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Radial symmetry of positive solutions to nonlinear polyharmonic Dirichlet problems

By *Elvise Berchio* at Torino, *Filippo Gazzola* at Milano, and *Tobias Weth* at Giessen

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Abstract. We extend the symmetry result of Gidas-Ni-Nirenberg [12] to semilinear polyharmonic Dirichlet problems in the unit ball. In the proof we develop a new variant of the method of moving planes relying on fine estimates for the Green function of the polyharmonic operator. We also consider minimizers for subcritical higher order Sobolev embeddings. For embeddings into weighted spaces with a radially symmetric weight function, we show that the minimizers are at least axially symmetric. This result is sharp since we exhibit examples of minimizers which do not have full radial symmetry.

1. Introduction and results

In their celebrated paper, Gidas-Ni-Nirenberg [12] proved that every positive smooth solution of the semilinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases}$$

is radially symmetric and decreasing in the radial variable. Here $\mathbf{B}$ is the unit ball of $\mathbb{R}^n$ $(n \geqq 2)$ and f is a locally Lipschitz function. The proof relies on the so-called moving plane technique due to Serrin [22], see also previous work by Alexandrov [2]. A variant of this technique can be used to study *higher order* problems such as

$$\text{(1.1)} \qquad \begin{cases} (-\Delta)^m u = f(u) & \text{in } \mathbf{B}, \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\mathbf{B}, \end{cases}$$

where $m \geqq 2$. Indeed, under these boundary conditions (usually named after *Navier*), problem (1.1) can be reduced to a second order semilinear elliptic *system*. If f is nondecreasing

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in u, then this reduced system is cooperative and Troy's symmetry result [25] applies, showing that any positive smooth solution of (1.1) is radially symmetric and radially decreasing.

In the present paper we prove radial symmetry of positive solutions to the corresponding polyharmonic problem under *Dirichlet* boundary conditions

$$\begin{cases} (-\Delta)^m u = f(u) & \text{in } \mathbf{B}, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{1.2}$$

Here $r = |x|$ denotes the radial variable. We point out that (1.2) cannot be reduced to a second order problem. We make the following general assumption:

$$f : [0, \infty) \to \mathbb{R} \text{ is a continuous, nondecreasing function with } f(0) \geqq 0. \tag{1.3}$$

Before stating our main result, we recall the weak formulation of (1.2) within the Sobolev space $\mathscr{H}^m := H_0^m(\mathbf{B})$, which is a Hilbert space with the scalar product

$$\langle u, v \rangle_m = \begin{cases} \int_{\mathbf{B}} \Delta^{m/2} u \Delta^{m/2} v \, dx, & m \text{ even}, \\ \int_{\mathbf{B}} \nabla \Delta^{(m-1)/2} u \nabla \Delta^{(m-1)/2} v \, dx, & m \text{ odd}, \end{cases}$$

for $u, v \in \mathscr{H}^m$. We denote the induced norm by $\| \cdot \|_m$. A nonnegative function $u \in \mathscr{H}^m \cap L^\infty(\mathbf{B})$ is called a *strong solution* of (1.2) if

$$\langle u, v \rangle_m = \int_{\mathbf{B}} f(u) v \, dx \quad \text{for all } v \in \mathscr{H}^m$$

(the integral exists since $u \in L^\infty(\mathbf{B})$ and f is continuous). By elliptic regularity, any such solution u is contained in $C^{2m-1,\alpha}(\bar{\mathbf{B}})$, and all partial derivatives of order less than m vanish on $\partial\mathbf{B}$. Moreover, if f is Hölder continuous, then $u \in C^{2m,\alpha}(\bar{\mathbf{B}})$ is a classical solution. Our main result is the following.

Theorem 1. *If* (1.3) *holds, then every strong positive solution* $u \in \mathscr{H}^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ *of* (1.2) *is radially symmetric and strictly decreasing in the radial variable.*

Theorem 1 deserves several immediate comments.

Remark 1. (i) The sign assumption on f is necessary in order to have the radial monotonicity of u, see the counterexample by Sweers [24].

(ii) Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)^m$ in $\mathbf{B}$. Combined with [14], Theorem 1, and [15], Theorem 2, Theorem 1 shows that, for $n > 2m$, the problem

$$\begin{cases} (-\Delta)^m u = \lambda u + u^{\frac{n+2m}{n-2m}} & \text{in } \mathbf{B}, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B} \end{cases}$$

admits a positive solution for any $\lambda \in (0, \lambda_1)$ if and only if $n \geqq 4m$. This brings further evidence for a conjecture by Pucci-Serrin [19]. We thank Hans-Christoph Grunau for this remark.

(iii) Combined with [8], Theorem 1, our Theorem 1 shows that if $m = 2$ and $f(u) = u^p$, with $0 < p < \dfrac{n+4}{n-4}$ and $p \neq 1$, then there exists a unique positive solution to (1.2).

