A note on approximable solutions of 3D Navier-Stokes equations

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Abstract

We consider both the stationary and evolution 3D Navier-Stokes equations; we define approximable solutions and we study existence and uniqueness of these solutions.

1 Introduction

We study the problem of uniqueness of the solutions of the Navier-Stokes equations in a smooth bounded domain $\Omega \subset \mathbb{R}^3$: we deal with both the homogeneous Dirichlet stationary problem and the evolution Cauchy-Dirichlet problem. Some of our results apply as well to the 2D case but we will not concern ourselves with this problem.
We first consider the evolution problem

$$\begin{cases} \partial_t u - \eta\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in} \quad \Omega\times(0,T) \\ \nabla\cdot u = 0 \quad \text{in} \quad \Omega\times(0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u(0) = u_0 \quad \text{in} \quad \Omega \end{cases} \tag{1}$$

where the unknowns are the velocity vector function u and the pressure scalar function p, η is the fluid viscosity and f an external force acting on the fluid. It is known that (1) admits at least a weak global solution for all forcing term f in a suitable functional space. In the 2D case, it is also known that the global solution is unique: this follows by a suitable application of the Gagliardo-Nirenberg inequality. In the 3D case, such inequality is not sufficient to obtain uniqueness: hence, one usually restricts to functional spaces where existence is not guaranteed, see e.g. Theorem 6.9 p. 84 in [4]. From a mathematical point of view it looks therefore natural to try to approximate the solutions by means of functions in such spaces; to this end, we consider the modified Navier-Stokes equations introduced in [5] which enable to regularize the solution sufficiently to obtain uniqueness: approximable solutions of (1) are then defined to be these solutions which are the limit (in a suitable sense) of the sequence of

solutions of the modified equations when the modification tends to disappear. We will prove that the set of approximable solutions is either infinite or it contains a single function.
Next, we consider the stationary equations with non-slip boundary conditions, namely

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u + \nabla p = f \qquad \text{in} \quad \Omega \\ \nabla\cdot u = 0 \qquad \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \quad \partial\Omega\ ; \end{cases} \tag{2}$$

it is well-know that (2) admits at least a weak solution for all f in a certain functional space $\mathbf{V}'$, see next section; on the other hand, uniqueness of a solution is assured only if the external force f is small in a suitable sense, see e.g. [6]. A remarkable result by Foias-Temam [1, 2] states that there exists an open dense subset $\mathcal{O}\subset\mathbf{V}'$ such that for all forcing term $f\in\mathcal{O}$ the set of solutions of (2) is finite (we refer to Proposition 1 below for the exact statement of their result); the crucial tool in the proof is the Sard-Smale Theorem. Subsequently, by means of the Leray-Schauder degree theory, Temam (see Section 10.3 in [7]) proved that for all $f\in\mathcal{O}$ the number of solutions is odd in number. However, up to now, uniqueness of the solutions of (2) for general f has not yet been proved.
In this paper we prove that for all $f\in\mathcal{O}$ there exists at most an approximable solution of (2): the definition of *approximable solution* is given in next section. In other words, we prove that for "almost all" f, any "well behaved" approximation yields a unique solution of (2).
The results we expose here are not yet satisfactory. For the stationary problem (2) we obtain a uniqueness result in a class of functions (the class of approximable solutions) where existence seems difficult to prove: standard numerical approximations (see [6]) seem not to fit in our framework. For the evolution problem (1) we do have an existence result but Definition 2 below seems not powerful enough to ensure uniqueness of an approximable solution. However, we believe that our results may be a starting point for further research.

2 Notations and results

Bold capital letters ($\mathbf{L}^2$, $\mathbf{H}^1$,...) represent functional spaces of vector functions and usual capital letters (L^2, H^1,...) represent spaces of scalar functions: we set $L^2 := L^2(\Omega)$,.... and we specify the set only when it is not Ω. With H^m we represent the Hilbertian Sobolev spaces, $\mathbf{H}^1_0$ denotes the $\mathbf{H}^1$ closure of the space of smooth functions with compact support in Ω: we denote by $\|\cdot\|_p$ the L^p norm. We consider the spaces

$$\mathbf{G} := \{f\in\mathbf{L}^2;\ \nabla\cdot f = 0,\ \gamma_n f = 0\} \qquad \mathbf{V} := \{f\in\mathbf{H}^1_0;\ \nabla\cdot f = 0\}$$

