Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

POSITIVE SOLUTIONS TO A LINEARLY PERTURBED CRITICAL GROWTH BIHARMONIC PROBLEM

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(Communicated by ....)

Abstract. Existence and nonexistence results for positive solutions to a linearly perturbed critical growth biharmonic problem under Steklov boundary conditions, are determined. Furthermore, by investigating the critical dimensions for this problem, a Sobolev inequality with remainder terms, of both interior and boundary type, is deduced.

1. **Introduction.**

Let $B \subset \mathbb{R}^n$ $(n \geq 5)$ be the unit ball, $2^* = \frac{2n}{n-4}$ denote the critical Sobolev exponent for the embedding $H^2(B) \subset L^{2^*}(B)$, $\lambda \geq 0$ and $d \in \mathbb{R}$. We consider the following fourth order elliptic problem with linearly perturbed critical growth and Steklov boundary conditions:

$$\begin{cases} \Delta^2 u = \lambda u + u^{2^*-1} & \text{in } B \\ u > 0 & \text{in } B \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where u_ν denotes the outer normal derivative of u on ∂B.

When $\lambda = 0$, it is well-known that (1) admits no solutions if $d = 0$, namely under Navier boundary conditions ($u = \Delta u = 0$ on ∂B), or if $d = -\infty$, namely Dirichlet boundary conditions ($u = u_\nu = 0$ on ∂B), see [24, 26, 34].

On the other hand, under both Dirichlet and Navier boundary conditions, existence results have been obtained by modifying the geometry of the domain, see [2, 10, 14], or by perturbing the nonlinearity ($\lambda > 0$), see [8, 11, 12, 19, 21, 36]. We also refer to [16] for an exhaustive treatment of the subject.

In [6] general Steklov boundary conditions are considered first. Then, existence results are determined for problem (1), when $\lambda = 0$, without modifying the geometry of the domain, see [6, Theorem 1]. One of the purposes of the present paper is to combine both the contribution of the modification of the nonlinearity and of the boundary conditions. This gives rise to problem (1).

Existence results under linear perturbations λu of the critical nonlinearity u^{2^*-1} are quite sensitive to the space dimension n and led Pucci-Serrin [29] to define the so-called *critical dimensions*. In these dimensions, one has nonexistence of *radial solutions* to the Dirichlet problem in B for small linear perturbations (small $\lambda > 0$), whereas in the other dimensions existence of radial solutions is ensured for any

2000 *Mathematics Subject Classification.* Primary: 35J35, 35J40; Secondary: 35G30.

Key words and phrases. Biharmonic equations, critical exponent, Steklov boundary conditions.

positive linear perturbation with λ smaller than the first eigenvalue. Some attempts were made in order to explain this phenomenon by means of the local summability properties of the fundamental solution of the biharmonic operator [23, 25] or by means of summability properties of remainder terms in Sobolev inequality [13]. According to [11, Theorem 1.1] and [29, Theorem 3], the critical dimensions for the biharmonic operator under Dirichlet boundary conditions are $n = 5, 6, 7$. By [36, Theorem 1] and [14, Theorem 3], the same dimensions are also critical for the Navier problem, at least in a weak sense, see Definitions 2 and 3 in Section 2. Steklov boundary conditions exhibit an unexpected feature since, for $d \in [4, n)$, the critical dimensions do not exist, see Theorem 4.

On the other hand, for $d < 4$ critical dimensions do exist and coincide again with $n = 5, 6, 7$. In these dimensions we prove nonexistence results for (1) when λ is sufficient small. As a by-product of the nonexistence results, we deduce a Sobolev inequality with *remainder terms* of both interior and boundary type.

The paper is organized as follows: in Section 2 we state our main results, in Sections 3 and 4 we give the proofs.

2. **Results.** We denote by $\|\cdot\|_p$ the L^p-norm (both on B and on $\mathbb{R}^n$) and we put

$$\|u\|_{\partial_\nu}^2 = \int_{\partial B} u_\nu^2 \, d\omega \qquad \text{for } u \in H^2(B) \cap H_0^1(B).$$

By [4] we know that the following inequality holds:

$$\|\Delta u\|_2^2 \geq n \|u\|_{\partial_\nu}^2 \qquad \text{for all } u \in H^2(B) \cap H_0^1(B) \, . \tag{2}$$

For $d < n$, this allows to endow the Sobolev space $H^2(B) \cap H_0^1(B)$ with the scalar product

$$(u, v) := \int_B \Delta u \Delta v \, dx - d \int_{\partial B} u_\nu v_\nu \, d\omega$$

and with the induced norm, which is equivalent to the usual $H^2 \cap H_0^1-$norm $\|\Delta \cdot \|_2$ (see [16, Theorem 2.31]).

By *solutions* of (1) we mean functions $u \in H^2(B) \cap H_0^1(B)$ such that $u > 0$ a.e. in B and

$$(u, v) = \int_B (\lambda u + u^{2^*-1}) \, v \, dx \qquad \text{for all } v \in H^2(B) \cap H_0^1(B) \, . \tag{3}$$

A solution in this sense is in fact a classical solution, see [4, Proposition 23] and also [35].

For any $d \leq n$ we denote by $\lambda_1(d)$ the first eigenvalue of the operator Δ^2 under Steklov boundary conditions, namely

$$\lambda_1(d) := \inf_{H^2(B) \cap H_0^1(B) \setminus \{0\}} \frac{\|\Delta u\|_2^2 - d \|u\|_{\partial_\nu}^2}{\|u\|_2^2}. \tag{4}$$

We refer to the Appendix for a possible way to compute $\lambda_1(d)$. Since the map $H^2(B) \cap H_0^1(B) \ni u \mapsto u_\nu \in L^2(\partial B)$ is compact, the infimum in (4) is achieved by some function ϕ_1^d. Furthermore, the map $(-\infty, n] \ni d \mapsto \lambda_1(d)$ is decreasing, concave and $\lambda_1(n) = 0$. For any $d < n$, Δ^2 under Steklov boundary conditions enjoys the positivity preserving property in B, see [18]. Combining this fact with the Krein-Rutman Theorem, it follows that ϕ_1^d is strictly of one sign in B and $\lambda_1(d)$ is simple.

