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Blow-up oscillating solutions to some nonlinear fourth order differential equations describing oscillations of suspension bridges

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ABSTRACT: We discuss some nonlinear fourth order differential equations which describe oscillations in suspension bridges. We give strong theoretical and numerical evidence that solutions blow up in finite time with infinitely many wild oscillations. We exhibit an explicit example where this phenomenon occurs and we give a possible new explanation of the collapse of bridges.

1 INTRODUCTION

Under suitable boundary and initial conditions, the following nonlinear beam equation was proposed by Lazer & McKenna (1990) as a model for a suspension bridge, when $t > 0$

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x,t) \quad x\epsilon(0,L), \tag{1}$$

where $L > 0$ denotes the length of the bridge, $u^+ = max\{u, 0\}$, γu^+ represents the force due to the cables which are considered as a spring with a one-sided restoring force (equal to γu if u is downward positive and to 0 if u is upward negative), and W represents the forcing term acting on the bridge (including its own weight per unit length and the wind or other external sources). The solution u represents the vertical displacement when the beam is bending. Normalizing (1) by putting $\gamma = 1$ and W$\equiv 1$, and seeking traveling waves $u(x,t) = 1 + w(x - ct)$ to (1) leads to the equation

$$u''''^{(s)} + ku''^{(s)} + [w(s) + 1]^+ - 1 = 0, \tag{2}$$

$$(s \in \mathbb{R}, k = c^2)$$

In order to maintain the same behavior but with a smooth nonlinearity, Chen & McKenna (1997) consider also the equation

$$w''''(s) + kw''(s) + e^{w(s)} - 1 = 0 \tag{3}$$

where $\in \mathbb{R}$. We wish to suggest here a variant of this model and to consider the more general equation

$$w''''(s) + kw''(s) + f\big(w(s)\big) = 0, \tag{4}$$

where $\in \mathbb{R}$, $k \in \mathbb{R}$, and f is a locally Lipschitz function satisfying the sign condition

$$f(t)t > 0 \quad \text{for all } t \in \mathbb{R}\backslash\{0\}. \tag{5}$$

This assumption reflects the fact that the nonlinearity has the same sign as the vertical displacement w of the beam. Since the parameter k equals the squared velocity of the traveling wave, one usually assumes that $k > 0$. Nevertheless, a similar equation with a nonlocal term and with $k < 0$ is considered by Como et al. (2005) in order to describe the vertical/torsional flutter of suspension bridges. Hence, equation (3) has interesting applications also in the case where $k \leq 0$. Let us mention that, in the mathematical community, when k is negative (4) is known as the extended Fisher-Kolmogorov equation, whereas when k is positive it is referred to as Swift-Hohenberg equation (Peletier & Troy, 2001)

As pointed out by McKenna (2006, Section 6), one of the most interesting behaviors for suspension bridges (including the Golden Gate and the Tacoma Narrows Bridge) is that

large vertical oscillations can rapidly change, almost instantaneously, to a torsional oscillation.

A possible explanation to this fact is that *since the motion cannot be continued downwards due to the cables, when the bridge reaches its equilibrium position the existing energy generates a crossing wave and, subsequently, a torsional oscillation.*

Since the Tacoma Bridge collapse was due to a wide torsional motion of the bridge, the bridge cannot be considered as a one dimensional beam.

This problem was overcome by Drábek et al. (2003, Section 2.3) by introducing the *deflection from horizontal* as a second unknown function (besides the vertical displacement). Alternatively, we suggest to maintain the one dimensional model provided one also allows displacements *below the equilibrium position*; in other words, the unknown function w represents the sum of two terms, the vertical displacement of the side of the bridge plus the deflection from horizontal. Hence, in our model, the nonlinearity f in (4) should be unbounded also from below.

In Figure 1 the dotted line represents the theoretical position of the suspension bridge in absence of the action of the cables whereas the horizontal line represents the limit position of the bridge when stopped by the cables.

