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ON SUBCRITICALITY ASSUMPTIONS FOR THE EXISTENCE OF GROUND STATES OF QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We study conditions on f which ensure the existence of nonnegative, nontrivial radial solutions vanishing at infinity of the quasilinear elliptic equation $-\Delta_p u = f(u)$ in $\mathbb{R}^n$, with $n > p$. Both the behaviors of f at the origin and at infinity are important. We discuss several different subcritical growth conditions at infinity, and we show that it is possible to obtain existence of solutions also in some supercritical cases. We also show that, after an arbitrarily small L^q perturbation ($1 \leq q < \infty$) on f, solutions can be obtained without any restrictions on the behavior at infinity. In our proofs we use techniques from calculus of variations and arguments from the theory of ordinary differential equations such as shooting methods and the Emden-Fowler inversion.

1. Introduction

We are interested in existence of radial ground states of the following quasilinear elliptic equation,

$$-\Delta_p u = f(u) \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the degenerate p-Laplace operator and $n > p > 1$. Here, by a ground state we mean a $C^1(\mathbb{R}^n)$ nonnegative, nontrivial distribution solution of (1.1) which tends to zero as $|x| \to \infty$. Since we deal only with *radial* solutions of (1.1), from now on by a ground state we mean precisely a radial ground state. Clearly, existence results strongly depend on the function f. Roughly speaking, here we deal with the case where

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$f(s)$ is initially negative and ultimately positive. This is usually called the *normal case* [11].

When $n > p = 2$ there are in literature a number of well-known existence results for ground states of (1.1); see in particular [1, 2, 5, 14] and references therein. In [2] critical-point methods are employed and the radial symmetry of the ground states is recovered thanks to symmetrization. Another approach to (1.1) is to reduce it to an ordinary differential equation and to use a shooting method as was first suggested in [3] in the case $p = 2$; then, thanks to an Emden inversion, existence results are obtained in [1]. Much less is known about ground states for the degenerate equation (1.1) when $p \neq 2$. Citti [4] has proved existence when $1 < p < n$, $f(0) = 0$, and f is bounded in $[0, \infty)$, while Franchi-Lanconelli-Serrin [8] have considered the case when $f(s)$ is "sublinear" at infinity. General existence results for (1.1) for any $p > 1$ were obtained in [10], where (1.1) is also reduced to an ordinary differential equation and is studied with the shooting method but without the Emden inversion.

Our first purpose is to extend the existence results in [2] to the case of *any* $p > 1$. To this end, we seek a ground state of (1.1) by solving a constrained minimization problem in $D^{1,p}(\mathbb{R}^n)$. The difficulty is that the constraint is defined as a level set of a nonsmooth functional (it is not even continuous or locally bounded!). This difficulty was overcome in [2] by requiring an additional natural constraint and by solving a kind of obstacle problem. The non-Hilbertian framework of equation (1.1) suggests arguing differently. We use a weak version of the Lagrange-multiplier method which enables us to obtain a distribution solution of (1.1). The precise statement of this result is given in Theorem 1. It requires restrictions for f both at zero and at infinity.

Our second purpose is precisely to discuss the assumptions needed to prove Theorem 1. In fact, both the assumptions (at zero and at infinity) on f may be removed. To show this, we extend some results in [1] to the quasilinear case; see Theorems 2 and 3. In this extension, we also drop the assumption that the ground state is positive, and we allow it to have compact support; see Remark 1 (i) for the precise definition. Then, the usual interval for the shooting levels (the one giving radial solutions of (1.1) in some finite ball) is not necessarily open: we solve this inconvenience with a new proof; see Lemma 5 and Remark 2 below.

Our final purpose is to make some criticism on the growth restrictions needed to prove existence results. A common feature of all these just-mentioned approaches is that they need some kind of subcriticality assumption on f at infinity. In particular, for the power case $f(s) = -s^{r-1} + s^{q-1}$

where $1 < r < q$, all these methods require that $q < p^*$; here $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent. For such f, the results in [11] show that (1.1) admits no ground states whenever $q \geq p^*$. But as soon as we allow more general functions f, subcriticality seems not so important for existence results; see Theorems 3, 5, and 6. And indeed, since the ground states u of (1.1) are bounded functions, it is not clear at all which is the role played by the behavior of f at infinity (which is outside the range of u!). In this spirit, we prove Theorem 6 below. It states that the set of functions f for which (1.1) admits a ground state is "dense in L^q" for any $1 \leq q < \infty$; more precisely, for any function f and any $\varepsilon > 0$ there exists $g \in L^q(0,\infty)$ such that $\int_0^\infty |g|^q < \varepsilon$ and $-\Delta_p u = (f+g)(u)$ admits a ground state. The proof of this statement also emphasizes that in the shooting method not only the shooting level γ but also the "shooting strength" $f(\gamma)$ plays a crucial role.

As a conclusion, we may say that further research and perhaps different tools are necessary in order to determine the exact class of functions f for which existence of ground states holds true for (1.1). Both the behaviors of f at zero and at infinity are important for existence results, and probably existence of ground states depends on a suitable combination of these two behaviors.

2. Existence results

We list here different assumptions which are needed in the statements of the existence results. For each statement we require a suitably combined number of them.

As we are interested in nonnegative solutions, with no restrictions we put $f(s) = 0$ for $s \leq 0$. First of all, we require some regularity of the function f, either

$$f \in C[0,\infty) \cap \mathrm{Lip}_{\mathrm{loc}}(0,\infty) \ , \qquad f(0) = 0 \tag{2.1}$$

or the less stringent

$$f \in \mathrm{Lip}_{\mathrm{loc}}(0,\infty) \ , \qquad \limsup_{s\to 0} f(s) \geq 0 \ , \quad \int_0 |f(s)|ds < \infty. \tag{2.2}$$

Clearly, (2.1) implies (2.2); moreover, by (2.2) it is clear that $F(s) = \int_0^s f(t)dt$ exists and is continuous on $[0,\infty)$, and $F(0) = 0$. We then assume further

$$\exists \zeta > 0 \quad \text{such that } f(\zeta) > 0 \ , \quad F(\zeta) = 0 \ , \quad F(s) < 0 \quad \text{for } 0 < s < \zeta. \tag{2.3}$$

Let ζ be as in (2.3); if f annihilates at some point $\overline{\zeta} > \zeta$ then (1.1) admits a ground state without any further assumption; see [10, Theorem 1, case (C1)]. For this reason we also require that

$$f(s) > 0 \qquad \text{for } \zeta \leq s < +\infty. \tag{2.4}$$

For our first existence result we also need the following behavior for f at 0:

$$\limsup_{s \to 0^+} \frac{f(s)}{s^{p^*-1}} \leq 0. \tag{2.5}$$

Finally, we need some growth restrictions at infinity. These restrictions are essentially of subcritical growth type, but in some statements we also allow supercritical growth. The standard subcriticality assumption is

$$\lim_{s \to +\infty} \frac{f(s)}{s^{p^*-1}} = 0. \tag{2.6}$$

In order to give an alternative subcriticality assumption we introduce a function related to the Pohozaev identity, namely $Q(s) = npF(s) - (n-p)sf(s)$, and we assume that

$$\begin{cases} Q(s) \text{ is locally bounded below near } s = 0 \\ \exists b > \zeta \text{ such that } Q(s) \geq 0 \quad \forall s \geq b \\ \exists k \in (0,1) \text{ such that } \displaystyle\limsup_{s \to \infty} Q(s_2) \Big(\frac{s^{p-1}}{f(s_1)} \Big)^{n/p} = \infty, \ \forall s_1, s_2 \in [ks, s]. \end{cases} \tag{2.7}$$

A slightly different growth constraint (not subcritical!) is given by

$$\liminf_{s \to +\infty} \frac{F(s)}{s^{(np-p)/(n-p)}} < +\infty. \tag{2.8}$$

Our first result is obtained thanks to critical-point theory; we extend some results by Berestycki-Lions [2] to the quasilinear equation (1.1).

Theorem 1. *Assume* (2.1), (2.3), (2.4), (2.5), *and* (2.6). *Then there exists a ground state* u *of* (1.1) *such that* $u \in D^{1,p}(\mathbb{R}^n)$.