where γ_n denotes the normal trace operator; moreover, we consider the dual space $\mathbf{V}'$ of $\mathbf{V}$. The space $\mathbf{V}$ is a Hilbert space when endowed with the scalar product $(u,v)_{\mathbf{V}} := (\nabla u, \nabla v)_2$; we denote by $\mathbf{V}_w$ the space $\mathbf{V}$ endowed with its weak topology. We also introduce the spaces $L^2(0,T;\mathbf{V})$ and $L^\infty(0,T;\mathbf{G})$ of functions defined in $\Omega\times(0,T)$ whose $\mathbf{V}$-norm is square integrable on $(0,T)$ (resp. whose $\mathbf{G}$-norm is essentially bounded on $(0,T)$): we denote by $\|\cdot\|_{L^2(\mathbf{V})}$ (resp. $\|\cdot\|_{L^\infty(\mathbf{G})}$) their norms. Similarly, for any couple of spaces B and $\mathbf{W}$ we define $B(0,T;\mathbf{W})$: if it is a normed space we denote its norm by $\|\cdot\|_{B(\mathbf{W})}$.
Finally, throughout this paper we denote by I the interval $I = [1,+\infty)$.

2.1 The evolution equations

Let $\mathcal{D} = \mathcal{D}_\Omega[0,T) := \{\phi \in C_c^\infty(\Omega \times [0,T));\ \nabla \cdot \phi = 0 \text{ in } \Omega\}$; we say that u is a *solution* of (1) if

$$\int_0^T \Big[-(u, \partial_t \phi)_{\mathbf{G}} + \eta (u, \phi)_{\mathbf{V}} + \langle (u \cdot \nabla) u, \phi \rangle \Big]\, dt = \int_0^T \langle f, \phi \rangle dt + (u_0, \phi(0))_{\mathbf{G}} \qquad \forall \phi \in \mathcal{D}\ , \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathbf{V}'$ and $\mathbf{V}$: even if solutions may be intended in a stronger sense, namely for a wider class of test functions, we will not concern ourselves with this problem. For all $f \in L^2(0,T;\mathbf{V}')$ there exists at least a solution u of (3) such that $u \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{G})$; moreover, $\partial_t u \in L^{4/3}(0,T;\mathbf{V}')$, see e.g. [6]. When no confusion arises, instead of (3) we simply write

$$\partial_t u - \eta \Delta u + (u \cdot \nabla) u = f \qquad \text{in } W_T^\Omega\ .$$

In the sequel we denote by S_f the set of solutions of (3): let us now explain what we mean by approximable solution.

Definition 1 *Let $f \in L^2(0,T;\mathbf{V}')$ and $u_0 \in \mathbf{G}$; we say that $\mathcal{A} = \mathcal{A}(u_0, f) = (\{u_s\}, \{f_s\})_{s \in I}$ is an* **approximating set of data** *if $\{f_s\}_{s\in I} \subset L^2(0,T;\mathbf{V}')$, $\{u_s\}_{s \in I} \subset \mathbf{G}$ and*
(i) $f_s \to f_{\bar{s}}$ in $L^2(0,T;\mathbf{V}')$ as $s \to \bar{s} \in I$, $\qquad f_s \rightharpoonup f$ in $L^2(0,T;\mathbf{V}')$ as $s \to \infty$
(ii) $u_s \to u_{\bar{s}}$ in $\mathbf{G}$ as $s \to \bar{s} \in I$, $\qquad u_s \rightharpoonup u_0$ in $\mathbf{G}_w$ as $s \to \infty$.

For all $s \geq 1$, consider the function $\sigma_s : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\sigma_s(x) = \begin{cases} \eta & \text{if} \quad x \leq s \\ x^3 & \text{if} \quad x \geq s+1 \\ \Psi_s(x) & \text{if} \quad s \leq x \leq s+1 \end{cases}$$

where Ψ_s is a C^2 function such that $\sigma_s \in C^2[0,+\infty)$, and $\Psi_s'(x) > 0$ for all $x \in (s, s+1)$; moreover, we choose Ψ_s so that

$$\forall \bar{s} \in I \qquad \lim_{s \to \bar{s}} \sigma_s = \sigma_{\bar{s}} \qquad \text{uniformly on } \mathbb{R}^+\ . \tag{4}$$

