Bounds for Sobolev embedding constants in non-simply connected planar domains

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Abstract

In a bounded non-simply connected planar domain Ω, with a boundary split in an interior part and an exterior part, we obtain bounds for the embedding constants of some subspaces of $H^1(\Omega)$ into $L^p(\Omega)$ for any $p > 1$, $p \neq 2$. The subspaces contain functions which vanish on the interior boundary and are constant (possibly zero) on the exterior boundary. We also evaluate the precision of the obtained bounds in the limit situation where the interior part tends to disappear and we show that it does not depend on p. Moreover, we emphasize the failure of symmetrization techniques in these functional spaces. In simple situations, a new phenomenon appears, namely the existence of a break even surface separating masses for which symmetrization increases/decreases the Dirichlet norm. The question whether a similar phenomenon occurs in more general situations is left open.
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1 Introduction

In the plane $\mathbb{R}^2$ we consider an open, bounded, connected, and simply connected domain K, with Lipschitz boundary ∂K. Then we remove K, seen as an obstacle, from a larger square Q such that $\partial K \cap \partial Q = \emptyset$, and we define the domain

$$Q = (-L, L)^2 , \qquad \Omega = Q \setminus \overline{K},$$

where $L > \mathrm{diam}(K)$, as shown in Figure 1.1.

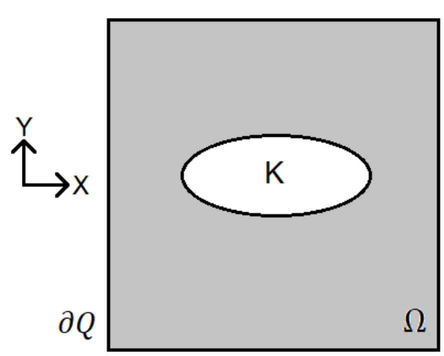

Figure 1.1: The planar domain Ω with a smooth obstacle K.

We focus our attention on the first order Hilbertian Sobolev space of functions vanishing on ∂K, which is a proper part of $\partial\Omega$ having positive 1D-measure:

$$H^1_*(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \;\text{ on }\; \partial K\} .$$

This space is rigorously defined as the closure of the space $\mathcal{C}^\infty_c(\overline{Q}\setminus\overline{K})$ with respect to the Dirichlet norm: this is legitimate since $|\partial K| > 0$ and the Poincaré inequality holds in $H^1_*(\Omega)$.

Motivated by the target of finding explicit thresholds for bifurcation from uniqueness in stationary Navier-Stokes equations modeling a flow around an obstacle, in a recent paper [7] we bounded some Sobolev embedding constants for $H^1_*(\Omega) \subset L^4(\Omega)$. We obtained a universal bound on the flow velocity for the appearance of a lift force on the obstacle K exerted by a fluid entering Q with constant velocity. In the present paper we drop this physical motivation and we focus our attention on the functional analytic aspect and on the possibility of obtaining similar inequalities in $L^p(\Omega)$ for any $p > 1$, $p \neq 2$. To the best of our knowledge, bounds in spaces of functions vanishing on a proper part of the boundary were obtained in the past only for the critical Sobolev embedding [1, 9] (thereby in space dimension $n \geq 3$), where one can exploit scaling methods since the optimal constant does not depend on the domain.

Given any subset $D \subset \mathbb{R}^2$ and $p > 1$, throughout the paper we denote by $\|\cdot\|_{p,D}$ the norm of the space $L^p(D)$. The relative capacity of K with respect to Q is defined by

$$\mathrm{Cap}_Q(K) = \min_{v \in H^1_0(Q)} \left\{ \int_Q |\nabla v|^2 \, dx \;\middle|\; v = 1 \text{ in } K \right\} \tag{1.1}$$

and the relative capacity potential $\psi \in H^1_0(Q)$, which achieves the minimum in (1.1), satisfies

$$\Delta \psi = 0 \ \text{ in } \ \Omega, \qquad \psi = 0 \ \text{ on } \ \partial Q, \qquad \psi = 1 \ \text{ in } \ K, \qquad \mathrm{Cap}_Q(K) = \|\nabla \psi\|^2_{2,\Omega}.$$

Then we consider a proper subspace of $H^1_*(\Omega)$, namely

$$H^1_c(\Omega) = \{ v \in H^1_*(\Omega) \mid v \text{ is constant on } \partial Q \}, \tag{1.2}$$

that can be rigorously characterized by using the relative capacity potential ψ. Indeed,

$$H^1_c(\Omega) = H^1_0(\Omega) \oplus \mathbb{R}(\psi - 1), \quad H^1_0(\Omega) \perp \mathbb{R}(\psi - 1),$$

so that $H^1_0(\Omega)$ has codimension 1 within $H^1_c(\Omega)$ and the "missing dimension" is spanned by the function $\psi - 1$, see [7] for the details. Since Ω is a planar domain, the embedding $H^1_*(\Omega) \subset L^p(\Omega)$ holds for any $1 < p < \infty$, and we define the Sobolev constants

$$\mathcal{S}_p = \min_{w \in H^1_*(\Omega) \setminus \{0\}} \frac{\|\nabla w\|^2_{2,\Omega}}{\|w\|^2_{p,\Omega}}, \qquad \mathcal{S}^0_p = \min_{w \in H^1_0(\Omega) \setminus \{0\}} \frac{\|\nabla w\|^2_{2,\Omega}}{\|w\|^2_{p,\Omega}}, \qquad \mathcal{S}^1_p = \min_{w \in H^1_c(\Omega) \setminus \{0\}} \frac{\|\nabla w\|^2_{2,\Omega}}{\|w\|^2_{p,\Omega}}. \tag{1.3}$$

Due to the inclusions $H^1_0(\Omega) \subset H^1_c(\Omega) \subset H^1_*(\Omega)$, we have $\mathcal{S}_p \leq \mathcal{S}^1_p \leq \mathcal{S}^0_p$, for every $p > 1$.

In Section 2 we obtain bounds for the constants $\mathcal{S}^0_p$ and $\mathcal{S}^1_p$, extending the results in [7] where only the case $p = 4$ was considered. To this end, we repeatedly use some sharp Gagliardo-Nirenberg inequalities due to del Pino-Dolbeault [5] and the behavior of pyramidal functions introduced in [7]. It turns out that the cases $p > 2$ and $p < 2$ require slightly different approaches. We obtain both lower and upper bounds for the constants $\mathcal{S}^0_p$ and $\mathcal{S}^1_p$ defined in (1.3) and we show that they are quite precise. In particular, we analyze the case where the obstacle tends to vanish ($|K| \to 0$) and we show that the ratio between these bounds converges to a *universal constant* $\pi/4 \approx 0.79$, independently of the value of $p > 1$ ($p \neq 2$), see Theorem 2.3. Our bounds do not depend on the position of the obstacle and it is therefore natural to expect that they might be improved, see Problem 2.1.

In Section 3 we address the question whether symmetrization techniques might be employed to obtain additional bounds. It turns out that, at least in its simplest forms, symmetrization is of no help in annuli, see Theorem 3.1. In its proof we exhibit examples where any of the possible inequalities may hold: in case of different (constant) conditions on the two connected components of the boundary

there is no a priori monotonicity of the Dirichlet norm under decreasing rearrangement neither from an annulus into itself, nor from an annulus into a disk with the same measure.

Moreover, we determine explicitly a "break even surface" which separates the cases where the mass of the gradient increases or decreases after symmetrization. We believe that this phenomenon deserves further investigation, see Problem 3.1.

2 Bounds for the Sobolev embedding constants

In Subsection 2.1 we provide lower bounds for the Sobolev embedding constants (1.3), for a general Lipschitz obstacle K. Then, in Subsection 2.2 we derive upper bounds for these constants and quantify the accuracy of our estimates when K is a square.