Note that in the statement of Theorem 1 assumptions on both the behavior of f at zero and at infinity are required. This is due to the (critical-point) tools used in the proof. In some cases we may drop assumption (2.5); we do so by strengthening (2.6) and by extending to the quasilinear case a result by Atkinson-Peletier [1]:

Theorem 2. *Assume* (2.1), (2.3), *and* (2.4). *Furthermore, suppose that there exist* $s_0 \geq \max\left\{\zeta, e^2\right\}$ *and* $m > \frac{p}{n-p}$ *such that*

$$f(s) = \frac{s^{p^*-1}}{(\ln s)^m} \qquad \forall s \geq s_0. \tag{2.9}$$

Then (1.1) *admits a ground state.*

We also extend another result from [1]: it states that we may drop assumption (2.5) and also (2.6) (allowing supercritical growth!) provided F has an "oscillating rate of blow-up" at infinity:

Theorem 3. *Assume* (2.1), (2.3), (2.4), *and* (2.8). *Then* (1.1) *admits a ground state.*

The assumptions in Theorems 1–3 should be compared with the ones in the following result, taken from [10]. Simple examples show that none of them is more powerful than the others.

Theorem 4. [10] *Assume* (2.2), (2.3), (2.4), *and* (2.7). *Then* (1.1) *admits a ground state* u.

Next, in a simple situation, we show that f may have any growth at infinity provided the subcritical term is sufficiently large.

Theorem 5. *Assume that* $p < r < p^* \leq q$, *and for all* $\lambda > 0$, *let*

$$f_\lambda(s) = -s^{p-1} + \lambda s^{r-1} + s^{q-1}.$$

Then there exists $\overline{\lambda} > 0$ *such that if* $\lambda > \overline{\lambda}$ *then the equation*

$$-\Delta_p u = f_\lambda(u) \quad in \quad \mathbb{R}^n \tag{2.10}$$

admits at least a ground state. Moreover, if $q > p^*$, *then there exists* $0 < \underline{\lambda} < \overline{\lambda}$ *such that if* $\lambda < \underline{\lambda}$, *then* (2.10) *admits no ground states.*

Theorem 5 is not a perturbation result, as one may find explicit values for $\overline{\lambda}$ and $\underline{\lambda}$; see the proof below. An interesting open problem is to understand if $\underline{\lambda} = \overline{\lambda}$, but this is beyond the scope of this paper. Note also that thanks to rescaling, Theorem 5 has several equivalent formulations concerning source terms as $f(s) = -s^{p-1} + s^{r-1} + \varepsilon s^{q-1}$ or $f(s) = -\beta u^{p-1} + u^{r-1} + \alpha u^{q-1}$.

Finally, in Section 6 we prove the following simple and striking statement:

Theorem 6. *Let* $q \in [1, \infty)$. *Assume that* f *satisfies* (2.2), (2.3), *and* (2.4). *Then for all* $\varepsilon > 0$ *there exists* f_ε *satisfying* (2.2), (2.3), *and* (2.4) *such that*

$$\int_0^\infty |f(s) - f_\varepsilon(s)|^q ds < \varepsilon$$

and such that the equation

$$-\Delta_p u = f_\varepsilon(u) \qquad \text{in } \mathbb{R}^n \tag{2.11}$$

admits at least a ground state.

Remark 1. (Further properties of the solutions)

(i) If $\int_0 |F(s)|^{-1/p} ds < \infty$ then ground states of (1.1) have compact support (a ball), whereas if $\int_0 |F(s)|^{-1/p} ds = \infty$ plus a further condition then ground states are positive; see [8].

(ii) If f has (near 0) the same homogeneity as Δ_p, then any ground state of (1.1) has exponential decay at infinity [9, Theorem 8], while for other behaviors of f ground states may have just polynomial decay [11, Proposition 5.1].

(iii) For the uniqueness of the ground states of (1.1) we refer to [13].

(iv) If $r = |x|$, the ground states $u = u(r)$ of (1.1) satisfy (see [10])

$$u'(r) < 0 \text{ for all } r > 0 \text{ such that } u(r) > 0. \tag{2.12}$$

(v) A ground state u is a classical solution in $\mathbb{R}^n \setminus \{0\}$, and it is as smooth as f permits in $\mathbb{R}^n \setminus \{0\}$. Moreover, by Theorem 2 in [6] and Theorem 1 in [15], any ground state of (1.1) is of class $C^{1,\alpha}(\mathbb{R}^n)$.

3. Proof of Theorem 1

The proof of Theorem 1 follows the same lines as that of Theorems 2 and 4 in [2] except for some significant changes. Consider on $D^{1,p}(\mathbb{R}^n)$ the two functionals

$$T(w) = \int_{\mathbb{R}^n} |\nabla w|^p \, dx \qquad\qquad V(w) = \int_{\mathbb{R}^n} F(w)\, dx; \tag{3.1}$$

note that $V(w)$ may not be finite for all $w \in D^{1,p}(\mathbb{R}^n)$. Let

$$\mathbf{K} = \left\{ w \in D^{1,p}(\mathbb{R}^n),\ F(w) \in L^1(\mathbb{R}^n),\ V(w) = 1 \right\},$$

and consider the following constrained minimization problem:

$$\min_{w \in \mathbf{K}} T(w). \tag{3.2}$$

We show the existence of a minimizing function:

Lemma 1. *Problem* (3.2) *has a solution* u *which is radially symmetric and nonincreasing.*

Proof. By arguing exactly as in Step 1, p. 324 in [2], one sees that $\mathbf{K} \neq \emptyset$. Take a sequence $\{u_m\} \subset \mathbf{K}$ such that $\lim_{m\to+\infty} T(u_m) = I$. Let u_m^* denote the Schwarz spherical rearrangement of $|u_m|$. We have $u_m^* \in \mathbf{K}$ (since $V(u_m^*) = V(u_m)$) and $T(u_m^*) \leq T(u_m)$; this means that $\{u_m^*\}$ is also a minimizing sequence. We have so found a minimizing sequence which is nonnegative, spherically symmetric and nonincreasing with respect to $r = |x|$.

Since $\{u_m\}$ is bounded in $D^{1,p}(\mathbb{R}^n)$, up to a subsequence (still denoted by $\{u_m\}$) it converges weakly in $D^{1,p}$ and almost everywhere in $\mathbb{R}^n$ to some $u \in D^{1,p}(\mathbb{R}^n)$ which is nonnegative, radially symmetric, and nonincreasing. Moreover, by the weak lower semicontinuity of the $D^{1,p}$ norm, we also know that

$$T(u) \leq \liminf_{m\to+\infty} T(u_m) = I. \tag{3.3}$$

As $\{u_m\}$ is bounded in $L^{p^*}(\mathbb{R}^n)$, by [2, Lemma A.IV] we have

$$|u_m(x)| \leq c\,|x|^{-(n-p)/p} \qquad \forall x \neq 0 \tag{3.4}$$

where $c > 0$ is independent of m. By (2.6) we have $|f(s)| \leq c + s^{p^*-1}$ for all $s \geq 0$, and hence, by (3.4), for any $R > 0$ there exists a positive constant C_R such that

$$\int_{B_R} |F(u_m)| \; dx \leq C_R \qquad \forall m \in \mathbb{N}. \tag{3.5}$$

Put $f^+ = \max(f, 0)$, $\;f^- = (-f)^+$, and

$$F_1(s) = \int_0^s f^+(t) \; dt \quad \text{ and } \quad F_2(s) = \int_0^s f^-(t) \; dt.$$

Then, by (2.5) and (3.4), for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$0 \leq F_1(u_m(r)) \leq \varepsilon\,|u_m(r)|^{p^*}, \quad \text{ for any } r \geq R_\varepsilon,\; m \in \mathbb{N},$$

and hence, for all m, we have

$$\int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} |F_1(u_m)| \; dx \leq \varepsilon \int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} |u_m(r)|^{p^*} dx \leq c\,\varepsilon. \tag{3.6}$$

Similarly, by lower semicontinuity of the $D^{1,p}$ norm with respect to weak convergence, u also satisfies a decay condition like (3.4), and therefore

$$\int_{\mathbb{R}^n\setminus B_{R_\varepsilon}} |F_1(u)| \; dx \leq c\,\varepsilon; \tag{3.7}$$

in particular, this tells us that $F_1(u) \in L^1(\mathbb{R}^n)$. Moreover, since $\{u_m\}$ is bounded in $W^{1,p}(B_R)$ for all $R > 0$, using (2.6) we infer that

$$\int_{B_R} F_1(u_m)\ dx \to \int_{B_R} F_1(u)\ dx \quad \text{as} \quad m \to +\infty \qquad \forall R > 0. \tag{3.8}$$

By arbitrariness of ε in (3.6) and (3.7), and by (3.8), we obtain

$$\int_{\mathbb{R}^n} F_1(u_m)\ dx \to \int_{\mathbb{R}^n} F_1(u)\ dx \quad \text{as} \quad m \to +\infty. \tag{3.9}$$