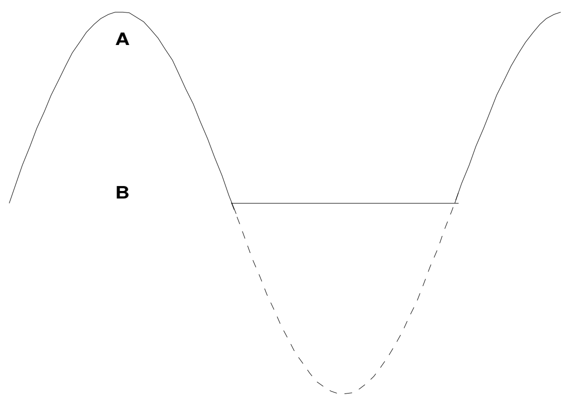

Figure 1. Theoretical position of the suspension bridge without cables (dotted line) and with cables (solid line)

A further remark concerns the source *f*. It is clear that more the position of the bridge is far from the horizontal equilibrium position, more the action of the wind becomes relevant because the wind hits transversally the surface of the bridge. For instance, in Figure 1 the action of the wind is more powerful in position A than in position B. If ever the bridge would reach the limit vertical position, the wind would hit it orthogonally. Hence, the forcing term *f* is superlinear, becoming more powerful for large displacements from the horizontal position. As our theoretical and numerical results seem to suggest, traveling waves with superlinear nonlinearities *f* blow up in finite time after wide oscillations.
Is this the explanation of the Tacoma collapse?

2 NUMERICAL RESULTS

In order to numerically evaluate zeros of the computed solution *w* to (4), we checked where the computed discrete function changed sign.
Then, for each detected interval, we used two different methods:
a) one step of bisection method;
b) computation of exact zero of the linear interpolation polynomial.
From the known bisection error and the comparison between the two computed values of each zero, we obtain the estimated correct digits of values reported in the next tables.
Concerning the computation of the solution *w*, we chose to use standard numerical methods for stiff equations, i.e. we used the MATLAB solvers `ode15s`, `ode23s`, `ode23tb`, according to the required efficiency and accuracy. We remind that `ode15s` is a variable order solver with low/medium order of accuracy; `ode23s` is a one-step solver which can be in some case more effective than `ode15s`; `ode23tb` is an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. Solutions computed by different methods were compared and then a reliable tolerance was chosen, in the sense that we used relative error threshold which revealed neither too tight nor too crude in order to guarantee the same results by different methods.
Let us now describe our numerical results.

Numerical Results 1. *When* $f(t) = 64t + t^3, k = -20$, *the first* 18 *zeros of the solution w to* (4) *satisfying* $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ *are given by:*

$z_1 = 0.716$, $z_2 = 1.7977$, $z_3 = 2.13827$,
$z_4 = 2.17358$, $z_5 = 2.18718$, $z_6 = 2.192412$,
$z_7 = 2.194429$, $z_8 = 2.1952053$, $z_9 = 2.1955044$,
$z_{10} = 2.1956196$, $z_{11} = 2.19566400$,
$z_{12} = 2.19568109$, $z_{13} = 2.195687680$,
$z_{14} = 2.195690216$, $z_{15} = 2.1956911931$,
$z_{16} = 2.1956915694$, $z_{17} = 2.19569171433$,
$z_{18} = 2.19569177015$.

Moreover the first 16 *critical levels (ordered on columns from left to right and then on consecutive lines) are given Table 1, where the levels of the relative maxima are highlighted in bold face.*

1.0000e+00	-7.28173e+01	**5.54303e+02**	-3.79831e+03
2.56635e+04	-1.73041e+05	**1.16639e+06**	-7.86188e+06
5.29910e+07	-3.57173e+08	**2.40743e+09**	-1.62267e+10
1.09371e+11	-7.37197e+11	**4.96887e+12**	-3.34914e+13

Table 1. First critical levels of the solution w to (4) satisfying $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ and $f(t) = 64t + t^3, k = -20$.