2.1 Lower bounds

As mentioned in the introduction, the cases $p \lesseqgtr 2$ are different and we consider first the case $p < 2$. Let

$$\mu_0 = \text{the first zero of the Bessel function of first kind of order zero} \approx 2.40483\,. \tag{2.1}$$

Then we have

Theorem 2.1. *For any $1 < p < 2$ and $u \in H^1_0(\Omega)$ one has*

$$\|u\|^2_{p,\Omega} \le \frac{|\Omega|^{\frac{2}{p}}}{\pi} \min\left\{\frac{1}{\mu_0^2},\, \frac{1}{2\pi}\frac{|Q|}{|\Omega|},\, \left(\frac{p}{2}\right)^{\frac{4-p}{2-p}}\right\} \|\nabla u\|^2_{2,\Omega}\,. \tag{2.2}$$

For any $1 < p < 2$ and $u \in H^1_c(\Omega)$ one has

$$\begin{aligned} \|u\|^2_{p,\Omega} \quad \le \quad & \frac{|\Omega|^{\frac{2}{p}}}{\pi}\left(\frac{p}{2}\right)^{\frac{4-p}{2-p}}\left(1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{\frac{2(p-1)}{p}} \\ & \times\left[1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}+\frac{p}{2-p}\,\frac{|K|}{|\Omega|}\,\left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right]^{\frac{2}{p}}\|\nabla u\|^2_{2,\Omega}\,. \end{aligned} \tag{2.3}$$

Proof. We begin by proving the following Poincaré inequality in Ω:

$$\|u\|_{2,\Omega} \le \min\left\{\frac{1}{\mu_0}\sqrt{\frac{|\Omega|}{\pi}},\, \frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\right\}\|\nabla u\|_{2,\Omega} \qquad \forall u \in H^1_0(\Omega). \tag{2.4}$$

Through the Faber-Krahn inequality [6, 8] we first bound the $L^2(\Omega)$-norm of functions in terms of their Dirichlet norm by using the Poincaré inequality in Ω^*, namely a disk having the same measure as Ω. Since $|\Omega| = |Q| - |K|$, the radius of Ω^* is given by

$$R = \sqrt{\frac{|\Omega|}{\pi}} = \sqrt{\frac{|Q|-|K|}{\pi}}\,.$$

Since the Poincaré constant (least eigenvalue of $-\Delta$) in the unit disk is given by μ_0^2, see (2.1), the Poincaré constant in Ω^* is given by μ_0^2/R^2, which means that

$$\min_{w\in H^1_0(\Omega)} \frac{\|\nabla w\|_{2,\Omega}}{\|w\|_{2,\Omega}} \ge \min_{w\in H^1_0(\Omega^*)} \frac{\|\nabla w\|_{2,\Omega^*}}{\|w\|_{2,\Omega^*}} = \frac{\mu_0}{R}\,.$$

Therefore,

$$\|u\|_{2,\Omega} \le \frac{R}{\mu_0}\|\nabla u\|_{2,\Omega} = \frac{1}{\mu_0}\sqrt{\frac{|\Omega|}{\pi}}\,\|\nabla u\|_{2,\Omega} \qquad \forall u \in H^1_0(\Omega),$$

which provides the first bound in (2.4). On the other hand, the least eigenvalue for the problem $-\Delta v = \lambda v$ in $H^1_0(Q)$ is given by $\lambda = \pi^2/2L^2$. Therefore, the Poincaré inequality in Q reads

$$\|u\|_{2,Q} \le \frac{\sqrt{2}L}{\pi}\,\|\nabla u\|_{2,Q} = \frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\,\|\nabla u\|_{2,Q} \qquad \forall u \in H^1_0(Q),$$

yielding the second bound in (2.4) since any function of $H^1_0(\Omega)$ can be extended by 0 in K, thereby becoming a function in $H^1_0(Q)$.

The first two bounds in (2.2) are obtained after applying both Hölder's inequality and (2.4)

$$\|u\|^p_{p,\Omega} \le |\Omega|^{\frac{2-p}{2}} \|u\|^p_{2,\Omega} \le \min\left\{ \frac{|\Omega|}{(\mu_0\sqrt{\pi})^p},\ |\Omega|^{\frac{2-p}{2}} \left(\frac{1}{\pi}\sqrt{\frac{|Q|}{2}}\right)^p \right\} \|\nabla u\|^p_{2,\Omega} \qquad \forall u \in H^1_0(\Omega).$$

To prove the third bound in (2.2), we recall the following (optimal) Gagliardo-Nirenberg inequality in $\mathbb{R}^2$ given by del Pino-Dolbeault [5, Theorem 2]:

$$\|u\|_{p,\Omega} \le \pi^{\frac{p-2}{2p}} \left(\frac{p}{2}\right)^{\frac{4-p}{2p}} \|\nabla u\|^{\frac{2-p}{p}}_{2,\Omega} \|u\|^{\frac{2(p-1)}{p}}_{2(p-1),\Omega} \qquad \forall u \in H^1_0(\Omega) \qquad \forall p \in (1,2). \tag{2.5}$$

Since functions in $H^1_0(\Omega)$ may be extended by zero outside Ω, they can be seen as functions defined over the whole plane. An application of the Hölder inequality shows that

$$\|u\|_{2(p-1),\Omega} \le |\Omega|^{\frac{2-p}{2p(p-1)}} \|u\|_{p,\Omega} \qquad \forall u \in H^1_0(\Omega)$$

which, combined with (2.5), yields the third bound in (2.2).

In order to prove (2.3) we restrict our attention to functions $u \in H^1_c(\Omega) \setminus H^1_0(\Omega)$: this restriction will be justified a posteriori because, if we manage proving (2.3) for these functions, then it will also hold for functions in $H^1_0(\Omega)$ since the constant in (2.2) is smaller, see also Figure 2.2 below. For functions $u \in H^1_c(\Omega) \setminus H^1_0(\Omega)$, it suffices to analyze the case where $u \ge 0$ in Ω (by replacing u with $|u|$), $u = 1$ on ∂Q (by homogeneity), and we define a.e. in Q the function

$$v(x,y) = \begin{cases} 1 - u(x,y) & \text{if } (x,y) \in \Omega \\ 1 & \text{if } (x,y) \in K, \end{cases}$$

so that $v \in H^1_0(Q)$ and, after a zero extension outside Q, v satisfies (2.5). Let us put

$$A_p = A_p(u) \doteq \pi^{\frac{p-2}{2}} \left(\frac{p}{2}\right)^{\frac{4-p}{2}} \|\nabla v\|^{2-p}_{2,Q} = \pi^{\frac{p-2}{2}} \left(\frac{p}{2}\right)^{\frac{4-p}{2}} \|\nabla u\|^{2-p}_{2,\Omega},$$

so that (2.5) reads

$$\int_Q |v|^p \le A_p \int_Q |v|^{2(p-1)} \implies \int_\Omega \left[|1-u|^p + \frac{|K|}{|\Omega|} - A_p\left(|1-u|^{2(p-1)} + \frac{|K|}{|\Omega|} \right) \right] \le 0. \tag{2.6}$$

The next step consists in finding $\alpha \in (0,1)$ and $\beta > 0$ (possibly depending on p, but having ratio independent of u) for which

$$|1-s|^p - A_p|1-s|^{2(p-1)} + (1-A_p)\frac{|K|}{|\Omega|} \ge \alpha s^p - \beta A_p^{\frac{p}{2-p}} \qquad \forall s \ge 0. \tag{2.7}$$