From $V(u_m) = 1$ we infer that $\int_{\mathbb{R}^n} F_1(u_m) = 1 + \int_{\mathbb{R}^n} F_2(u_m)$. Then, it follows from (3.9) and Fatou's lemma that $\int_{\mathbb{R}^n} F_1(u) \geq 1 + \int_{\mathbb{R}^n} F_2(u)$ so that also $F_2(u) \in L^1(\mathbb{R}^n)$; moreover,

$$V(u) = \int_{\mathbb{R}^n} F_1(u)\ dx - \int_{\mathbb{R}^n} F_2(u)\ dx\ \geq 1.$$

Suppose for contradiction that $V(u) > 1$; then by the scale change $u_\sigma(x) = u(x/\sigma)$ we have $V(u_\sigma) = \sigma^n V(u) = 1$, for some $\sigma \in (0,1)$. Moreover, $T(u_\sigma) = \sigma^{n-p}T(u) \leq \sigma^{n-p}I \leq I$, and this implies $I = 0$ (if $I > 0$, then $T(u_\sigma) < I$, and this is absurd because $u_\sigma \in \mathbf{K}$). Then, by (3.3) we have $T(u) = 0$; that is, $u \equiv 0$, contradicting $V(u) > 0$. Thus, we have $V(u) = 1$ and $T(u) = I$, so that u is a solution of the minimization problem (3.2). □

The proof of the Theorem 1 is complete once we prove

Lemma 2. *Let u be the (radial) solution of problem* (3.2) *found in Lemma* 1. *Let*

$$\overline{u}(x) = u\left(\frac{x}{\theta^{1/p}}\right) \qquad \text{where} \quad \theta = \frac{n-p}{np}\int_{\mathbb{R}^n} |\nabla u|^p\, dx.$$

Then $\overline{u} \in C^1(\mathbb{R}^n)$ and $\overline{u}$ is a ground state of (1.1).

Proof. The fact that $\theta > 0$ follows from $V(u) = 1$. Fix $\varphi \in C_c^\infty(\mathbb{R}^n)$ and define the function

$$\alpha_\varphi(t) = V(u + t\varphi) = \int_{\mathbb{R}^n} F(u + t\varphi)\ dx \qquad t \in \mathbb{R}. \tag{3.10}$$

Note that $\alpha_\varphi \in C^1(\mathbb{R})$ and

$$\alpha'_\varphi(t) = \int_{\mathbb{R}^n} f(u + t\varphi)\varphi\ dx \qquad t \in \mathbb{R}. \tag{3.11}$$

We set $u_t = u + t\varphi$ and $\sigma_t = [\alpha_\varphi(t)]^{-1/n}$; clearly, σ_t is well-defined for small t (recall $\alpha_\varphi(0) = 1$). Set also $\overline{u}_t(x) = u_t(x/\sigma_t)$. With a change of variable we find $V(\overline{u}_t) = 1$, and hence

$$T(u) \leq T(\overline{u}_t) \qquad \forall t. \tag{3.12}$$

Next, note that $T(\overline{u}_t) = \sigma_t^{n-p}\, T(u_t)$ and that

$$T(u_t) = \int_{\mathbb{R}^n} |\nabla(u + t\varphi)|^p \, dx = \int_{\mathbb{R}^n} |\nabla u|^p \, dx + tp \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + o(t)$$

as $t \to 0$; therefore, we get

$$T(\overline{u}_t) = \sigma_t^{n-p} \Big(T(u) + tp \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \Big) + o(t) \quad \text{as } t \to 0. \tag{3.13}$$

On the other hand, we also have

$$\sigma_t = [\alpha_\varphi(t)]^{-1/n} = \big[1 + \alpha'_\varphi(0)\, t\big]^{-1/n} + o(t) \qquad \text{as} \;\; t \to 0,$$

and hence, in view of (3.11),

$$\sigma_t = 1 - \frac{t}{n} \int_{\mathbb{R}^n} f(u)\, \varphi \; dx + o(t) \qquad \text{as} \;\; t \to 0. \tag{3.14}$$

Inserting (3.14) into (3.13) gives

$$T(\overline{u}_t) = T(u) + tp \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \varphi \; dx + \frac{p-n}{n} T(u) t \int_{\mathbb{R}^n} f(u)\, \varphi \; dx + o(t)$$

as $t \to 0$. Since (3.12) holds for all t, by switching t into $-t$ and letting $t \to 0$, the previous equality becomes

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \nabla \varphi \; dx = \theta \int_{\mathbb{R}^n} f(u) \varphi \; dx.$$

By arbitrariness of φ and introducing the function $\overline{u}$ we finally obtain

$$\int_{\mathbb{R}^n} |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \; dx = \int_{\mathbb{R}^n} f(\overline{u}) \varphi \; dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^n);$$

that is, $\overline{u}$ is a distributional solution of (1.1). The fact that $\overline{u} \in L^\infty \cap C^1$ is standard; see e.g. Propositions A.1 and A.5 in [7] and Proposition 1 in [12]. This ends the proof. □

4. Proof of Theorems 2 and 3

4.1. **The Emden inversion and the shooting method.** A nonnegative, nontrivial radial solution $u=u(r)$ of (1.1) is also a solution of the ordinary differential initial-value problem

$$\begin{cases} \left(|u'|^{p-2}\,u'\right)' + \dfrac{n-1}{r}\,|u'|^{p-2}\,u' + f(u) = 0, & r>0 \\ u(0)=\gamma\,, \quad u'(0)=0 \end{cases} \tag{4.1}$$

for some $\gamma>0$. We make the Emden inversion $t := \left(\frac{n-p}{p-1}\right)^{\frac{n-p}{p-1}} r^{-\frac{n-p}{p-1}}$, so that (4.1) becomes

$$\begin{cases} \left(|u'|^{p-2}\,u'\right)' + t^{-\frac{np-p}{n-p}} f(u) = 0, \quad t\in\mathbb{R}^+ \\ \lim\limits_{t\to+\infty} u(t)=\gamma, \quad \lim\limits_{t\to+\infty} u'(t)=0. \end{cases} \tag{4.2}$$

A ground state u of (4.1) satisfies (4.2) and the following condition:

$$\lim_{t\to 0} u(t) = 0. \tag{4.3}$$

Let E be the Lyapunov function for u defined as follows:

$$E(t) = \frac{p-1}{p} t^{\frac{np-p}{n-p}} \left|u'(t)\right|^p + F(u(t))\,, \qquad t>0. \tag{4.4}$$

By differentiating and using (4.2), we obtain

$$E'(t) = \frac{(n-1)(p-1)}{n-p}\; t^{\frac{n(p-1)}{n-p}} \left|u'(t)\right|^p \geq 0 \tag{4.5}$$

so that E is nondecreasing. We first restate [1, Lemma 4] in our setting:

Lemma 3. *Assume u solves (4.2), and suppose that $u(t)\to\delta\geq 0$ as $t\to 0$. Then $f(\delta)=0$ and*

$$t^{\frac{np-p}{n-p}} \left|u'(t)\right|^p \to 0 \qquad E(t)\to F(\delta) \qquad \text{as } t\to 0.$$

Concerning the asymptotic behavior of u and E at infinity, we prove

Lemma 4. *For a given $\gamma\in(0,\infty)$, a solution u of (4.2) satisfies*

$$t^{\frac{np-p}{n-p}} \left|u'(t)\right|^p \to 0 \qquad E(t)\to F(\gamma) \qquad \text{as } t\to +\infty.$$

Proof. Since u is bounded at infinity, by (4.2) we have

$$|u'(t)|^{p-1}\, t^{\frac{np-n}{n-p}} = \frac{\left|\int_t^{+\infty} s^{-\frac{np-p}{n-p}}\; f(u(s))\; ds\right|}{t^{\frac{-np+n}{n-p}}} \leq \frac{n-p}{n(p-1)} \sup_{s\in(t,+\infty)} |f(u(s))|\,,$$

and this proves that

$$\left|u'(t)\right|^{p-1} = O\left(t^{\frac{-np+n}{n-p}}\right) \quad \text{as } t \to +\infty.$$

Therefore,

$$\lim_{t\to+\infty} t^{\frac{np-p}{n-p}} \left|u'(t)\right|^{p} = \lim_{t\to+\infty} \left(|u'(t)|^{p-1} t^{\frac{np-n}{n-p}}\right)^{\frac{p}{p-1}} t^{\frac{-p}{n-p}} = 0,$$

and the first limit is proved. The second limit follows since $F(u(t)) \to F(\gamma)$ as $t \to +\infty$. □

We now introduce the tools for the shooting method. Given $\gamma > 0$, there exists a unique solution u_γ of (4.2) in a neighborhood of $t = +\infty$ (see [1, Lemma 1] for the case $p = 2$ and [8, Proposition A4] for the general case $p > 1$ without the Emden inversion). Therefore, the number

$$T(\gamma) := \inf\{T \geq 0;\ u_\gamma(t) > 0\ \forall t > T\}$$

is well-defined: it represents the infimum of the maximal interval of backward continuation for $u_\gamma(t)$ under the restriction that it remains positive. By (4.5) and Lemma 4 we have $E(t) \leq F(\gamma)$ and $F(u(t)) \leq F(\gamma)$ for all $t \in (T(\gamma), \infty)$, that is,

$$\left|u'(t)\right| \leq \left(\frac{p}{p-1}\right)^{1/p} \left[F(\gamma) - F(u(t))\right]^{1/p}\ t^{-\frac{n-1}{n-p}},$$

and from this, after a change of variable (u is monotone increasing by (2.12)), we deduce

$$\int_{u(t)}^{\gamma} \left[F(\gamma) - F(u)\right]^{-1/p} du \leq \left(\frac{p}{p-1}\right)^{1/p} \int_t^{\infty} s^{-\frac{n-1}{n-p}} ds \quad \forall t \in \left(T\left(\gamma\right), \infty\right). \tag{4.6}$$

Consider the set $S := \{\gamma \in (\zeta, \infty);\ T(\gamma) > 0\}$. The next result gives a sufficient condition for (1.1) to admit a ground state:

Lemma 5. *Suppose that the set S is not empty; then there exists a ground state of* (4.2)–(4.3).

