptotic behavior of $\mathcal{E}(P_b)$ for $b \to \infty$ depends on the choice of the function $\theta(b)$. If $\theta(b) = \frac{\pi}{2}$, then P_b is a rectangle and $\mathcal{E}(P_b) \to 1$. If $\theta(b) = \arcsin(\pi/b^2)$, then P_b is a rhombus and the triangular effect prevails so that $\mathcal{E}(P_b) \to \frac{3}{4}$. Finally, for different choices of θ, the limit of $\mathcal{E}(P_b)$ may take all the intermediate values between $\frac{3}{4}$ and 1.

4. Proof of Theorem 1

Thanks to Proposition 4, in order to complete the proof of Theorem 1 we have to show that

$$\mathcal{E}(\Omega) > \tfrac{3}{4}, \qquad \forall \Omega \in \mathcal{C}. \tag{17}$$

This is the goal of the section. Since the proof of (17) is delicate and covers a number of pages, we divide it into several steps.

4.1. Two fundamental tools

In this subsection we define the piercing function λ mentioned in the introduction and we prove an isoperimetric inequality for convex polygons.

For a.e. $y \in \partial\Omega$, the outer unit normal is well defined and it will be denoted by n_y. For a.e. $x \in \Omega$, the point $\Pi(x) \in \partial\Omega$ such that $|x - \Pi(x)| = d(x, \partial\Omega)$ is uniquely determined. Then we set

$$\lambda(y) = \sup\{k \geqq 0;\ \Pi(y - kn_y) = y\} \qquad \text{for a.e. } y \in \partial\Omega. \tag{18}$$

We clearly have $0 \leqq \lambda(y) \leqq W_\Omega$ on $\partial\Omega$. In what follows, we also make use of the following extension of λ to points $x \in \Omega$:

$$\lambda(x) = \lambda(\Pi(x)) - |x - \Pi(x)| \qquad \text{for a.e. } x \in \Omega. \tag{19}$$

Note that, for convex polygons $\Omega \subset \mathbb{R}^2$, (19) enables us to write the measure of the parallel set Ω_t as

$$|\Omega_t| = \int_{\partial\Omega_t} \lambda(y)\, dy. \tag{20}$$

Now we establish an isoperimetric inequality for convex polygons which will be used to estimate the term $\Delta(\Omega)$ in (33). In the case of polygons with 4 sides this inequality appears, for instance, in [4, (23)]; in the case of arbitrary polygons, see [21] and references therein. For sake of completeness we give here a full proof.

Theorem 2 (Isoperimetric inequality for convex polygons**).** *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Then*

$$|\Omega| \leqq \frac{|\partial\Omega|^2}{4C}, \tag{21}$$

where $C := \sum_{i=1}^N \operatorname{cotan} \frac{\theta_i}{2}$, and $\theta_1, \ldots, \theta_N$ denote the inner angles of the polygon. Equality in (21) *holds if and only if Ω is a circumscribed polygon.*

Remark 3. We have $C \geqq N \operatorname{cotan}\left[\frac{(N-2)}{2N}\pi\right] = N \tan\frac{\pi}{N} > \pi$. Thus (21) improves (for convex polygons) the usual isoperimetric inequality which holds for any set in the plane with $C = \pi$, see also [20, Section 12.4].

Proof. Let $\alpha(t) := |\partial\Omega_t|$, $t \in [0, W_\Omega]$. For t small enough we have $|\Omega_t| = |\Omega| - |\partial\Omega|t + Ct^2$, hence

$$\alpha(t) = -\frac{d}{dt}|\Omega_t| = |\partial\Omega| - 2Ct \qquad (t \text{ small}).$$

As a consequence of the Brunn–Minkowski Theorem, α is a concave function in $[0, W_\Omega]$ (see [5, Section 24, Section 55] and [8, Lemma 4.2]). Hence

$$\alpha(t) \leqq |\partial\Omega| - 2Ct \qquad \forall t \in [0, W_\Omega].$$

Integrating this inequality in $[0, W_\Omega]$ gives

$$|\Omega| = \int_0^{W_\Omega} \alpha(t)\,dt \leqq |\partial\Omega| W_\Omega - C W_\Omega^2,$$

and (21) follows maximizing the last term with respect to W_Ω. We remark that equality in (21) holds if and only if $\alpha(t) = |\partial\Omega| - 2Ct$ for every $t \in [0, W_\Omega]$ and $W_\Omega = |\partial\Omega|/2C$. These two conditions are simultaneously satisfied if and only if the polygon Ω circumscribed a disk. $\square$