When $\lambda = 0$, problem (1) was studied in [6] and [17]. We recall the known results:

Proposition 1. [6, 17] *For $\lambda = 0$ the following statements hold:*

(i) *if $d \le 4$ or $d \ge n$, (1) admits no solutions;*
(ii) *if $4 < d < n$, (1) admits a unique radially symmetric solution.*

For completeness we remark that, even if Proposition 1-(i) is proved in [6] only for $d > 0$, the same proof extends to the case $d \le 0$.

As already mentioned in the introduction, when $\lambda > 0$, the equation in (1) has been extensively studied under Navier and Dirichlet boundary conditions, corresponding to $d = 0$ and $d = -\infty$ in (1). We complement the known results by Theorems 1 and 4 below:

Theorem 1. *For $n \ge 8$ and $\lambda > 0$ the following statements hold:*

(i) *if $d \ge n$ or $d < n$ and $\lambda \ge \lambda_1(d)$, (1) admits no solutions;*
(ii) *if $d < n$, then (1) admits a radially symmetric solution for all $\lambda \in (0, \lambda_1(d))$.*

According to [29] we recall

Definition 2. The dimension n is called *critical* for problem (1) if there exists $\overline{\lambda} = \overline{\lambda}(d) > 0$ such that a necessary condition for a *radial* solution to (1) (without the positivity assumption) to exist is $\lambda > \overline{\lambda}$.

By [11] and [29], the critical dimensions for the Dirichlet problem are known to be $n = 5, 6, 7$. More precisely, when $5 \le n \le 7$, by [11, Theorem 1.6] there exist $0 < \overline{\lambda} \le \lambda_*(n) < \lambda_1(-\infty)$ such that problem (1) with $d = -\infty$ admits no radial solution if $\lambda \in (0, \overline{\lambda})$ and admits a radial solution if $\lambda \in (\lambda_*(n), \lambda_1(-\infty))$. The values of both $\lambda_*(n)$ and $\lambda_1(-\infty)$ are explicitly given in terms of the first positive roots of certain functions related to Bessel functions. By means of some numerical computations with Mathematica the following approximations hold

n	5	6	7
$\lambda_1(-\infty)$	769.93	1216.3	1818.1
$\lambda_*(n)$	373.28	267.59	140.67

Table 1. The bounds of the intervals where existence is known when $d = -\infty$.

In order to study higher order polyharmonic equations for which the determination of the critical dimensions is more difficult to handle, see [20], a notion of weakly critical dimensions was introduced in [22]:

Definition 3. The dimension n is called *weakly critical* for problem (1) if there exists $\overline{\lambda}_+ = \overline{\lambda}_+(d) > 0$ such that a necessary condition for a *positive radial* solution to (1) to exist is $\lambda > \overline{\lambda}_+$.

In [14] the dimensions $n = 5, 6, 7$ are shown to be weakly critical also for the Navier problem ($d = 0$). For the more general problem (1) we prove that the weakly critical dimensions are still $n = 5, 6, 7$, when $d < 4$. When $4 \le d < n$, something somehow surprising happens: the critical dimensions do not exist.

Theorem 4. *For $n \in \{5, 6, 7\}$ and $\lambda > 0$, the following statements hold:*

(i) *if $d \ge n$ or $d < n$ and $\lambda \ge \lambda_1(d)$, (1) admits no solutions;*

(ii) *if* $4 \leq d < n$, *then* (1) *admits a radially symmetric solution for all* $\lambda \in (0, \lambda_1(d))$.

(iii) *If* $d < 4$, *there exist* $C(n) > 0$ *such that problem* (1) *admits:*

- *no radially symmetric solution if* $\lambda < C(n)\,\frac{4-d}{n-d}$;
- *a radially symmetric solution if*

$$\lambda > \min\left\{3(8-n)(n+4)(4-d), \lambda_*(n)\right\}, \tag{5}$$

with $\lambda_*(n)$ *as defined in Table 1.*

It is clear that for d close to 4 the minimum in (5) is given by $3(8-n)(n+4)(4-d)$ whereas for $d < 4$ far away from 4 the minimum is given by $\lambda_*(n)$.

When $d = -\infty$ or $d = 0$, by [7] and [33] we know that any solution to (1) is radially symmetric. A similar statement is not known under Steklov boundary conditions. Then, in view of Theorem 4-(*iii*), it is natural to wonder if the upper bound for the nonexistence of *radial* solutions to (1), is also an upper bound for the nonexistence of *any* solution.

We observe that $\lambda_1(0) = Z^4$, where Z is the first zero of the Bessel function $J_{\frac{n-2}{2}}$. According to [1] we have:

n	5	6	7
$\lambda_1(0)$	407.6653	695.6191	1103.3996
$12(8-n)(n+4)$	324	240	132

TABLE 2. The lower bound for existence in (5) when $d = 0$.

By Tables 1 and 2, we see that when $d = 0$ the best lower bound for existence in (5) is $12(8-n)(n+4)$.

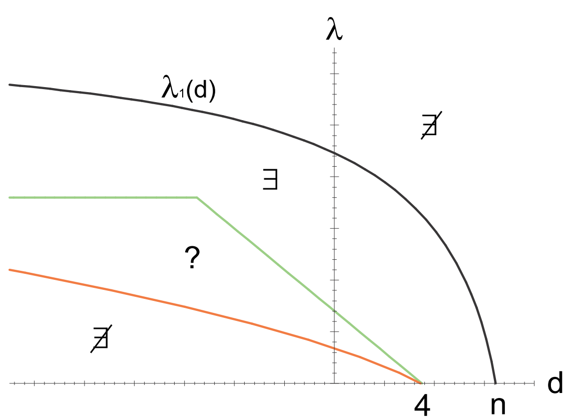

FIGURE 1. The existence and nonexistence regions when $n = 5, 6, 7$.