Proof. Let $\gamma_0 = \inf S$. Consider first the case where $\gamma_0 \notin S$. By arguing as in [1, Lemma 6], we infer that if $\gamma \in S$, then $u'_\gamma(t) > 0$ and $E(t) > 0$ on $(T(\gamma), \infty)$. Therefore, by continuity we also have $u'_{\gamma_0}(t) \geq 0$ and $E(t) \geq 0$ for any $t \in (0, \infty)$; hence u_{γ_0} converges monotonically to some $\delta \in [0, \zeta)$ as $t \to 0$ (see Lemma 2.4 in [10]). If $\delta > 0$, then Lemma 3 and (2.3) yield $E(t) \to F(\delta) < 0$ as $t \to 0$, contradicting $E(t) \geq 0$. Hence, u_{γ_0} tends monotonically to 0 as $t \to 0$ and is therefore a ground state of (4.2) (4.3).

Assume now that $\gamma_0 \in S$. Then $T(\gamma_0) > 0$ and two cases may occur:

$$u'_{\gamma_0}(T(\gamma_0)) = 0 \qquad \text{or} \qquad u'_{\gamma_0}(T(\gamma_0)) > 0.$$

In fact, the second case cannot occur: otherwise, by the continuous dependence of u_γ from γ (see Propositions A3 and A4 in [8]) we would find $\gamma < \gamma_0$ such that $\gamma \in S$ contradicting the definition of γ_0. Hence, $u'_{\gamma_0}(T(\gamma_0)) = 0$ and the function

$$v(t) = \begin{cases} 0 & \text{if } t \le T(\gamma_0) \\ u_{\gamma_0}(t) & \text{if } t \ge T(\gamma_0) \end{cases}$$

is a ground state of (4.2) (4.3). □

Remark 2. If we know a priori that ground states are positive, then the set S is open and $\gamma_0 \notin S$. Otherwise, S may also not be open: in such a case one may give a different characterization of S (see the definition of I^- in [10]) and still find an open set. Here, we avoid this topological argument.

Theorems 3 and 2 will be proved in the following subsections by showing that $S \neq \emptyset$.

4.2. **Proof of Theorem 3.** We first prove a sufficient condition for Lemma 5 to hold:

Lemma 6. *Assume that for some $\gamma > 0$ there exists $T_1 > 0$ such that the corresponding solution u_γ of* (4.2) *satisfies*

$$u_\gamma(T_1) > \zeta \quad \text{and} \quad u'_\gamma(T_1) > 0, \tag{4.7}$$

and assume that for some $T' \in (0, T_1)$ there results

$$\sup_{0 \le \tau \le u_\gamma(T_1)} |f(\tau)| \int_{T'}^{T_1} s^{-\frac{np-p}{n-p}} \, ds \le \frac{1}{2} |u'_\gamma(T_1)|^{p-1} \tag{4.8}$$

and

$$2^{1/(p-1)} u_\gamma(T_1) < u'_\gamma(T_1)(T_1 - T'). \tag{4.9}$$

Then $\gamma \in S$.

Proof. By definition of S it is sufficient to show that u_γ cannot stay positive in (T', T_1). For contradiction, assume that u_γ remains positive in (T', T_1) and let $t \in (T', T_1)$; integrating (4.2) over $[t, T_1]$ and using (4.8) gives

$$|u'_\gamma(T_1)|^{p-1} - |u'_\gamma(t)|^{p-1} \le \sup_{0 \le \tau \le u_\gamma(T_1)} |f(\tau)| \int_t^{T_1} s^{-\frac{np-p}{n-p}} ds \le \frac{1}{2} \left| u'_\gamma(T_1) \right|^{p-1},$$

that is,

$$|u_\gamma'(t)|^{p-1} \geq \frac{1}{2}\left|u_\gamma'(T_1)\right|^{p-1} > 0,$$

and hence $u_\gamma'(t) \geq 2^{-1/(p-1)}\ u_\gamma'(T_1)$, which contradicts (4.9) after integration over $[T', T_1]$. □

Let

$$\overline{F} = -\min_{s\geq 0} F(s); \tag{4.10}$$

then the following holds:

Lemma 7. *Assume that for some $\gamma > 0$ there exists $T_1 > 0$ such that the corresponding solution u_γ of* (4.2) *satisfies* (4.7) *and*

$$\left(\frac{p-1}{p}\right)^{1/p}\frac{p(n-1)}{n-p}\, T_1^{\frac{p-1}{n-p}} u_\gamma(T_1) < \frac{E(T_1)}{\left[E(T_1)+\overline{F}\right]^{\frac{p-1}{p}}}. \tag{4.11}$$

Then $\gamma \in S$.

Proof. By definition of S, we need to show that as u_γ is continued backwards for $t < T_1$, it reaches the value zero in the interval $(0, T_1)$. By (4.5), $E(t) \leq E(T_1)$ for any $t \in (T(\gamma), T_1)$, and so

$$\frac{p-1}{p}\, t^{\frac{np-p}{n-p}} |u_\gamma'(t)|^p = E(t) - F(u(t)) \leq E(T_1) + \overline{F}.$$

This, combined with (4.5) and the fact that $u_\gamma' > 0$ in $(T(\gamma), T_1)$ (see (2.12)), gives

$$\begin{aligned} E(T_1) - E(T') &= \int_{T'}^{T_1} \left(\frac{p-1}{p}\, t^{\frac{np-p}{n-p}} |u_\gamma'(t)|^p\right)^{\frac{p-1}{p}} \left(\frac{p-1}{p}\right)^{1/p} \frac{np-p}{n-p} t^{\frac{p-1}{n-p}} u_\gamma'(t)\, dt \\ &< \left(E(T_1)+\overline{F}\right)^{\frac{p-1}{p}} \sigma(T_1) \qquad \forall T' \in (T(\gamma), T_1), \end{aligned}$$

where

$$\sigma(t) := \left(\frac{p-1}{p}\right)^{1/p}\ \frac{p(n-1)}{n-p}\ \ t^{\frac{p-1}{n-p}} u_\gamma(t). \tag{4.12}$$

Hence, by (4.11),

$$E(T') > E\left(T_1\right) - \left(E(T_1)+\overline{F}\right)^{\frac{p-1}{p}} \sigma\left(T_1\right) > 0 \qquad \forall T' \in (T(\gamma), T_1). \tag{4.13}$$

Next we claim that $|u_\gamma'(t)|^p\, t^{\frac{np-p}{n-p}}$ has a positive lower bound. For contradiction, assume that

$$\exists T \in [0, T_1] \qquad \text{such that} \qquad \lim_{t\to T}\ |u_\gamma'(t)|^p\, t^{\frac{np-p}{n-p}} = 0\ .$$

Then, since $u_\gamma \in C^1[T(\gamma), T_1]$, by what we just proved we necessarily have $T = T(\gamma) = 0$. Moreover, since u'_γ remains positive, the limit δ of u_γ as $t \to 0$ exists. Hence, Lemma 3 applies and $E(t) \to F(\delta) \le 0$ since $f(\delta) = 0$; see assumptions (2.3) and (2.4). But this contradicts (4.13), which gives a positive lower bound for E on the interval $[T(\gamma), T_1]$. Thus, there exists $\nu_1 > 0$ such that $|u'_\gamma(t)|^p t^{\frac{np-p}{n-p}} \ge \nu_1$ for all $t \in [T(\gamma), T_1]$, and so, by (4.5),

$$E'(t) \ge \frac{(n-1)(p-1)}{n-p}\, \nu_1\, t^{-1} \qquad \forall t \in [T(\gamma), T_1];$$

if $T(\gamma) = 0$, this implies that $E(t) \to -\infty$ as $t \to 0$, and we contradict Lemma 3. This shows that $T(\gamma) > 0$ and concludes the proof. □

In order to apply Lemma 6 we take $\zeta_1 \in (\zeta, \gamma)$ and $T_1 = T_1(\gamma)$ determined by

$$u_\gamma(T_1(\gamma)) = \zeta_1. \tag{4.14}$$

Lemma 8. *Let $\{\gamma_m\}$ be a sequence, with $\gamma_m > \zeta_1$ for any $m \in \mathbb{N}$ and such that one of the following cases holds:*

Case A: $T_1(\gamma_m) \to 0$ as $m \to +\infty$.