4.2. An upper bound for $\mathcal{D}$

In this subsection we prove the following upper bound for the torsion of a polygon in terms of the piercing function:

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and let $\overline{u}$ be the minimizer of J in $H_0^1(\Omega)$. Then there exists $\delta = \delta(\Omega) > 0$ such that*

$$\mathcal{D}(\Omega) = \frac{1}{3} \int_{\partial\Omega} \lambda^3(y)\,dy - \delta(\Omega) \tag{22}$$

and

$$\delta(\Omega) \geqq \int_\Omega \left[|\nabla\overline{u}(x)|^2 - |\nabla\overline{u}(x) \cdot n(\Pi(x))|^2 \right] dx \tag{23}$$

where $n(\Pi(x))$ is the unit outer normal to $\partial\Omega$ at the point $\Pi(x)$ (when it exists).

Proof. We first prove (22). Assume that Ω has N sides and denote them by $F_1, \dots, F_N$. For simplicity, for all $j = 1, \dots, N$ we denote by F_j the *open* segment, namely the j-th side of Ω *without* its endpoints. Note that the function λ introduced in (18) is defined in every point of $\partial\Omega$ except for the N vertices. Moreover, $n_y \equiv n_j$ is a constant vector on F_j. We take a partition of Ω into N open subpolygons $P_1, \dots, P_N$ defined as follows:

$$P_j = \{y - tn_j;\ y \in F_j,\ 0 < t < \lambda(y)\}.$$

Each polygon P_j may also be seen as the (open) epigraph Z_j of the function λ on F_j, namely

$$Z_j = \{(y, t);\ y \in F_j,\ 0 < t < \lambda(y)\}.$$

For all $j \in \{1, \dots, N\}$ let

$$H_*^1(P_j) := \{v \in H^1(P_j);\ v = 0 \text{ on } F_j\},$$
$$H_*^1(Z_j) := \{v \in H^1(Z_j);\ v(y,0) = 0\ \forall y \in F_j\}$$

and consider the functional

$$J_j(v) = \int_{P_j} \left(\tfrac{1}{2} |\nabla v|^2 - v \right) \qquad \forall v \in H_*^1(P_j).$$

Note that

$$J_j(v) = \int_{F_j} \int_0^{\lambda(y)} \left[\tfrac{1}{2} |\nabla v(y - t n_j)|^2 - v(y - t n_j) \right] dt\, dy \qquad \forall v \in H^1_*(P_j). \tag{24}$$

Let $\overline{u}$ be the minimizer of J. Let also u_j denote the restrictions of $\overline{u}$ to P_j ($j = 1, \ldots, N$) and set

$$u_j^*(y, t) = u_j(y - t n_j) \qquad \forall (y, t) \in Z_j. \tag{25}$$

Since $u_j \in C^1 \cap H^1_*(P_j)$, we have $u_j^* \in H^1_*(Z_j)$ and $\partial u_j^*/\partial t = -\nabla u_j \cdot n_j$ so that

$$\left| \frac{\partial u_j^*}{\partial t}(y, t) \right| \leqq |\nabla u_j(y - t n_j)| \qquad \forall (y, t) \in Z_j. \tag{26}$$

Therefore we get

$$J_j(u_j) \geqq I_j(u_j^*) \qquad (j = 1, \ldots, N) \tag{27}$$

where

$$I_j(v) := \int_{F_j} \int_0^{\lambda(y)} \left[\frac{1}{2} \left(\frac{\partial v}{\partial t} \right)^2 - v \right] dt\, dy \qquad \forall v \in H^1_*(Z_j).$$

On the other hand, at each fixed $y \in F_j$, we have

$$\min \left\{ \int_0^{\lambda(y)} \left[\tfrac{1}{2} |g'(t)|^2 - g(t) \right] dt;\; g \in H^1(0, \lambda(y)),\; g(0) = 0 \right\} = -\tfrac{1}{6} \lambda^3(y).$$