Let $\varphi_s : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by $\varphi_s(\underline{x}) = \sigma_s(|\underline{x}|)\underline{x}$; for all $s \in I$ consider the problem

$$(P_s) \qquad \begin{cases} \partial_t u - \Delta \varphi_s(u) + (u \cdot \nabla) u = f_s & \text{in} \quad \Omega \times (0,T) \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_s \quad \text{in} \quad \Omega\ : \end{cases}$$

we say that u is a *solution* of (P_s) if

$$\int_0^T \Big[-(u, \partial_t \phi)_{\mathbf{G}} - (\varphi_s(u), \Delta \phi)_2 + \langle (u \cdot \nabla) u, \phi \rangle \Big]\, dt = \int_0^T \langle f, \phi \rangle dt + (u_s, \phi(0))_{\mathbf{G}} \qquad \forall \phi \in \mathcal{D}\ .$$

Note that for all s, the function σ_s satisfies the assumptions a) b) c$_1$) of Theorem 5 in [5]: therefore, there exists a unique $v_s \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{G}) \cap L^5(0,T;\mathbf{L}^5)$, solution of (P_s); moreover, in [3] it is proved that v_s is continuous in $[0,T]$ with respect to the weak $\mathbf{G}$ topology, so that the initial condition makes sense.

Definition 2 *Given an approximating set of data* $\mathcal{A} = (\{u_s\}, \{f_s\})_{s\in I}$*, denote by* v_s *the unique solution of* (P_s)*; we say that* $u \in S_f$ *is* **approximable** *if there exists a subsequence* $\{v^k\}_{k\in\mathbf{N}} \subset \{v_s\}_{s\in I}$ *such that*

$$v^k \rightharpoonup u \text{ in } L^2(0,T;\mathbf{V}) \qquad v^k \rightharpoonup^* u \text{ in } L^\infty(0,T;\mathbf{G}) \ .$$

We denote by $\mathcal{U}_\mathcal{A}$ *the set of approximable solutions of (1).*

We prove the following alternative result for approximable solutions:

Theorem 1 *Let* $f \in L^2(0,T;\mathbf{V}')$*,* $u_0 \in \mathbf{G}$ *and let* $\mathcal{A}$ *be an approximating set of data; then,* $\mathcal{U}_\mathcal{A} \neq \emptyset$*. Moreover, if (1) admits more than one approximable solution then the set of approximable solutions* $\mathcal{U}_\mathcal{A}$ *relative to (1) is infinite (at least a continuum).*

Remark. It will be clear from the proof that Theorem 1 continues to hold if instead of $f \in L^2(0,T;\mathbf{V}')$ we take $f = f_1 + f_2$ with $f_1 \in L^2(0,T;\mathbf{V}')$ and $f_2 \in L^1(0,T;\mathbf{G})$, see p.264 in [6] and Theorem 2.2 in [3]. □

2.2 The stationary equations

We assume that $f \in \mathbf{V}'$ and we define implicitly the operator $A : \mathbf{V} \to \mathbf{V}'$ by

$$\langle Au, \phi\rangle = \langle -\eta\Delta u + (u\cdot\nabla)u, \phi\rangle \qquad \forall\phi \in \mathbf{V} \tag{5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{V}'$ and $\mathbf{V}$; hence, $A \in C(\mathbf{V}, \mathbf{V}')$. We say that u is a *solution* of (2) if

$$\langle Au - f, \phi\rangle = 0 \qquad \forall\phi \in \mathbf{V} \ ; \tag{6}$$

in the sequel we denote by S_f the set of solutions of (6): as already mentioned, we have $S_f \neq \emptyset$ for all $f \in \mathbf{V}'$.

The definition of approximable solution is different from that in the evolution case.

Definition 3 *We say that the set* $\{(A_h, f_h)\}_{h\in I}$ *is an* **approximating scheme** *for (6) if* $\{A_h\}_{h\in I} \subset C(\mathbf{V}_w, \mathbf{V}'_w)$*,* $\{f_h\}_{h\in I} \subset \mathbf{V}'$ *and*
(i) $A_h u_h \rightharpoonup Au$ *in* $\mathbf{V}'_w$ *whenever* $u_h \rightharpoonup u$ *in* $\mathbf{V}_w$
(ii) $f_h \rightharpoonup f$ *in* $\mathbf{V}'_w$*.*

