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On Radially Symmetric Minima of Nonconvex Functionals[1]

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We study a minimization problem in the space $W_0^{1,1}(B_R)$ where B_R is the ball of radius R with center at the origin; the functional considered is not necessarily convex. Under suitable assumptions, we prove the existence of a radially symmetric (decreasing) solution. By strengthening the assumptions we obtain uniqueness results. Finally, we study under which assumptions and in which sense the solutions found solve the corresponding Euler equation. The proofs are very direct and simple: they only make use of the functions $T_n^{\pm}$ introduced by the author [*Arch. Rational Mech. Anal.*, 1999].

1. INTRODUCTION

Let B_R be the open ball of radius R centered at the origin in $\mathbf{R}^n$ ($n \geq 2$); we are interested in existence, uniqueness, and qualitative properties (such as L^∞ and $W^{1,\infty}$ estimates) of nonnegative radially symmetric solutions of the minimization problem

$$\min_{u \in W_0^{1,1}(B_R)} \int_{B_R} \left[h(|\nabla u|) - f(|x|)G(u) \right] dx \qquad (1)$$

under general assumptions on h, f, and G. In particular, since we require no convexity on h, it is not at all obvious that the minimum does exist. Minimization problems of the kind in (1) are motivated by their applications in optimal design [14–17] and nonlinear elasticity [1, 2].

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In some cases, the solutions of the above problem may be seen as ground state solutions of the equation

$$\begin{aligned} -\operatorname{div}\{A(|\nabla u|)\nabla u\} &= f(|x|)g(u) \qquad \text{in } B_R \\ u = 0 \qquad &\text{on } \partial B_R; \end{aligned} \tag{2}$$

we call a *ground state solution* of (2) a nonnegative radially symmetric solution having the least action among all possible solutions of (2). We similarly define ground state solutions of (1): the term ground state is improperly introduced here since it usually refers to entire functions. Formally, the functions g and A are related to G and h by means of the relations $G(s) = \int_0^s g(t)\,dt$ and $h(s) = \int_0^s tA(t)\,dt$, up to the addition of constants. If $A(s) \equiv 1$ then (2) reduces to the classical scalar field equation $-\Delta u = f(|x|)g(u)$; if $A(s) = s^{p-2}$ ($p > 1$, $p \neq 2$) then we obtain the degenerate p-Laplace operator; if $A(s) = (1 + s^2)^{-1/2}$ then we obtain the mean curvature operator. These cases have been widely studied in the literature [3, 10, 11, 13, 18] and in the following we use them in several examples in order to illustrate our results. In this paper we consider more general (possibly irregular) functions h: therefore, a major problem we have to face is understanding if the solutions of the minimum problem (1) are indeed solutions (and in which sense) of the corresponding Euler equation (2).

Under suitable assumptions on f, g, and h we study the existence, uniqueness, and qualitative properties of the ground state solutions of (1). This study is performed by an extensive use of the functions $T_n^{\pm}$ that we introduced in [12] (in the case $n = 2$); see (5) below. The existence result we obtain (see Theorem 2 below) is essentially known [5, 7, 8], but, in our opinion, our proof is simpler and more direct: all our proofs are obtained by arguing by contradiction. Further, the functions $T_n^{\pm}$ allow us to obtain upper pointwise estimates for the length of the gradient of any ground state solution; then, we show that among all possible solutions there exists one "privileged" solution which also satisfies suitable lower pointwise estimates. The functions $T_n^{\pm}$ also play a crucial role in uniqueness results: under the same assumptions which yield existence of a solution, we prove the uniqueness of the solution in the class of privileged solutions. Uniqueness of the ground state solution is obtained under further assumptions: in particular, we determine a necessary and sufficient condition on the function f which seems to be new; see $(f2)$ below. Finally, the functions $T_n^{\pm}$ also enable us to study the regularity of the ground state solution and to determine in which sense it satisfies the Euler equation (2). The assumptions we make are not the most general possible but we preferred to avoid nonessential complications in order to better illustrate our method; for more general assumptions we refer the reader to [7, 8].

This paper is organized as follows. In Section 2 we define and characterize the functions $T_n^{\pm}$. Furthermore, we recall two results by Cellina-Perrotta [5]: the first one allows us to reduce (1) to a one-dimensional problem, the second one yields the existence of a ground state solution of a relaxed problem. In Section 3.1 we determine sufficient conditions for the existence of a ground state solution of (1). In particular, we establish the existence of a "privileged" solution which satisfies some lower estimates. Moreover, we show that any ground state solution of (1) belongs to $W^{1,\infty}$ and satisfies suitable upper estimates. By strengthening the assumptions, in Section 3.2 we state that the ground state solution of (1) is unique: of course, it is precisely the above-mentioned privileged solution. In Section 3.3 we study the Euler equation (2) and we determine sufficient conditions for its ground state solutions to be smooth and to satisfy the equation in a suitable sense. All the proofs of our results are given in Section 4. Finally, Section 5 is devoted to some remarks. Throughout the paper we give examples and counterexamples which illustrate and justify the assumptions.

2. PRELIMINARIES

Let $B_R \subset \mathbf{R}^n$ $(n \geq 2)$ be the open ball centered at the origin, of given radius $R > 0$. Without loss of generality we may assume that $h(0) = G(0) = 0$. Then we have the following formal relations between the functions in Eq. (1) and (2): $h(s) = \int_0^s tA(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$.

Throughout this paper we require the function f to be continuous and nonnegative,

$$f \in C([0, R]; \mathbf{R}_+). \qquad (f)$$

In the following we will need the two nonnegative numbers

$$\underline{F} = \min_{r \in [0, R]} f(r), \qquad \overline{F} = \max_{r \in [0, R]} f(r).$$

We assume that $G \in C^1$ and we require that

$$g \in C(\mathbf{R}; \mathbf{R}_+),\ g(0) = g_0 > 0, \text{ and } g \text{ is nonincreasing} \qquad (g)$$

so that we may define $g_\infty := \lim_{s \to +\infty} g(s)$ and $0 \leq g_\infty \leq g_0$. Note that (g) implies that $s \mapsto G(s)$ is nondecreasing and concave.

Finally, we assume that the function h is proper (i.e., $h \not\equiv +\infty$) and satisfies the conditions

$$h : \mathbf{R}_+ \to (-\infty, +\infty] \text{ is l.s.c.} \qquad \text{and} \qquad \lim_{s \to +\infty} \frac{h(s)}{s} = +\infty. \qquad (h)$$

Under the above assumptions, we consider the functional

$$J(u) = \int_{B_R} [h(|\nabla u|) - f(|x|)G(u)]\, dx.$$

DEFINITION 1. We say that $u \in W_0^{1,1}(B_R)$ is a *ground state solution* of (1) if u is nonnegative and radially symmetric and u minimizes the functional J in the space $W_0^{1,1}(B_R)$.