Therefore the minimum of I_j on $H^1_*(Z_j)$ (which is attained by the function $w(y, t) = t[2\lambda(y) - t]/2$) may be evaluated as

$$\min\{I_j(v);\; v \in H^1_*(Z_j)\} = -\frac{1}{6} \int_{F_j} \lambda^3(y)\, dy. \tag{28}$$

Then, by (27) and (28) we have

$$J(\overline{u}) = \sum_{j=1}^N J_j(u_j) \geqq \sum_{j=1}^N I_j(u_j^*) \geqq -\frac{1}{6} \sum_{j=1}^N \int_{F_j} \lambda^3(y)\, dy = -\frac{1}{6} \int_{\partial\Omega} \lambda^3(y)\, dy. \tag{29}$$

This yields

$$\mathcal{D}(\Omega) = -2J(\overline{u}) \leqq \frac{1}{3} \int_{\partial\Omega} \lambda^3(y)\, dy. \tag{30}$$

To complete the proof of (22), it remains to show that the inequality in (30) is strict. We may have equality in (30) only if we have equalities in (26) for all $j = 1, \ldots, N$.

But this is equivalent to $\overline{u} \in \mathcal{K}(\Omega)$ (i.e., $\overline{u}$ web function), and in turn, to $\mathcal{E}(\Omega) = 1$. By Proposition 1, this contradicts the assumption that Ω is a polygon.

We now prove (23). By using (29), (24) and (25) we obtain

$$\begin{aligned}
\delta(\Omega) &= \frac{1}{3}\int_{\partial\Omega} \lambda^3 - \mathcal{D}(\Omega) = 2\Big[J(\overline{u}) + \frac{1}{6}\int_{\partial\Omega}\lambda^3\Big] \geqq 2\sum_j [J_j(u_j) - I_j(u_j^*)] \\
&= 2\sum_j \int_{F_j}\int_0^{\lambda(y)} \Bigg[\frac{1}{2}|\nabla u_j(y - tn_j)|^2 - u_j(y - tn_j) \\
&\quad - \frac{1}{2}\left|\frac{\partial u_j^*}{\partial t}(y,t)\right|^2 + u_j^*(y,t)\Bigg]\, dt\, dy \\
&= \sum_j \int_{F_j}\int_0^{\lambda(y)} \Big[|\nabla u_j(y - tn_j)|^2 - |\nabla u_j(y - tn_j)\cdot n_j|^2\Big] dt\, dy \\
&= \int_\Omega \Big[|\nabla\overline{u}(x)|^2 - |\nabla\overline{u}(x)\cdot n(\Pi(x))|^2\Big]\, dx
\end{aligned}$$

and (23) follows. $\square$

4.3. *A lower bound for $\mathcal{N}$*

In this subsection, for polygons Ω, we obtain the following lower bound for $\mathcal{N}(\Omega)$ in terms of the piercing function:

Theorem 4. *For any convex polygon $\Omega \subset \mathbb{R}^2$,*

$$\mathcal{N}(\Omega) \geqq \frac{1}{4}\int_{\partial\Omega} \lambda^3(y)\, dy.$$

Remark 4. Theorem 4 *cannot be extended to any convex domain*. To see this, just consider the case $\Omega = B$. Simple calculations give

$$\mathcal{N}(B) = \frac{\pi}{8}, \qquad \frac{1}{4}\int_{\partial B}\lambda^3(y)\, dy = \frac{1}{4}\int_{\partial B} dy = \frac{\pi}{2}\, .$$

The explanation of this fact is that the map

$$\Omega \mapsto \int_{\partial\Omega} \lambda^3(y)\, dy$$

is not continuous with respect to the Hausdorff distance d_{H}. For this reason the piercing function is a tool which should be handled very carefully!

Theorem 4 follows directly from Lemmas 1 and 2 below.