Given any $p \in (0,1)$ and $\gamma \in \mathbb{R}$, the function $s \mapsto |1-s|^p - A_p|1-s|^{2(p-1)} + \gamma$ is symmetric with respect to $s = 1$, so it suffices to find $\alpha \in (0,1)$ and $\beta > 0$ ensuring (2.7) for every $s \ge 1$. Thus, for all such α and β we define the function

$$\varphi_p(s) = (s-1)^p - A_p(s-1)^{2(p-1)} - \alpha s^p + (1-A_p)\frac{|K|}{|\Omega|} + \beta A_p^{\frac{p}{2-p}} \qquad \forall s \ge 1,$$

and we seek $\alpha \in (0,1)$ and $\beta > 0$ in such a way that φ_p has a non-negative minimum value at some $s > 1$. Equivalently, we seek $\gamma > 0$ such that φ_p attains its minimum at $s_0 = 1 + \gamma A_p$, that is,

$$\varphi'(s_0) = \gamma^{p-1} A_p^{p-1} \left[p - 2(p-1)\gamma^{p-2} A_p^{p-1} \right] - p\alpha(1+\gamma A_p)^{p-1} = 0,$$

thus fixing α in dependence of u through the expression

$$\alpha = \frac{1}{p}\left(\frac{\gamma A_p}{1+\gamma A_p}\right)^{p-1}\left[p-2(p-1)\gamma^{p-2}A_p^{p-1}\right] \in (0,1) \quad\iff\quad \gamma > \left(\frac{2p-2}{p}\right)^{\frac{1}{2-p}} A_p^{\frac{p-1}{2-p}}.$$

By imposing $\varphi(s_0) \geq 0$, we obtain the following lower bound for β:

$$\beta \geq A_p^{\frac{p}{p-2}}\left[\alpha\left(1+\gamma A_p\right)^p + \gamma^{2p-2}A_p^{2p-1}\left(1-\gamma^{2-p}A_p^{1-p}\right)\right] + \frac{A_p-1}{A_p^{\frac{p}{2-p}}}\frac{|K|}{|\Omega|}.$$

This condition is certainly satisfied if we choose

$$\beta = A_p^{\frac{p}{p-2}}\left[\alpha\left(1+\gamma A_p\right)^p + \gamma^{2p-2}A_p^{2p-1}\left(1-\gamma^{2-p}A_p^{1-p}\right)\right] + A_p^{\frac{2p-2}{p-2}}\frac{|K|}{|\Omega|} \geq 0,$$

where one should take $\gamma \leq A_p^{\frac{p-1}{2-p}}$ in order to ensure that $\beta \geq 0$. With the above choices of α and β we obtain the ratio

$$\begin{aligned}\frac{\beta}{\alpha} \quad = \quad & \frac{pA_p^{\frac{p}{p-2}}\left(\frac{\gamma A_p}{1+\gamma A_p}\right)^{1-p}}{p-2(p-1)\gamma^{p-2}A_p^{p-1}} \\ & \times\left\{\frac{1+\gamma A_p}{p}(\gamma A_p)^{p-1}\left(p-2(p-1)\gamma^{p-2}A_p^{p-1}\right) + \gamma^{2p-2}A_p^{2p-1}\left(1-\gamma^{2-p}A_p^{1-p}\right) + A_p\frac{|K|}{|\Omega|}\right\},\end{aligned} \tag{2.8}$$

which depends on u and on $\gamma > 0$ such that

$$\left(\frac{2p-2}{p}\right)^{\frac{1}{2-p}} A_p^{\frac{p-1}{2-p}} < \gamma \leq A_p^{\frac{p-1}{2-p}}.$$

Hence, we still have the freedom of choosing γ. By taking $\gamma = A_p^{\frac{p-1}{2-p}}$ (which, numerically, appears to be close to the global minimum of the right-hand side of (2.8)), we obtain

$$\frac{\beta}{\alpha} = \left(1+\frac{1}{A_p(u)^{\frac{1}{2-p}}}\right)^{p-1}\left(1+\frac{1}{A_p(u)^{\frac{1}{2-p}}} + \frac{p}{2-p}\frac{1}{A_p(u)^{\frac{2p-2}{2-p}}}\frac{|K|}{|\Omega|}\right), \tag{2.9}$$

where we emphasized the dependence of A on u. In order to obtain an upper bound for the ratio β/α independent of u, we use [7, Remark 2.1] which states that

$$\|\nabla u\|_{2,\Omega}^2 \geq \frac{4\pi}{\log(|Q|)-\log(|K|)} \qquad \forall u \in H_c^1(\Omega) \;\text{ s.t. }\; u=1 \text{ on } \partial Q,\; u \geq 0 \text{ in } \Omega,$$

thus yielding

$$A_p(u) \geq 2^{2-p}\left(\frac{p}{2}\right)^{\frac{4-p}{2}}\left(\log\left(\frac{|Q|}{|K|}\right)\right)^{\frac{p-2}{2}} \qquad \forall u \in H_c^1(\Omega) \;\text{ s.t. }\; u=1 \text{ on } \partial Q,\; u \geq 0 \text{ in } \Omega.$$

Hence, from (2.9) we obtain the following uniform bound (independent of u)

$$\begin{aligned}\frac{\beta}{\alpha} \leq & \left(1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{p-1} \\ & \times\left[1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)} + \frac{p}{2-p}\frac{|K|}{|\Omega|}\left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right].\end{aligned}$$

In turn, from (2.6), by replacing s with $|u|$ in (2.7) and integrating over Ω, we obtain

$$\begin{aligned}\|u\|_{p,\Omega}^p &\le \frac{\beta}{\alpha}\, A_p(u)^{\frac{p}{2-p}}\,|\Omega| \\ &\le \pi^{-\frac{p}{2}}\,\left(\frac{p}{2}\right)^{\frac{p(4-p)}{2(2-p)}}\,|\Omega|\,\left(1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{p-1} \\ &\qquad \times\left[1+\frac{1}{2}\sqrt{\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)}+\frac{p}{2-p}\,\frac{|K|}{|\Omega|}\,\left(\frac{1}{4}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log\left(\frac{|Q|}{|K|}\right)\right)^{p-1}\right]\|\nabla u\|_{2,\Omega}^p\,,\end{aligned}$$

for every $u\in H_c^1(\Omega)$ such that $u=1$ on ∂Q and $u\ge 0$ in Ω. The bound in (2.3) follows by taking the p-roots in the last inequality. □

Remark 2.1. *We point out that* (2.4) *provides an upper bound for the Poincaré constant in* $H_0^1(\Omega)$ *(for* $p=2$*). On the other hand, a bound for the Poincaré constant in* $H_c^1(\Omega)$ *for* $p=2$ *cannot be obtained by taking the limit in* (2.3) *when* $p\to 2$*, because the right-hand side of* (2.3) *blows up. This is the reason why the analysis of the case* $p=2$ *has been excluded in the present article.*

We now turn to the case $p>2$.

Theorem 2.2. *For any* $p>2$ *and* $u\in H_0^1(\Omega)$ *one has*

$$\|u\|_{p,\Omega}^2\le\frac{|\Omega|^{\frac{2}{p}}}{\pi}\left(\frac{p+2}{4}\right)^{\frac{p-6}{p}}\min\left\{\frac{1}{\mu_0^2},\,\frac{1}{2\pi}\frac{|Q|}{|\Omega|}\right\}^{\frac{2}{p}}\|\nabla u\|_{2,\Omega}^2\,. \tag{2.10}$$

For any $p>2$ *and* $u\in H_c^1(\Omega)$ *one has*

$$\begin{aligned}\|u\|_{p,\Omega}^2 \;&\le\; \frac{|\Omega|^{\frac{2}{p}}}{\pi}\left(\frac{p+2}{4}\right)^{\frac{p-6}{p-2}}\left(1+\frac{1}{2}\sqrt{\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)}\right)^{\frac{2(p-1)}{p}} \qquad (2.11)\\ &\times\;\left[1+\frac{1}{2}\sqrt{\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)}+\frac{2p}{p-2}\frac{|K|}{|\Omega|}\left(\frac{1}{4}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}}\log\left(\frac{|Q|}{|K|}\right)\right)^{\frac{p+2}{4}}\right]^{\frac{2}{p}}\|\nabla u\|_{2,\Omega}^2\,.\end{aligned}$$