Figure 1 represents the existence and nonexistence regions, as d and λ vary, for radial solutions to problem (1) as stated by Theorem 4. The question mark indicates the region not covered by our results.

Let $\mathcal{D}^{2,2}(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm $\|\Delta\,.\,\|_2$. We recall that the best constant for the embedding $\mathcal{D}^{2,2}(\mathbb{R}^n)\subset L^{2^*}(\mathbb{R}^n)$ may be characterized by

$$S=\inf_{u\in\mathcal{D}^{2,2}(\mathbb{R}^n)\setminus\{0\}}\frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2}. \tag{6}$$

It is shown in [35], see also [15], that for any smooth domain $\Omega\subset\mathbb{R}^n$ we have

$$\inf\{\|\Delta u\|_2^2;\ u\in H^2(\Omega)\cap H_0^1(\Omega),\ \|u\|_{2^*}=1\}=S$$

although the infimum is not achieved if $\Omega\neq\mathbb{R}^n$. This suggests to try to improve the Sobolev inequality by adding remainder terms. In [14, Theorem 5], the remainder term added was of interior L^p-type whereas in [6, Corollary 3] it was of H^1 boundary type. Here, from Theorem 4-(iii), we deduce a Sobolev inequality with both interior and boundary remainder terms:

Theorem 5. *Let $d\le 4$, there exists an optimal $\Lambda(d)\ge 0$ such that for all $u\in H^2(B)\cap H_0^1(B)$ we have*

$$\|\Delta u\|_2^2\ge S\|u\|_{2^*}^2+d\|u\|_{\partial_\nu}^2+\Lambda(d)\|u\|_2^2. \tag{7}$$

If $n\ge 8$, $\Lambda(d)\equiv 0$. If $n\in\{5,6,7\}$, the map $d\mapsto\Lambda(d)$ is nonincreasing and strictly positive on $(-\infty,4)$. Furthermore, $\Lambda(d)\to 0$ as $d\to 4$.

3. Existence and nonexistence for $n\in\{5,6,7\}$.

3.1. Existence.

Let S be as in (6). Up to translations and nontrivial real multiples, the infimum in (6) is achieved only by the functions

$$u_\varepsilon(x):=\frac{1}{(\varepsilon^2+|x|^2)^{\frac{n-4}{2}}} \tag{8}$$

for any $\varepsilon>0$, see [11, Theorem 2.1] and [32, Theorem 4]. From (7.3) and (7.4) in [6] we have

$$\int_{\mathbb{R}^n}|u_\varepsilon|^{2^*}=\frac{\omega_n}{2\varepsilon^n}\frac{[\Gamma(\frac{n}{2})]^2}{\Gamma(n)}=:\frac{K_2}{\varepsilon^n}$$

and

$$\int_{\mathbb{R}^n}|\Delta u_\varepsilon|^2=S\frac{K_2^{2/2^*}}{\varepsilon^{n-4}}=:\frac{K_1}{\varepsilon^{n-4}}. \tag{9}$$

Here and in the sequel, ω_n denotes the surface measure of the unit ball in $\mathbb{R}^n$:

$$\omega_n:=|\partial B|=\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{10}$$

$r:=|x|$ denotes the radial variable. Set

$$\mathcal{H}=\{u\in H^2(B)\cap H_0^1(B);\ u=u(r)\}$$

and consider the minimization problem

$$\Sigma_{d,\lambda}\ :=\ \inf_{u\in\mathcal{H}\setminus\{0\}}\ Q_{d,\lambda}(u), \tag{11}$$

where

$$Q_{d,\lambda}:H^2(B)\cap H_0^1(B)\setminus\{0\}\to\mathbb{R},\qquad Q_{d,\lambda}(u)=\frac{\|\Delta u\|_2^2-d\|u\|_{\partial_\nu}^2-\lambda\|u\|_2^2}{\|u\|_{2^*}^2}. \tag{12}$$

We have

Proposition 2. *If $\Sigma_{d,\lambda} < S$ the infimum in (11) is achieved. Moreover, up to a change of sign and to a Lagrange multiplier, any minimizer is a radial solution to (1).*

The proof of Proposition 2 is given in [6, Proposition 13] for $\lambda = 0$ but it directly extends to the case $\lambda > 0$.

The purpose of this section is to prove

Proposition 3. *Let $n \in \{5, 6, 7\}$ and $d \le 4$. If $\lambda_1(d) > 3(8-n)(n+4)(4-d)$ and*

$$3(8-n)(n+4)(4-d) < \lambda < \lambda_1(d) \tag{13}$$

then (1) admits a radially symmetric solution. In particular, if $d = 4$, (1) admits a radial solution for all $\lambda \in (0, \lambda_1(d))$.

As shown by Table 1, it turns out that $\lambda_1(0) > 12(8-n)(n+4)$, for any $n \in \{5, 6, 7\}$. Since the map $d \mapsto \lambda_1(d)$ is concave, this allows to conclude that

$$\lambda_1(d) > 3(8-n)(n+4)(4-d) \qquad \text{for all } \overline{d} \le d \le 4,$$

for some $\overline{d} < 0$. Hence, the assumptions of Proposition 3 make sense.