Lemma 1. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon, then*

$$\mathcal{N}(\Omega) = \frac{1}{4}\int_{\partial\Omega}\lambda^3(y)\, dy + \int_0^{W_\Omega}\left[\frac{|\Omega_t|^2}{|\partial\Omega_t|} - \frac{3}{4}\int_{\partial\Omega_t}\lambda^2\right] dt. \tag{31}$$

Proof. Let $\overline{v} \in \mathcal{K}(\Omega)$ be the (unique) minimizing web function, i.e.,

$$J(\overline{v}) = \min_{u \in \mathcal{K}(\Omega)} J(u).$$

By [8, Theorem 3.1] we have

$$\overline{v}(x) = \int_0^{d(x,\partial\Omega)} \nu(t)\, dt\,, \qquad \nu(t) := \frac{|\Omega_t|}{|\partial\Omega_t|}.$$

Then, since $\mathcal{N}(\Omega) = -2J(\overline{v})$, using the coarea formula and an integration by parts we infer

$$\begin{aligned}
\mathcal{N}(\Omega) &= -2\int_{\partial\Omega}\int_0^{\lambda(y)} \left[\frac{\nu^2(t)}{2} - \int_0^t \nu(s)ds\right] dt\, dy \\
&= \int_{\partial\Omega}\int_0^{\lambda(y)} \left[-\nu(t)^2 + 2(\lambda(y) - t)\nu(t)\right] dt\, dy \\
&= \int_{\partial\Omega}\int_0^{\lambda(y)} [\lambda(y) - t]^2\, dt\, dy - \int_{\partial\Omega}\int_0^{\lambda(y)} [\nu(t) - (\lambda(y) - t)]^2\, dt\, dy \\
&= \frac{1}{4}\int_{\partial\Omega} \lambda^3(y)\, dy + \Delta(\Omega),
\end{aligned} \tag{32}$$

where

$$\Delta(\Omega) := \int_{\partial\Omega}\int_0^{\lambda(y)} \left[\tfrac{1}{4}[\lambda(y) - t]^2 - [\nu(t) - (\lambda(y) - t)]^2\right] dt\, dy.$$

By Fubini's Theorem and recalling that (19) defines λ in the whole Ω, we may rewrite $\Delta(\Omega)$ as

$$\Delta(\Omega) = \int_0^{W_\Omega}\int_{\partial\Omega_t} \left[\tfrac{1}{4}\lambda^2(z) - [\nu(t) - \lambda(z)]^2\right] dz\, dt.$$

Finally, by (20), we have that $\nu(t)$ is the integral mean value of λ in $\partial\Omega_t$, and the above equation becomes

$$\Delta(\Omega) = \int_0^{W_\Omega} \left[\frac{|\Omega_t|^2}{|\partial\Omega_t|} - \frac{3}{4}\int_{\partial\Omega_t} \lambda^2\right] dt, \tag{33}$$

which, combined with (32), proves (31). □

We now show that the term $\Delta(\Omega)$ in (33) is nonnegative:

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Then, the function*

$$\psi(t) := |\Omega_t|^2 - \tfrac{3}{4}|\partial\Omega_t| \int_{\partial\Omega_t} \lambda^2, \qquad t \in [0, W_\Omega], \tag{34}$$

is nonnegative.

Proof. Denote by $0 < t_1 < \cdots < t_h = W_\Omega$ the set of *interaction times,* defined by recurrence as follows. The first interaction time t_1 is the smallest value $t \in (0, W_\Omega]$ for which (at least) one vertex of the parallel polygon Ω_t is the intersection between two or more bisecting lines of inner angles of Ω. Assume now we have defined $t_i \in (0, W_\Omega]$, for $i \leqq k$. If $t_k = W_\Omega$, then $h = k$, namely there are k interaction times $t_1, \dots, t_k$. Otherwise, we define t_{k+1} as the smallest value $t \in (t_k, W_\Omega]$ for which (at least) one vertex of the parallel polygon Ω_t is the intersection between two or more bisecting lines of inner angles of Ω_{t_k}. This means that, for every interaction time t_i and for every $\varepsilon > 0$, $\Omega_{t_i-\varepsilon}$ has (at least) one side more than Ω_{t_i}. Note in particular that the set of interaction times is finite and is reduced to $t_1 = W_\Omega$ in the case of circumscribed polygons.