Let Γ be the set of absolute minima of the function h: by (h) we infer that $\Gamma \neq \emptyset$ and that Γ is bounded. Define

$$\gamma = \max \Gamma, \qquad \eta = \min \Gamma. \tag{3}$$

In order to minimize J we introduce the function h^{**}, the convexification of h (the pointwise supremum of the convex functions less than or equal to h). Then, we define the (non-decreasing) function

$$h_{**}(s) = \begin{cases} h^{**}(\gamma) & \text{if } 0 \leq s \leq \gamma \\ h^{**}(s) & \text{if } s \geq \gamma \end{cases}$$

and the corresponding relaxed functional

$$J_{**}(u) = \int_{B_R} [h_{**}(|\nabla u|) - f(|x|)G(u)]\, dx.$$

In the following we deal with the minimization problem

$$\min_{u \in W_0^{1,1}(B_R)} \int_{B_R} [h_{**}(|\nabla u|) - f(|x|)G(u)]\, dx. \tag{4}$$

Denote by Σ the support of h_{**}, namely, $\Sigma := \{t \geq 0;\ h_{**}(t) < +\infty\}$. If γ is as in (3), we clearly have $[0, \gamma] \subseteq \Sigma$. Consider the functions

$$\begin{aligned} T_n^-(\sigma) &= \min\left\{t \in \Sigma;\ \frac{h_{**}(t+\varepsilon) - h_{**}(t)}{\varepsilon} \geq \frac{\sigma}{n}\ \forall \varepsilon > 0\right\}, \\ T_n^+(\sigma) &= \max\left\{t \in \Sigma;\ \frac{h_{**}(t) - h_{**}(t-\varepsilon)}{\varepsilon} \leq \frac{\sigma}{n}\ \forall \varepsilon > 0\right\}, \end{aligned} \tag{5}$$

where we use the conventions that $h_{**}(s) = +\infty$ for all $s < 0$ and that $h_{**}(t) - h_{**}(t-\varepsilon) = +\infty$ for all $\varepsilon > 0$ and all t strictly greater than any element of Σ. In particular, note that $T_n^+(0) = \gamma$. As we will see, this number plays a crucial role in the existence and uniqueness results. It is not difficult to verify that an equivalent definition of the functions introduced in (5) is

$$T_n^-(\sigma) = \min\left\{t \in \Sigma;\ (h_{**})'_+(t) \geq \frac{\sigma}{n}\right\}$$

$$T_n^+(\sigma) = \max\left\{t \in \Sigma;\ (h_{**})'_-(t) \leq \frac{\sigma}{n}\right\}$$

Hence, these functions are related to the left and right derivatives of the polar function h^*; see [9]. We also refer the reader to [12] for some properties of the functions $T_n^{\pm}$ (in fact, in [12] the functions $T_n^{\pm}$ are defined with respect to h^{**} but it makes no difference). The features which are needed here may be summarized in the following.

Proposition 1. *The functions $T_n^{\pm}$ defined in* (5) *are nondecreasing*, T_n^- *is left continuous*, *and T_n^+ is right continuous*. *Furthermore*,

$$T_n^-(\sigma) \leq T_n^+(\sigma) \qquad \forall\sigma \geq 0,$$

$$T_n^-(\sigma) = T_n^+(\sigma) \qquad \text{for a.e. } \sigma \geq 0,$$

$$T_n^-(\sigma) < T_n^+(\sigma) \Rightarrow h_{**} \text{ is affine in the interval } \left[T_n^-(\sigma), T_n^+(\sigma)\right].$$

Finally, *if h_{**} is strictly convex and $h_{**} \in C^1(R_+)$*, *then* $T_n^+(\sigma) \equiv T_n^-(\sigma) = [h'_{**}]^{-1}(\frac{\sigma}{n})$.

If h satisfies (h) then T_n^- and T_n^+ are locally bounded on $\mathbf{R}_+$. Clearly, this is not the case if h is only asymptotically linear at infinity; see Section 5.2.

In order to reduce the study of (4) to a one-dimensional problem, we recall a result in [5].

Proposition 2. *Assume* (f), (g), (h). *The function* $u \in W_0^{1,1}(B_R)$ *is a ground state solution of* (4) *if and only if* $u = u(r)$ *minimizes the functional* (*denoted again by* J_{**})

$$J_{**}(u) = \int_0^R r^{n-1}\left[h_{**}(|u'(r)|) - f(r)G(u(r))\right] dr$$

in the space $W = \{v \in AC_{loc}(0, R];\ v(R) = 0\}$.

This result allows us to argue directly in radial coordinates. Nevertheless, as pointed out in [8], this one-dimensional problem may not be treated in a standard fashion because of the "singular term" r^{n-1} and the fact that there is no constraint on the initial point $u(0)$. From now on we denote by u both the function $u = u(x)$ (defined on B_R) and the function $u = u(r)$ (defined on $[0, R]$). In particular, the statements of the results will be in Cartesian coordinates while their proofs will often be in radial coordinates. Similarly, the functional J_{**} will be evaluated in both ways. Thanks to this change of variables, Cellina and Perrotta [5] prove the following:

PROPOSITION 3. *Assume* (f), (g), (h); *then* (4) *admits at least a ground state solution* $\bar{u}$.

Finally, let us explain what we mean by a solution of (2).

DEFINITION 2. We say that $u \in W_0^{1,1}(B_R)$ is a *ground state solution* of (2) if u is a ground state solution of (1), if $u = u(r)$ is differentiable a.e. in $[0, R]$ and if $|u'(r)| \in [T_n^+(\underline{F}g_\infty r), T_n^-(\overline{F}g_0 r)]$ for a.e. $r \in [0, R]$.

This notion of a solution of (2) is very weak; nevertheless, we recall that for nonsmooth minimization problems the classical necessary condition that a minimum u satisfies in some sense the Euler equation is replaced by the condition that u satisfies some differential inclusion; see e.g. [6]. As we will see, this definition implies directly that any ground state solution of (1) is also a ground state solution of (2): this result may also be obtained as a consequence of (11) in [8].

3. MAIN RESULTS

3.1. *Existence of a Ground State Solution*

We first consider the simpler case where $f \equiv 0$ so that the minimizing problem (1) reduces to

$$\min_{u \in W_0^{1,1}(B_R)} \int_{B_R} h(|\nabla u|)\, dx. \tag{6}$$

In this case we have a trivial result which we quote for completeness and because it is somehow a simplified version of Theorem 2 below.