Proof. In view of Proposition 2, we are led to exhibit a nontrivial radial function $U_{\varepsilon,\delta} \in \mathcal{H}$ such that

$$Q_{d,\lambda}(U_{\varepsilon,\delta}) < S. \tag{14}$$

Our construction of this function $U_{\varepsilon,\delta}$ depends on two parameters ε and δ and follows the lines of [17]. First, for $\delta \in (0,1)$ we define

$$a := \frac{2(n-2)}{2 - n\delta^{n-2} + (n-2)\delta^n}$$

and consider the function

$$\begin{aligned} \Phi(\delta) \quad := \quad & a^2(1-\delta^n)\Big[(4-d)(1-\delta^n) + n\delta^n\Big] \\ & -\lambda a^2 \int_\delta^1 \left(\frac{2+(n-2)\delta^n}{2(n-2)} - \frac{r^{n-2}}{n-2} - \frac{\delta^n}{2r^2}\right)^2 \frac{dr}{r^{n-7}} - \frac{\lambda\,\delta^{8-n}}{8-n}. \end{aligned} \tag{15}$$

Some tedious computations show that

$$\lim_{\delta \to 0} \Phi(\delta) = (n-2)^2 \left[4 - d - \frac{\lambda}{3(8-n)(n+4)}\right] < 0$$

since (13) holds. Hence, we may fix $\delta > 0$ such that

$$\Phi(\delta) < 0\ . \tag{16}$$

For such δ, let

$$g_\delta(r) := \begin{cases} 1 & \text{for } r \in [0, \delta] \\ a\left(\dfrac{2+(n-2)\delta^n}{2(n-2)} - \dfrac{r^{n-2}}{n-2} - \dfrac{\delta^n}{2r^2}\right) & \text{for } r \in (\delta, 1], \end{cases} \tag{17}$$

so that $g_\delta \in C^1[0,1] \cap W^{2,\infty}(0,1)$ and $g_\delta(1) = 0$. The explicit form (17) for g_δ will be used at the very end of this proof.

Consider the family of functions

$$U_{\varepsilon,\delta}(x) = g_\delta(|x|)u_\varepsilon(x) = \frac{g_\delta(|x|)}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}}$$

where, again, $\delta > 0$ is fixed and satisfies (16). Then, $U_{\varepsilon,\delta} \in \mathcal{H}$ and

$$U_{\varepsilon,\delta}(x) = u_\varepsilon(x) = \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}} \qquad \text{in } B_\delta = \{x \in \mathbb{R}^n;\ |x| < \delta\}.$$

In what follows we let ε vary and we show that for ε sufficiently small (14) holds.

The asymptotic behavior of the denominator in (12) is readily obtained:

$$\begin{aligned}
&\int_B |U_{\varepsilon,\delta}(x)|^{2^*} \\
&= \int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*} - \int_{\mathbb{R}^n\setminus B} |u_\varepsilon(x)|^{2^*} - \int_{B\setminus B_\delta} \frac{1 - g_\delta(|x|)^{2^*}}{(\varepsilon^2 + |x|^2)^n} \\
&= \frac{K_2}{\varepsilon^n} + O(1).
\end{aligned} \tag{18}$$

Here and below, $O(1)$ and $o(1)$ are intended as $\varepsilon \to 0$. Next, we seek an upper bound for the numerator. By (9) we infer

$$\int_B |\Delta u_\varepsilon|^2 = \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 - \int_{\mathbb{R}^n\setminus B} |\Delta u_\varepsilon|^2$$

$$= \frac{K_1}{\varepsilon^{n-4}} - (n-4)^2 \int_{\mathbb{R}^n\setminus B} \frac{(n\varepsilon^2 + 2|x|^2)^2}{(\varepsilon^2 + |x|^2)^n} = \frac{K_1}{\varepsilon^{n-4}} - 4(n-4)\,\omega_n + o(1).$$

Therefore, we may split the integral as follows

$$\begin{aligned}
\int_B |\Delta U_{\varepsilon,\delta}|^2 &= \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 - \int_{B\setminus B_\delta} |\Delta u_\varepsilon|^2 + \int_{B\setminus B_\delta} |\Delta U_{\varepsilon,\delta}|^2 - \int_{\mathbb{R}^n\setminus B} |\Delta u_\varepsilon|^2 \\
&= \frac{K_1}{\varepsilon^{n-4}} - 4(n-4)\omega_n + o(1) + \int_{B\setminus B_\delta} \Big(|\Delta U_{\varepsilon,\delta}|^2 - |\Delta u_\varepsilon|^2\Big).
\end{aligned} \tag{19}$$

In radial coordinates, after some computations we find

$$\Delta U_{\varepsilon,\delta}(r) = U''_{\varepsilon,\delta}(r) + \frac{n-1}{r}\, U'_{\varepsilon,\delta}(r)$$

$$= \frac{g''_\delta(r)}{(\varepsilon^2 + r^2)^{(n-4)/2}} + \frac{g'_\delta(r)}{r(\varepsilon^2 + r^2)^{(n-2)/2}}\Big[(7-n)r^2 + (n-1)\varepsilon^2\Big]$$

$$-(n-4)\frac{g_\delta(r)}{(\varepsilon^2 + r^2)^{n/2}}(2r^2 + n\varepsilon^2)\ .$$

Let us recall that $g'_\delta(r) = g''_\delta(r) = 0$ for $r < \delta$. Furthermore, as $\varepsilon \to 0$, we have

$$\Delta U_{\varepsilon,\delta}(r) = \frac{g''_\delta(r)}{r^{n-4}} + (7-n)\frac{g'_\delta(r)}{r^{n-3}} - 2(n-4)\frac{g_\delta(r)}{r^{n-2}} + o(1)$$

uniformly with respect to $r \in [\delta, 1]$. By squaring, we get

$$|\Delta U_{\varepsilon,\delta}(r)|^2 = \frac{g''_\delta(r)^2}{r^{2n-8}} + (7-n)^2\,\frac{g'_\delta(r)^2}{r^{2n-6}} + 4(n-4)^2\,\frac{g_\delta(r)^2}{r^{2n-4}} +$$

$$+2(7-n)\,\frac{g''_\delta(r)g'_\delta(r)}{r^{2n-7}} - 4(n-4)\,\frac{g''_\delta(r)g_\delta(r)}{r^{2n-6}} + 4(n-4)(n-7)\,\frac{g'_\delta(r)g_\delta(r)}{r^{2n-5}} + o(1).$$