Definition 4 *Given an approximating scheme* $\{(A_h, f_h)\}_{h\in I}$*, we say that* $\{u_h\}_{h\in I} \subset \mathbf{V}$ *is an* **approximating family** *for (6) if:*
(i) for all $h \in I$*,* u_h *is the unique solution of the equation* $A_h u_h = f_h$
(ii) for all sequence $h_n \to +\infty$ *there exist a subsequence* $\{h'_n\}$ *and* $u \in S_f$ *such that* $u_{h'_n} \rightharpoonup u$ *in* $\mathbf{V}_w$
(iii) for all $\bar{h} \in I$*,* $\lim_{h\to\bar{h}} \|u_h - u_{\bar{h}}\|_2 = 0$*.*

Definition 5 *Given an approximating scheme* $\{(A_h, f_h)\}_{h\in I}$*, we say that a solution* u *of (6) is* **approximable** *(with respect to* $\{(A_h, f_h)\}_{h\in I}$*) if there exist an approximating family* $\{u_h\}_{h\in I}$ *and a sequence* $h_n \to +\infty$ *such that* $u_{h_n} \rightharpoonup u$ *in* $\mathbf{V}_w$*. We denote by* $\mathcal{U}_f$ *the set of approximable solutions of (6).*

Clearly, there may exist different choices of the approximating scheme, that is, different choices of the operators A_h and of the functions f_h; we will prove the following uniqueness criterion:

Theorem 2 *There exists a dense open set $\mathcal{O} \subset \mathbf{V}'$ such that for all $f \in \mathcal{O}$ and for all approximating scheme $\{(A_h, f_h)\}_{h\in I}$ there exists at most one approximable solution u of (2).*

Theorem 2 is related to the following extension of a result by Foias-Temam [1, 2]:

Proposition 1 *Let $f \in \mathbf{V}'$, then S_f is homeomorphic to a compact set of $\mathbb{R}^m$ for a suitable m. Moreover, there exists a dense open set $\mathcal{O} \subset \mathbf{V}'$ such that for all $f \in \mathcal{O}$ the set S_f is finite.*

The original proof of Foias-Temam is performed under the assumption that $f \in \mathbf{G}$; however, it can be easily extended to the case $f \in \mathbf{V}'$: for sake of completeness we quote the proof in the appendix.

3 Proof of Theorem 1

Let $f \in L^2(0,T;\mathbf{V}')$, $u_0 \in \mathbf{G}$ and let $\mathcal{A} = (\{u_s\},\{f_s\})_{s\in I}$ be an approximating set of data; we first prove

Lemma 1 *For all $s, \bar{s} \in I$ let v_s and $v_{\bar{s}}$ denote the solutions of (P_s) and $(P_{\bar{s}})$ respectively; then*

$$v_s \rightharpoonup^* v_{\bar{s}} \quad \text{in } L^\infty(0,T;\mathbf{G}) \quad \text{and} \quad v_s \rightharpoonup v_{\bar{s}} \quad \text{in } L^2(0,T;\mathbf{V}) \qquad \text{as } s \to \bar{s}\ . \tag{7}$$

Proof. We first prove that $\|v_s\|_{L^5(\mathbf{L}^5)}$ remains bounded as $s \to \bar{s}$. By standard energy estimates (see (4.49) in [5]) one gets

$$\frac{1}{2}\|v_s\|^2_{L^\infty(\mathbf{G})} + \frac{\eta}{2}\|v_s\|^2_{L^2(\mathbf{V})} \le \frac{1}{2}\|u_s\|^2_{\mathbf{G}} + \frac{1}{2\eta}\|f_s\|^2_{L^2(\mathbf{V}')} \qquad \forall s \in I\ ; \tag{8}$$

therefore, from (4.51) and Lemma 1 in [5] one gets

$$\forall \delta > 0 \qquad \exists C_\delta(\bar{s}) > 0\ , \qquad \|v_s\|_{L^5(\mathbf{L}^5)} \le C_\delta \qquad \forall s \in [1, \bar{s}+\delta]\ . \tag{9}$$

Let $w_s = v_s - v_{\bar{s}}$: by subtracting the the two equations relative to (P_s) and $(P_{\bar{s}})$ we get

$$\partial_t w_s - \Delta[\varphi_s(v_s) - \varphi_s(v_{\bar{s}})] + (v_s\cdot\nabla)v_s - (v_{\bar{s}}\cdot\nabla)v_{\bar{s}} = f_s - f_{\bar{s}} + \Delta[\varphi_s(v_{\bar{s}}) - \varphi_{\bar{s}}(v_{\bar{s}})]\ .$$