THEOREM 1. *Let* $B_R \subset \mathbf{R}^n$ *be the ball centered at the origin with radius* $R > 0$, *and assume* (h). *Then* (6) *admits at least a ground state solution* $\bar{u} = \bar{u}(|x|)$ *which satisfies*

$$|\nabla \bar{u}(x)| \in \Gamma \qquad \text{for a.e. } x \in B_R$$

$$\eta(R - |x|) \le \bar{u}(x) \le \gamma(R - |x|) \qquad \forall x \in B_R.$$

In particular, $\bar{u} \in W^{1,\infty}(B_R)$, *and we have*

$$\eta R \le \|\bar{u}\|_\infty \le \gamma R \qquad \eta \le \|\bar{u}\|_{1,\infty} \le \gamma.$$

Furthermore, $\bar{u}$ *is radially decreasing*.

Moreover, any ground state solution u of (6) *satisfies $u \in W^{1,\infty}(B_R)$ and* $\|u\|_{1,\infty} \le \gamma$.

Next, we deal with the more interesting case where f is nontrivial, namely

$$\overline{F} > 0. \qquad (f1)$$

Then we establish a slightly modified version of Theorem 3.4 from [7].

THEOREM 2. *Let $B_R \subset \mathbf{R}^n$ be the ball centered at the origin with radius $R > 0$, and assume* (f), $(f1)$, (g), (h). *Then* (1) *admits at least a ground state solution $\bar{u} = \bar{u}(|x|)$ which satisfies*

$$\begin{aligned} &T_n^+\left(\underline{F}g_\infty|x|\right) \le |\nabla\bar{u}(x)| \le T_n^-\left(\overline{F}g_0|x|\right) \qquad \text{for a.e. } x \in B_R \\ &\int_{|x|}^R T_n^+(\underline{F}g_\infty\sigma)\, d\sigma \le \bar{u}(x) \le \int_{|x|}^R T_n^-\left(\overline{F}g_0\sigma\right) d\sigma \qquad \forall x \in B_R. \end{aligned} \tag{7}$$

In particular, $\bar{u} \in W^{1,\infty}(B_R)$, and we have

$$\begin{aligned} \int_0^R T_n^+(\underline{F}g_\infty\sigma)\, d\sigma \le \|\bar{u}\|_\infty \le \int_0^R T_n^-\left(\overline{F}g_0\sigma\right) d\sigma \\ T_n^+(\underline{F}g_\infty R) \le \|\bar{u}\|_{1,\infty} \le T_n^-\left(\overline{F}g_0 R\right). \end{aligned} \tag{8}$$

Furthermore, $\bar{u}$ is radially decreasing.

Moreover, any ground state solution u of (1) *satisfies $u \in W^{1,\infty}(B_R)$ and the upper estimates in* (7) *and* (8).

Theorem 2 just guarantees the existence of a ground state solution of (1): one is often interested in determining *nontrivial* solutions. If $f \equiv 0$ the situation is simple: (6) admits a nontrivial solution if and only if $\gamma > 0$. If $(f1)$ holds, we define

$$H := \inf_{t > s \ge 0} \frac{h_{**}(t) - h_{**}(s)}{t - s}$$

with the convention that if $h_{**}(t) = +\infty$ and $s < t$ then $h_{**}(t) - h_{**}(s) = +\infty$, independently of the value of $h_{**}(s)$. Note that it could be $H = +\infty$: this happens if and only if $\Sigma = \{0\}$. Note also that if h_{**} is smooth in a neighborhood of 0, then $H = h'_{**}(0)$. We have

THEOREM 3. *Let $B_R \subset \mathbf{R}^n$ be the ball centered at the origin with radius $R > 0$ and assume* (f), $(f1)$, (g), (h). *Then*:

(*i*) *if* $T_n^-(\overline{F}g_0R) = 0$, *then* (1) *only admits the trivial solution*;

(ii) *if* $T_n^-(\underline{F}g_\infty R) > 0$, *then* (1) *admits at least a nontrivial ground state solution*;

(iii) *if* $H < +\infty$, $\underline{F} > 0$, *and*

$$R > \frac{(n+1)H}{\underline{F}g_0}, \tag{9}$$

then any solution of (1) *is nontrivial.*

Note that $\underline{F}$ depends on R: if $\inf_{\mathbb{R}_+} f > 0$ and either $g_\infty > 0$ or $H < +\infty$, then (ii) and (iii) state that (1) admits a nontrivial ground state solution for sufficiently large R. On the other hand, if $\inf_{\mathbb{R}_+} f > 0$ and $H = 0$, then by (iii) the ground state solution of (1) is nontrivial for all $R > 0$: this occurs, for instance, for the p-Laplace operator ($h(s) = s^p/p$, $p > 1$).

In case (ii), the assumption $T_n^-(\underline{F}g_\infty R) > 0$ cannot be relaxed to $T_n^+(\underline{F}g_\infty R) > 0$: to see this, take $R = n$ and

$$f \equiv 1 \qquad g \equiv 1 \qquad h(s) = \begin{cases} s & \text{if } 0 \le s \le 1 \\ +\infty & \text{if } s > 1. \end{cases}$$

Then, $T_n^+(R) = 1$ while $T_n^-(\sigma) \equiv 0$ on $[0, R]$ which, by (7), entails that (1) only admits the trivial solution.

Finally, it will be clear from the proof that sufficient conditions, other than (iii), may be obtained by arguing similarly.

3.2. *Uniqueness of the Ground State Solution*

Again, we first consider the case where $f \equiv 0$. In this case, it is not difficult to see that (6) admits infinitely many solutions for any function h satisfying (h).

On the other hand, as we will show in Theorem 6 below, the assumptions of Theorem 2 are not enough to ensure the uniqueness of the ground state solution of (1). Therefore, we first strengthen ($f1$) with

$$\forall \rho \in (0, R]\ \exists \bar{\rho} \in [0, \rho) \text{ s.t. } f(\bar{\rho}) > 0. \tag{$f2$}$$

In other words, we assume that the function f does not vanish identically in any right neighborhood of 0: in particular, ($f2$) is satisfied if $f(0) > 0$. Then, we also strengthen (g) with

$$g \in C(\mathbf{R}; (0, +\infty)). \tag{$g1$}$$

This, together with (g), implies that $s \mapsto G(s)$ is strictly increasing. Then we have

THEOREM 4. *Let $B_R \subset \mathbf{R}^n$ be the ball centered at the origin with radius $R > 0$. Assume (f), $(f2)$, (g), $(g1)$, (h). Then* (1) *admits a unique ground state solution. By Theorem* 2, *such a solution is radially nonincreasing and satisfies* (7) *and* (8).

In fact, a stronger result holds: by arguing as in [5], one can prove that there exists a unique solution of (1) in the whole space $W_0^{1,1}$ and that such a solution is a ground state solution. However, since we are only interested in ground state solutions and since the proof is simpler we preferred to state Theorem 4 as above.

As a by product of the proof of Theorem 4 we get the following uniqueness criterion *without* assumptions $(f2)$ and $(g1)$.