(iv) The monotonicity assumption on f enters crucially in our arguments. On the other hand, if instead of (1.3) we assume that f is differentiable and satisfies $f'(s) < \lambda_1$ for every $s \geqq 0$, then it is easy to see that (1.2) admits at most one strong solution which is necessarily radially symmetric. Indeed, assuming by contradiction that $u, v \in \mathscr{H}^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ are different strong solutions of (1.2), we find

$$\langle u - v, u - v \rangle_m = \int_{\mathbf{B}} \big(f(u) - f(v)\big)(u - v)\, dx < \lambda_1 \int_{\mathbf{B}} (u - v)^2\, dx$$

contrary to the variational characterization of λ_1. In fact, symmetry is also ensured if $f'(s) < \lambda_2$ (see [9]) and uniqueness is guaranteed for sublinear f (see [10]).

(v) It will be evident from our proof that Theorem 1 is also valid for (1.2) with $f(u)$ replaced by the nonautonomous radial nonlinearity $f(|x|, u)$ provided that $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous, nonincreasing in the first variable and nondecreasing in the second one.

(vi) One of the crucial steps in the moving plane technique consists in comparing the solution u in a segment of the ball with its reflection u^r through the hyperplane which bounds the segment (see e.g. [12], Lemma 2.2). For second order problems the comparison follows from suitable versions of the maximum principle since $u^r \geqq u$ holds a priori on the boundary of this segment. This information however is not enough for higher order problems, and therefore the classical moving plane method fails for (1.2). In this paper, we employ a different technique to carry out the moving plane mechanism, using the integral representation of u in terms of the Green function of the polyharmonic operator $(-\Delta)^m$ in $\mathbf{B}$ under Dirichlet boundary conditions. This function is explicitly known since the pioneering work by Boggio [5] and has been widely studied more recently by Grunau-Sweers [17]. In the recent paper [11], Ferrero-Gazzola-Weth established monotonoticy properties of the biharmonic Green function with respect to reflections at hyperplanes. Here we generalize the estimates of [11] and add further inequalities which are essential to our approach.

To state our second main result, we recall that, by the Rellich-Sobolev embedding theorem, $\mathscr{H}^m$ is compactly embedded in $L^p(\mathbf{B})$ for $1 \leqq p < 2^*$. Here $2^* = 2^*(m)$ is the corresponding higher order critical exponent, i.e., $2^* = \dfrac{2n}{n-2m}$ for $n > 2m$ and $2^* = \infty$ for $n \leqq 2m$. The best constants for these embeddings are

$$S_p^m = \inf_{u \in \mathscr{H}^m \setminus \{0\}} \frac{\|u\|_m^2}{|u|_p^2}, \tag{1.4}$$

so these are the largest constants such that $S_p^m |u|_p^2 \leqq \|u\|_m^2$ for all $u \in \mathscr{H}^m$. Here and in the following, we denote by $|u|_p$ the usual L^p-norm of a function $u \in L^p(\mathbf{B})$. Since the embeddings $\mathscr{H}^m \hookrightarrow L^p(\mathbf{B})$ are compact for $1 \leqq p < 2^*$, there exists a minimizer u for the minimization problem (1.4), namely there exists a nontrivial function $u \in \mathscr{H}^m$ such that $S_p^m |u|_p^2 = \|u\|_m^2$. Since up to sign and normalization these minimizers are positive solutions of (1.2) with $f(u) = |u|^{p-2}u$, Theorem 1 implies that they are radial and monotone in the radial variable. For $m = 2$ this has already been proved in [11]. In the present paper we study the effect of radial weight functions on the symmetry of minimizers, complementing the results in [18], [20], [21] for the harmonic case $m = 1$. We assume that

$$\tau : \bar{\mathbf{B}} \to [0, \infty) \text{ is a continuous and a.e. positive radially symmetric function,} \tag{1.5}$$

and we study the minimization problem

$$S(m,p,\tau) = \inf_{u \in \mathscr{H}^m, u \not\equiv 0} \frac{\|u\|_m^2}{|\tau^{1/p}u|_p^2} = \inf_{u \in \mathscr{H}^m, u \not\equiv 0} \frac{\|u\|_m^2}{\left(\int_{\mathbf{B}} \tau(x)|u|^p\,dx\right)^{2/p}}. \tag{1.6}$$

By the same compactness argument as above, the infimum is attained for $p < 2^*$. Moreover, the minimizers of (1.6), when normalized such that $|\tau^{1/p}u|_p = 1$, are weak solutions of the boundary value problem

$$\begin{cases} (-\Delta)^m u = S(m,p,\tau)\tau(x)|u|^{p-2}u & \text{in } \mathbf{B}, \\ u = \dfrac{\partial u}{\partial r} = \cdots = \dfrac{\partial^{m-1} u}{\partial r^{m-1}} = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{1.7}$$

If τ is nonincreasing in the radial variable, then every weak positive solution of (1.7) is radial by Remark 1(iv). In the general case we have the following partial symmetry result for minimizers of (1.6).