Next, let $G = (-\Delta)^{-1}$ denote Green's operator from $\mathbf{V}'$ into $\mathbf{V}$: by (3.1)-(3.2) and Lemma 3 in [5], for all $\omega > 0$ there exists $C_\omega > 0$ such that if we test the above expression with $Gw_s(t)$, integrate over Ω we obtain for a.e. $t \in [0,T]$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_s(t)\|^2_{\mathbf{V}'} + \eta\|w_s(t)\|^2_{\mathbf{G}} \\ &\le C_\omega(\|v_s(t)\|_5^5 + \|v_{\bar{s}}(t)\|_5^5)\|w_s(t)\|^2_{\mathbf{V}'} + \omega\|w_s(t)\|^2_{\mathbf{G}} + \|f_s(t) - f_{\bar{s}}(t)\|_{\mathbf{V}'}\|Gw_s(t)\|_{\mathbf{V}} + \\ &\quad + \|\varphi_s(v_{\bar{s}}(t)) - \varphi_{\bar{s}}(v_{\bar{s}}(t))\|_{\mathbf{L}^2}\|w_s(t)\|_{\mathbf{G}}\ . \end{aligned} \tag{10}$$

Let $\Phi_s(t) = \|v_s(t)\|_5^5 + \|v_{\bar{s}}(t)\|_5^5$; then, by (9), we have $\Phi_s \in L^1(0,T)$ for all s sufficiently close to $\bar{s}$. Hence, choose $\omega = \eta$ in (10), multiply by $2\exp(-2C_\eta \int \Phi_s(\tau)d\tau)$ and integrate over $[0,t]$ (for some $t \in (0,T]$) to infer

$$\begin{aligned}&\|w_s(t)\|_{\mathbf{V}'}^2 \exp\Big(-2C_\eta \int_0^t \Phi_s(\tau)d\tau\Big) - \|u_s - u_{\bar{s}}\|_{\mathbf{V}'}^2 \\ &\le 2\int_0^t \Big(\|f_s(\tau) - f_{\bar{s}}(\tau)\|_{\mathbf{V}'}\|w_s(\tau)\|_{\mathbf{V}'} + \|\varphi_s(v_{\bar{s}}(\tau)) - \varphi_{\bar{s}}(v_{\bar{s}}(\tau))\|_{\mathbf{L}^2}\|w_s(\tau)\|_{\mathbf{G}}\Big)d\tau \ ;\end{aligned}$$

by (9), there exists $C > 0$ such that

$$\exp\Big(2C_\eta \int_0^T \Phi_s(\tau)d\tau\Big) \le C$$

for all s sufficiently close to $\bar{s}$; hence, we obtain

$$\|w_s(t)\|_{\mathbf{V}'}^2 \le C\Big(\|u_s - u_{\bar{s}}\|_{\mathbf{V}'}^2 + 2\|f_s - f_{\bar{s}}\|_{L^2(\mathbf{V}')}\|w_s\|_{L^2(\mathbf{V}')}\Big) \ .$$

Therefore, by arbitrariness of t, by taking into account (4) (8) and that $(\{u_s\},\{f_s\})_{s\in I}$ is an approximating set of data, we deduce

$$\|w_s\|_{L^\infty(\mathbf{V}')} \to 0 \ :$$

this, together with (8), yields (7). □

We are now ready to give the

Proof of Theorem 1. Let us first prove that $\mathcal{U}_\mathcal{A} \neq \emptyset$: by (8) we infer that there exists a sequence $\{v^k\} \subset \{v_s\}$ and a function $\bar{u} \in L^\infty(0,T;\mathbf{G}) \cap L^2(0,T;\mathbf{V})$ such that

$$v^k \rightharpoonup^* \bar{u} \qquad \text{in } L^\infty(0,T;\mathbf{G}) \qquad \text{and} \qquad v^k \rightharpoonup \bar{u} \qquad \text{in } L^2(0,T;\mathbf{V}) \ ;$$