We may now rewrite in simplified radial form the terms contained in the last integral in (19). With some integrations by parts, and taking into account the behavior of $g_\delta(r)$ for $r \in \{1, \delta\}$, we obtain

$$\int_\delta^1 \frac{g''_\delta(r)g'_\delta(r)}{r^{n-6}}\,dr = \frac{n-6}{2}\int_\delta^1 \frac{g'_\delta(r)^2}{r^{n-5}}\,dr + \frac{g'_\delta(1)^2}{2}\ , \tag{20}$$

$$\int_\delta^1 \frac{g''_\delta(r)g_\delta(r)}{r^{n-5}}\,dr=-\int_\delta^1 \frac{g'_\delta(r)^2}{r^{n-5}}\,dr+(n-5)\int_\delta^1 \frac{g'_\delta(r)g_\delta(r)}{r^{n-4}}\,dr\ , \tag{21}$$

$$\int_\delta^1 \frac{g'_\delta(r)g_\delta(r)}{r^{n-4}}\,dr=\frac{n-4}{2}\int_\delta^1 \frac{g_\delta(r)^2}{r^{n-3}}\,dr-\frac{1}{2\delta^{n-4}}\ . \tag{22}$$

Using (20), (21) and (22) we find

$$\int_{B\setminus B_\delta}\Big(|\Delta U_{\varepsilon,\delta}|^2-|\Delta u_\varepsilon|^2\Big)=\omega_n\int_\delta^1\left(\frac{g''_\delta(r)^2}{r^{n-7}}+3(n-3)\,\frac{g'_\delta(r)^2}{r^{n-5}}\right)dr \tag{23}$$
$$+(7-n)\omega_n g'_\delta(1)^2+4(n-4)\omega_n.$$

Let us now estimate the L^2-norm for $n\in\{5,6,7\}$. With the change of variables $r=\varepsilon s$ we obtain

$$\int_B |U_{\varepsilon,\delta}|^2=\omega_n\varepsilon^{8-n}\int_0^{\delta/\varepsilon}\frac{s^{n-1}}{(1+s^2)^{n-4}}\,ds+\omega_n\int_\delta^1\frac{r^{n-1}\,g_\delta(r)^2}{(\varepsilon^2+r^2)^{n-4}}\,dr\ .$$

Calculus arguments show that, as $\varepsilon\to0$,

$$\int_0^{\delta/\varepsilon}\frac{s^4}{1+s^2}\,ds=\left[\frac{s^3}{3}-s+\arctan s\right]_0^{\delta/\varepsilon}=\frac{\delta^3}{3\varepsilon^3}+o(\varepsilon^{-3})\ ,$$

$$\int_0^{\delta/\varepsilon}\frac{s^5}{(1+s^2)^2}\,ds=\left[s^2-\log(1+s^2)-\frac{s^4}{2(1+s^2)}\right]_0^{\delta/\varepsilon}=\frac{\delta^2}{2\varepsilon^2}+o(\varepsilon^{-2})\ ,$$

$$\int_0^{\delta/\varepsilon}\frac{s^6}{(1+s^2)^3}\,ds=\left[\frac{15}{8}(s-\arctan s)-\frac{5}{8}\frac{s^3}{1+s^2}-\frac{1}{4}\frac{s^5}{(1+s^2)^2}\right]_0^{\delta/\varepsilon}=\frac{\delta}{\varepsilon}+o(\varepsilon^{-1})\ .$$

Summarizing, we get

$$\int_B |U_{\varepsilon,\delta}|^2=\frac{\omega_n\,\delta^{8-n}}{8-n}+\omega_n\int_\delta^1\frac{g_\delta(r)^2}{r^{n-7}}dr+o(1). \tag{24}$$

Finally, simple computations show that

$$\int_{\partial B}(U_{\varepsilon,\delta})^2_\nu=\omega_n g'_\delta(1)^2+o(1)$$

which, combined with (19) (23) (24), yields

$$\int_B|\Delta U_{\varepsilon,\delta}|^2-d\int_{\partial B}(U_{\varepsilon,\delta})^2_\nu-\lambda\int_B U^2_{\varepsilon,\delta}$$
$$=\frac{K_1}{\varepsilon^{n-4}}+\omega_n\int_\delta^1\left(\frac{g''_\delta(r)^2}{r^{n-7}}+3(n-3)\,\frac{g'_\delta(r)^2}{r^{n-5}}-\lambda\frac{g_\delta(r)^2}{r^{n-7}}\right)dr$$
$$+\omega_n(7-n-d)g'_\delta(1)^2-\frac{\omega_n\,\delta^{8-n}}{8-n}\lambda+o(1)\ .$$

At this point of the proof we use the explicit form (17) of g_δ. Then, some lengthy computations show that the last equality may be rewritten as

$$\int_B|\Delta U_{\varepsilon,\delta}|^2-d\int_{\partial B}(U_{\varepsilon,\delta})^2_\nu-\lambda\int_B U^2_{\varepsilon,\delta}=\frac{K_1}{\varepsilon^{n-4}}+\omega_n\Phi(\delta)+o(1)\,,$$

where $\Phi(\delta)$ is as in (15). Therefore, by (16) and (18), we get

$$Q_{d,\lambda}(U_{\varepsilon,\delta})=\frac{\frac{K_1}{\varepsilon^{n-4}}+\omega_n\Phi(\delta)+o(1)}{\left(\frac{K_2}{\varepsilon^n}+O(1)\right)^{2/2^*}}=S+\frac{\omega_n\Phi(\delta)}{K_2}\,\varepsilon^{n-4}+o(\varepsilon^{n-4})<S \tag{25}$$

for sufficiently small ε. Hence, (14) follows and, by Proposition 2, we infer that there exists a positive radial solution to (1). Proposition 3 is so proved. □

3.2. **Nonexistence.** First we prove

Lemma 6. *If $u = u(r)$ is a radially symmetric solution to (1), then $(-\Delta u)(r)$ and $u(r)$ are radially decreasing for $r \in (0,1)$ and $(\Delta u)'(1) > 0$, $u'(1) < 0$.*

Proof. The proof follows the same idea of [30, Proposition 1], where Dirichlet boundary conditions are considered.