Theorem 2. *If* (1.5) *holds, then every minimizer u of the minimization problem* (1.6) *is foliated Schwarz symmetric with respect to some unit vector e, i.e., it is axially symmetric with respect to the axis $\mathbb{R}e \subset \mathbb{R}^n$ and nonincreasing in the polar angle $\theta = \arccos \frac{x}{|x|} \cdot e$. Moreover, u is of one sign, and either u is radial or u is strictly decreasing in $\theta \in [0, \pi]$ for $0 < |x| < 1$.*

In the case $m = 1$, this has been proved in [18], see also [21]. The proof in [18] relies on the maximum principle for general second order operators and does not carry over to the polyharmonic Dirichlet problem. The approach in [21] is based on polarization, a simple two-point rearrangement for functions which is well defined in first order Sobolev spaces, spaces of continuous functions or L^p-spaces, see [3], [4], [6], [7], [21] and (4.6) below. However, for $m \geqq 2$, the polarization of an $\mathscr{H}^m$-function is not contained in $\mathscr{H}^m$ anymore. To prove Theorem 2, we first establish a duality principle (see Proposition 12) which reduces the minimization problem (1.6) to a maximization problem in $L^{p'}(\mathbf{B})$. We then study maximizers of the reduced problem with the help of polarization. We note that in [11] polarization was already applied in a different way to biharmonic problems.

By the results in [20], it is known that the symmetry statement of Theorem 2 is optimal in the second order case $m = 1$. More precisely, considering the weight functions

$\tau_\alpha : \mathbf{B} \to \mathbb{R}_+$, $\tau_\alpha(x) = |x|^\alpha$ for $\alpha > 0$, it is proved in [20] that some parameter values of α and p lead to nonradial minimizers. Here we note a similar statement for general m.

Theorem 3. *Suppose that $n > 2m$. Then for every $\alpha > 0$ there exists $p(\alpha) \in (2, 2^*)$ such that no minimizer of*

$$S(m,p,\alpha) := S(m,p,\tau_\alpha) = \inf_{u \in \mathscr{H}^m \setminus \{0\}} \frac{\|u\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |u|^p \, dx\right)^{2/p}} \tag{1.8}$$

is radially symmetric if $p \in [p(\alpha), 2^*)$.

The paper is organized as follows. In Section 2 we establish some inequalities for the polyharmonic Green function relative to Dirichlet boundary conditions and its derivative. In Section 3 we carry out the moving plane procedure and complete the proof of Theorem 1. In Section 4 we consider the minimization problem (1.6) and prove Theorem 2. Finally, in Section 5 we prove Theorem 3.

2. Green function inequalities

In this section we derive some pointwise inequalities for the Green function $G = G(x, y)$ of Δ^m on $\mathbf{B}$ relative to Dirichlet boundary conditions. For $x, y \in \mathbb{R}^n$, it is convenient to introduce the quantities

$$d(x,y) = |x-y|^2 \tag{2.1}$$

and

$$\theta(x,y) = \begin{cases} (1-|x|^2)(1-|y|^2) & \text{if } x, y \in \mathbf{B}, \\ 0 & \text{if } x \notin \mathbf{B} \text{ or } y \notin \mathbf{B}. \end{cases} \tag{2.2}$$

Then for $x, y \in \mathbf{B}$, $x \neq y$ we have the following representation due to Boggio, see [5], p. 126:

$$\begin{aligned} G(x,y) &= k_n^m |x-y|^{2m-n} \int_1^{\left(\frac{\theta(x,y)}{|x-y|^2}+1\right)^{1/2}} \frac{(z^2-1)^{m-1}}{z^{n-1}} \, dz \\ &= \frac{k_n^m}{2} |x-y|^{2m-n} \int_0^{\frac{\theta(x,y)}{|x-y|^2}} \frac{z^{m-1}}{(z+1)^{n/2}} \, dz = \frac{k_n^m}{2} H\big(d(x,y), \theta(x,y)\big). \end{aligned} \tag{2.3}$$

Here k_n^m is a positive constant, and

$$H : (0,\infty) \times [0,\infty) \to \mathbb{R}, \quad H(s,t) = s^{m-\frac{n}{2}} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^{n/2}} \, dz. \tag{2.4}$$

The following lemma is a direct consequence of Boggio's work [5], elliptic regularity (see [1]) and the estimates in [17].