Here and in what follows only the estimated correct digits are reported. We quote an increasing number of digits in the zeros z_k in order to emphasize the small differences which appear between two consecutive zeros.
From the reported data, it appears that the amplitude of oscillations is increasing and that the length of the cycles (the distance between two consecutive zeros of the solution) is decreasing with *s*, until a threshold value where numerical computation stops because of the impossibility to reach the required accuracy by the variable step integrator in use. This may be a symptom of a vertical asymptote.
We denote such threshold by *T*. In the Numerical Results 1 we have *T*=2.1957 (rounded to 5 significant digits). Figure 2 plots the computed solution until *s* = 2.05281, that is, just a little after the second relative maximum point.

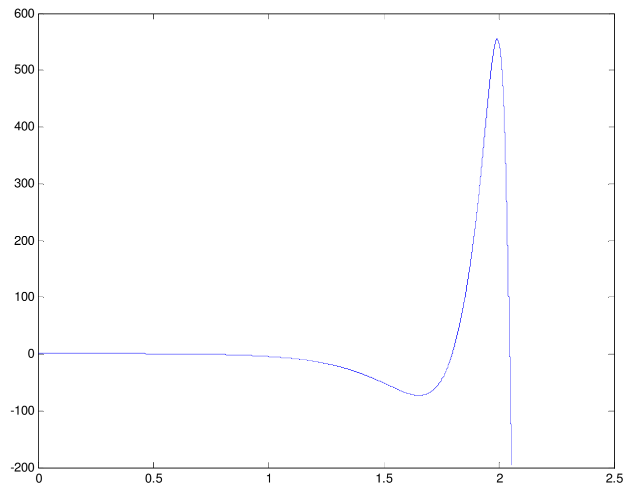

Figure 2. The computed solution until $s = 2.05281$, referring to Numerical results 1

Next, we slightly change the equation and we obtain

Numerical Results 2. *When* $f(t)=t+t^3$, $k=0$ *the first* 20 *zeros of the solution* w *to* (4) *satisfying* $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ *are given by:*

$z_1 = 1.9$, $z_2 = 4.0$, $z_3 = 4.77$, $z_4 = 5.08$,
$z_5 = 5.20$, $z_6 = 5.242$, $z_7 = 5.259$, $z_8 = 5.2661$,
$z_9 = 5.2687$, $z_{10} = 5.26973$, $z_{11} = 5.27012$,
$z_{12} = 5.27027$, $z_{13} = 5.270328$, $z_{14} = 5.270350$,
$z_{15} = 5.2703590$, $z_{16} = 5.2703622$,
$z_{17} = 5.27036356$, $z_{18} = 5.27036406$,
$z_{19} = 5.27036424$, $z_{20} = 5.270364321$.

Moreover, the first 20 critical levels (ordered on columns from left to right and then on consecutive lines) are given in Table 2, where the levels of the relative maxima are highlighted in bold face.

1.0000e+00	-7.3459e+00	**4.9789e+01**	-3.3565e+02
2.2622e+03	-1.5251e+04	**1.0279e+05**	-6.9287e+05
4.6701e+06	-3.1478e+07	**2-1216e+08**	-1.4299e+09
9.6376e+09	-6.4961e+10	**4.3788e+11**	-2.9514e+12
1.9895e+13	-1.3410e+14	**9.0384e+14**	-6.0917e+15

Table 1. First critical levels of the solution w to (4) satisfying [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0] and $f(t) = t + t^3$, $k = 0$.