Proof. For $p>2$, del Pino-Dolbeault [5, Theorem 1] obtained the optimal constant for the following Gagliardo-Nirenberg inequality in $\mathbb{R}^2$:

$$\|u\|_{p,\Omega}\le\pi^{\frac{2-p}{4p}}\left(\frac{p+2}{4}\right)^{\frac{p-6}{4p}}\|\nabla u\|_{2,\Omega}^{\frac{p-2}{2p}}\,\|u\|_{\frac{p}{2}+1,\Omega}^{\frac{p+2}{2p}}\qquad\forall u\in H_0^1(\Omega). \tag{2.12}$$

As in Theorem 2.1, we notice that functions in $H_0^1(\Omega)$ may be extended by zero outside Q, so they can be seen as functions defined over the whole plane. For general exponents, the optimal constant in the Gagliardo-Nirenberg inequality is not known, this is why we introduce the $L^{\frac{p}{2}+1}$-norm. By combining (2.12) with the following form of the Hölder inequality

$$\|u\|_{\frac{p}{2}+1,\Omega}^{\frac{p}{2}+1}\le\|u\|_{2,\Omega}\,\|u\|_{p,\Omega}^{p/2}\qquad\forall u\in L^p(\Omega)\,,$$

we infer that

$$\|u\|_{p,\Omega}^2\le\pi^{\frac{2-p}{p}}\left(\frac{p+2}{4}\right)^{\frac{p-6}{p}}\|\nabla u\|_{2,\Omega}^{\frac{2(p-2)}{p}}\,\|u\|_{2,\Omega}^{4/p}\qquad\forall u\in H_0^1(\Omega)\,. \tag{2.13}$$

Then (2.10) is obtained after inserting (2.4) into (2.13).

The proof of (2.11) follows exactly the procedure employed in the proof of inequality (2.3) given in Theorem 2.1, and therefore is omitted here. □

Remark 2.2. *Notice that the minimum in* (2.2) *and* (2.10) *is the consequence of the Poincaré inequality* (2.4) *that we only use in the space* $H^1_0(\Omega)$*. In particular, from* (2.10) *we deduce that*

$$\|u\|^2_{p,\Omega} \le \frac{|\Omega|^{2/p}}{\pi \mu_0^{4/p}} \left(\frac{p+2}{4}\right)^{\frac{p-6}{p}} \|\nabla u\|^2_{2,\Omega} \qquad \text{for all } u \in H^1_0(\Omega) \text{ and } p > 2. \tag{2.14}$$

On the other hand, by applying firstly (2.12) *and then Hölder's inequality, we also have*

$$\|u\|^2_{p,\Omega} \le \frac{|\Omega|^{2/p}}{\pi} \left(\frac{p+2}{4}\right)^{\frac{p-6}{p-2}} \|\nabla u\|^2_{2,\Omega} \qquad \text{for all } u \in H^1_0(\Omega) \text{ and } p > 2. \tag{2.15}$$

The ratio between the constants appearing in the right-hand sides of (2.14) *and* (2.15) *is plotted in Figure 2.1 as a function of* $p > 2$*, showing that the smallest constant corresponds to* (2.14)*.*

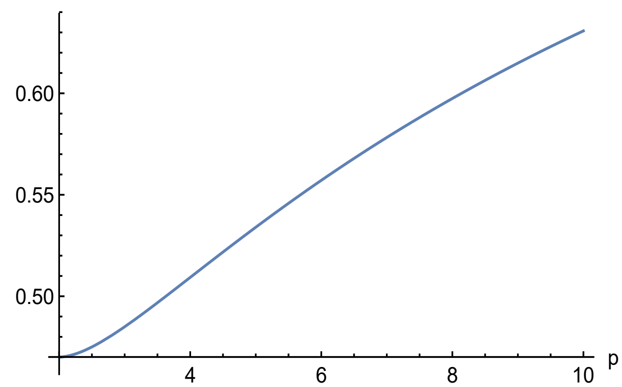

Figure 2.1: Ratio between the embedding constants given in (2.14) and (2.15).

Theorems 2.1 and 2.2 yield (unpleasant) lower bounds for the Sobolev constants in (1.3): it suffices to take the inverse of the constants appearing in (2.2), (2.3), (2.10) and (2.11). The lower bounds for $\mathcal{S}^1_p$ may be treated as functions of $|Q|/|K| \in [1,\infty)$: regardless of the value of $p > 1$, they vanish like $[\log(|Q|/|K|)]^{-1}$ as $|Q|/|K| \to \infty$, see the plots in Figure 2.2 where we also compare them with the (larger) lower bound for $\mathcal{S}^0_p$. One should also compare this uniform asymptotic behavior with the result of Theorem 2.3.

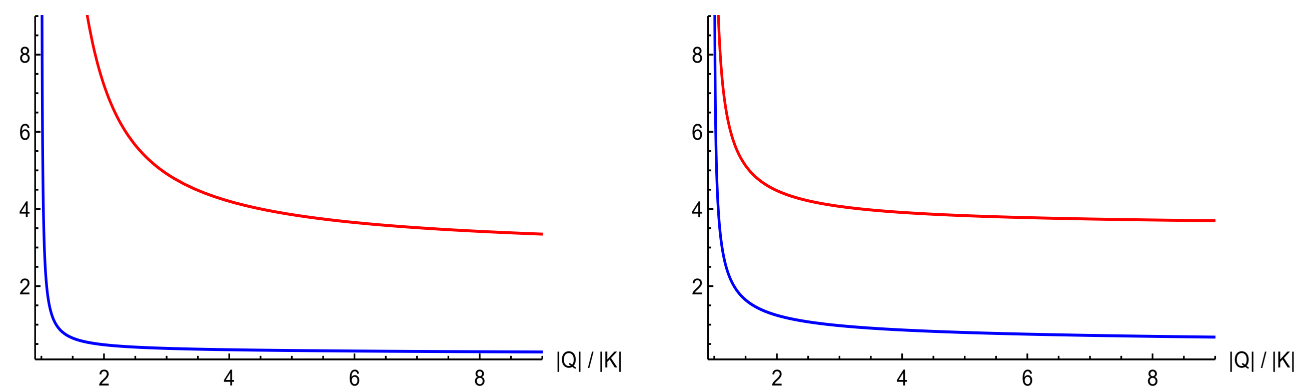

Figure 2.2: Behavior of the lower bounds for $\mathcal{S}^p_0$ (red) and $\mathcal{S}^p_1$ (blue) as functions of $|Q|/|K|$, when $p = 3/2$ (left) and $p = 6$ (right).

In the case when K is a square, the explicit lower bounds for $\mathcal{S}^1_p$ are as follows:

Corollary 2.1. *For $0 < a < L$, suppose that $K = (-a, a)^2$. Then, for every $1 < p < 2$ we have*

$$\mathcal{S}_p^1 \geq L^{-4/p} \frac{\pi}{4^{2/p}} \left[1 - \left(\frac{L}{a}\right)^{-2}\right]^{-\frac{2}{p}} \left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \left(1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)}\right)^{\frac{2(1-p)}{p}}$$

$$\times \left[1 + \sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)} + \frac{p}{2-p} \frac{1}{\left(\frac{L}{a}\right)^2 - 1} \left(\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}} \log\left(\frac{L}{a}\right)\right)^{p-1}\right]^{-\frac{2}{p}},$$

and for every $p > 2$ we have

$$\mathcal{S}_p^1 \geq L^{-4/p} \frac{\pi}{4^{2/p}} \left[1 - \left(\frac{L}{a}\right)^{-2}\right]^{-\frac{2}{p}} \left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \left(1 + \sqrt{\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)}\right)^{\frac{2(1-p)}{p}}$$

$$\times \left[1 + \sqrt{\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)} + \frac{2p}{p-2} \frac{1}{\left(\frac{L}{a}\right)^2 - 1} \left(\frac{1}{2}\left(\frac{p+2}{4}\right)^{\frac{6-p}{p-2}} \log\left(\frac{L}{a}\right)\right)^{\frac{p+2}{4}}\right]^{-\frac{2}{p}}.$$