Lemma 1. *Let $h \in L^\infty(\mathbf{B})$, and let $u \in \mathscr{H}^m$ satisfy*

$$\langle u, v\rangle_m = \int_{\mathbf{B}} hv\,dx \quad \text{for all } v \in \mathscr{H}^m, \tag{2.5}$$

i.e., u is a weak solution of $(-\Delta)^m u = h$ in $\mathbf{B}$ under Dirichlet boundary conditions. Then $u \in C^{2m-1,\alpha}(\overline{\Omega})$, and u satisfies

$$D^k u(x) = \int_{\mathbf{B}} D_x^k G(x,y)h(y)\,dy \quad \text{for every } x \in \overline{\mathbf{B}}, \tag{2.6}$$

where D^k stands for any partial derivative of order $|k| < 2m$. In particular, $D^k u \equiv 0$ on $\partial\mathbf{B}$ for $|k| \leqq m-1$.

We need the following inequalities for the function H defined in (2.4).

Lemma 2. *For all $s, t > 0$ we have*

$$\partial_s H(s,t) < 0, \quad \partial_t H(s,t) > 0, \quad \partial_s\partial_t H(s,t) < 0.$$

Proof. We compute

$$\partial_t H(s,t) = \frac{t^{m-1}}{(t+s)^{n/2}}, \quad \partial_s\partial_t H(s,t) = -\frac{nt^{m-1}}{2(t+s)^{n/2+1}}$$

and

$$\partial_s H(s,t) = \left(m - \frac{n}{2}\right) s^{m-\frac{n}{2}-1} \int_0^{\frac{t}{s}} \frac{z^{m-1}}{(z+1)^{n/2}}\,dz - \frac{t^m}{s(t+s)^{n/2}}.$$

Hence the last two inequalities follow. Also the first inequality follows in case $n \geqq 2m$ while in the remaining case $n < 2m$, we rewrite $\partial_s H(s,t)$ as

$$\begin{aligned}
\partial_s H(s,t) &= \left(m - \frac{n}{2}\right) \int_0^t \frac{x^{m-1}}{s(x+s)^{n/2}}\,dx - \frac{t^m}{s(t+s)^{n/2}} \\
&= \left(m - \frac{n}{2}\right) \int_0^t \frac{x^{m-\frac{n}{2}-1}}{s} \left(\frac{x}{x+s}\right)^{n/2} dx - \frac{t^m}{s(t+s)^{n/2}} \\
&< \left(m - \frac{n}{2}\right) \left(\frac{t}{t+s}\right)^{n/2} \int_0^t \frac{x^{m-\frac{n}{2}-1}}{s}\,dx - \frac{t^m}{s(t+s)^{n/2}} = 0.
\end{aligned}$$

This completes the proof. □

In the following, we will assume that G is extended in a trivial way to $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$, i.e., $G(x,y) = 0$ if $|x| \geqq 1$ or $|y| \geqq 1$. Then formula (2.3) is valid for all $x, y \in \mathbb{R}^n$, $x \neq y$. We introduce some more notation. For all $\lambda \in [0,1]$ we put

$$H_\lambda := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 < \lambda\}, \quad T_\lambda := \partial H_\lambda \quad \text{and} \quad \Sigma_\lambda := \mathbf{B} \cap H_\lambda. \tag{2.7}$$

Moreover, for any $x \in \mathbb{R}^n$ we denote by $\bar{x}$ its reflection through T_λ.

Lemma 3. *Let $\lambda \in [0,1)$. Then for every $x \in \mathbf{B} \cap T_\lambda$ and $y \in \Sigma_\lambda$ we have*

$$\partial_{x_1} G(x,y) < 0 \tag{2.8}$$

and

$$\partial_{x_1} G(x,y) + \partial_{x_1} G(x,\bar{y}) \leqq 0. \tag{2.9}$$

Moreover, the second inequality is strict if $\lambda > 0$.

Proof. For abbreviation, we put $d := d(x,y) = d(x,\bar{y}) > 0$, $\theta = \theta(x,y) > 0$ and $\bar{\theta} = \theta(x,\bar{y}) \geqq 0$. Then

$$\partial_{x_1} G(x,y) = k_n^m \big(\partial_s H(d,\theta)(x_1 - y_1) - \partial_t H(d,\theta)(1-|y|^2)x_1\big) < 0,$$

by Lemma 2, since $x_1 \geqq 0$ and $x_1 > y_1$. Moreover

$$\begin{aligned} &\partial_{x_1}[G(x,y) + G(x,\bar{y})] \\ &\quad = k_n^m \big(\partial_s H(d,\theta)(x_1 - y_1) + \partial_s H(d,\bar{\theta})\big(x_1 - (\bar{y})_1\big) \\ &\qquad - [\partial_t H(d,\theta)(1-|y|^2) + \partial_t H(d,\bar{\theta})(1-|\bar{y}|^2)]x_1\big) \\ &\quad \leqq k_n^m [\partial_s H(d,\theta) - \partial_s H(d,\bar{\theta})](x_1 - y_1), \end{aligned} \tag{2.10}$$

where we used Lemma 2 and the fact that $x_1 - \bar{y}_1 = y_1 - x_1$. Since moreover $\bar{\theta} \leqq \theta$ and $\partial_t \partial_s H < 0$ in $(0,\infty)^2$ by Lemma 2, we conclude that $\partial_s H(d,\theta) - \partial_s H(d,\bar{\theta}) \leqq 0$. Hence (2.9) follows from (2.10). Finally, if $\lambda > 0$, then we have the strict inequality $\bar{\theta} < \theta$, so that we obtain a strict inequality in (2.10). □