In the case considered in the Numerical Results 2 it appears that $T = 5.2704$ (rounded to 5 significant digits). In Figure 3 the second minimum point and the third maximum point can be easily recognized. It is worth noticing that the third maximum point is obtained at about $s = 5.17$ and then between this value

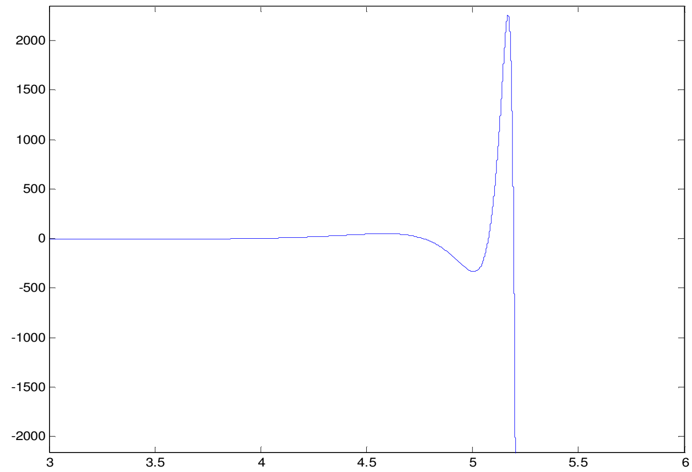

Figure 3. The computed solution referring to Numerical results 2, until the third maximum point

and T, further 7 relative maxima and 8 relative minima were computed.
It appears that the behavior of oscillations is similar

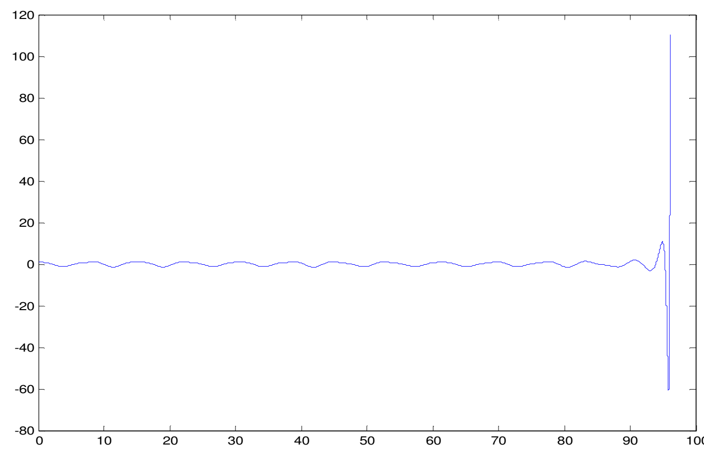

Figure 4. The computed solution to (4) with $f(t)=t+t^3$, $k=3.6$, $[w(0), w'(0), w''(0), w'''(0)] = [0.9, 0, 0, 0]$

to the Numerical Result 1. Analogous behaviors of solutions were found when computing the solution to (4) for many different values of k (including large negative values) and with different initial conditions.

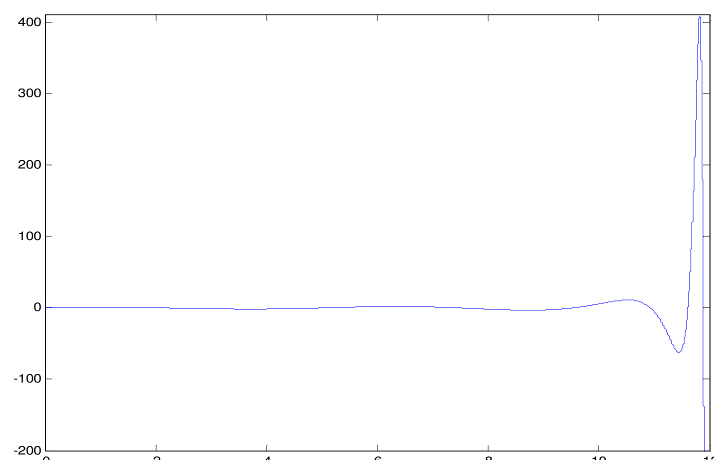

Figure 5. The computed solution to (4) with $f(t)=t+t^3$, $k=3.5$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$, just until the fourth maximum