Problem 2.1. *The bounds obtained in Theorems 2.1 and 2.2 for the Sobolev constants merely depend on the measure of the obstacle K but they do not depend on its position nor on its shape. It is natural to conjecture that obstacles close to ∂Q might generate larger Sobolev constants. Moreover, it is well-known that Steiner symmetrization [10] preserves the L^p norms of functions and reduces their Dirichlet norm, see [2, 3, 4, 11] and references therein. In our 2D setting, the Steiner symmetrization produces rearrangements that gain symmetry about a line. We are here interested in a finite number of iterations by symmetrizing about the four lines $x = 0$, $y = 0$ and $y = \pm x$, namely the axes of symmetry of Q. Then, it appears interesting to find the shape and the position of the optimal obstacle minimizing the Sobolev constants among obstacles K of given measure.*

2.2 Upper bounds

It is natural to wonder whether the lower bounds for $\mathcal{S}_p^0$ and $\mathcal{S}_p^1$ so far obtained are accurate. This can be tested through suitable upper bounds. For $\mathcal{S}_p^0$ we take the function $w(x, y) = \cos(\frac{\pi x}{2L}) \cos(\frac{\pi y}{2L})$, defined for $(x, y) \in \overline{Q}$, so that $w \in H_0^1(Q)$ and

$$\|w\|_{p,Q}^2 = \left[\frac{2L}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(1+\frac{p}{2}\right)}\right]^{4/p}, \quad \|\nabla w\|_{2,Q}^2 = \frac{\pi^2}{2} \quad \Longrightarrow \quad \mathcal{S}_p^0 \leq \frac{\pi^2}{2} \left[\frac{\sqrt{\pi}}{2L} \frac{\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{1+p}{2}\right)}\right]^{4/p}. \tag{2.16}$$

Notice that the upper bound (2.16) holds for any obstacle K.

In order to derive an upper bound for $\mathcal{S}_p^1$, we recall the definition of *pyramidal function*, introduced in [7, Theorem 2.2]. For $0 < d \leq a < L$, suppose that $K = (-a, a) \times (-d, d)$ and divide the domain Ω into four trapezia T_1, T_2, T_3, T_4 as in the left picture in Figure 2.3. By *pyramidal function* we mean any function having the level lines as in the right picture of Figure 2.3, namely level lines parallel to ∂Q (and to the rectangle K) in each of the trapezia. In particular, pyramidal functions are constant on ∂K and constitute the following convex subset of $H_0^1(Q)$:

$$\mathcal{P}(Q) = \{u \in H_0^1(Q) \mid u = 1 \text{ in } K,\ u = u(y) \text{ in } T_1 \cup T_3,\ u = u(x) \text{ in } T_2 \cup T_4\}. \tag{2.17}$$

Now, any $V^\phi \in \mathcal{P}(Q)$ is fully characterized by a (continuous) function

$$\phi \in H^1([0, 1]; \mathbb{R}) \quad \text{such that} \quad \phi(0) = 1, \quad \phi(1) = 0,$$

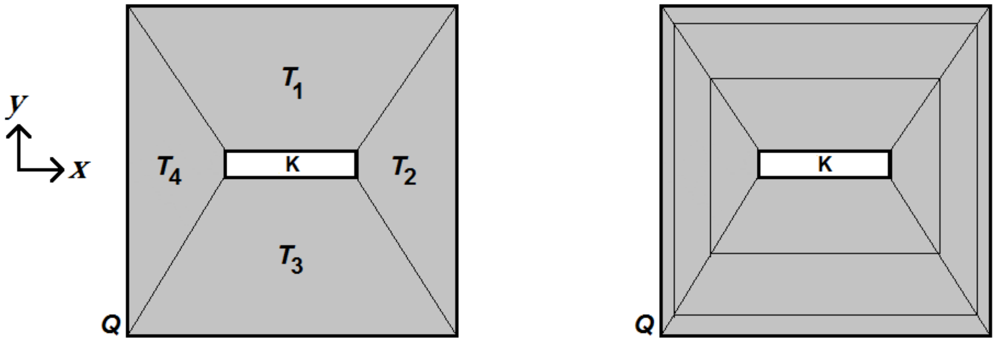

Figure 2.3: The domain Ω (left) and the level lines of pyramidal functions (right).

giving the values of V^ϕ on the oblique edges of the trapezia. For instance, consider the right trapezia $T_5, T_6 \subset Q$ being, respectively, half of the trapezia T_1 and T_2, defined by

$$T_5 = \left\{ (x,y) \in Q \;\middle|\; d < y < L,\ 0 < x < a + \frac{L-a}{L-d}(y-d) \right\},$$
$$T_6 = \left\{ (x,y) \in Q \;\middle|\; a < x < L,\ 0 < y < d + \frac{L-d}{L-a}(x-a) \right\}.$$

Since V^ϕ is a function of y in T_1 and a function of x in T_2, ϕ and V^ϕ are linked through the formulas

$$V^\phi(x,y) = \phi\left(\frac{y-d}{L-d}\right) \qquad \forall (x,y) \in T_5, \qquad\qquad V^\phi(x,y) = \phi\left(\frac{x-a}{L-a}\right) \qquad \forall (x,y) \in T_6. \tag{2.18}$$

Whence,

$$\frac{\partial V^\phi}{\partial y}(x,y) = \frac{1}{L-d}\phi'\left(\frac{y-d}{L-d}\right) \quad \forall (x,y) \in T_5, \quad \frac{\partial V^\phi}{\partial x}(x,y) = \frac{1}{L-a}\phi'\left(\frac{x-a}{L-a}\right) \quad \forall (x,y) \in T_6. \tag{2.19}$$

To avoid tedious computations, we restrict again our attention to the case $d = a$ (squared obstacle). The next result gives an upper bound for the constants $\mathcal{S}_p^1$ and measures the precision of the bounds in the limit situation where K is a vanishing square. Interestingly, the ratio between our lower and upper bounds for $\mathcal{S}_p^1$ converges to a limit that is independent of p.

Theorem 2.3. *For $0 < a < L$, suppose that $K = (-a,a)^2$. Then, for every $p > 1$ ($p \neq 2$) we have*

$$\mathcal{S}_p^1 \leq \frac{8^{1-\frac{2}{p}}}{L^{4/p}} \left(\frac{L}{a}\right)^{4/p} \log\left(\frac{L}{a}\right) \left(\int_1^{L/a} t \log^p(t)\, dt\right)^{-2/p}. \tag{2.20}$$

Moreover, the ratio between the lower bounds in Corollary 2.1 and the upper bound (2.20) *tends to* $\pi/4 \approx 0.79$ *as* $L/a \to \infty$, ***independently of the value of*** $p > 1$ ***(****$p \neq 2$****).***

Proof. Let $\mathcal{P}(Q)$ be as in (2.17) and let $V^\phi \in \mathcal{P}(Q)$ be defined by (2.18) with

$$\phi(s) = \log\left(\frac{a + (L-a)s}{L}\right) \Big/ \log\left(\frac{a}{L}\right) \qquad \forall s \in [0,1].$$

For symmetry reasons, the contribution of $|\nabla V^\phi|$ over $T_1 \cup T_3$ is four times the contribution over T_5, whereas the contribution of $|\nabla V^\phi|$ over $T_2 \cup T_4$ is four times the contribution over T_6. By taking into account all these facts, in particular (2.19), we infer that

$$
\begin{aligned}
\|\nabla V^\phi\|_{2,\Omega}^2 &= 4\int_a^L \int_0^y \left|\frac{\partial V^\phi}{\partial y}\right|^2 dx\,dy + 4\int_a^L \int_0^x \left|\frac{\partial V^\phi}{\partial x}\right|^2 dy\,dx \\
&= 4\int_a^L y\left|\frac{\partial V^\phi}{\partial y}\right|^2 dy + 4\int_a^L x\left|\frac{\partial V^\phi}{\partial x}\right|^2 dx = \frac{8}{L-a}\int_0^1 [a+(L-a)s]\,\phi'(s)^2\,ds \\
&= 8\left[\log\left(\frac{L}{a}\right)\right]^{-1}. \qquad (2.21)
\end{aligned}
$$