We conclude this section with a further consequence of Lemma 2 which was already obtained in [11] in the case $m = 2$. Arguing exactly as in the proof of [11], Lemma 3, we obtain

Lemma 4 (Ferrero-Gazzola-Weth [11]). *Let $\lambda \in (0,1)$. For all $x, y \in \Sigma_\lambda$, $x \neq y$, we have*

$$G(x,y) > \max\{G(x,\bar{y}), G(\bar{x},y)\}, \tag{2.11}$$

$$G(x,y) - G(\bar{x},\bar{y}) > |G(x,\bar{y}) - G(\bar{x},y)|. \tag{2.12}$$

3. The moving plane argument

In this section we complete the proof of Theorem 1. Throughout this section we consider a fixed strong positive solution $u \in \mathscr{H}^m(\mathbf{B}) \cap L^\infty(\mathbf{B})$ of (1.2), recalling from the introduction that $u \in C^{2m-1,\alpha}(\bar{\mathbf{B}})$. In the following we let H_λ, T_λ, Σ_λ and $\bar{x}$ be defined as in the previous section, see (2.7). We first provide three crucial estimates for directional derivatives which are related to the Hopf boundary lemma for second order problems.

Lemma 5 (Grunau-Sweers [16]). *Let $x_0 \in \partial\mathbf{B}$, and let μ be a unit vector with $\mu \cdot x_0 < 0$. Then* $\left(\frac{\partial}{\partial\mu}\right)^m u(x_0) > 0$.

Proof. The statement follows by noticing that the Green function vanishes precisely of order m on $\partial\mathbf{B}$, see [16], Theorem 3.2. □

In the following we extend u by zero outside of $\mathbf{B}$ so that it is defined on the whole of $\mathbb{R}^n$ and we put

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s > 0, \\ 0 & \text{if } s = 0, \end{cases} \tag{3.1}$$

so that $\tilde{f} : [0, \infty) \to [0, \infty)$ is still nondecreasing while it may lose continuity at $s = 0$.

For the next estimate we need the following technical result:

Lemma 6. *Let $0 < \lambda < 1$, and suppose that $u(x) \geqq u(\bar{x})$ for all $x \in \Sigma_\lambda$. Then, $f\big(u(y)\big) \geqq \tilde{f}\big(u(\bar{y})\big) \geqq 0$ for all $y \in \Sigma_\lambda$, and there exists a nonempty open set $\mathcal{O}_\lambda \subset \Sigma_\lambda$ such that $f\big(u(y)\big) > \tilde{f}\big(u(\bar{y})\big)$ or $\tilde{f}\big(u(\bar{y})\big) > 0$ for all $y \in \mathcal{O}_\lambda$.*

Proof. The inequalities $f\big(u(y)\big) \geqq \tilde{f}\big(u(\bar{y})\big) \geqq 0$ for all $y \in \Sigma_\lambda$ follow from assumption (1.3). For the second statement it then suffices to show that $f\big(u(y)\big) \not\equiv 0$ in Σ_λ since then one of the two above inequalities would become strict in a nonempty open set $\mathcal{O}_\lambda \subset \Sigma_\lambda$. For contradiction, if $f(u) \equiv 0$ in Σ_λ then the above inequalities would imply $f(u) \equiv 0$ in $\mathbf{B}$. In turn, this implies $(-\Delta u)^m \equiv 0$ which contradicts the positivity of u. □

Thanks to Lemmas 3 and 6 we prove

Lemma 7. *Let $0 < \lambda < 1$, and suppose that $u(x) \geqq u(\bar{x})$ for all $x \in \Sigma_\lambda$. Then $\frac{\partial u}{\partial x_1} < 0$ on $T_\lambda \cap \mathbf{B}$.*

Proof. For all $x \in T_\lambda \cap \mathbf{B}$ we have:

$$\begin{aligned} \frac{\partial u}{\partial x_1}(x) &= \int_{\mathbf{B}} \partial_{x_1} G(x, y) f\big(u(y)\big)\, dy \\ &= \int_{\Sigma_\lambda} \big[\partial_{x_1} G(x, y) f\big(u(y)\big) + \partial_{x_1} G(x, \bar{y}) \tilde{f}\big(u(\bar{y})\big)\big]\, dy. \end{aligned} \tag{3.2}$$