THEOREM 5. *Let $B_R \subset \mathbf{R}^n$ be the ball centered at the origin with radius $R > 0$. Assume (f), (g), (h); then* (1) *admits a unique a ground state solution $\bar{u}$ which satisfies*

$$|\nabla \bar{u}(x)| \geq T_n^+(0) \qquad \text{for a.e. } x \in B_R.$$

Now we state that $(f2)$ is a necessary condition in order to have uniqueness results.

THEOREM 6. *Let $B_R \subset \mathbf{R}^n$ be the ball centered at the origin with radius $R > 0$. Assume that f satisfies (f), $(f1)$, but not $(f2)$. Then there exists a function h satisfying (h) such that for any g satisfying (g) and $(g1)$ problem* (1) *admits infinitely many ground state solutions.*

In the particular case where $f(s) \equiv 1$ and $g(s) \equiv 1$, Theorems 2 and 4 (and Proposition 1) yield the following.

COROLLARY 1. *Assume (h); then the problem*

$$\min_{u \in W_0^{1,1}(B_R)} \int_{B_R} [h(|\nabla u|) - u]\, dx$$

admits a unique ground state solution u which is explicitly expressed by

$$u(x) = \int_{|x|}^{R} T_n^-(\sigma)\, d\sigma.$$

This result is proved in [12] in the case $n = 2$ and by means of a proof involving web functions. Here, we obtain it for any dimension $n \geq 2$ and by means of a more direct proof.

Moreover, as a further consequence, we have

COROLLARY 2. *Assume* (h) *and that* h *is a nonnegative function. Define*

$$\rho := \sup\{s \geq 0;\, h(s) = 0\} \qquad \text{and}$$
$$\Lambda := \sup\{a \geq 0;\, h(s) \geq a(s - \rho)\ \forall s \geq 0\}.$$

If $\rho > 0$ *and* $R \leq \Lambda$, *then the problem*

$$\min_{u \in W_0^{1,1}(B_R)} \int_{B_R} [h(|\nabla u|) - u]\, dx \tag{10}$$

admits a unique ground state solution u *which is explicitly expressed by* $u(x) = \rho(R - |x|)$.

The previous result is proved in [4, 19] where also more general domains (other than balls) are considered. We state it here as a consequence of Theorem 2 because it shows how the theorem may be applied: note also that the *linear term u in the functional J does not alter the minimum as long as the radius R is sufficiently small*. In other words, the ground state solution of problem (10) is also a ground state solution of (6).

3.3. *Ground State Solutions of the Corresponding Euler Equation*

According to Theorems 1 and 2 and Definition 2 we can say that if (f), (g), (h) hold, then (2) admits a ground state solution; since Definition 2 gives only a very weak notion of a solution, we study whether (2) is in fact satisfied in a stronger sense. This problem is often related to the regularity of the minimizer. To this end, we first state the following result.

THEOREM 7. *Let* $B_R \subset \mathbf{R}^n$ *be the ball centered at the origin with radius* $R > 0$. *Assume* (f), (g), (h) *and that* $T_n^+(0) > 0$; *then* (1) *admits at least a ground state solution* $u \notin C^1(B_R)$.

Therefore, even if f, g, and h are smooth, the ground state solution needs not be smooth.

Let $\Lambda = \sup \Sigma$: then, by its definition, we clearly have $T_n^-(\sigma) \leq \Lambda$ for all $\sigma \in \mathbf{R}_+$. Moreover, if Σ is open in $\mathbf{R}_+$ (i.e., $\Lambda \notin \Sigma$), the strict inequality is also automatically satisfied. Let $\psi : \mathbf{R}^n \to \mathbf{R}$ the map defined by $\psi(\xi) = \mathrm{h}_{**}(|\xi|)$ and assume that

$$\xi \mapsto \psi(\xi) \text{ is differentialbe for all } \xi \in \mathbf{R}^n \text{ such that } |\xi| < \Lambda. \tag{$h1$}$$

Then (2) is satisfied in the following sense.

THEOREM 8. *Let* $B_R \subset \mathbf{R}^n$ *be the ball centered at the origin with radius* $R > 0$ *and assume* (f), $(f1)$, (g) (h), $(h1)$. *Assume moreover that* $T_n^-(\overline{F} g_0 R) < \Lambda$; *then* (2) *admits at least a ground state solution* $\bar{u} = \bar{u}(|x|)$

which satisfies

$$\int_{B_R} A(|\nabla \bar{u}|)\, \nabla \bar{u}\, \nabla v = \int_{B_R} f(|x|) g(\bar{u}) v \qquad \forall v \in W_0^{1,\infty}(B_R).$$

As mentioned above, the strict inequality $T_n^-(\overline{F}g_0 R) < \Lambda$ is automatically satisfied if Σ is open in $\mathbf{R}_+$. Otherwise, it is satisfied only for small values of R. In order to justify the previous result we give an example where (8) allows us to establish that the Euler equation (2) is satisfied in the classical sense only "locally," i.e., in a suitable subset of B_R. Consider

$$h(s) = \begin{cases} \dfrac{s^2}{2} & \text{if } s \in [0,1] \\ +\infty & \text{if } s \in (1 + \infty), \end{cases}$$

so that

$$T_n^-(\sigma) = T_n^+(\sigma) = \begin{cases} \dfrac{\sigma}{n} & \text{if } \sigma \in [0,n] \\ 1 & \text{if } \sigma \in [n + \infty). \end{cases}$$

For simplicity, take f satisfying (f), $(f2)$ and g satisfying (g), $(g1)$ with $g_\infty > 0$. In this case, Theorem 4 yields a unique ground state solution $\bar{u}$ of (1). Assume first that $R < n/\overline{F}g_0$: then (8) yields

$$\|\bar{u}\|_{1,\infty} \le T_n^-\left(\overline{F}g_0 R\right) < T_n^-(n) = 1.$$

Hence, $\bar{u}$ is a classical solution of the Euler equation

$$\begin{aligned} -\Delta u &= f(|x|) g(u) \qquad \text{in } B_R \\ u &= 0 \qquad \text{on } \partial B_R. \end{aligned} \tag{11}$$

Assume now that $R > n/\underline{F}g_\infty$; then (7) and Proposition 1 yield

$$|\nabla \bar{u}(x)| \ge T_n^+\left(\underline{F}g_\infty |x|\right) \ge T_n^+(n) = 1 \qquad \text{for a.e. } |x| \ge \frac{n}{\underline{F}g_\infty}.$$

On the other hand, since $\bar{u}$ minimizes J, we also have $|\nabla \bar{u}(x)| \le 1$ for a.e. $x \in B_R$. Then, if $R > n/\underline{F}g_\infty$, we obtain a ground state solution of (1) which satisfies (11) for $|x| \le n/\overline{F}g_0$ and satisfies $\bar{u}(x) = R - |x|$ if $|x| \ge n/\underline{F}g_\infty$ Hence, there remains only the annulus $n/\underline{F}g_\infty \le |x| \le n/\overline{F}g_0$ where we cannot explicitly write $\bar{u}$: clearly, if $f(s) \equiv 1$ and $g(s) \equiv 1$ such an annulus has zero measure.