Case B: $T_1(\gamma_m)\left|u'_{\gamma_m}(T_1(\gamma_m))\right|^p \to +\infty$ as $m \to +\infty$.

Then, $\gamma_m \in S$ for all m large enough.

Proof. Let σ be as in (4.12). If Case A holds, then for any m large enough we have (4.11); indeed, if $T_1(\gamma_m) \to 0$ then $\sigma(T_1(\gamma_m)) \to 0$ as $m \to +\infty$, since ζ_1 is fixed. Furthermore, the right-hand side of (4.11) is bounded away from zero, since

$$E(T_1(\gamma_m))\, F(\zeta_1) > 0.$$

As (4.11) holds, Lemma 7 applies and the lemma is proved in Case A.

In Case B, we can take $T_1(\gamma_m)$ bounded away from zero since otherwise Case A applies.

If $T_1(\gamma_m)$ is bounded above, then necessarily $u'_{\gamma_m}(T_1(\gamma_m)) \to +\infty$ as $m \to +\infty$; in such a case, take $T' \in (0, \inf_m T_1(\gamma_m))$ and apply Lemma 6 with $T_1 = T_1(\gamma_m)$. It is easy to see that, with these choices, (4.8) and (4.9) hold for sufficiently large m.

Consider now the case where $T_1(\gamma_m) \to +\infty$ as $m \to +\infty$. We wish here to apply Lemma 6 with

$$T' = T_1(\gamma_m) - T_1(\gamma_m)^{1/p}.$$

By the assumption of Case B and with our choice of T' we readily see that (4.9) holds for m large enough. Moreover, with this choice we also have

$$\int_{T'}^{T_1} s^{-\frac{np-p}{n-p}}\,ds = (T_1(\gamma_m))^{\frac{(1-p)[n(p+1)-p]}{p(n-p)}} + o\Big((T_1(\gamma_m))^{\frac{(1-p)[n(p+1)-p]}{p(n-p)}} \Big) \quad (4.15)$$

as $m \to +\infty$. Finally, the original assumption also entails

$$(T_1(\gamma_m))^{\frac{n(p+1)-p}{n-p}} \left|u'_{\gamma_m}(T_1(\gamma_m))\right|^p \to +\infty \quad \text{as } m \to +\infty$$

since $\frac{n(p+1)-p}{n-p} > 1$. Thus,

$$(T_1(\gamma_m))^{\frac{(p-1)[n(p+1)-p]}{p(n-p)}} |u'_{\gamma_m}(T_1(\gamma_m))|^{p-1} \to +\infty \quad \text{as } m \to +\infty,$$

and using (4.15) we obtain (4.8) for sufficiently large m. Hence, Lemma 6 applies and the proof of Case B is complete. □

Lemma 9. *Make the same assumptions as in Theorem* 3. *Let* $\{\gamma_m\}$ *be a sequence such that* $\gamma_m \to +\infty$ *as* $m \to +\infty$; *suppose that there exist functions* ϕ *and* ψ *such that*

(*i*) $\inf \phi(\gamma_m) > \zeta$ *and* $\psi(\gamma_m) < \gamma_m$ *for any* $m \geq 1$;
(*ii*) $\phi(\gamma_m)/\psi(\gamma_m) \to 0$ *as* $m \to +\infty$;
(*iii*) $\int_{\psi(\gamma_m)}^{\gamma_m} [F(\gamma_m) - F(s)]^{-1/p}\,ds \geq \left(\frac{p}{p-1}\right)^{1/p} \int_{\phi(\gamma_m)}^{\infty} s^{-\frac{n-1}{n-p}} ds.$

Then $S \neq \emptyset$ *and* (1.1) *has a solution.*

Proof. Clearly, we may assume that $\gamma_m > \zeta$ for all m. Let $\zeta_1 \in (\zeta, \inf\ \phi(\gamma_m))$ and for m large enough so that $\psi(\gamma_m) > \phi(\gamma_m) > \zeta_1$, take $T_1(\gamma_m)$ as in (4.14). Let t_m be determined by $u_{\gamma_m}(t_m) = \psi(\gamma_m)$. Then by (4.6) we have

$$\int_{\psi(\gamma_m)}^{\gamma_m} [F(\gamma_m) - F(u_{\gamma_m})]^{-1/p}\,du_{\gamma_m} \leq \left(\frac{p}{p-1}\right)^{1/p} \int_{t_m}^{\infty} s^{-\frac{n-1}{n-p}} ds,$$

and so by (iii), $t_m \leq \phi(\gamma_m)$. We can take $T_1(\gamma_m)$ bounded away from zero; otherwise, Case A of Lemma 8 applies. As $(|u'_{\gamma_m}(t)|^{p-1})' < 0$ in $(T_1(\gamma_m), t_m)$, u_{γ_m} is concave, and hence

$$u'_{\gamma_m}(T_1(\gamma_m)) \geq \frac{\psi(\gamma_m) - \zeta_1}{t_m - T_1(\gamma_m)} \to +\infty \qquad \text{as } m \to +\infty.$$

Thus, we are either in Case A or in Case B of Lemma 8; this completes the proof. □

We are now ready to give the proof of Theorem 3. We distinguish three cases; in each of them we assume (2.1), (2.3), (2.4), and (2.8), plus a further requirement.

Case 1. This is the "sublinear" case.

$$\liminf_{s\to+\infty} s^{-p}F(s) < +\infty. \tag{4.16}$$

If (4.16) holds, then we find a divergent sequence $\{\gamma_m\}$ such that

$$\limsup_{m\to+\infty} \gamma_m^{-p}\, F(\gamma_m) < +\infty.$$

Then we also have

$$\liminf_{m\to+\infty} \int_{\frac{1}{2}\gamma_m}^{\gamma_m} [F(\gamma_m) - F(s)]^{-1/p}\, ds > 0. \tag{4.17}$$

Now take $\phi(\gamma) \equiv C =$ constant and $\psi(\gamma) = \frac{\gamma}{2}$. If C is large enough, (i) and (ii) in Lemma 9 are satisfied. Moreover, condition (iii) follows from (4.17). Hence, thanks to Lemma 9, Theorem 3 is proved under the additional assumption (4.16).

Case 2.

$$\liminf_{s\to+\infty} s^{-\frac{np-p}{n-p}} F(s) = 0. \tag{4.18}$$

Again we want to construct a divergent sequence $\{\gamma_m\}$ in order to apply (iii) in Lemma 9 with $\psi(\gamma) = \frac{1}{2}\gamma$. This yields the condition

$$\frac{\gamma_m}{2F(\gamma_m)^{1/p}} \geq \left(\frac{p}{p-1}\right)^{1/p} \frac{n-p}{p-1} \left[\phi(\gamma_m)\right]^{\frac{1-p}{n-p}}. \tag{4.19}$$

Now we want to determine ϕ so that equality in (4.19) holds:

$$\phi(\gamma_m) = C\left[\gamma_m^{-p} F(\gamma_m)\right]^{\frac{n-p}{p(p-1)}}.$$

We may suppose that $\gamma_m^{-p}F(\gamma_m) \to +\infty$, since otherwise (4.16) holds and this yields the conclusion. Thus $\phi(\gamma_m) \to +\infty$, and hence $\phi(\gamma_m) > \zeta$ for large m. By (4.18) we can choose $\{\gamma_m\}$ so that

$$\gamma_m^{-\frac{np-p}{n-p}} F(\gamma_m) \to 0 \quad \text{as } m \to +\infty. \tag{4.20}$$

This implies that $\phi(\gamma_m)/\gamma_m \to 0$ as $m \to +\infty$; thus (i), (ii), and (iii) in Lemma 9 are satisfied, and this completes the proof of Theorem 3 under the additional assumption (4.18).

Case 3. (4.16) and (4.18) do not hold.