According to Lemma 6 we have $f\big(u(y)\big) \geqq \tilde{f}\big(u(\bar{y})\big) \geqq 0$ for all $y \in \Sigma_\lambda$ and two cases may occur. In the first case, $f\big(u(y)\big) > \tilde{f}\big(u(\bar{y})\big)$ for all $y \in \mathcal{O}_\lambda$; in this case, (3.2) yields

$$\frac{\partial u}{\partial x_1}(x) < \int_{\Sigma_\lambda} [\partial_{x_1} G(x, y) + \partial_{x_1} G(x, \bar{y})] \tilde{f}\big(u(\bar{y})\big)\, dy \leqq 0,$$

where in the first inequality we used (2.8) and in the second we used (2.9). In the second case, $\tilde{f}\big(u(\bar{y})\big) > 0$ for all $y \in \mathcal{O}_\lambda$; in this case,

$$\frac{\partial u}{\partial x_1}(x) \leqq \int_{\Sigma_\lambda} [\partial_{x_1} G(x,y) + \partial_{x_1} G(x,\bar{y})] \tilde{f}\big(u(\bar{y})\big)\, dy < 0,$$

where in the first inequality we used (2.9) which is strict for $\lambda > 0$. In any case, $\frac{\partial u}{\partial x_1}(x) < 0$, as claimed. □

The third estimate for directional derivatives reads:

Lemma 8. *Let* $0 < \lambda < 1$, *and suppose that* $\frac{\partial u}{\partial x_1} < 0$ *on* $T_\lambda \cap \mathbf{B}$. *Then, there exists* $\gamma \in (0, \lambda)$ *such that* $\frac{\partial u}{\partial x_1} < 0$ *on* $T_\ell \cap \mathbf{B}$ *for all* $\ell \in (\lambda - \gamma, \lambda)$.

Proof. For any $y \in \mathbb{R}^n$ and any $a > 0$ consider the hypercube centered at y, namely

$$\mathscr{U}_a(y) := \left\{ x \in \mathbb{R}^n;\ \max_{1 \leqq i \leqq n} |x_i - y_i| < a \right\}.$$

In view of Lemma 5, for any $x_0 \in T_\lambda \cap \partial\mathbf{B}$ we know that

$$(-1)^m \left(\frac{\partial}{\partial x_1}\right)^{m-1} \frac{\partial u}{\partial x_1}(x_0) = (-1)^m \left(\frac{\partial}{\partial x_1}\right)^m u(x_0) > 0.$$

Since from the boundary conditions we also know that $\left(\frac{\partial}{\partial x_1}\right)^k u(x_0) = 0$ for all $k = 0, \dots, m-1$, there exists $a = a(x_0) > 0$ such that

$$\frac{\partial u}{\partial x_1}(x) < 0 \quad \text{for all } x \in \mathscr{U}_a(x_0) \cap \mathbf{B}. \tag{3.3}$$

Then, by compactness of $T_\lambda \cap \partial\mathbf{B}$, there exists $\bar{a} > 0$ such that

$$\frac{\partial u}{\partial x_1}(x) < 0 \quad \text{for all } x \in A := \bigcup_{x_0 \in T_\lambda \cap \partial\mathbf{B}} \big(\mathscr{U}_{\bar{a}}(x_0) \cap \mathbf{B}\big). \tag{3.4}$$

Consider now the compact set $K := (T_\lambda \cap \mathbf{B}) \setminus A$ and for $d > 0$ consider $K_d := K - de_1$, where $e_1 = (1, 0, \dots, 0)$. Since by assumption $\frac{\partial u}{\partial x_1} < 0$ on K, there exists $\delta > 0$ such that

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{on } K_d \quad \text{for all } d \in [0, \delta]. \tag{3.5}$$

Let $\gamma := \min\{\bar{a}, \delta\} > 0$. Then, the statement follows from (3.4)–(3.5). □

We are now ready to start the moving plane procedure.

Lemma 9. *There exists* $\varepsilon > 0$ *such that for all* $\lambda \in [1 - \varepsilon, 1)$ *we have*

$$u(x) > u(\bar{x}) \quad \text{for } x \in \Sigma_\lambda, \qquad \frac{\partial u}{\partial x_1}(x) < 0 \quad \text{for } x \in T_\lambda \cap \mathbf{B}. \tag{3.6}$$

Proof. Note that $T_1 \cap \partial B = \{e_1\}$, where $e_1 = (1, 0, \dots, 0)$. By arguing as for (3.3), we infer that there exists $\varepsilon > 0$ such that $\frac{\partial u}{\partial x_1}(x) < 0$ for $x \in \mathbf{B} \backslash \Sigma_{1-2\varepsilon}$. In turn, from this we infer that (3.6) holds for all $\lambda \in [1-\varepsilon, 1)$. □

We consider

$$\Lambda := \left\{ \lambda \in (0,1); u(x) > u(\bar{x}) \text{ for all } x \in \Sigma_\lambda, \frac{\partial u}{\partial x_1}(x) < 0 \text{ for all } x \in T_\lambda \cap \mathbf{B} \right\} \tag{3.7}$$

and we prove

Lemma 10. *Let* Λ *be as in* (3.7). *Then,* $\Lambda = (0,1)$.