In Figures 4 and 5 we display the plot of two solutions which have some small oscillations on a somehow large interval of time and then wide oscillations in a very small interval of time. Numerical results suggest that the blow up time for the solutions occur respectively for $T = 96.5947$ and for $T = 12.06618$.
Finally, for $f(t) = t+t^3$, Table 2 shows the numerically found blow up time $T(k)$ (depending on k) of the solution w to (4) satisfying the initial conditions $[w(0), w'(0), w''(0), w'''(0)] = [\alpha, 0, 0, 0]$

α	1	2	20	200	1000
T(-3.5)	4.2958	3.5266	1.3064	0.42008	0.18815
T(-2)	4.6209	3.7144	1.3164	0.42041	0.18818
T(0)	5.2704	4.0442	1.3303	0.42086	0.18822
T(2.1)	6.6400	4.5684	1.3455	0.42133	0.18826
T(2.9)	7.8974	4.8612	1.3516	0.42152	0.18827
T(3.5)	12.066	5.1458	1.3562	0.42165	0.18829

Table 2. The numerically found blow up time $T(k)$ (depending on k) of the solution w to (4) satisfying the initial conditions $[w(0), w'(0), w''(0), w'''(0)] = [\alpha, 0, 0, 0]$

This table suggests that T is decreasing with respect to α (as expected) and increasing with respect to k. Other values of α and k display similar behaviors. However, for very large (positive) values of k and/or for very small (positive) values of α our numerical procedure does not show blow up but a somehow periodic behavior. In these cases we do not know if the solution is indeed periodic or if the blow up time T is so large that the numerical procedure does not reach it with sufficient precision.

We also tried some asymmetric nonlinearity. In the case where

$$f(t) = (t + t^3 + e^t - 1)/2, \quad (7)$$

$k = 2$, and $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ the computed solution exhibits a threshold value $T = 6.3009$. The relative maximum and minimum values of the solution were estimated numerically and are reported in Table 3. The maximum points and values are written in bold characters.

s	**0.0**	3.8532	**5.5342**	6.1571
$w(s)$	**1.0**	- 3.3786	**11.055**	-184.06
s	**6.2754**	6.29695	**6.3009**	6.3009
$w(s)$	**33.554**	-1.5026e+5	**73.377**	-2.318e+10

Table 3. Maximum and minima of a solution to (4) in case (7)

The last two extremal values are obtained for values of s which differ less than 10^{-4}. Figure 6 displays the solution until the third maximum and shows that maxima values are much smaller than the absolute value of minima.

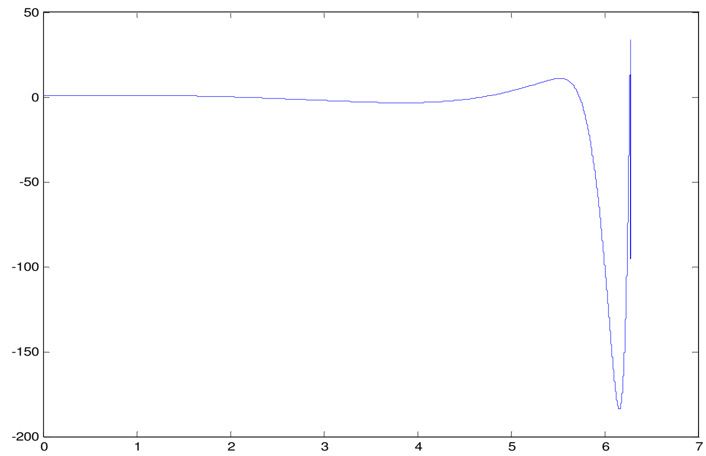

Figure 6. Solution referring to Table 3 until the third maximum

Our Numerical Results suggest that fourth order equations with superlinear nonlinearities exhibit traveling waves which *blow up in finite time after a long waiting time of apparent calm and sudden wide oscillations*. This seems to be a serious obstruction to real projects where numerical and/or physical experiments are performed on a finite interval of time. Even if these experiments would show that no serious oscillations arise up to some large instant T, how can one be sure that they still do not arise at a slightly larger instant $T + \varepsilon$ with ε small?