In what follows we denote by C positive constants which may vary from line to line. Thanks to the additional assumption we have

$$F(s) \geq C s^{\frac{np-p}{n-p}} \quad \text{for } s \text{ large enough}. \tag{4.21}$$

Furthermore, by (2.8), we can choose a sequence $\{\gamma_m\}$, $\gamma_m \to +\infty$, such that

$$F(\gamma_m) \le C\gamma_m^{\frac{np-p}{n-p}}. \tag{4.22}$$

We may assume that

$$T_1(\gamma_m) \ge C \tag{4.23}$$

since otherwise Case A in Lemma 8 applies. Choosing $\gamma_m > 4\zeta_1$, we define $T_0(\gamma_m)$ by the relation $u_{\gamma_m}(T_0(\gamma_m)) = \frac{1}{2}\gamma_m$. Now, by (4.6) and (4.22) we have

$$T_0(\gamma_m) \le C\gamma_m. \tag{4.24}$$

Since, by the concavity of u_{γ_m} we have

$$u'_{\gamma_m}(T_1(\gamma_m)) \ge \frac{\frac{1}{2}\gamma_m - \zeta_1}{T_0(\gamma_m) - T_1(\gamma_m)}\ , \tag{4.25}$$

it follows that $u'_{\gamma_m}(T_1(\gamma_m)) \ge C$ for m large enough; here we used (4.24) and the fact that $\gamma_m \to +\infty$. This implies that we can take $T_1(\gamma_m)$ bounded above, i.e.,

$$T_1(\gamma_m) \le C, \tag{4.26}$$

since otherwise Case B of Lemma 8 applies. For the same reason we assume that

$$u'_{\gamma_m}(T_1(\gamma_m)) \le C. \tag{4.27}$$

Therefore, by (4.23), (4.25), and (4.27), for m large enough we have

$$T_0(\gamma_m) \ge C\gamma_m. \tag{4.28}$$

Now, integrating (4.2) over the interval $(T_1(\gamma_m), T_0(\gamma_m))$ we obtain

$$\left|u'_{\gamma_m}(T_1(\gamma_m))\right|^{p-1} - \left|u'_{\gamma_m}(T_0(\gamma_m))\right|^{p-1} = \int_{T_1(\gamma_m)}^{T_0(\gamma_m)} t^{-\frac{np-p}{n-p}} f(u_{\gamma_m}(t))\ dt,$$

and since u'_{γ_m} is bounded above in $[T_1(\gamma_m), \infty)$ we have

$$\begin{aligned} C &\ge \int_{T_1(\gamma_m)}^{T_0(\gamma_m)} t^{-\frac{np-p}{n-p}} f(u_{\gamma_m}(t))\, u'_{\gamma_m}(t)\ dt \\ &= (T_0(\gamma_m))^{-\frac{np-p}{n-p}} F\left(\tfrac{1}{2}\gamma_m\right) - (T_1(\gamma_m))^{-\frac{np-p}{n-p}} F(\zeta_1) \\ &\quad + \frac{np-p}{n-p}\int_{T_1(\gamma_m)}^{T_0(\gamma_m)} t^{-\frac{np-p}{n-p}-1} F(u_{\gamma_m}(t))\ dt \end{aligned} \tag{4.29}$$

uniformly with respect to m thanks to (4.27). Thus, by (4.23) and (4.29) we have

$$\int_{T_1(\gamma_m)}^{T_0(\gamma_m)} t^{-\frac{np-p}{n-p}-1} F\left(u_{\gamma_m}(t)\right)\, dt \le C,$$

and using (4.21) we obtain

$$\int_{T_1(\gamma_m)}^{T_0(\gamma_m)} t^{-\frac{np-p}{n-p}-1} \left(u_{\gamma_m}(t)\right)^{\frac{np-p}{n-p}}\, dt \le C. \tag{4.30}$$

From the concavity of u_{γ_m} when $u_{\gamma_m} > \zeta$ we have

$$u_{\gamma_m}(t) \ge \zeta_1 + \frac{\frac{1}{2}\gamma_m - \zeta_1}{T_0(\gamma_m) - T_1(\gamma_m)} \left(t - T_1(\gamma_m)\right) \quad \text{for} \quad T_1(\gamma_m) \le t \le T_0(\gamma_m).$$

Since $\gamma_m > 4\zeta_1$ (and hence $\frac{1}{2}\gamma_m - \zeta_1 > \frac{1}{4}\gamma_m$), it follows from (4.23) and (4.24) that for large m

$$u_{\gamma_m}(t) \ge \zeta_1 + C\left(t - T_1(\gamma_m)\right) \ge Ct, \tag{4.31}$$

where in the last inequality we used (4.26). Now by (4.30) and (4.31) we have

$$\ln \frac{T_0(\gamma_m)}{T_1(\gamma_m)} \le C,$$

which contradicts (4.26) and (4.28) for large m. This completes the proof of Theorem 3.

4.3. **Proof of Theorem 2.** Consider again the function $Q(s) = npF(s) - (n-p)sf(s)$; then by (2.9) we have

$$Q(s) = Q(s_0) + m(n-p)\int_{s_0}^{s} \frac{t^{p^*-1}}{(\ln t)^{m+1}}\, dt \qquad \forall s \ge s_0\ .$$

In particular, there exists $s_1 \ge s_0$ such that

$$Q(s) \ge 0 \qquad \forall s \ge s_1. \tag{4.32}$$

Now, we claim that there exists $\zeta_1 \ge s_1$ such that

$$\frac{sf(s)}{F(s)} \le p^* - \frac{m}{2\ln s} \qquad \forall s \ge \zeta_1. \tag{4.33}$$

Indeed,

$$F(s) = F(s_1) + \frac{1}{p^*}\left[sf(s) - s_1 f(s_1) + m\int_{s_1}^{s} \frac{t^{p^*-1}}{(\ln t)^{m+1}} dt\right] \qquad \forall s \ge s_1$$

and, by (4.32), we obtain

$$p^* \geq \frac{sf(s)}{F(s)} + \frac{m}{\ln s}\Big[1 - \frac{F(s_1)}{F(s)}\Big] \qquad \forall s \geq s_1,$$

and hence, since $F(s) \to +\infty$ as $s \to +\infty$, we get (4.33) for large s, say $s \geq \zeta_1$, and (4.33) follows.

Let $\overline{F}$ be as in (4.10), and consider the function

$$\varphi(z) = \frac{z}{\left(z + \overline{F}\right)^{\frac{p-1}{p}}}, \qquad z \geq 0;$$

since φ is strictly increasing in $[0, \infty)$ we may define $\omega(y) = \varphi^{-1}(y)$ for all $y \geq 0$. Note that $\omega(y) \sim y^p$ as $y \to +\infty$, and therefore for all $T > 0$ the number

$$M_T = \sup_{t \geq T} t^{\frac{n-np}{n-p}}\ \omega\Big(\Big(\frac{p-1}{p}\Big)^{1/p}\Big(\frac{np-p}{n-p}\Big)\zeta_1\, t^{\frac{p-1}{n-p}}\Big)$$

is finite and positive; here $\zeta_1 > \zeta$ is defined by (4.33).

From now on we take $\gamma > 4\zeta_1$, we denote by u the solution of (4.2) (we omit the subindex γ), and we define $T_1(\gamma)$ by $u(T_1(\gamma)) = \zeta_1$. We introduce the following Lyapunov function:

$$\begin{aligned} H(t) &= \frac{p-1}{p} t|u'|^p - \frac{p-1}{p} u|u'|^{p-1} + t^{-\frac{n(p-1)}{n-p}} F(u) \\ &= t^{-\frac{n(p-1)}{n-p}} E(t) - \frac{p-1}{p} u|u'|^{p-1}. \end{aligned}$$

We claim that the proof of Theorem 2 follows if there exists $T \leq T_1(\gamma)$ such that

$$H\left(T_1\left(\gamma\right)\right) \geq M_T. \tag{4.34}$$

Indeed, since $H(t) < t^{\frac{-n(p-1)}{n-p}} E(t)$, by (4.34) we have

$$t^{\frac{-n(p-1)}{n-p}} E(t) > \sup_{t \geq T} t^{\frac{-n(p-1)}{n-p}} \omega\Big(\Big(\frac{p-1}{p}\Big)^{1/p} \frac{np-p}{n-p}\, \zeta_1\, t^{\frac{p-1}{n-p}}\Big),$$

and hence, since $T_1(\gamma) \geq T$,

$$\begin{aligned} E\left(T_1\left(\gamma\right)\right) &\sup_{t \geq T}\Big\{\omega\Big(\Big(\frac{p-1}{p}\Big)^{1/p} \frac{np-p}{n-p}\, \zeta_1\, t^{\frac{p-1}{n-p}}\Big)\Big\} \\ &\geq \omega\Big(\Big(\frac{p-1}{p}\Big)^{1/p} \frac{np-p}{n-p}\, \zeta_1 \left(T_1(\gamma)\right)^{\frac{p-1}{n-p}}\Big); \end{aligned}$$

recalling that $\omega = \varphi^{-1}$, this means that (4.11) holds, and the result follows from Lemma 7.