In a similar fashion, for every $p>1$ we have

$$
\begin{aligned}
\|1-V^\phi\|_{p,\Omega}^p &= 4\int_a^L \int_0^y \left|1-V^\phi(y)\right|^p dx\,dy + 4\int_a^L \int_0^x \left|1-V^\phi(x)\right|^p dy\,dx \\
&= 4\int_a^L y\left|1-V^\phi(y)\right|^p dy + 4\int_a^L x\left|1-V^\phi(x)\right|^p dx \\
&= 8(L-a)\int_0^1 [a+(L-a)s]\,|1-\phi(s)|^p\,ds.
\end{aligned}
$$

Through the change of variable $t=a+(L-a)s$, for $s\in[0,1]$, we then obtain

$$
\|1-V^\phi\|_{p,\Omega}^p = 8a^2\left[\log\left(\frac{L}{a}\right)\right]^{-p}\int_1^{L/a} t\log^p(t)\,dt. \qquad (2.22)
$$

We finally notice that if $v\in\mathcal{P}(Q)$, then $1-v\in H_c^1(\Omega)$ with $v=1$ on ∂Q. Therefore,

$$
\mathcal{S}_p^1 \le \min_{v\in\mathcal{P}(Q)} \frac{\|\nabla v\|_{2,\Omega}^2}{\|1-v\|_{p,\Omega}^2} \le \frac{\|\nabla V^\phi\|_{2,\Omega}^2}{\|1-V^\phi\|_{p,\Omega}^2} \qquad \forall p>1,
$$

which yields (2.20) in view of (2.21) and (2.22).

Next, for any $1<p<2$, denote by $\mathcal{R}(z)$ the ratio between the lower bound for $\mathcal{S}_p^1$ given in Corollary 2.1 and the just proved upper bound (2.20), as a function of $z=L/a$:

$$
\mathcal{R}(z) = \frac{\pi\, 2^{\frac{2}{p}-3}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\left(\frac{1}{z^2-1}\int\limits_1^z t\log^p(t)\,dt\right)^{\frac{2}{p}}\left(1+\sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log(z)}\right)^{\frac{2(1-p)}{p}}}{\log(z)\left[1+\sqrt{\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log(z)}+\frac{p}{2-p}\,\frac{1}{z^2-1}\left(\frac{1}{2}\left(\frac{2}{p}\right)^{\frac{4-p}{2-p}}\log(z)\right)^{p-1}\right]^{\frac{2}{p}}} \qquad \forall z>1,
$$

so that

$$
\mathcal{R}(z) \sim \pi\, 2^{\frac{2}{p}-2}\left(\frac{1}{z^2\log^p(z)}\int\limits_1^z t\log^p(t)\,dt\right)^{\frac{2}{p}} \qquad \text{as} \quad z\to\infty.
$$

An application of L'Hôpital's rule yields

$$
\lim_{z\to\infty}\frac{1}{z^2\log^p(z)}\int\limits_1^z t\log^p(t)\,dt = \frac{1}{2},
$$

which concludes the proof, since the limit in the case $p>2$ can be treated exactly in the same way. □

Remark 2.3. *If $p > 1$ is an integer we may explicitly compute*

$$\int_1^{L/a} t\log^p(t)\,dt = p!\left[\left(\frac{L}{a}\right)^2\sum_{k=0}^{p}\frac{(-1)^k}{2^{k+1}}\frac{1}{(p-k)!}\left(\log\left(\frac{L}{a}\right)\right)^{p-k}-\frac{(-1)^p}{2^{p+1}}\right].$$

By dropping the multiplicative term $L^{-4/p}$, the lower and upper bounds for $\mathcal{S}_p^1$ in Corollary 2.1 and in Theorem 2.3 can be treated as functions of $L/a \in (1,\infty)$. The plots in Figure 2.4 describe the overall behavior for $p = 3$. Qualitatively, the same plots are found for any value of $p > 1$ ($p \neq 2$).

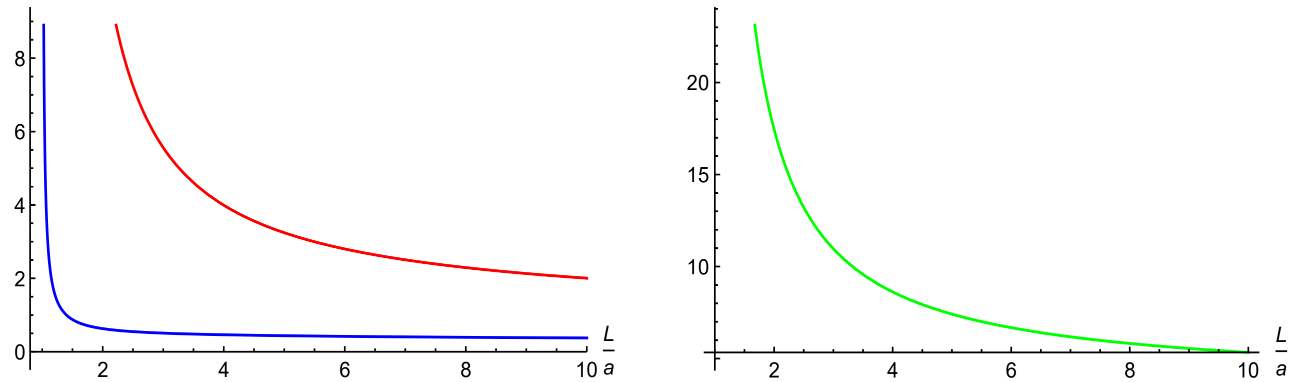

Figure 2.4: On the left: behavior of the lower and upper bounds for $\mathcal{S}_3^1$ as a function of L/a. On the right: ratio between the upper and lower bounds for $\mathcal{S}_3^1$ as a function of L/a.

3 Failure of elementary symmetrization methods

Theorems 2.1 and 2.2 may be extended to any space dimension $n \geq 3$ and any $1 < p < \frac{2n}{n-2}$ but the question whether they might be improved arises naturally. In particular, one wonders whether some symmetrization techniques [11] could be used. In this section we show that, at least in its simplest forms, symmetrization is of no help: we argue in any space dimension $n \geq 2$ because this creates no additional difficulties.

For any $R > 0$ we denote by $B_R \subset \mathbb{R}^n$ the n-dimensional ball of radius R centered at the origin. In the next statement we show that if we compare the Dirichlet norm of a (radial) function in this annulus with that of its decreasing rearrangement, nothing can be said a priori: both inequalities may occur.

Theorem 3.1. *There exist radial functions $f_1, f_2, f_3, f_4 \in H_c^1(B_2 \setminus B_1)$ such that*

$$\|\nabla f_1\|_{2,\,B_2\setminus B_1} < \|\nabla g_1\|_{2,\,B_2\setminus B_1}\,, \qquad \|\nabla f_2\|_{2,\,B_2\setminus B_1} > \|\nabla g_2\|_{2,\,B_2\setminus B_1} \tag{3.1}$$

and, for $R = \sqrt[n]{2^n - 1}$,

$$\|\nabla f_3\|_{2,\,B_2\setminus B_1} < \|\nabla g_3\|_{2,\,B_R}\,, \qquad \|\nabla f_4\|_{2,\,B_2\setminus B_1} > \|\nabla g_4\|_{2,\,B_R}, \tag{3.2}$$

where g_i denotes the decreasing rearrangement of f_i, for $i \in \{1,2,3,4\}$.