Now fix $t \notin \{t_1, \dots t_h\}$ and take $\varepsilon > 0$ small enough such that the interval $(t-\varepsilon, t+\varepsilon)$ does not contain any interaction time. Let C_t denote the constant in (21) relative to the polygon Ω_t. It follows by some geometrical arguments that

$$|\partial\Omega_{t+\varepsilon}| = |\partial\Omega_t| - 2C_t\varepsilon, \quad |\Omega_{t+\varepsilon}| = |\Omega_t| - |\partial\Omega_t|\varepsilon + o(\varepsilon),$$
$$\int_{\partial\Omega_{t+\varepsilon}} \lambda^2 = \int_{\partial\Omega_t} \lambda^2 - 2\varepsilon|\Omega_t| + o(\varepsilon),$$

hence

$$\psi(t+\varepsilon) = \psi(t) + \left[\tfrac{3}{2}C_t \int_{\partial\Omega_t} \lambda^2 - \tfrac{1}{2}|\partial\Omega_t||\Omega_t|\right]\varepsilon + o(\varepsilon).$$

Since the same argument can be repeated replacing $t+\varepsilon$ by $t-\varepsilon$, we deduce that the function ψ is differentiable at every $t \notin \{t_1, \dots, t_h\}$, and

$$\psi'(t) = \tfrac{3}{2}C_t \int_{\partial\Omega_t} \lambda^2 - \tfrac{1}{2}|\partial\Omega_t||\Omega_t|.$$

Recalling the definition of ψ, we have $\int_{\partial\Omega_t} \lambda^2 = \frac{4}{3}\,|\partial\Omega_t|^{-1}\,[|\Omega_t|^2 - \psi(t)]$, then

$$\psi'(t) = 2C_t \frac{|\Omega_t|}{|\partial\Omega_t|}\left[|\Omega_t| - \frac{|\partial\Omega_t|^2}{4C_t}\right] - 2\frac{C_t}{|\partial\Omega_t|}\psi(t). \tag{35}$$

Now, using the isoperimetric inequality (21), we obtain

$$\psi'(t) \leqq -2\frac{C_t}{|\partial\Omega_t|}\psi(t) \qquad \forall\, t \notin \{t_1, \dots, t_h\}.$$

Since ψ is a continuous function, vanishing at $t = W_\Omega$, the above inequality means that $\psi(t) \geqq 0$ for every $t \in [0, W_\Omega]$. □

Remark 5. Lemma 2 may be complemented with the statement that for a convex polygon Ω we have $\psi(t) \equiv 0$ if and only if Ω circumscribes a disk. Indeed, for such Ω, for every $t \in [0, W_\Omega]$ the following equalities hold:

$$|\partial\Omega_t| = \frac{|\partial\Omega|}{W_\Omega}(W_\Omega - t),\ |\Omega_t| = \frac{|\partial\Omega|}{2W_\Omega}(W_\Omega - t)^2,\ \int_{\partial\Omega_t}\lambda^2 = \frac{|\partial\Omega|}{3W_\Omega}(W_\Omega - t)^3.$$

Therefore, by using these identities in (34), it follows that $\psi(t) \equiv 0$. Conversely, assume that $\psi(t) \equiv 0$. By (35) and Theorem 2, we deduce that Ω_t must be a circumscribed polygon for all $t \in [0, W_\Omega]$.

4.4. Conclusion

In this subsection we prove inequality (17): by Proposition 1 we may restrict our attention to the case where Ω is not a disk. Then, denoting by d_{H} the Hausdorff distance of domains, we make precise the behavior of the function δ (found in Theorem 3) on converging sequences of polygons.

Lemma 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex set (different from a disk). Then there exists a sequence $\{P_h\}_h$ of convex polygons such that $d_H(P_h, \Omega) \to 0$ and $\delta(P_h) \to C_\Omega > 0$, where C_Ω is a constant depending only on Ω.*

Proof. Take a sequence of polygons $\{P_h\}_h$ such that $d_{\mathrm{H}}(P_h, \Omega) \to 0$ and $P_h \subset \Omega$ for all h. Then, extending by zero on $\Omega \setminus P_h$ functions in $H_0^1(P_h)$, we have the usual embedding $H_0^1(P_h) \subset H_0^1(\Omega)$. We may also extend by zero the distance function from ∂P_h:

$$d_h(x) = \begin{cases} d(x, \partial P_h) & \text{if } x \in P_h, \\ 0 & \text{if } x \in \Omega \setminus P_h. \end{cases}$$

Also define $d(x) = d(x, \partial\Omega)$.