Let u be a smooth radially symmetric solution to (1), then

$$r^{n-1}(\Delta u)'(r) = \int_0^r \left(s^{n-1}(\Delta u)'(s)\right)' ds = \int_0^r s^{n-1}\left(\lambda u + u^{2^*-1}\right) ds > 0$$

for all $r \in (0,1]$. Hence, $(\Delta u)'(r) > 0$ in $(0,1]$. Now we set

$$v(r) := \begin{cases} \dfrac{u'(r)}{r} & \text{for } r \in (0,1]\,, \\ u''(0) & \text{for } r = 0\,. \end{cases}$$

Then, v is smooth in $[0,1]$ and satisfies

$$\begin{cases} (r^{n+1}v'(r))' = r^n(\Delta u)'(r) \geq 0 \quad r \in [0,1]\,, \\ v'(0) = 0\,, \\ v(1) = u'(1)\,. \end{cases}$$

By integrating we deduce that $v'(r) \geq 0$ in $[0,1]$. Since $v(1) = u'(1) < 0$, this yields $v(r) < 0$ in $(0,1]$ and we conclude. □

As expected, for nonexistence results to problem (1), a key tool is a *Pohozaev-type identity* [27, 28] in the spirit of the one noted by Mitidieri [24]. More precisely, by arguing as in [6, Section 6], one sees that the following identity holds

$$\int_{\partial B} [2(\Delta u)_\nu + d(n-d)u_\nu]u_\nu \, d\omega = -4\lambda \int_B u^2 \, dx$$

for any solution to (1). If we additionally require u to be radially symmetric, then we obtain

$$2(\Delta u)'(1)u'(1) + d(n-d)(u'(1))^2 = -\frac{4\lambda}{\omega_n}\int_B u^2 \, dx = -4\lambda \int_0^1 r^{n-1}u(r)^2 \, dr\,, \quad (26)$$

with ω_n as in (10). Note that (26), combined with Lemma 6, readily implies that (1) admits no radial solutions if $\lambda = 0$ and $d < 0$. Moreover, (26) is the key ingredient in the proof of the following

Proposition 4. *Let $n \in \{5,6,7\}$ and $d < 4$. There exists $C(n) > 0$ such that problem (1) admits no radially symmetric solution for every $\lambda < C(n)\,\frac{4-d}{n-d}$.*

Proof. By the divergence Theorem we have

$$u'(1) = \frac{1}{\omega_n}\int_B \Delta u \quad \text{and} \quad (\Delta u)'(1) = \frac{1}{\omega_n}\int_B \Delta^2 u.$$

Hence, (26) becomes

$$-4\lambda\,\omega_n \int_B u^2 = 2\left(\int_B \Delta^2 u\right)\left(\int_B \Delta u\right) + d(n-d)\left(\int_B \Delta u\right)^2. \quad (27)$$

Let $w(x) := (1-|x|^2)/(2n)$, with $x \in B$. Then, $-\Delta w = 1$ in B and $w = 0$ on ∂B. Next, if u is a radial solution to (1), integrating by parts we deduce

$$-\int_B \Delta u = \int_B \Delta w \Delta u = \int_B w\Delta^2 u + \int_{\partial B} w_\nu \Delta u$$
$$= \int_B w\Delta^2 u - \frac{d}{n}\int_{\partial B} u_\nu = \int_B w\Delta^2 u - \frac{d}{n}\int_B \Delta u,$$

namely

$$-\int_B \Delta u = \frac{n}{n-d}\int_B w\Delta^2 u.$$

This, inserted into (27), gives

$$\frac{4\,\lambda\omega_n(n-d)}{n}\int_B u^2 = \left(2\int_B \Delta^2 u - nd\int_B w\Delta^2 u\right)\left(\int_B w\Delta^2 u\right). \tag{28}$$

Since

$$\int_B w\Delta^2 u \le \frac{1}{2n}\int_B \Delta^2 u, \tag{29}$$

the right hand side of (28) is positive for any $d < 4$. Denote by $B_{1/2}$ the ball of radius $1/2$. By Lemma 6, u is radially decreasing and so is $\Delta^2 u$, hence

$$\int_B \Delta^2 u = \int_{B_{1/2}} \Delta^2 u + \int_{B\setminus B_{1/2}} \Delta^2 u \le \int_{B_{1/2}} \Delta^2 u + |B\setminus B_{1/2}|\,\Delta^2 u(1/2)$$
$$\le \frac{1}{w(1/2)}\left(1 + \frac{|B\setminus B_{1/2}|}{|B_{1/2}|}\right)\int_{B_{1/2}} w\Delta^2 u = \frac{n\,2^{n+3}}{3}\int_B w\Delta^2 u.$$

Hence,

$$\int_B w\Delta^2 u \ge \frac{3}{n\,2^{n+3}}\int_B \Delta^2 u =: K(n)\int_B \Delta^2 u. \tag{30}$$

In view of (29) and (30), by setting $s := \int_B w\Delta^2 u$ and $A := \int_B \Delta^2 u$, the right hand side of (28) corresponds to the positive function

$$\psi(s) = 2\,As - nds^2\,, \qquad \text{with } s \in \left[K(n)\,A\,,\,\frac{A}{2n}\right].$$

The function ψ is concave so that the following estimate holds

$$\psi(s) \ge \min\left\{\psi\left(K(n)A\right), \psi\left(\frac{A}{2n}\right)\right\}$$
$$= A^2 \min\left\{2K(n) - ndK^2(n), \frac{4-d}{4n}\right\} \ge \frac{3A^2}{n2^{n+4}}(4-d).$$

This, inserted into (28), gives

$$\lambda\,\|u\|_2^2 \ge \frac{3}{2^{n+6}\omega_n}\,\frac{4-d}{n-d}\,\|\Delta^2 u\|_1^2.$$