Proof. First we prove (3.1). In the annulus $B_2 \setminus B_1$, take any positive strictly increasing radial function $f = f(r)$ over the interval $[1,2]$ such that $f(1) = 0$ and $f(2) = 1$. Its decreasing rearrangement within the annulus is given by

$$g(r) = f\left(\sqrt[n]{2^n + 1 - r^n}\right) \qquad \forall r \in (1,2).$$

Hence, as expected, we have

$$\int_1^2 r^{n-1} g(r)^p\,dr = \int_1^2 r^{n-1} f\left(\sqrt[n]{2^n+1-r^n}\right)^p dr = \int_1^2 t^{n-1} f(t)^p\,dt \qquad \forall p > 1,$$

where we used the change of variables

$$t = \sqrt[n]{2^n + 1 - r^n} \quad \Longleftrightarrow \quad r = \sqrt[n]{2^n + 1 - t^n}. \tag{3.3}$$

On the other hand, we have

$$g'(r) = -r^{n-1}\Big(2^n + 1 - r^n\Big)^{\frac{1}{n}-1} f'\Big(\sqrt[n]{2^n + 1 - r^n}\Big) \qquad \forall r \in (1,2),$$

so that, using again (3.3),

$$\int_1^2 r^{n-1} f'(r)^2\, dr = \int_1^2 t^{n-1} f'\Big(\sqrt[n]{2^n + 1 - t^n}\Big)^2 dt = \int_1^2 \frac{\Big(2^n + 1 - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} g'(t)^2\, dt.$$

The "break even" in the integral occurs whenever

$$\frac{\Big(2^n + 1 - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} = t^{n-1} \iff t = r^* \doteq \Big(2^{n-1} + \frac{1}{2}\Big)^{\frac{1}{n}}.$$

Let us consider first the function (see Figure 3.1 when $n = 2$)

$$f_1(r) = \begin{cases} \frac{r-1}{r^*-1} & \text{if } \ 1 < r \le r^* \\ 1 & \text{if } \ r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_1'(r) = \begin{cases} \frac{1}{r^*-1} & \text{if } \ 1 < r < r^* \\ 0 & \text{if } \ r^* < r < 2 \end{cases},$$

so that

$$g_1(r) = \begin{cases} 1 & \text{if } \ 1 < r \le r^* \\ \frac{\sqrt[n]{2^n+1-r^n}-1}{r^*-1} & \text{if } \ r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad g_1'(r) = \begin{cases} 0 & \text{if } \ 1 < r < r^* \\ \frac{-r^{n-1}}{r^*-1}\,\frac{1}{(2^n+1-r^n)^{1-1/n}} & \text{if } \ r^* < r < 2\,. \end{cases}$$

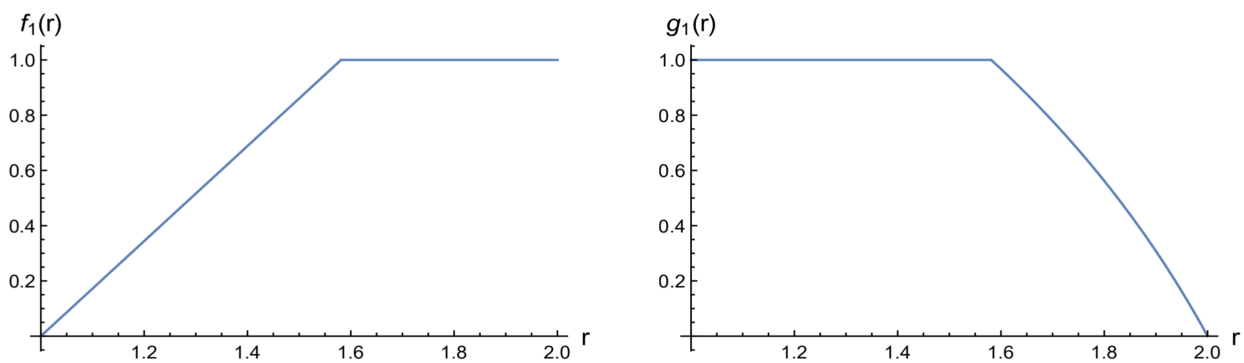

Figure 3.1: Plot of f_1 and of its symmetric decreasing rearrangement g_1, with $n = 2$ and $r^* = \sqrt{5/2}$.

Then we have

$$\int_1^2 r^{n-1} f_1'(r)^2\, dr < \int_1^2 t^{n-1} g_1'(t)^2\, dt\,,$$

which proves the first of (3.1).

Next, consider the function (see Figure 3.2 when $n = 2$)

$$f_2(r) = \begin{cases} 0 & \text{if } \ 1 < r \le r^* \\ \frac{r-r^*}{2-r^*} & \text{if } \ r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_2'(r) = \begin{cases} 0 & \text{if } \ 1 < r < r^* \\ \frac{1}{2-r^*} & \text{if } \ r^* < r < 2 \end{cases},$$

so that

$$g_2(r) = \begin{cases} \frac{\sqrt[n]{2^n+1-r^n}-r^*}{2-r^*} & \text{if } 1 < r \le r^* \\ 0 & \text{if } \ r^* \le r < 2 \end{cases} \quad \Longrightarrow \quad g_2'(r) = \begin{cases} \frac{-r^{n-1}}{2-r^*} \frac{1}{(2^n+1-r^n)^{1-1/n}} & \text{if } 1 < r < r^* \\ 0 & \text{if } \ r^* < r < 2\,. \end{cases}$$

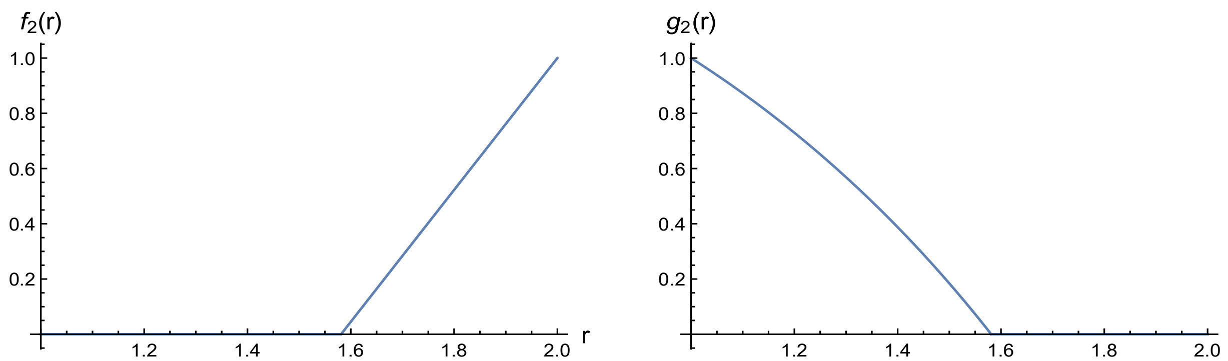

Figure 3.2: Plot of f_2 and of its symmetric decreasing rearrangement g_2, with $n = 2$ and $r^* = \sqrt{5/2}$.

Then we have

$$\int_1^2 r^{n-1} f_2'(r)^2\, dr > \int_1^2 t^{n-1} g_2'(t)^2\, dt\,,$$

which proves the second inequality in (3.1).