Let $\overline{u}$ and u_h be the minimizers of J over $H_0^1(\Omega)$ and $H_0^1(P_h)$ respectively, that is, $\mathcal{D}(\Omega) = -2J(\overline{u})$ and $\mathcal{D}(P_h) = -2J(u_h)$ for all h. Since for a.e. $x \in \Omega$ we have $n(\Pi_h(x)) = \nabla d_h(\Pi_h(x)) = \nabla d_h(x)$ (where $\Pi_h(x)$ is the projection point of x onto ∂P_h), by (23) and Fatou's Lemma we infer

$$\liminf_h \delta(P_h) \geqq \int_\Omega \liminf_h \left[|\nabla u_h(x)|^2 - |\nabla u_h(x) \cdot \nabla d_h(x)|^2 \right] dx. \tag{36}$$

We first claim that, up to a subsequence, we have

$$\nabla u_h(x) \to \nabla \overline{u}(x) \qquad \text{for a.e. } x \in \Omega. \tag{37}$$

By using the Euler equations (1) in Ω and P_h, we find that $u_h \rightharpoonup \overline{u}$ in $H_0^1(\Omega)$ and $\|\nabla u_h\|_{L^2(\Omega)} \to \|\nabla \overline{u}\|_{L^2(\Omega)}$ so that $\nabla u_h \to \nabla \overline{u}$ in $L^2(\Omega)$ and (37) follows, up to a subsequence.

Next we claim that, up to a subsequence,

$$\nabla d_h(x) \to \nabla d(x) \qquad \text{for a.e. } x \in \Omega. \tag{38}$$

To show this, note that $d_h(x) \to d(x)$ for all $x \in \Omega$. When this is combined with the uniform estimate $\|\nabla d_h\|_2^2 = |P_h| \leqq |\Omega|$, we infer that, up to subsequences, $\{d_h\}$ converges weakly to d in $H_0^1(\Omega)$. Moreover, since $|P_h| \to |\Omega|$ we also have $\|\nabla d_h\|_2 \to \|\nabla d\|_2$. Therefore $\nabla d_h \to \nabla d$ in $L^2(\Omega)$ and (38) follows, up to a subsequence.

For contradiction, assume that $\delta(P_h) \to 0$. Then, by (36), (37) and (38) we conclude that

$$\int_\Omega \left[|\nabla \overline{u}(x)|^2 - |\nabla \overline{u}(x) \cdot \nabla d(x)|^2 \right] dx = 0$$

which implies that $\overline{u}$ is a web function ($\overline{u} \in \mathcal{K}(\Omega)$) and hence $\mathcal{E}(\Omega) = 1$. But this is impossible in view of Proposition 1 since we assumed that Ω is not a disk. Then the statement of the lemma follows, possibly by extracting a subsequence from P_h.
□

Proof of (17). If Ω is a convex polygon, (17) is a straightforward consequence of Theorems 3 and 4. If $\Omega \in \mathcal{C}$ is not a polygon or a disk, consider the sequence $\{P_h\}_h$ of convex polygons determined in Lemma 3. By Theorems 3 and 4 we have

$$\mathcal{E}(P_h) = \frac{\mathcal{N}(P_h)}{\mathcal{D}(P_h)} \geqq \frac{3}{4}\,\frac{\mathcal{D}(P_h) + \delta(P_h)}{\mathcal{D}(P_h)} \qquad \forall h \in \mathbb{N}\,. \tag{39}$$

Let $C_\Omega > 0$ be as in Lemma 3. Then, recalling the continuity of $\mathcal{E}$ with respect to d_{H} (see [8]) and letting $h \to \infty$ in (39) we obtain

$$\mathcal{E}(\Omega) \geqq \frac{3}{4}\,\frac{\mathcal{D}(\Omega) + C_\Omega}{\mathcal{D}(\Omega)} > \frac{3}{4}$$

and (17) is proved for all Ω.