By a duality argument and elliptic estimates (see e.g. [9, Appendix Chapter IX] for the second order case) we know that, if $\Delta^2 u \in L^1(B)$ and the boundary conditions satisfy the complementing condition (see [4, Lemma 15]), then $u \in L^q(B)$ for all $q < \frac{n}{n-4}$ and

$$\|u\|_q \ge c(q)\|\Delta^2 u\|_1.$$

Since $n \in \{5, 6, 7\}$ we have $\dfrac{n}{n-4} > 2$ and therefore there exists $c(n) > 0$, independent of u, such that

$$\|\Delta^2 u\|_1^2 \geq c(n)\|u\|_2^2.$$

Summarizing, if a radial solution of (1) exists we necessarily have that

$$\lambda \geq C(n)\,\frac{4-d}{n-d},$$

for a suitable constant $C(n) > 0$. Hence, no solution exists if $\lambda < C(n)\,\frac{4-d}{n-d}$. □

4. Proof of Theorems 1, 4 and 5.

4.1. **Proof of Theorem 1.** *Proof of* (i). Assume first that (1) admits a solution u for $d \geq n$. Then, let $\phi_1(x) = 1 - |x|^2$ be the eigenfunction corresponding to the first Steklov boundary eigenvalue $d = n$ of Δ^2 in B, see [4]. We recall that ϕ_1 is the unique function, up to a multiplicative constant, for which the equality holds in (2). By writing (3) with $v = \phi_1$, we deduce that

$$(n-d)\int_{\partial B} u_\nu\,(\phi_1)_\nu > (n-d)\int_{\partial B} u_\nu\,(\phi_1)_\nu - \lambda\int_B u\,\phi_1 = \int_B u^{2^*-1}\,\phi_1 > 0$$

and we immediately get a contradiction. Similarly, for $d < n$, we write (3) with $v = \phi_1^d$, the first eigenfunction corresponding to $\lambda_1(d)$, and we deduce that

$$(\lambda_1(d) - \lambda)\int_B u\,\phi_1^d = \int_B u^{2^*-1}\phi_1^d.$$

Since $\phi_1^d > 0$ in B, this concludes the proof of (i).

Proof of (ii). We use the notations introduced in Section 3.1. By [11] we know that

$$\inf_{u\in\mathcal{H}\cap H_0^2(B)\setminus\{0\}} Q_{0,\lambda}(u) < S, \qquad \text{for all } 0 < \lambda < \lambda_1(-\infty),$$

where $\lambda_1(-\infty)$ is the first Dirichlet eigenvalue of Δ^2. Since $\mathcal{H}\cap H_0^2(B) \subset \mathcal{H}$, this readily implies that

$$\Sigma_{d,\lambda} = \inf_{u\in\mathcal{H}\setminus\{0\}} Q_{d,\lambda}(u) \leq \inf_{u\in\mathcal{H}\cap H_0^2(B)\setminus\{0\}} Q_{d,\lambda}(u) = \inf_{u\in\mathcal{H}\cap H_0^2(B)\setminus\{0\}} Q_{0,\lambda}(u) < S,$$

for all $0 < \lambda < \lambda_1(d) \leq \lambda_1(-\infty)$. By Proposition 2 this gives the statement. □

4.2. **Proof of Theorem 4.** The proof of (i) is the same of Theorem 1-(i).

Proof of (ii). For $4 < d < n$, by Proposition 1-(ii) we know that

$$\inf_{u\in\mathcal{H}\setminus\{0\}} Q_{d,0}(u) < S,$$

see [6, 17] for the details. This implies that

$$\Sigma_{d,\lambda} = \inf_{u\in\mathcal{H}\setminus\{0\}} Q_{d,\lambda}(u) \leq \inf_{u\in\mathcal{H}\setminus\{0\}} Q_{d,0}(u) < S,$$

for all $4 < d < n$ and for all $0 < \lambda < \lambda_1(d)$. Then statement (ii) follows from Proposition 2.

For $d = 4$, the statement follows from Proposition 3.

Proof of (iii). For $d < 4$, the nonexistence for $\lambda < C(n)\,\frac{4-d}{n-d}$ comes from Proposition 4.

Now, by [11, Theorem 1.6], we deduce

$$\Sigma_{d,\lambda} = \inf_{u\in\mathcal{H}\setminus\{0\}} Q_{d,\lambda}(u) \le \inf_{u\in\mathcal{H}\cap H^2_0(B)\setminus\{0\}} Q_{d,\lambda}(u) = \inf_{u\in\mathcal{H}\cap H^2_0(B)\setminus\{0\}} Q_{0,\lambda}(u) < S,$$

for all $\lambda \in (\lambda_*(n), \lambda_1(-\infty))$.

Combining the estimates so far collected with the statement of Proposition 3, with the aid of Table 1 and 3, we finally obtain the proof. □

4.3. **Proof Theorem 5.** For any $d \le 4$, we set

$$\Lambda(d) \ := \inf_{u\in H^2(B)\cap H^1_0(B)\setminus\{0\}} F_d(u),$$

where

$$F_d : H^2(B)\cap H^1_0(B)\setminus\{0\} \to \mathbb{R}, \qquad F_d(u) = \frac{\|\Delta u\|_2^2 - d\|u\|^2_{\partial_\nu} - S\|u\|^2_{2^*}}{\|u\|_2^2}.$$

By [6, Corollary 3] we know that

$$\|\Delta u\|_2^2 \ge S\|u\|^2_{2^*} + 4\|u\|^2_{\partial_\nu},$$

for all $u \in H^2(B)\cap H^1_0(B)$. Hence, $F_d(u) \ge 0$ for all $u \in H^2(B)\cap H^1_0(B)$. This makes $\Lambda(d)$ well-defined and implies $\Lambda(d) \ge 0$. Furthermore, by definition, the map $d \mapsto \Lambda(d)$ is nonincreasing. On the other hand, recalling (4), we deduce that

$$\Lambda(d) \le \lambda_1(d) - \frac{S}{|B|^{4/n}} \le \lambda_1(-\infty) - \frac{S}{|B|^{4/n}}, \quad \text{for all } d \ge 4.$$

Assume that $n \ge 8$. For any $\lambda > 0$ there exists $u_\lambda \in H^2(B)\cap H^1_0(B)$ such that $Q_{d,\lambda}(u_\lambda) < S$, that is

$$F_d(u_\lambda) < \lambda,$$

where u_λ is the least energy solution to problem (1) as given by Theorem 1. This readily implies that $\Lambda(d) \equiv 0$.