Proof. By Lemma 9, we know that $[1-\varepsilon, 1) \subset \Lambda$. Let $\bar{\lambda} \in [0,1)$ be the smallest number such that $(\bar{\lambda}, 1) \subset \Lambda$; the proof will be complete once we show that $\bar{\lambda} = 0$. By continuity we have

$$u(x) \geqq u(\bar{x}) \quad \text{for all } x \in \Sigma_{\bar{\lambda}}. \tag{3.8}$$

We argue by contradiction and assume that $\bar{\lambda} > 0$. By Lemma 7 and (3.8), we have

$$\frac{\partial u}{\partial x_1}(x) < 0 \quad \text{for all } x \in T_{\bar{\lambda}} \cap \mathbf{B}.$$

Hence, by Lemma 8,

$$\exists \gamma \in (0, \bar{\lambda}) \quad \text{such that} \quad \frac{\partial u}{\partial x_1} < 0 \quad \text{on } T_\ell \cap \mathbf{B} \quad \text{for all } \ell \in (\bar{\lambda} - \gamma, \bar{\lambda}). \tag{3.9}$$

Consider the function $\tilde{f}$ defined in (3.1); for all $x \in \Sigma_{\bar{\lambda}}$ we compute

$$\begin{aligned} u(x) - u(\bar{x}) &= \int_{\mathbf{B}} [G(x,y) - G(\bar{x},y)] f(u(y))\, dy \\ &= \int_{\Sigma_{\bar{\lambda}}} [G(x,y) - G(\bar{x},y)] f(u(y))\, dy \\ &\quad + \int_{\Sigma_{\bar{\lambda}}} [G(x,\bar{y}) - G(\bar{x},\bar{y})] \tilde{f}(u(\bar{y}))\, dy. \end{aligned} \tag{3.10}$$

According to Lemma 6, two cases may occur. If $f(u(y)) > \tilde{f}(u(\bar{y}))$ for all $y \in \mathcal{O}_\lambda$, then (2.11) and (3.10) yield

$$u(x) - u(\bar{x}) > \int_{\Sigma_{\bar{\lambda}}} [G(x,y) - G(\bar{x},y) + G(x,\bar{y}) - G(\bar{x},\bar{y})] \tilde{f}(u(\bar{y}))\, dy \geqq 0,$$

where the last inequality follows from (2.12). If $\tilde{f}(u(\bar{y})) > 0$ for all $y \in \mathcal{O}_\lambda$, then again (2.11), (2.12) and (3.10) yield

$$u(x) - u(\bar{x}) \geqq \int_{\Sigma_{\bar{\lambda}}^-} [G(x,y) - G(\bar{x},y) + G(x,\bar{y}) - G(\bar{x},\bar{y})]\tilde{f}\big(u(\bar{y})\big)\,dy > 0.$$

Hence, in any case we have shown that

$$u(x) > u(\bar{x}) \quad \text{for all } x \in \Sigma_{\bar{\lambda}}^-. \tag{3.11}$$

From (3.9) and (3.11) we deduce by a standard compactness argument that there exists $0 < \gamma_1 < \gamma$ such that

$$u(x) > u(\bar{x}) \quad \text{for all } x \in \Sigma_\ell \text{ and } \ell \in (\bar{\lambda} - \gamma_1, \bar{\lambda}]. \tag{3.12}$$

This, combined with (3.9), shows that $(\bar{\lambda} - \gamma_1, \bar{\lambda}] \subset \Lambda$, contrary to the characterization of $\bar{\lambda}$. □

Now we complete the *proof of Theorem* 1. Since $0 \in \partial\Lambda$ by Lemma 10, the continuity of u implies that

$$u(-x_1, x_2, \dots, x_n) \geqq u(x_1, x_2, \dots, x_n) \quad \text{for } x = (x_1, \dots, x_n) \in \mathbf{B} \text{ with } x_1 \geqq 0. \tag{3.13}$$

Since, for a given rotation $A \in \mathrm{SO}(n)$, the function $u_A := u \circ A$ is also a strong positive solution of (1.2), the inequality (3.13) also holds for u_A in place of u. This readily implies that u is symmetric with respect to every hyperplane containing the origin. Consequently, u is radially symmetric. Moreover we have $\frac{\partial u}{\partial r} < 0$ in $\mathbf{B}\backslash\{0\}$, since $\frac{\partial u}{\partial x_1} < 0$ in $\{x \in \mathbf{B},\ x_1 > 0\}$ by definition of Λ.