5. Historical notes and open problems

5.1. A brief story of the torsion problem and parallel sets

The study of the torsion problem has a long history. In 1856, SAINT-VENANT [29] conjectured that among all cross-sections Ω of given area the disk has the maximal torsional rigidity. This conjecture was proved for simply connected regions by PÓLYA [24] in 1948, and then extended to multiply connected domains by PÓLYA & WEINSTEIN [27]. In [24] can also be found the uniform upper bound (independent of $\Omega \in \mathcal{C}$) $\|\overline{u}\|_\infty \leqq \frac{1}{4}$, where $\overline{u}$ is the warping function solving (1); for a lower bound of $\|\overline{u}\|_\infty$ in terms of the harmonic radius of Ω, see [22].

The sets Ω_t defined in (10) are known in the literature as *inner parallel sets*, whereas the sets $\Omega^t := \Omega + tB$ are called *outer parallel sets*. The latter seem have been first considered in 1882 by STEINER [33], who proved the relations between $|\Omega|$, $|\partial\Omega|$, $|\Omega^t|$, and $|\partial\Omega^t|$, nowadays known precisely as Steiner's formulas. The origin of inner parallel sets may be perhaps attributed to RIESZ [28], who used them in order to prove integral inequalities. A first study of the properties of inner parallel sets was developed by BOL [4]. The papers [11,14] focus attention on the piecewise regularity of $\partial\Omega_t$ for a.e. t; in particular, Steiner's formulas may be partially extended (for small t) to inner parallel sets, see e.g. [1, Section I.1.4].

The first flavor of the idea to use inner parallel sets in variational problems may be found in the monograph by PÓLYA & SZEGÖ [26, Section 1.29]: they introduce a new method which consists in restricting the class of admissible functions by prescribing the family of their level lines. Only some years later, MAKAI [18] used the distance function from the boundary $d(x)$ in order to give a lower bound for the torsional rigidity of planar domains. Shortly afterwards, by dealing with the whole class of functions depending only on $d(x)$, Makai's bounds where improved first by PÓLYA [25] and subsequently by PAYNE & WEINBERGER [23].

Further inequalities related to the torsion problem may be found in [26]. For more recent results, see the book [1] and references therein. For the history of the location of maxima for the gradient of solutions to (1) and related problems, we refer to [15, Section 4].

Some different applications of inner parallel sets, mainly in the framework of convex bodies, may be found in [31] (see in particular the bibliographic note 2 for Section 6.5).

5.2. The torsion problem with two different materials

This problem was first considered in the celebrated paper by Pólya & Weinstein [27] and subsequently studied by many authors, see [2, 7, 13, 16, 19] and references therein. Assume we wish to place two different linearly elastic materials (of different shear moduli) in the plane domain Ω so as to maximize the torsional rigidity of the resulting rod; moreover, the proportions of these materials are prescribed. After some calculations, the problem is reduced to one of minimizing the functional

$$I(u) = \int_\Omega f(|\nabla u|) - u, \qquad u \in H_0^1(\Omega),$$

where $f(t) = \min\{\alpha t^2, \beta t^2 + \gamma\}$ with $\alpha > \beta > 0, \gamma > 0$.

Since f is not convex, such a problem may not have a solution. Then we are led to introduce the relaxed functional which does have a minimum. From a physical point of view, this means that there exists an optimal design if we are allowed to incorporate composites by mixing the two materials on a microscopic scale. However, the resulting design may not be so easy to manufacture and therefore it may be necessary to try to find an optimal design in a simpler class of possible designs. Again, we could restrict ourselves to the class of web functions (where I admits a unique minimum) and try to determine a sharp lower bound for the relative error

$$\frac{\min_{u\in\mathcal{K}(\Omega)} I(u)}{\inf_{u\in H_0^1(\Omega)} I(u)}.$$

For an estimate of the lower bound when Ω is a square, see [9, Proposition 8].

5.3. Minimization of $\mathcal{E}$ in higher space dimensions

The value $\frac{3}{4}$ found in Theorem 1 is somehow puzzling as it is not intuitively obvious where it comes from. Since it is strictly related to the geometrical properties of convex sets in the plane $\mathbb{R}^2$, it would be interesting to find the corresponding value in higher dimensional spaces $\mathbb{R}^d$, $d \geqq 3$. Some of our results may be easily extended to this context but others seem to be less suitable. However, at least in $\mathbb{R}^3$, it should not be too difficult to figure out what minimizing sequences look like and to find a numerical approximation of the infimum of $\mathcal{E}$.

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