3 THEORETICAL RESULTS

We were able to find full theoretical evidence of the blow up in finite time only in particular situations. However, we believe that these results are enough to show that the numerical results explained in Section 2 are reliable and that the blow up is not caused by computational instability.
We first mention that Berchio at al. (2011) show that, under the sole assumption (5), the only way that finite time blow up can occur is with wide oscillations of the solution. More precisely, if a local solution w to (3) blows up at some finite $R \in \mathbb{R}$, then

$$\lim_{s\to R} \inf w(s) = -\infty \quad \text{and}$$

$$\lim_{s\to R} \sup w(s) = +\infty \quad (8)$$

Our first theoretical result (see Theorem 2, in Gazzola & Pavani (2011)) states that, under suitable assumptions on f and k, the solution to (4) exhibits "wide and thinning oscillations". By this we mean that the altitude of the oscillation increases and tends to infinity whereas its cycle (the distance between two consecutive zeros of the solution) decreases and tends to zero. Clearly, this *does not* prove that blow up occurs in finite time but, at least, it gives a strong hint in this direction.
In full detail, we assume that f is superlinear in the following sense

$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, there exists $\lambda,\delta,\gamma > 0$ such that

$$f(t) \geq \delta t^2 + \lambda |t|^{2+\gamma} \quad \text{for every } t \in \mathbb{R} \tag{9}$$

and we consider (3) in the limit situation where $k=0$

$$w''''(s) + f(w(s)) = 0 \quad (s \in \mathbb{R}). \tag{10}$$

Then we consider a local solution w to (10) in a neighborhood of $s = 0$ such that

$$w'(0)w''(0) - w(0)w'''(0) > 0\,. \tag{11}$$

If $R \in (0, +\infty]$ denotes the supremum of the maximal interval of continuation of w, then there exists an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that:

(i) $s_j \nearrow R$ as $j \to \infty$;

(ii) $w(s_j) = 0$ and w has constant sign in (s_j, s_{j+1}) f or all $j \in \mathbb{N}$;

(iii) $\lim_{j\to\infty}(s_{j+1} - s_j) = 0$;

(iv) $\max_{s\in[s_j,s_{j+1}]}|w(s)| \to +\infty$.

As already mentioned, this statement is not completely satisfactory since the assumptions on f and on k are quite restrictive and since it does not show that the solution w blows up in finite time since it could be $R = +\infty$. However, this statement gives a strong hint in favor of a blow up in finite time.

Our second theoretical result (see Theorem 3 in Gazzola & Pavani (2011)) gives an explicit example where the finite time blow up with wide and thinning oscillations indeed occurs. As far as we are aware, this is the first example which exhibits this phenomenon. Fix any integer $n \geq 5$, let $k = -\frac{n^2-4n+8}{2} < 0$ and let

$$f(t) = \left(\frac{n(n-4)}{4}\right)^2 t + |t|^{8/(n-4)}t \tag{12}$$

so that f is superlinear. In this case, there exists a solution $w = w(s)$ to the equation

$$w''''(s) - \frac{n^2-4n+8}{2}w''(s) + \left(\frac{n(n-4)}{4}\right)^2 w(s) +$$

$$+|w(s)|^{\frac{8}{n-4}}\, w(s) = 0 \tag{13}$$

which is defined in a neighborhood of $s = -\infty$ and such that (8) holds for some finite $R \in \mathbb{R}$.

Note that when $n = 8$, the above equation simply becomes

$$w''''(s) - 20w''(s) + 64w(s) + w(s)^3 = 0. \tag{14}$$

This result proves that fourth order equations such as (4) may exhibit finite time blow up with wide oscillations. As shown by Gazzola & Pavani (2011), this *is not* the case in lower order differential equations.

The proof of this result is based on three main ingredients: a Liouville-type nonexistence result for critical growth biharmonic partial differential equations, the radial version of this pde, and a suitable change of variables in the corresponding ode.

As we have seen in Section 2, finite time blow up in more general situations is supported by numerical evidence.

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