When $n \in \{5,6,7\}$, in view of Theorem 4-(ii), the same argument applied above allows to deduce that $\Lambda(4) = 0$. When $d = 0$, by [33] any positive solution to the Navier problem is radially symmetric. Thus, Theorem 4-(iii) implies that problem (1) admits no solution for all $\lambda < C(n)\,\frac{4}{n}$ and by Proposition 2 we have

$$\inf_{u\in H^2(B)\cap H^1_0(B)\setminus\{0\}} Q_{0,\lambda}(u) = S.$$

In particular, taking $\lambda = C(n)\,\frac{2}{n}$, this implies

$$\|\Delta u\|_2^2 \ge S\|u\|^2_{2^*} + C(n)\,\frac{2}{n}\|u\|_2^2,$$

for all $u \in H^2(B)\cap H^1_0(B)$. By this, $F_0(u) \ge C(n)\,\frac{2}{n}$ for all $u \in H^2(B)\cap H^1_0(B)$ and, in turn, we deduce that $\Lambda(0) > 0$. Since

$$F_d(u) \ge F_0(u) \ge \Lambda(0) \quad \text{for all } d < 0,$$

we also deduce that $\Lambda(d) > 0$ for all $d < 0$. It remains to show that $\Lambda(d) > 0$ for any $d \in (0,4)$. Let $d_1, d_2 \in [0,4]$, for any $t \in (0,1)$ there holds

$$F_{td_1+(1-t)d_2}(u) = tF_{d_1}(u) + (1-t)F_{d_2}(u) \ge t\Lambda(d_1) + (1-t)\Lambda(d_2),$$

for all $u \in H^2(B)\cap H^1_0(B)$. For $d_1 = 0$ and $d_2 = 4$ this gives

$$F_{(1-t)4}(u) \ge t\Lambda(0) > 0,$$

for all $t \in (0,1)$ and $u \in H^2(B)\cap H^1_0(B)$ and the statement follows. □

Appendix: computation of $\lambda_1(d)$. Since $\lambda_1(d)$ is simple, the corresponding eigenfunction is a radially symmetric function.

It is known that all the radial smooth solutions to

$$\Delta^2 y = y \quad \text{on } \mathbb{R}^n$$

are

$$y(r) = r^{1-\frac{n}{2}} \left(c_1 J_{\frac{n}{2}-1}(r) + c_2 I_{\frac{n}{2}-1}(r) \right) \qquad c_1, c_2 \in \mathbb{R},$$

where the $J_{\frac{n}{2}-1}$ and $I_{\frac{n}{2}-1}$ are, respectively, the Bessel and the Bessel modified functions, see [11, (4.19)] and [1]. We seek $r_0 > 0$ such that y solves the problem

$$\begin{cases} \Delta^2 y = y & \text{in } B_{r_0} \\ y = r_0 \Delta y - d y_\nu = 0 & \text{on } \partial B_{r_0}. \end{cases}$$

Writing the two boundary conditions in radial coordinates, we obtain the system

$$\begin{aligned} & r_0^{1-\frac{n}{2}} \left(c_1 J_{\frac{n}{2}-1}(r_0) + c_2 I_{\frac{n}{2}-1}(r_0) \right) = 0 \,, \\ & \left[r^{1-\frac{n}{2}} \left(c_1 J_{\frac{n}{2}-1}(r) + c_2 I_{\frac{n}{2}-1}(r) \right) \right]'' |_{r=r_0} \\ & + \tfrac{n-1-d}{r_0} \left[r^{1-\frac{n}{2}} \left(c_1 J_{\frac{n}{2}-1}(r) + c_2 I_{\frac{n}{2}-1}(r) \right) \right]' |_{r=r_0} = 0 \,. \end{aligned}$$

By exploiting the identity $F'_\nu(t) = F_{\nu-1}(t) - \frac{\nu}{t} F_\nu(t)$ which holds for all $\nu \in \mathbb{R}$, for all $t > 0$ and $F = J, I$, we deduce that nontrivial constants c_1 and c_2 can be determined provided

$$\det \begin{pmatrix} J_{\frac{n}{2}-1}(r_0) & I_{\frac{n}{2}-1}(r_0) \\ \\ \frac{4-n-d}{r_0} J_{\frac{n}{2}-2}(r_0) + J_{\frac{n}{2}-3}(r_0) & \frac{4-n-d}{r_0} I_{\frac{n}{2}-2}(r_0) + I_{\frac{n}{2}-3}(r_0) \end{pmatrix} = 0. \tag{31}$$

Once y is determined, we have that $u(s) := y(r_0 s)$ solves

$$\begin{cases} \Delta^2 u = r_0^4 u & \text{in } B \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial B. \end{cases}$$

Hence, if we put

$$\alpha(d) := \min\{ r_0 = r_0(d) > 0 : (31) \text{ holds} \},$$

then

$$\lambda_1(d) = \alpha^4(d).$$

The existence of such $\alpha(d)$ follows from the existence of $\lambda_1(d)$. For fixed d, the explicit value of $\alpha(d)$ as the first positive root of (31), can be determined numerically with Mathematica.

d	5	4	3	2	1	0	$-\infty$
$\lambda_1(d)$	0	133.95	231.84	305.55	362.53	407.67	769.93

TABLE 3. Some values of $\lambda_1(d)$ when $n = 5$.

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Received xxxx 20xx; revised xxxx 20xx.

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