clearly, $\bar{u} \in \mathcal{U}_\mathcal{A}$.
Next, let us prove that if there exist $u^1, u^2 \in \mathcal{U}_\mathcal{A}$ such that $u^1 \neq u^2$ then $\mathcal{U}_\mathcal{A}$ is at least a continuum. By (8), there exists $K > 0$ such that $\|v_s\|_{L^2(\mathbf{V})} \le K$ for all $s \in I$: therefore, on the $L^2(\mathbf{V})$-ball of radius K we can introduce a metric δ which defines a topology equivalent to the weak topology of $L^2(\mathbf{V})$. Let $\rho := \delta(u^1,u^2)$ (clearly $\rho > 0$) and define the function $\lambda(s) := \delta(u^1, v_s)$; from (7) we infer that $\lambda \in C(I;\mathbb{R}^+)$ and the limit class Λ of λ as $s \to +\infty$ is connected; moreover, $\{0,\rho\} \subset \Lambda$. Therefore, for all $r \in (0,\rho)$, there exists a sequence $\{s_n\} \subset I$ diverging to $+\infty$ such that $\lambda(s_n) \to r$. By (8), there exists $u_r \in L^\infty(0,T;\mathbf{G}) \cap L^2(0,T;\mathbf{V})$ such that

$$v_{s_n} \rightharpoonup^* u_r \qquad \text{in } L^\infty(0,T;\mathbf{G}) \qquad \text{and} \qquad v_{s_n} \rightharpoonup u_r \qquad \text{in } L^2(0,T;\mathbf{V}) \ , \tag{11}$$

up to a subsequence: then $\delta(u^1,u_r) = r$ and therefore $u_r \neq u^1$ and $u_r \neq u^2$. As $(\{u_s\},\{f_s\})_{s\in I}$ is an approximating set of data, by (11) and by letting $s_n \to +\infty$ we obtain that u_r solves (1) in W_T^Ω, that is, $u_r \in \mathcal{U}_\mathcal{A}$. If we repeat the above argument for all $r \in (0,\rho)$ we obtain a continuum of approximable solutions. □

4 Proof of Theorem 2

Theorem 2 will be proved by means of

Lemma 2 *Let $f \in \mathbf{V}'$; if $\mathcal{U}_f$ has a $\mathbf{V}$ isolated point then $\mathcal{U}_f$ is a singleton.*

Proof. Assume that $u \neq v$ and that $u, v \in \mathcal{U}_f$; the result follows if we prove that u is a $\mathbf{V}$ cluster point of $\mathcal{U}_f$: by Proposition 1, the set $\mathcal{U}_f$ lies in a finite dimensional subspace of $\mathbf{V}$ and therefore it suffices to prove that u is a $\mathbf{G}$ cluster point of $\mathcal{U}_f$.
By definition of $\mathcal{U}_f$, there exist an approximating family $\{u_h\}_{h\in I}$ and two increasing sequences $\{a_n\}$ and $\{b_n\}$ diverging to $+\infty$ and such that $u_{a_n} \rightharpoonup u$, $u_{b_n} \rightharpoonup v$ in $\mathbf{V}_w$. Let $\rho = \|u - v\|_2$ and define the function $\sigma(h) := \|u_h - u\|_2$; by (iii) in Definition 4, we have $\sigma \in C(I, \mathbb{R}^+)$ and the limit class Λ of σ as $h \to +\infty$ is connected. Therefore, as $0, \rho \in \Lambda$, if we take $r \in (0, \rho)$, there exists a sequence $\{h_n\}$ diverging to $+\infty$ such that $\sigma(h_n) \to r$: then, by (ii) in Definition 4, there exists $u_r \in S_f$ such that $u_{h_n} \rightharpoonup u_r$ in $\mathbf{V}_w$, up to a subsequence; by the compact imbedding $\mathbf{V} \subset \mathbf{L}^2$ we obtain $\|u - u_r\|_2 = r$. Obviously, $u_r \in \mathcal{U}_f$; if we repeat the above argument for all $r \in (0, \rho)$ and we let $r \to 0$, we obtain a sequence $\{u_r\} \subset \mathcal{U}_f$ such that $\|u - u_r\|_2 = r \to 0$ and the result is proved. □

Proof of Theorem 2. It follows directly from Lemma 2 and the second part of the statement of Proposition 1. □

5 Appendix: proof of Proposition 1

By taking $\phi = u$ in (6) and by Schwarz inequality one has

$$\|u\|_{\mathbf{V}} \leq \frac{\|f\|_{\mathbf{V}'}}{\eta} \qquad \forall u \in S_f \ . \tag{12}$$