Remark 1. If $h \in C^1(\mathbf{R}_+)$ and h is strictly increasing and strictly convex (so that $h \equiv h_{**}$), the Euler equation corresponding to the functional considered in (1) is (2) where $A(s) = h'(s)/s$. If we seek smooth radial solutions, we are led to integrate the corresponding ordinary differential equation,

$$\left(A(|u'(r)|)u'(r)\right)' + \frac{n-1}{r}A(|u'(r)|)u'(r) + f(r)g(u(r)) = 0,$$

together with the conditions $u(R) = u'(0) = 0$. If we multiply it by r^{n-1} we obtain

$$-\frac{d}{dr}\left(r^{n-1}A(|u'(r)|)u'(r)\right) = r^{n-1}f(r)g(u(r)).$$

Since $A(s) > 0$ for all $s > 0$, by integrating we see that u is radially nonincreasing and we get

$$h'(|u'(r)|) = A(|u'(r)|)|u'(r)| = \frac{1}{r^{n-1}}\int_0^r t^{n-1}f(t)g(u(t))\,dt. \quad (12)$$

Therefore, if h is smooth, strictly increasing, and strictly convex, by using the inverse of h' (and by Proposition 1) this equation allows us to obtain directly (7). Without these assumptions on h it does not seem possible to use such a simple method involving the Euler equation. In order to obtain (7), in the next section we argue instead on the functional J.

4. PROOFS

4.1. *Proof of the Existence Results*

Proof of Theorem 1. Let $\bar{h} = h(\gamma)$: it is easy to verify that $u \in W_0^{1,1}(B_R)$ is a solution of the minimum problem (6) if and only if $|\nabla u(x)| = \bar{h}$ for a.e. $x \in B_R$. All the results then follow readily.

Now we turn to the case where $\bar{F} > 0$. We first establish the following result.

LEMMA 1. *Assume* (f), $(f1)$, (g), (h). *If* $u \in W_0^{1,1}(B_R)$ *is a solution of the minimum problem* (4), *then also* $|u|$ (*which is nonnegative and belongs to* $W_0^{1,1}$) *is a solution of* (4). *Moreover, if u is a ground state solution of* (4) *then the function* $v(r) = \int_r^R |u'(t)|\,dt$ *is also a* (*radially nonincreasing*) *ground state solution of* (4).

Proof. By (g) we deduce that $G(s) > 0 > G(-s)$ for all $s > 0$: then, (f) yields $J_{**}(|u|) \le J_{**}(u)$ for all $u \in W_0^{1,1}(B_R)$, and the first statement follows.

Now let u be a ground state solution of (4): if $u'(r) \le 0$ for a.e. $r \in [0, R]$ there is nothing to prove. Otherwise, by the definition of v we have $v'(r) = -|u'(r)|$ and $h_{**}(|v'(r)|) = h_{**}(|u'(r)|)$ for a.e. $r \in [0, R]$; furthermore, since $v(r) \ge u(r)$, by (f), (g) we get $f(r)G(v(r)) \ge f(r)G(u(r))$ in $[0, R]$: this proves that $J_{**}(v) \le J_{**}(u)$ and that v is a ground state solution of (4). ∎

Next, we give an upper estimate for the length of the gradient of the ground state solutions of (4).

LEMMA 2. *Assume* (f), $(f1)$, (g), (h) *and let* $\bar{u}$ *be a ground state solution of* (4). *Let* $\sigma \le R$; *then*

$$|\nabla \bar{u}(x)| \le T_n^-\big(\bar{F} g_0 \sigma\big) \qquad \textit{for a.e. } x \textit{ such that } |x| \le \sigma .$$

Proof. By Proposition 2 we may proceed in radial coordinates.

By contradiction, assume that there exist $\sigma \le R$, $\varepsilon > 0$, and a set $I \subset [0,\sigma]$ of positive measure such that $|\bar{u}'(r)| \ge T_n^-(\bar{F} g_0 \sigma) + \varepsilon$ for a.e. $r \in I$: by Lemma 1 we may assume that $\bar{u}'(r) \le 0$ for a.e. $r \in [0, R]$ and hence $\bar{u}'(r) \le -T_n^-(\bar{F} g_0 \sigma) - \varepsilon$ for a.e. $r \in I$. Define the function $v = v(r)$ by

$$v'(r) = \begin{cases} \bar{u}'(r) & \text{if } r \notin I \\ \bar{u}'(r) + \varepsilon & \text{if } r \in I \end{cases} \qquad v(r) = \int_r^R |v'(t)|\,dt .$$

Then, by Lemma 1 and the Lagrange and Fubini Theorems we get

$$\begin{aligned} J_{**}(v) - J_{**}(\bar{u}) &= \int_0^R r^{n-1}\big[h_{**}(|v'(r)|) - h_{**}(|\bar{u}'(r)|) \\ &\qquad + f(r)\big(G(\bar{u}(r)) - G(v(r))\big)\big]\,dr \\ &\le \int_I r^{n-1}\big[h_{**}(|\bar{u}'(r)| - \varepsilon) - h_{**}(|\bar{u}'(r)|)\big]\,dr \\ &\qquad + \bar{F} g_0 \int_0^R r^{n-1}[\bar{u}(r) - v(r)]\,dr \\ &= \varepsilon \int_I r^{n-1}\,\frac{h_{**}(|\bar{u}'(r)| - \varepsilon) - h_{**}(|\bar{u}'(r)|)}{\varepsilon}\,dr \\ &\qquad + \bar{F} g_0 \int_0^R r^{n-1} \int_r^R \big[|\bar{u}'(t)| - |v'(t)|\big]\,dt\,dr \end{aligned}$$

$$\leq -\varepsilon \frac{\overline{F}g_0}{n} \int_I \sigma r^{n-1}\, dr$$

$$+ \overline{F}g_0 \int_I \left[|\overline{u}'(t)| - |v'(t)| \right] \int_0^t r^{n-1}\, dr\, dt$$

$$= \frac{\varepsilon}{n} \overline{F}g_0 \int_I r^{n-1}(r - \sigma)\, dr < 0,$$

the latter inequality being a consequence of $I \in [0, \sigma]$ and $\overline{F}g_0 > 0$. This contradicts the assumption that $\overline{u}$ minimizes J_{**}. ∎

By arguing as in the proof of Lemma 2 we may also obtain a lower estimate for the length of the gradient of ground state solutions of the relaxed problem: nevertheless, the situation is more delicate and we have to distinguish two cases.