Let us now prove (3.2). Notice that $|B_2 \setminus B_1| = \omega_n(2^n - 1)$, where ω_n is the measure of the unit ball B_1. Hence, the disk D of radius $R = \sqrt[n]{2^n - 1}$ has the same measure as $B_2 \setminus B_1$ so that $B_R = (B_2 \setminus B_1)^*$. Consider a positive strictly increasing radial function $f = f(r)$ over the interval $(1, 2)$, then its decreasing rearrangement within the disc B_R is given by

$$g(r) = f\Big(\sqrt[n]{2^n - r^n}\Big) \qquad \forall r \in \Big(0, \sqrt[n]{2^n - 1}\Big)\,.$$

We have again

$$\int_0^{\sqrt[n]{2^n-1}} r^{n-1} g(r)^p\, dr = \int_1^2 t^{n-1} f(t)^p\, dt \qquad \forall p > 1.$$

where we used the change of variables

$$t = \sqrt[n]{2^n - r^n} \quad \Longleftrightarrow \quad r = \sqrt[n]{2^n - t^n}. \tag{3.4}$$

On the other hand, we have

$$g'(r) = -r^{n-1}\Big(2^n - r^n\Big)^{\frac{1}{n}-1} f'\Big(\sqrt[n]{2^n - r^n}\Big) \qquad \forall r \in \Big(0, \sqrt[n]{2^n - 1}\Big)\,.$$

so that, using again (3.4),

$$\int_1^2 r^{n-1} f'(r)^2\, dr = \int_0^{\sqrt[n]{2^n-1}} \frac{\Big(2^n - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} g'(t)^2\, dt.$$

The "break even" in the integral occurs whenever

$$\frac{\Big(2^n - t^n\Big)^{2-\frac{2}{n}}}{t^{n-1}} = t^{n-1} \iff t = r^* \doteq 2^{1-\frac{1}{n}}.$$

Let us consider first the function (see Figure 3.3 when $n = 2$)

$$f_3(r) = \begin{cases} \frac{r-1}{r^*-1} & \text{if } \ 1 < r \le r^* \\ 1 & \text{if } \ r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_3'(r) = \begin{cases} \frac{1}{r^*-1} & \text{if } \ 1 < r < r^* \\ 0 & \text{if } \ r^* < r < 2 \end{cases},$$

so that

$$g_3(r) = \begin{cases} 1 & \text{if } \ 0 < r < r^* \\ \frac{\sqrt[n]{2^n - r^n} - 1}{r^* - 1} & \text{if } \ r^* < r < \sqrt[n]{2^n - 1} \end{cases}$$

and

$$g_3'(r) = \begin{cases} 0 & \text{if } \ 0 < r < r^* \\ \frac{-r^{n-1}}{r^*-1} \, \frac{1}{(2^n - r^n)^{1-1/n}} & \text{if } \ r^* < r < \sqrt[n]{2^n - 1}\,. \end{cases}$$

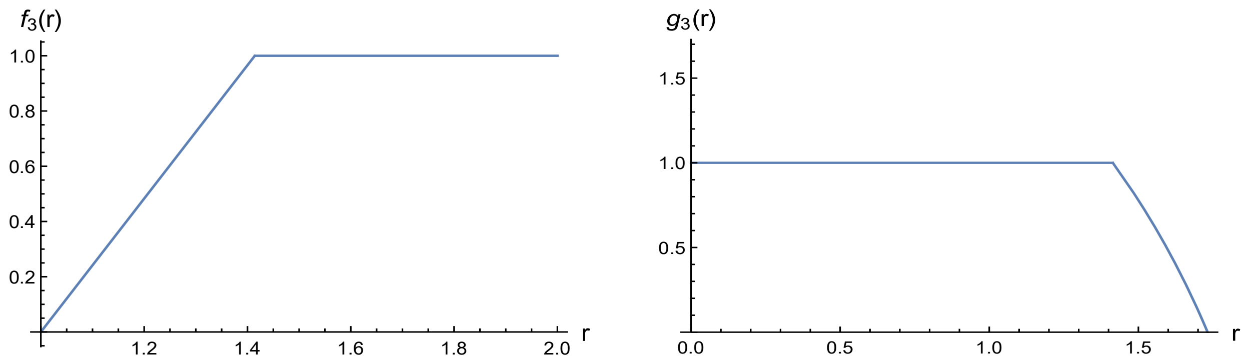

Figure 3.3: Plot of f_3 and of its symmetric decreasing rearrangement g_3, with $n = 2$ and $r^* = \sqrt{2}$.

Then

$$\int_1^2 r^{n-1} f_3'(r)^2 \, dr < \int_0^{\sqrt[n]{2^n - 1}} t^{n-1} g_3'(t)^2 \, dt\,,$$

thereby proving the first inequality in (3.2).

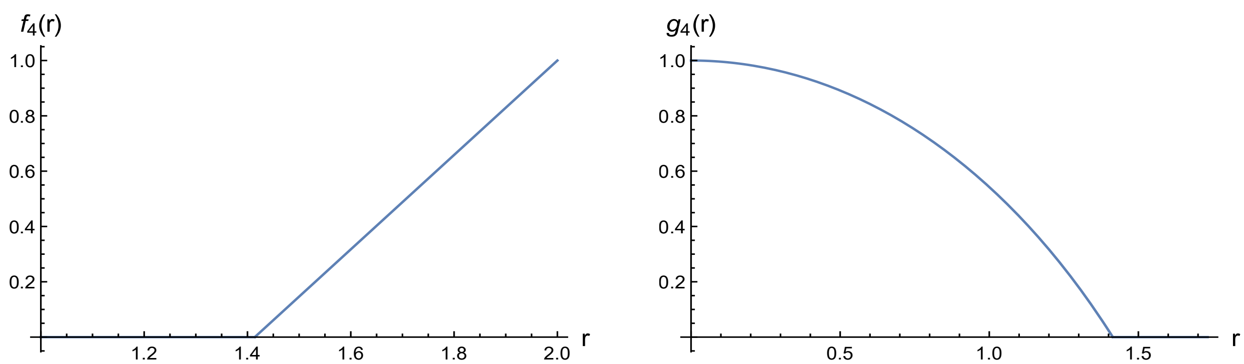

Figure 3.4: Plot of f_4 and of its symmetric decreasing rearrangement g_4, with $n = 2$ and $r^* = \sqrt{2}$.

Finally, consider the function (see Figure 3.4 when $n = 2$)

$$f_4(r) = \begin{cases} 0 & \text{if } \ 1 < r \le r^* \\ \frac{r - r^*}{2 - r^*} & \text{if } \ r^* \le r < 2 \end{cases} \qquad \Longrightarrow \qquad f_4'(r) = \begin{cases} 0 & \text{if } \ 1 < r < r^* \\ \frac{1}{2-r^*} & \text{if } \ r^* < r < 2 \end{cases},$$

so that

$$g_4(r) = \begin{cases} \frac{\sqrt[n]{2^n - r^n} - r^*}{2 - r^*} & \text{if } \ 0 < r \le r^* \\ 0 & \text{if } \ r^* \le r < \sqrt[n]{2^n - 1} \end{cases}$$

and

$$g_4'(r) = \begin{cases} \frac{-r^{n-1}}{2-r^*} \frac{1}{(2^n-r^n)^{1-1/n}} & \text{if } 0 < r < r^* \\ 0 & \text{if } r^* < r < \sqrt[n]{2^n-1}\,. \end{cases}$$

Then we have

$$\int_1^2 r^{n-1} f_4'(r)^2\, dr > \int_0^{\sqrt[n]{2^n-1}} t^{n-1} g_4'(t)^2\, dt\,,$$

proving also the second inequality in (3.2). □

One then naturally wonders if a result similar to Theorem 3.1 holds in *any* non-simply connected domain, that is

Problem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be the difference between two simply connected bounded convex domains Q and K such that $K \subset Q$ and $\partial K \cap \partial Q = \emptyset$. Define $H_c^1(\Omega)$ as in (1.2), and for any $f \in H_c^1(\Omega)$, let f^* be the symmetric decreasing rearrangement of f on Ω^*, the n-dimensional ball having the same measure as Ω. Does there exists a break even $(n-1)$-dimensional surface such that if $f \in H_c^1(\Omega)$ concentrates its mass inside (resp. outside) this surfaces, then the Dirichlet norm of f in Ω is strictly smaller (resp. larger) than the Dirichlet norm of f^* in Ω^*? The same question may be formulated by considering the symmetric decreasing rearrangement of f on the annulus whose inner ball has the same measure of K and whose outer ball has the same measure as Q.*

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