Before proving (4.34) we make a few remarks about the function H. First of all note that

$$H(t) \to 0 \quad \text{as} \quad t \to +\infty \tag{4.35}$$

since $E(t)$ is bounded by Lemma 4 and $u'(t) \to 0$. Furthermore, for any $t > T_1(\gamma)$, we have

$$\begin{aligned} H'(t) &= \frac{p-1}{p} t^{-\frac{p(n-1)}{n-p}} F(u(t)) \Big[\frac{u(t)f(u(t))}{F(u(t))} - p^* \Big] \\ &\leq -\frac{(p-1)m}{2p} t^{-\frac{np-p}{n-p}} \frac{F(u(t))}{\ln(u(t))} < 0 \end{aligned} \tag{4.36}$$

in view of (4.33). By (4.35) and (4.36) we have

$$H(t) \geq \frac{m(p-1)}{2p} \int_t^{+\infty} s^{-\frac{np-p}{n-p}} \frac{F(u(s))}{\ln(u(s))} ds \qquad \forall t \geq T_1(\gamma). \tag{4.37}$$

We now turn to the proof of (4.34). We define $T_0(\gamma)$ by $u(T_0(\gamma)) = \frac{1}{2}\gamma$ so that $T_0(\gamma) < T_1(\gamma)$. By (4.33), $F(s) > sf(s)/p^*$ for any $s \geq \zeta_1$. Using this in (4.37) we obtain

$$\begin{aligned} H(T_1(\gamma)) &> \frac{(p-1)(n-p)m}{2np^2} \int_{T_1(\gamma)}^{+\infty} t^{-\frac{np-p}{n-p}} \frac{(u(t))^{p^*}}{[\ln(u(t))]^{m+1}}\, dt \\ &= -\frac{(p-1)(n-p)m}{2np^2} \int_{T_1(\gamma)}^{+\infty} \left(|u'(t)|^{p-1} \right)' \frac{u(t)}{\ln(u(t))}\, dt \\ &= \frac{(p-1)(n-p)m}{2np^2} \Big\{ |u'(T_1(\gamma))|^{p-1} \frac{u(T_1(\gamma))}{\ln(u(T_1(\gamma)))} \\ &\qquad + \int_{T_1(\gamma)}^{+\infty} |u'(t)|^p \frac{\ln(u(t)) - 1}{[\ln(u(t))]^2}\, dt \Big\} \\ &> \frac{(p-1)(n-p)m}{2np^2} \int_{T_1(\gamma)}^{T_0(\gamma)} |u'(t)|^p \frac{\ln(u(t)) - 1}{[\ln(u(t))]^2}\, dt \\ &> \frac{(p-1)(n-p)m}{2np^2} \frac{\ln\gamma - 1}{(\ln\gamma)^2} \int_{T_1(\gamma)}^{T_0(\gamma)} |u'(t)|^p\, dt \end{aligned}$$

where in the last inequality we use the fact that the map $s \mapsto (\ln s)^{-1} - (\ln s)^{-2}$ is decreasing in the interval (e^2, ∞) (recall $s_0 \geq e^2$). Using the positivity of u' in the interval $[T_1(\gamma), T_0(\gamma)]$ we obtain

$$H(T_1) > \frac{(p-1)(n-p)m}{2np^2} \frac{\ln\gamma - 1}{(\ln\gamma)^2} \int_{T_1(\gamma)}^{T_0(\gamma)} |u'(t)|^p\, dt$$

$$\geq \frac{(p-1)(n-p)m}{2np^2} \frac{\ln\gamma - 1}{(\ln\gamma)^2} \frac{\left(\frac{1}{2}\gamma - \zeta_1\right)^p}{(T_0(\gamma) - T_1(\gamma))^{p-1}}, \tag{4.38}$$

where the last inequality is a consequence of the fact that among functions $v \in C^1[T_0, T_1]$ satisfying $v(T_0) = \frac{\gamma}{2}$ and $v(T_1) = \zeta_1$, the minimum of the functional $\int_{T_1}^{T_0} |v'|^p$ is attained by the affine function w (which satisfies $w'' \equiv 0$). Thus, (4.34) holds if

$$T_0(\gamma) < T_1(\gamma) + \Big(\frac{(p-1)(n-p)m(\ln\gamma - 1)\left(\frac{1}{2}\gamma - \zeta_1\right)^p}{2np^2 M_T (\ln\gamma)^2}\Big)^{1/(p-1)};$$

since $\gamma > 4\zeta_1$, it is sufficient to show that

$$\begin{aligned} T_0(\gamma) &< T_1(\gamma) + \Big(\frac{(p-1)(n-p)m(\ln\gamma - 1)\gamma^p}{2np^2 M_T (\ln\gamma)^2 4^p}\Big)^{1/(p-1)} \\ &=: T_1(\gamma) + K_T \Big(\frac{(\ln\gamma - 1)\gamma^p}{(\ln\gamma)^2}\Big)^{1/(p-1)}, \end{aligned} \tag{4.39}$$

where K_T is a well-defined constant depending on T which is bounded away from 0 and ∞ for any $T \leq T_1(\gamma)$ (this follows from the definition of M_T). Integrating twice (4.2) and using (4.33), we have

$$\begin{aligned} \frac{\gamma}{2} &\leq \Big(\sup_{s\in(T_0,+\infty)} f(u(s))\Big)^{\frac{1}{p-1}} \int_{T_0}^{+\infty} \Big(\int_t^{+\infty} s^{-\frac{np-p}{n-p}} ds\Big)^{\frac{1}{p-1}} dt \\ &\leq (f(\gamma))^{\frac{1}{p-1}} \Big(\frac{(n-p)^p}{n(p-1)p^{p-1}}\Big)^{\frac{1}{p-1}} T_0(\gamma)^{-\frac{p}{n-p}}. \end{aligned}$$

Using (2.9), this implies (for a suitable $C = C(n,p) > 0$)

$$T_0(\gamma) < C \frac{[f(\gamma)]^{(n-p)/p(p-1)}}{\gamma^{(n-p)/p}} = C \frac{\gamma^{p/(p-1)}}{(\ln\gamma)^{m(n-p)/p(p-1)}}. \tag{4.40}$$

Since $m > \frac{p}{n-p}$, it is clear that (4.40) implies (4.39) for large enough γ. This completes the proof of (4.34) and, consequently, also of Theorem 2.

5. Proof of Theorem 5

Let $F_\lambda(s) = \int_0^s f_\lambda(t)dt$ and $Q_\lambda(s) = npF_\lambda(s) - (n-p)sf_\lambda(s)$. Since

$$F_\lambda(s) > \frac{\lambda}{r} s^r - \frac{s^p}{p} \qquad \forall s > 0,$$

by minimizing the right-hand side we obtain

$$\overline{F}_\lambda = -\min_{s\geq 0} F_\lambda(s) \leq \Big(\frac{1}{p} - \frac{1}{r}\Big)\lambda^{-\frac{p}{r-p}}. \tag{5.1}$$

We can now prove

Lemma 10. *There exists $C>0$ such that for λ sufficiently large we have* $Q_\lambda(s)\ge 0\ \forall s\in[C\lambda^{-\frac{1}{r-p}},1]$.