LEMMA 3. *Assume* (f), (g), (h) *and that* $\underline{F}g_\infty > 0$; *let* $\overline{u}$ *be a ground state solution of* (4). *Let* $\sigma \leq R$; *then*

$$|\nabla \overline{u}(x)| \geq T_n^+(\underline{F}g_\infty \sigma) \qquad \text{for } a.e.\ x \text{ such that } \sigma \leq |x| \leq R.$$

Proof. By contradiction, assume that there exist $\sigma \leq R$, $\varepsilon \in (0, T_n^+(\underline{F}g_\infty \sigma)]$, and $I \subset [\sigma, R]$ of positive measure such that $|\overline{u}'(r)| \leq T_n^+(\underline{F}g_\infty \sigma) - \varepsilon$ for a.e. $r \in I$. By Lemma 1 we may assume that $0 \geq \overline{u}'(r) \geq -T_n^+(\underline{F}g_\infty \sigma) + \varepsilon$ for a.e. $r \in I$. Now define the function v by

$$v'(r) = \begin{cases} \overline{u}'(r) & \text{if } r \notin I \\ \overline{u}'(r) - \varepsilon & \text{if } r \in I \end{cases} \qquad v(r) = \int_r^R |v'(t)|\, dt.$$

By arguing as in the proof of Lemma 2 we arrive at

$$J_{**}(v) - J_{**}(\overline{u}) \leq \frac{\varepsilon}{n} \underline{F}g_\infty \int_I r^{n-1}(\sigma - r)\, dr < 0,$$

the strict inequality being a consequence of the facts that $\underline{F}g_\infty > 0$ and $I \subset [\sigma, R]$. Again, this contradicts the assumption that $\overline{u}$ minimizes J_{**}. ∎

In the case $\underline{F}g_\infty = 0$, we obtain a weaker result:

Lemma 4. *Assume* (f), (g), (h) *and that* $\underline{F}g_\infty = 0$; *then* (4) *admits a ground state solution* $\overline{u}$ *satisfying*

$$|\nabla \overline{u}(x)| \geq T_n^+(0) \qquad \textit{for } a.e.\ x \in B_R. \tag{13}$$

Proof. By Proposition 3 we know that (4) admits a ground state solution u: if u already satisfies (13), we are done. So, assume that there exists $B \subseteq B_R$ of positive measure such that $|\nabla u(x)| < T_n^+(0)$ for all $x \in B$ (we denote by B the largest subset of B_R having this property): define the function $\bar{u} \in W_0^{1,1}(B_R)$ by $\nabla \bar{u}(x) = -T_n^+(0)\frac{x}{|x|}$ for all $x \in B \setminus \{0\}$ and by $\nabla \bar{u}(x) = \nabla u(x)$ for a.e. $x \in B_R \setminus B$. Then, $h_{**}(|\nabla \bar{u}(x)|) = h_{**}(|\nabla u(x)|)$ in B and $\bar{u}(x) \geq u(x)$ in B_R which, by (f), (g), yield $J_{**}(\bar{u}) \leq J_{**}(u)$ and prove that $\bar{u}$ is also a ground state solution of (4). Moreover, $\bar{u}$ satisfies (13). ∎

Finally, we return to the original problem; we have

LEMMA 5. *There exists a ground state solution $\bar{u}$ of problem* (1). *Moreover, any ground state solution of* (1) *is also a ground state solution of* (4).

Proof. By Proposition 3, we know that (4) admits a ground state solution: then, by Lemmas 3 and 4, (4) admits a ground state solution $\bar{u}$ satisfying (13). Hence, $\bar{u}$ is also a ground state solution of (1) if, for a.e. $x \in B_R$, $|\nabla \bar{u}(x)|$ does not take its values on any interval where h_{**} is affine and strictly increasing: this follows by arguing as in the proof of Theorem 2 in [5].

Now take any ground state solution w of (1); then, since $h_{**} \leq h$ and since $\bar{u}$ solves both (1) and (4), we have

$$J_{**}(w) \leq J(w) = J(\bar{u}) = J_{**}(\bar{u}),$$

which proves that w is also a ground state solution of (4). ∎

Proof of Theorem 2. The existence of a ground state solution $\bar{u}$ of (1) satisfying (7) follows by combining the results of Lemmas 2, 3, 4, and 5: Indeed, by Proposition 1 we know that the functions $T_n^\pm$ are continuous a.e. and that the pointwise estimates of Lemmas 2–4 yield the pointwise estimate for $|\nabla \bar{u}(x)|$ in (7). Lemma 1 ensures that $\bar{u} \geq 0$ and that $\bar{u}$ may be chosen radially decreasing. Finally, by Lemma 5 any ground state solution of (1) is also a ground state solution of (4): hence, by Lemma 2, it belongs to $W^{1,\infty}(B_R)$ and it satisfies the upper estimates in (7) and (8). ∎

Proof of Theorem 3. Since the function T_n^- is nondecreasing by Proposition 1, Case (i) follows directly from (7).

In order to prove (ii) we distinguish two cases. If $\underline{F}g_\infty = 0$ then $T_n^+(0) \geq T_n^-(0) > 0$ and the result follows directly from Theorem 2. If $\underline{F}g_\infty > 0$, since by Proposition 1 the function T_n^- is left continuous, there exists $\varepsilon > 0$ such that $T_n^-(\underline{F}g_\infty(R-\varepsilon)) > 0$. Then, again by Proposition 1, we have

$$T_n^+\left(\underline{F}g_\infty|x|\right) \geq T_n^+\left(\underline{F}g_\infty(R-\varepsilon)\right) \geq T_n^-\left(\underline{F}g_\infty(R-\varepsilon)\right) > 0 \qquad \text{for a.e. } R-\varepsilon \leq |x| \leq R.$$

The result then follows by the first lower estimate in (7).

In order to prove (iii), we can argue directly on the functional J_{**}. For all $\varepsilon > 0$ consider the function $u_\varepsilon(r) = \varepsilon(R - r)$ and note that

$$J_{**}(0) = h_{**}(0)\frac{R^n}{n}$$

$$J_{**}(u_\varepsilon) = h_{**}(\varepsilon)\frac{R^n}{n} - \int_0^R r^{n-1} f(r) G(\varepsilon(R-r))\, dr.$$

Then, as $\varepsilon \to 0$, we have

$$\begin{aligned} J_{**}(0) - J_{**}(u_\varepsilon) &\sim -\varepsilon\frac{HR^n}{n} + g_0\varepsilon\int_0^R r^{n-1} f(r)(R-r)\, dr \\ &\geq \frac{\varepsilon R^n}{n}\left(\frac{\underline{F}g_0 R}{n+1} - H\right). \end{aligned}$$

By (9), this shows that $J_{**}(0) > J_{**}(u_\varepsilon)$ for sufficiently small ε and that $u \equiv 0$ does not minimize J_{**}. ∎

4.2. *Proof of the Uniqueness Results*

In order to prove Theorem 4 we need the following.

LEMMA 6. *Assume* (f), $(f2)$, (g), $(g1)$, (h); *then, any ground state solution* u *of* (4) *satisfies* $|\nabla u(x)| \geq T_n^+(0)$ *for a.e.* $x \in B_R$.