We recall that the Stokes operator $-\mathcal{P}\Delta$ has a compact selfadjoint inverse (here $\mathcal{P}$ denotes the orthogonal projector of $\mathbf{L}^2$ onto $\mathbf{G}$); therefore, there exists an orthonormal basis $\{e_m\}_{m=1}^{\infty}$ and a positive diverging sequence $\{\lambda_m\}_{m=1}^{\infty}$ such that $-\mathcal{P}\Delta e_m = \lambda_m e_m$: it is also well-known that $\lambda_m \asymp m^{2/3}$. In the sequel P_m denotes the orthogonal projection onto the space spanned by e_1,...,e_m (either in $\mathbf{V}$, $\mathbf{G}$ or $\mathbf{V}'$) and $Q_m = Id - P_m$.
We claim that if m is large enough then there exists $c_m > 0$ such that

$$\|u - v\|_{\mathbf{V}} \leq c_m \|P_m(u - v)\|_{\mathbf{G}} \qquad \forall u, v \in S_f \ . \tag{13}$$

Indeed, take $u, v \in S_f$ and let $w = u - v$: by subtracting the two relations (6) corresponding to u and v we obtain

$$\eta \int_\Omega \nabla w \nabla \phi = -\int_\Omega (u \cdot \nabla) u \cdot \phi + \int_\Omega (v \cdot \nabla) v \cdot \phi \qquad \forall \phi \in \mathbf{V} \ ;$$

choosing $\phi = Q_m w$ and using well-known properties of the trilinear form $(u, v, w) \mapsto (u \cdot \nabla) v \cdot w$ we obtain

$$\begin{aligned}
\eta \|Q_m w\|_{\mathbf{V}}^2 &= -\int_\Omega (u \cdot \nabla) w \cdot Q_m w - \int_\Omega (w \cdot \nabla) v \cdot Q_m w \\
&= -\int_\Omega (u \cdot \nabla) P_m w \cdot Q_m w - \int_\Omega (P_m w \cdot \nabla) v \cdot Q_m w - \int_\Omega (Q_m w \cdot \nabla) v \cdot Q_m w \\
&\leq c\|u\|_{\mathbf{V}} \cdot \|P_m w\|_{\mathbf{V}} \cdot \|Q_m w\|_{\mathbf{V}} + c\|P_m w\|_{\mathbf{V}} \cdot \|v\|_{\mathbf{V}} \cdot \|Q_m w\|_{\mathbf{V}} + c\|v\|_{\mathbf{V}} \cdot \|Q_m w\|_4^2 \ .
\end{aligned}$$

Taking into account (12), the generalized Poincaré inequality $\lambda_{m+1}^{1/2}\|Q_m\phi\|_{\mathbf{G}} \leq \|Q_m\phi\|_{\mathbf{V}}$ (which holds for all $\phi \in \mathbf{V}$), the Gagliardo-Nirenberg inequality (see e.g. Lemma 3.5 p.296 in [6]) and the equivalence of the norms in finite-dimensional spaces we obtain

$$\eta\|Q_m w\|_{\mathbf{V}}^2 \leq c_m\|P_m w\|_2 \cdot \|Q_m w\|_{\mathbf{V}} + c\lambda_{m+1}^{-1/4}\|Q_m w\|_{\mathbf{V}}^2 \ ; \tag{14}$$

finally, by Young inequality, we infer

$$\eta\|Q_m w\|_{\mathbf{V}}^2 \leq \Big(\frac{\eta}{4} + c\lambda_{m+1}^{-1/4}\Big)\|Q_m w\|_{\mathbf{V}}^2 + c_m\|P_m w\|_2^2 \ .$$

Therefore, if m is large enough we get

$$\|Q_m w\|_{\mathbf{V}}^2 \leq c_m\|P_m w\|_2^2 \ ;$$

hence,

$$\|w\|_{\mathbf{V}}^2 = \|P_m w\|_{\mathbf{V}}^2 + \|Q_m w\|_{\mathbf{V}}^2 \leq c_m\|P_m w\|_2^2$$

and (13) is proved.

The first part of Proposition 1 is now easily obtained. Indeed, by (12) and the compact imbedding $\mathbf{V} \subset \mathbf{G}$ we infer that S_f is $\mathbf{G}$ compact, and hence it is $\mathbf{V}$ compact by (13). Moreover, by (13), if $u, v \in S_f$ satisfy $P_m u = P_m v$ then $u = v$: hence, S_f is contained in a space isomorphic to span$\{e_1, ..., e_m\}$. Therefore, the set S_f is homeomorphic to a compact set of $\mathbb{R}^m$.