Proof. For all $c>0$ we have

$$Q_\lambda\big(c\lambda^{-\frac{1}{r-p}}\big)=pc^p\Big[\Big(\frac{n}{r}-\frac{n-p}{p}\Big)c^{r-p}-1\Big]\lambda^{-\frac{p}{r-p}}+np\Big(\frac{1}{q}-\frac{1}{p^*}\Big)c^q\lambda^{-\frac{q}{r-p}}.$$

Take $c>2(\frac{n}{r}-\frac{n-p}{p})^{-\frac{1}{r-p}}$ and let $\lambda\to+\infty$; then

$$Q_\lambda\big(c\lambda^{-\frac{1}{r-p}}\big)>0 \qquad \text{for } \lambda \text{ large enough.} \tag{5.2}$$

Moreover,

$$Q_\lambda(1)=-p+\lambda np\big(\frac{1}{r}-\frac{1}{p^*}\big)+np\big(\frac{1}{q}-\frac{1}{p^*}\big)>0 \quad \text{for } \lambda \text{ large enough.} \tag{5.3}$$

It is not difficult to verify that for λ large enough there exist $c_2(\lambda)>c_1(\lambda)>0$ such that

$$Q_\lambda(s)>0 \Longleftrightarrow s\in(c_1(\lambda),c_2(\lambda))$$

so that the positivity set of Q_λ is an interval (note that if $q=p^*$, then $c_2(\lambda)=+\infty$). This, together with (5.2) and (5.3), proves the statement. □

We are now in position to give the

Proof of Theorem 5. Let $C>0$ be the constant found in Lemma 10 and let

$$b_\lambda=C\lambda^{-\frac{1}{r-p}}; \tag{5.4}$$

then there exists $c>0$ such that

$$F_\lambda(b_\lambda)=c\lambda^{-\frac{p}{r-p}}+o\big(\lambda^{-\frac{p}{r-p}}\big) \quad \text{as } \lambda\to+\infty. \tag{5.5}$$

With the choice (5.4) for b_λ we estimate the constant $C(b_\lambda)$ defined in [10, (3.1)], namely,

$$C(b_\lambda)=(n-1)\big(\frac{p}{p-1}\big)^{\frac{p-1}{p}}\frac{b_\lambda}{F_\lambda(b_\lambda)}\big[\,\overline{F}_\lambda+F_\lambda(b_\lambda)\big]^{\frac{p-1}{p}}. \tag{5.6}$$

By (5.1), (5.4), (5.5), and (5.6) we infer that there exists $\overline{C}>0$ such that

$$C(b_\lambda)\le\overline{C} \qquad \text{for } \lambda \text{ large enough.} \tag{5.7}$$

Next, we claim that there exists $c>0$ such that

$$\overline{Q}_\lambda=-\min_{0\le s\le 1}Q_\lambda(s)\le c\lambda^{-\frac{p}{r-p}} \qquad (\lambda \text{ large}). \tag{5.8}$$

Indeed, take λ large enough so that Lemma 10 holds. Then Q_λ attains its unique local minimum in the interval $[0,1]$ (in fact, in $(0, C\lambda^{-\frac{1}{r-p}})$). In such an interval, since $q>p$, we have

$$Q_\lambda(s) \geq -p\Big[1+n\Big(\frac{1}{p^*}-\frac{1}{q}\Big)\Big]s^p + \lambda np\Big(\frac{1}{r}-\frac{1}{p^*}\Big)s^r =: -\gamma_1 s^p + \lambda\gamma_2 s^r.$$

Hence, $\overline{Q}_\lambda \leq -\min\limits_{0\leq s\leq 1} (-\gamma_1 s^p + \lambda\gamma_2 s^r) = c\lambda^{-\frac{p}{r-p}}$, and (5.8) follows.

We now verify (4.2) in [10] and apply Theorem 2 in [10]. To this end, we remark that by (5.4), (5.5), (5.7), and (5.8), we get

$$[\,\overline{Q}_\lambda + npF_\lambda(b_\lambda) + n(p-1)b_\lambda^p][C(b_\lambda)+1]^n \leq C\lambda^{-\frac{p}{r-p}} \quad (\lambda \text{ large}). \tag{5.9}$$

Now take $\alpha=1$ and $k=\frac{1}{2}$ in (4.2) in [10]. We have to estimate from below $\frac{Q_\lambda(k_2)}{[f_\lambda(k_1)]^{n/p}}$ for all $k_1, k_2 \in \left[\frac{1}{2},1\right]$. Using the same argument as for Lemma 10 we have

$$Q_\lambda(k_2) \geq c\lambda \qquad \forall k_2 \in \left[\tfrac{1}{2},1\right] \text{ and } \lambda \text{ large}. \tag{5.10}$$

Moreover,

$$f_\lambda(k_1) \leq c\lambda \qquad \forall k_1 \in [\tfrac{1}{2},1] \text{ and } \lambda \text{ large}. \tag{5.11}$$

By combining (5.10) and (5.11) we obtain

$$\frac{Q_\lambda(k_2)}{[f_\lambda(k_1)]^{n/p}} \geq c\lambda^{1-\frac{n}{p}} \qquad \forall k_1, k_2 \in [\tfrac{1}{2},1] \text{ and } \lambda \text{ large}. \tag{5.12}$$

Then, combining (5.9) and (5.12), we obtain for large λ

$$\frac{Q_\lambda(k_2)}{[f_\lambda(k_1)]^{n/p}} \geq c\lambda^{-\frac{n-p}{p}} \gg c\lambda^{-\frac{p}{r-p}} \geq [\,\overline{Q}_\lambda + npF_\lambda(b_\lambda) + n(p-1)b_\lambda^p][C(b_\lambda)+1]^n$$

so that (4.2) in [10] is satisfied. Recalling that $Q_\lambda(s) \geq 0$ if $s \in [b_\lambda, 1]$ (see Lemma 10) we may apply Theorem 2 in [10] and obtain the existence of a ground state u for (2.10).

It remains to prove that if $q > p^*$ and λ is small, then (2.10) admits no ground states. To see this, note that there exists $\underline{\lambda}$ such that for any $\lambda < \underline{\lambda}$, $Q_\lambda(s) < 0$ for all $s > 0$. By Proposition 3 in [9] we know that

$$\int_0^\infty Q_\lambda(u(t))\, t^{n-1} dt = 0$$

for any ground state u of (2.10). So if Q_λ is everywhere negative, then ground states cannot exist. This completes the proof of Theorem 5.

6. Proof of Theorem 6

If the equation $-\Delta_p u = f(u)$ admits a ground state, there is nothing to prove. Otherwise, fix $\varepsilon > 0$, fix $\sigma > \zeta$, and consider the sequence of functions $\{g_m\}$ defined by

$$g_m(s) = \begin{cases} f(s) & \text{if } s \in (0,\sigma] \cup [\sigma + \frac{1}{m}, \infty) \\ 0 & \text{if } s = \sigma + \frac{1}{2m} \\ \text{affine continuous connection} & \text{if } s \in (\sigma, \sigma + \frac{1}{2m}) \cup (\sigma + \frac{1}{2m}, \sigma + \frac{1}{m}). \end{cases}$$

Then we obviously have

$$\lim_{m\to\infty} \int_0^\infty |g_m(s) - f(s)|^q ds = 0.$$

So, there exists $M \in \mathbb{N}$ such that

$$\int_0^\infty |g_M(s) - f(s)|^q ds < \frac{\varepsilon}{2^q}. \tag{6.1}$$

Clearly, g_M satisfies (2.2) and (2.3) but not (2.4). Moreover, by Theorem 1 (C1) in [10], the equation $-\Delta_p u = g_M(u)$ admits a ground state $\overline{u}$ which satisfies $\|\overline{u}\|_\infty = \overline{u}(0) = \gamma < \sigma + \frac{1}{2M}$.

It remains to fulfill condition (2.4). Define another sequence $\{f_k\}$ by

$$f_k(s) = \begin{cases} f(s) & \text{if } s \in (0,\sigma] \cup [\sigma + \frac{1}{M}, \infty) \\ \max\{\frac{1}{k}, g_M(s)\} & \text{if } s \in (\sigma, \sigma + \frac{1}{M}). \end{cases}$$

For large enough k, the function f_k satisfies (2.2), (2.3), and (2.4). Furthermore, for sufficiently large k, say $k = K$, we have

$$f_K(s) = g_M(s) \qquad \forall s \le \gamma \tag{6.2}$$

and

$$\int_0^\infty |g_M(s) - f_K(s)|^q ds < \frac{\varepsilon}{2^q}. \tag{6.3}$$

Set $f_\varepsilon = f_K$; then by (6.2) also $\overline{u}$ is a ground state of (2.11) and by (6.1), (6.3), and the convexity of the q-th power we have

$$\int_0^\infty |f_\varepsilon(s) - f(s)|^q ds \le 2^{q-1}\Big(\int_0^\infty |g_M(s) - f(s)|^q ds + \int_0^\infty |g_M(s) - f_\varepsilon(s)|^q ds\Big) < \varepsilon,$$

and the proof is complete. □

Remark 3. At first glance, Theorem 6 seems a simple joke: of course, the behavior at infinity of f_ε (and hence of f) has no role in the existence of ground states for (2.11) as the shooting level γ is smaller than $\sigma + \frac{1}{2M}$. However, a careful analysis of the proof shows that γ may be chosen as large as desired, since σ (which is fixed arbitrarily) gives a lower bound for γ. Therefore, one may take a function f having any behavior at infinity, even supercritical no matter in which sense. Then one may "almost" obtain a ground state u of (1.1): roughly speaking, starting from $u = 0$ at infinity one may initially allow the solution u to "follow f" on an arbitrarily large interval $(0, \sigma]$ and next force the solution u to bend and to reach the origin with zero slope (i.e., $u'(0) = 0$) by modifying f in some small interval $(\sigma, \sigma + \delta]$. In other words, not only is the shooting level γ important, but also the "shooting strength" $f(\gamma)$.

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