Proof. If $T_n^+(0) = 0$ there is nothing to prove. So, let $T_n^+(0) > 0$ and let u be a ground state solution of (4): by (8) and Proposition 1 we know that

$$\|u\|_\infty \leq RT_n^-\left(\overline{F}g_0 R\right). \tag{14}$$

Let $\bar{g} := g(RT_n^-(\overline{F}g_0 R) + RT_n^+(0))$ and note that by $(g1)$ we have $\bar{g} > 0$.

Assume by contradiction that there exist $\varepsilon \in (0, T_n^+(0)]$ and $I \subset [0, R]$ of positive measure such that $|u'(r)| \leq T_n^+(0) - \varepsilon$ for a.e. $r \in I$. By eventually restricting I we may assume that $\inf I = \bar{r} > 0$. By Lemma 1 we may suppose that $0 \geq u'(r) \geq -T_n^+(0) + \varepsilon$ for a.e. $r \in I$. Define the function v by

$$v'(r) = \begin{cases} u'(r) & \text{if } r \notin I \\ -T_n^+(0) & \text{if } r \in I \end{cases} \qquad v(r) = \int_r^R |v'(t)|\, dt.$$

Note that (14) entails

$$\|v\|_\infty \leq RT_n^-\left(\overline{F}g_0 R\right) + RT_n^+(0). \tag{15}$$

Since $\bar{r} > 0$, by (f), (f1) there exist $0 \leq \rho_1 < \rho_2 \leq \bar{r}$ and $\delta > 0$ such that $f(r) \geq \delta$ for all $r \in [\rho_1, \rho_2]$. Hence, as $T_n^+(0) \in \Gamma$ and $u(r) \leq v(r)$ in $[0, R]$ (which by (g), yields $G(u(r)) \leq G(v(r))$), we have

$$\begin{aligned} J_{**}(v) - J_{**}(u) &= \int_I r^{n-1}\left[h_{**}(T_n^+(0)) - h_{**}(|u'(r)|)\right] dr \\ &\quad + \int_0^R r^{n-1} f(r)\left[G(u(r)) - G(v(r))\right] dr \\ &\leq \int_{\rho_1}^{\rho_2} r^{n-1} f(r)\left[G(u(r)) - G(v(r))\right] dr \\ &\leq \delta \int_{\rho_1}^{\rho_2} r^{n-1}\left[G(u(r)) - G(v(r))\right] dr. \end{aligned}$$

Using (15), the Lagrange Theorem, and the facts that $r \leq \bar{r}$ and g is nonincreasing, we have

$$\begin{aligned} |G(u(r)) - G(v(r))| \geq \bar{g}(v(r) - u(r)) &= \bar{g}\int_I \left[T_n^+(0) - |u'(t)|\right] dt \\ &\geq \bar{g}\varepsilon|I| > 0. \end{aligned}$$

Therefore, $G(u(r)) - G(v(r)) < 0$ on $[\rho_1, \rho_2]$, and we get $J_{**}(v) - J_{**}(u) < 0$ which contradicts the assumption that u minimizes J_{**}. ∎

Remark 2. A pointwise inequality of the kind just proved was also obtained in [5] in order to prove that every solution of (1) is radially symmetric. ∎

Remark 3. By arguing as in the proof of Lemma 6 we get the following improved version of the lower estimates in (7) and (8):

$$|\nabla u(x)| \geq T_n^+\left(\underline{F}\bar{g}|x|\right) \qquad \text{for a.e. } x \in B_R,$$

$$u(x) \geq \int_{|x|}^R T_n^+\left(\underline{F}\bar{g}\sigma\right) d\sigma \qquad \forall x \in B_R.$$

Proof of Theorem 4. By Lemma 5, to prove the result it suffices to prove that (4) admits a unique ground state solution. By contradiction, let u and v be two ground state solutions of (4): since h_{**} is nondecreasing and J_{**} is convex, for all $t \in [0, 1]$ also the function $w_t = tu + (1 - t)v$ is a ground state solution of (4). By the monotonicity and convexity of h_{**}

and $-G$ we get

$$\begin{aligned} J_{**}(w_t) &= \int_{B_R} \big[h_{**}(|t\,\nabla u + (1-t)\,\nabla v|) \\ &\qquad -f(|x|)G(tu + (1-t)v)\big]\,dx \\ &\le \int_{B_R} \big[h_{**}(t|\nabla u| + (1-t)|\nabla v|) \\ &\qquad -f(|x|)G(tu + (1-t)v)\big]\,dx \\ &\le \int_{B_R} \big[t h_{**}(|\nabla u|) + (1-t)h_{**}(|\nabla v|) \\ &\qquad -f(|x|)(tG(u) + (1-t)G(v))\big]\,dx \\ &= tJ_{**}(u) + (1-t)J_{**}(v) = J_{**}(u). \end{aligned}$$

As u is a ground state solution, all the previous inequalities are in fact equalities: in particular, we have

$$\begin{aligned} &h_{**}(t|\nabla u(x)| + (1-t)|\nabla v(x)|) \\ &\quad = t h_{**}(|\nabla u(x)|) + (1-t)h_{**}(|\nabla v(x)|) \qquad \text{for a.e. } x \in B_R. \end{aligned} \tag{16}$$

By contradiction, assume that $|\nabla u(x)| < |\nabla v(x)|$ in a subset $B \subseteq B_R$ of positive measure; then, in B we have

$$|\nabla u(x)| < t|\nabla u(x)| + (1-t)|\nabla v(x)| < |\nabla v(x)| \qquad \forall t \in (0,1),$$

and the only possibility to have (16) would then be that h_{**} is affine between $|\nabla u(x)|$ and $|\nabla v(x)|$ for a.e. $x \in B$ and that $|\nabla w_t(x)|$ belongs to the interior of the affine interval $(|\nabla u(x)|, |\nabla v(x)|)$. This contradicts either the arguments used in the proof of Lemma 5 (if the slope of the affine part is strictly positive) or Lemma 6 (if the slope of the affine part is zero). Therefore, $|\nabla u(x)| = |\nabla v(x)|$ for a.e. $x \in B_R$ and the result follows by (f), $(f2)$, (g), $(g1)$.