The second part of Proposition 1 can be obtained by applying a suitable form of the Sard-Smale Theorem, see e.g. Theorem 10.3 in [7]: it suffices to prove that the operator A defined in (5) is a proper Fredholm operator of index 0.

It is not difficult to verify that the operator A is a C^∞ mapping from $\mathbf{V}$ into $\mathbf{V}'$ and that for all $u \in \mathbf{V}$ its derivative $A'(u)$ is the linear operator from $\mathbf{V}$ into $\mathbf{V}'$ implicitly defined by

$$\langle A'(u)[v], \phi\rangle = \langle -\eta\Delta v + (u\cdot\nabla)v + (v\cdot\nabla)u, \phi\rangle \qquad \forall v, \phi \in \mathbf{V} \ ;$$

for all $u \in \mathbf{V}$ the linear maps $v \mapsto (u\cdot\nabla)v$ and $v \mapsto (v\cdot\nabla)u$ are continuous from $\mathbf{V}$ into $\mathbf{L}^{3/2}$ and hence, compact from $\mathbf{V}$ into $\mathbf{V}'$: since the Stokes operator is an isomorphism from $\mathbf{V}$ onto $\mathbf{V}'$, this proves that A is a Fredholm operator of index 0.

Now let K denote a compact subset of $\mathbf{V}'$: to prove that A is proper, we need to prove that $A^{-1}(K)$ is compact in $\mathbf{V}$. Since K is bounded in $\mathbf{V}'$, by (12) we know that there exists $C > 0$ such that

$$\|u\|_{\mathbf{V}} \leq C \qquad \forall u \in A^{-1}(K) \ . \tag{15}$$

Consider a sequence $\{u_k\} \subset A^{-1}(K)$; then there exists a sequence $\{f_k\} \subset K$ such that

$$\langle -\eta\Delta u_k + (u_k\cdot\nabla)u_k, \phi\rangle = \langle f_k, \phi\rangle \qquad \forall\phi \in \mathbf{V} \ . \tag{16}$$

Since K is compact, there exists $\bar f \in K$ such that $f_k \to \bar f$ in $\mathbf{V}'$, up to a subsequence; moreover, by (15) and by extracting a further subsequence, we infer that there exists $\bar u \in \mathbf{V}$ such that $u_k \rightharpoonup \bar u$ in $\mathbf{V}_w$ and $u_k \to \bar u$ in $\mathbf{L}^4$: we still denote by $\{u_k\}$ such subsequence. Consider (16) for two integers k and h, subtract and take $\phi = u_k - u_h$: then by well-known properties of the trilinear form $(u, v, w) \mapsto (u\cdot\nabla)v\cdot w$ we obtain

$$\eta\|u_k - u_h\|_{\mathbf{V}}^2 \leq \|f_k - f_h\|_{\mathbf{V}'}\|u_k - u_h\|_{\mathbf{V}} + \|u_h\|_{\mathbf{V}}\|u_k - u_h\|_4^2 \ ;$$

hence, by (15) we infer that

$$\eta \|u_k - u_h\|_{\mathbf{V}}^2 \leq 2C\|f_k - f_h\|_{\mathbf{V}'} + C\|u_k - u_h\|_4^2 \ .$$

Therefore, as $\{f_k\}$ and $\{u_k\}$ are Cauchy sequences in $\mathbf{V}'$ and $\mathbf{L}^4$ respectively, we infer that $\{u_k\}$ is also a Cauchy sequence in $\mathbf{V}$: hence, $u_k \to \bar{u}$ in $\mathbf{V}$. Since $A(\bar{u}) = \bar{f} \in K$, we infer that $\bar{u} \in A^{-1}(K)$: the operator A is proper and the second part of Proposition 1 is proved. □

Remark. In estimate (14) the term $\lambda_{m+1}^{-1/4}$ is involved while in the corresponding proof of Foias-Temam, the term involved is $\lambda_{m+1}^{-1/2}$; this could mean that the integer m found in Proposition 1 is larger than that of Theorem 1.2 in [1]: in this case, the set of weak solutions of (2) (corresponding to $f \in \mathbf{V}'$) is in fact larger (in the sense that it has "more dimensions") than the set of strong solutions (corresponding to $f \in \mathbf{G}$).

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