Proof of Theorem 5. By Theorems 1 and 2, (1) admits a ground state solution. If $f \equiv 0$, the function $\bar{u}(x) = T_n^+(0)(R - |x|)$ is the unique ground state solution of (1) which satisfies (13). If $(f1)$ holds, by Lemmas 3 and 4 we infer that (1) admits a ground state solution $\bar{u}$ satisfying (13). The uniqueness of such a solution follows by arguing as in the proof of Theorem 4. ∎

Proof of Theorem 6. If $(f2)$ does not hold, then there exists $\delta > 0$ such that $f(r) = 0$ for all $r \in [0, \delta]$: let δ denote the largest such number. Since f satisfies $(f1)$, we have $R > \delta$. Take

$$h(s) = \begin{cases} 0 & \text{if } s \leq 1 \\ +\infty & \text{if } s > 1 \end{cases}$$

so that $h \equiv h^{**} \equiv h_{**}$. It is not difficult to see that, independently of the choice of the function g, both of the functions $u(x) = R - |x|$ (privileged solution satisfying $|\nabla u(x)| \geq T_n^+(0) = 1$ a.e.) and

$$v(x) = \begin{cases} R - \delta & \text{if } |x| \leq \delta \\ R - |x| & \text{if } \delta \leq |x| \leq R \end{cases}$$

are two different ground state solutions of (1). Similarly, one may construct infinitely many ground state solutions. ∎

4.3. *Proof of the Results of Section* 3.1

Proof of Theorem 7. By Theorem 2, there exists a ground state solution u of (1) which satisfies $|\nabla u(x)| \geq T_n^+(0) > 0$ for a.e. $x \in B_R$. Moreover, since u is radially decreasing, if $u \in C^1(B_R)$ then $\nabla u(0) = 0$ and $\lim_{x \to 0} \nabla u(x) = 0$, which is impossible since $|\nabla u(x)| \geq T_n^+(0)$ a.e. in B_R. ∎

Proof of Theorem 8. Let $\delta = \Lambda - T_n^-(\bar{F} g_0 R)$; then $\delta > 0$. Take $v \in W_0^{1,\infty}(B_R)$ and let $\lambda > 0$ be sufficiently small so that $\lambda \|v\|_{1,\infty} < \delta$. By Theorem 2 we know that there exists a ground state solution $\bar{u}$ of (1), which is a solution of (4) also and, according to Definition 2 and (7), which is also a ground state solution of (2). In particular, for all $t \in [-1, 1]$ and a.e. $x \in B_R$ we have

$$|\nabla \bar{u}(x) + t\lambda \nabla v(x)| < T_n^-\left(\bar{F} g_0 R\right) + \delta = \Lambda$$

and $h_{**}(|\nabla \bar{u} + t\lambda \nabla v|)$ is well-defined. For all $t \in [-1, 1]$ we have $J_{**}(\bar{u} + t\lambda v) - J_{**}(\bar{u}) \geq 0$, that is,

$$\int_{B_R} \left[h_{**}(|\nabla \bar{u} + t\lambda \nabla v|) - h_{**}(|\nabla \bar{u}|)\right]$$

$$- \int_{B_R} f(|x|)\left[G(\bar{u} + t\lambda v) - G(\bar{u})\right] \geq 0.$$

If we let $t \to 0$ and we simplify by λ, the previous inequality becomes

$$\int_{B_R} h'_{**}(|\nabla \bar{u}|) \frac{\nabla \bar{u}}{|\nabla \bar{u}|} \nabla v - \int_{B_R} f(|x|) g(\bar{u}) v \geq 0.$$

Since this inequality also holds if we replace v with $-v$ it is an equality and the result follows. ∎

5. CONCLUDING REMARKS

5.1. *An Estimate of the Estimates*

How sharp are the estimates (8)? In this section we show that, in a simple case, the (8) estimates are "sufficiently good" for large values of R.

Let $p > 1$, $q > 0$, and consider the equation

$$\begin{aligned} -\Delta_p u &= |x|^q \qquad \text{in } B_R \\ u &= 0 \qquad \text{on } \partial B_R. \end{aligned} \tag{17}$$

This is just (2) with $A(s) = s^{p-2}$, $f(s) = s^q$, and $g(s) \equiv 1$ so that $\overline{F} = R^q$ and $g_0 = 1$. By taking into account that $(h')^{-1}(s) = s^{1/(p-1)}$ and by integrating (12), one finds that the unique ground state solution of (17) is given by

$$u(r) = \frac{p-1}{(p+q)(n+q)^{1/(p-1)}}\left(R^{(p+q)/(p-1)} - r^{(p+q)/(p-1)}\right)$$

and therefore

$$\begin{aligned} \|u\|_\infty = u(0) &= \frac{p-1}{(p+q)(n+q)^{1/(p-1)}} R^{(p+q)/(p-1)} \\ \|u\|_{1,\infty} = |u'(R)| &= \frac{R^{(q+1)/(p-1)}}{(n+q)^{1/(p-1)}}. \end{aligned} \tag{18}$$

On the other hand, by Proposition 1 we have $T_n^-(\sigma) = (\frac{\sigma}{n})^{1/(p-1)}$ and therefore (8) yields

$$\|u\|_\infty \le \frac{p-1}{pn^{1/(p-1)}} R^{(p+q)/(p-1)}, \qquad \|u\|_{1,\infty} \le \frac{R^{(q+1)/(p-1)}}{n^{1/(p-1)}}.$$

These estimates and (18) have the same rate of growth as $R \to \infty$. Clearly, as $q \to 0$ the estimates tend to the values in (18) since for $q = 0$ (i.e. $f(s) \equiv 1$) our method allows us to determine the explicit form of the solution (see Corollary 1) and, of course, gives sharp estimates.

5.2. *A Limit Case*: *The Mean Curvature Operator*

The function $h(s) = \sqrt{1 + s^2}$ *does not satisfy* (h): we show here that our results may not apply. Since h is smooth and strictly convex, Proposition 1 yields

$$T_n^-(\sigma) = T_n^+(\sigma) = \frac{\sigma}{\sqrt{n^2 - \sigma^2}}$$

and T_n^- is defined only if $\sigma < n$. Clearly, the upper estimates in (8) make sense only if $T_n^-(\overline{F}g_0 R)$ is well-defined. Thus, we have the restriction

$$\overline{F}g_0 R < n. \tag{19}$$

In order to justify this restriction consider the case $g(s) \equiv 1$ and $f(r) = r^q$ $(q \geq 0)$, namely the equation

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = |x|^q \qquad \text{in } B_R$$
$$u = 0 \qquad \text{on } \partial B_R.$$

Formally, by (12) we obtain

$$|u'(r)| = \frac{r^{q+1}}{\sqrt{(n+q)^2 - r^{2(q+1)}}} \qquad \text{for a.e. } r \in [0, R],$$

which proves that $u \notin W^{1,\infty}(B_R)$ if $R > (n+q)^{1/(q+1)}$. On the other hand, (19) becomes $R < n^{1/(q+1)}$. Therefore, even if the smallness assumption (19) on R is too strict, we cannot obtain $W^{1,\infty}$-estimates if R is too large.

More generally, we can weaken the growth assumption in (h) by merely requiring that

$$\liminf_{s \to +\infty} \frac{h(s)}{s} = \alpha > 0,$$

provided R is sufficiently small, that is, $R < n\alpha / \overline{F}g_0$.

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