4. Partial symmetry of minimizers for the weighted minimization problem

In this section we prove Theorem 2. We fix a continuous and almost everywhere positive radially symmetric function $\tau : \bar{\mathbf{B}} \to \mathbb{R}_+$ and $p \in (1, 2^*)$, and we let $p' = \frac{p}{p-1}$ be the conjugate exponent of p. We first note the following.

Lemma 11. *Any minimizer $u \in \mathscr{H}^m$ for* (1.6) *belongs to $C^{2m-1,\alpha}(\bar{\mathbf{B}})$, and up to a reflection $u \mapsto -u$ it is strictly positive in $\mathbf{B}$.*

Proof. We may normalize u such that $|\tau^{1/p}u|_p = 1$, so that u is a weak solution of (1.7). Since the nonlinearity in (1.7) is continuous and has subcritical growth, elliptic regularity implies that $u \in C^{2m-1,\alpha}(\bar{\mathbf{B}})$. The proof of the strict positivity of u is precisely the same as in [11], Section 2, where the statement was proved in the biharmonic case $m = 2$ for the nonlinearity without weight. □

Let $\mathscr{G} : L^{p'}(\mathbf{B}) \to \mathscr{H}^m(\mathbf{B})$ denote the solution operator for the polyharmonic equation under Dirichlet boundary conditions defined by

$$\langle \mathscr{G}w, v\rangle_m := \int_{\mathbf{B}} wv \quad \text{for } w \in L^{p'}(\mathbf{B}),\ v \in \mathscr{H}^m. \tag{4.1}$$

We note that, if $w \in L^\infty(\mathbf{B})$, then Lemma 1 yields the usual integral representation for $\mathscr{G}w$ in terms of the Green function, namely,

$$[\mathcal{G}w](x) = \int_{\mathbf{B}} G(x,y)w(y)\,dy \quad \text{for every } x \in \mathbf{B}. \tag{4.2}$$

Consider the maximization problem corresponding to

$$\Lambda(m,p,\tau) = \sup_{w \in L^{p'}(\mathbf{B}),\, w \not\equiv 0} \frac{\int_{\mathbf{B}} \big(\mathcal{G}(\tau w)\big)(x)\tau(x)w(x)\,dx}{|\tau^{1/p'}w|_{p'}^2}. \tag{4.3}$$

Note that, if w is a maximizer for (4.3), then (4.2) and the positivity of G imply that $|w|$ is also a maximizer. We need the following *duality principle*.

Lemma 12. (i) $\Lambda(m,p,\tau) = \dfrac{1}{S(m,p,\tau)}$.

(ii) *If* $u \in \mathcal{H}^m$ *is a positive minimizer for* (1.6) *with* $\int_{\mathbf{B}} \tau(x)u^p\,dx = 1$, *then* $w = u^{p-1} \in L^{p'}(\mathbf{B})$ *is a maximizer for* (4.3).

(iii) *If* $w \in L^{p'}(\mathbf{B})$ *is a nonnegative maximizer for* (4.3) *with* $\int_{\mathbf{B}} \tau(x)w^{p'}\,dx = 1$, *then* $u = w^{p'-1} \in \mathcal{H}^m$ *is a minimizer for* (1.6).

Proof. For abbreviation, we put $S := S(m,p,\tau)$ and $\Lambda := \Lambda(m,p,\tau)$. Let $u \in \mathcal{H}^m$ be a positive minimizer for (1.6) with $\int_{\mathbf{B}} \tau(x)u^p\,dx = 1$. Then u is a solution of problem (1.7). Consequently, $u = S\mathcal{G}(\tau u^{p-1})$, and therefore $S\mathcal{G}(\tau w) = w^{\frac{1}{p-1}}$ for $w = u^{p-1}$. Multiplying both the sides of this equality by τw and integrating over $\mathbf{B}$, we obtain

$$S\int_{\mathbf{B}} \big(\mathcal{G}(\tau w)\big)(x)\tau(x)w(x)\,dx = \int_{\mathbf{B}} \tau(x)w^{p'}\,dx = \int_{\mathbf{B}} \tau(x)u^p\,dx = 1,$$

hence

$$\Lambda \geqq \frac{\int_{\mathbf{B}} \big(\mathcal{G}(\tau w)\big)(x)\tau(x)w(x)\,dx}{|\tau^{1/p'}w|_{p'}^2} = \frac{1}{S}. \tag{4.4}$$

Next let $w \in L^{p'}(\mathbf{B})$ be a nonnegative maximizer for (4.3) with $\int_{\mathbf{B}} \tau(x)w^{p'}\,dx = 1$. By the corresponding Euler-Lagrange equation in weak form,

$$\int_{\mathbf{B}} \big(\mathcal{G}(\tau w)\big)(x)\tau(x)z(x)\,dx = \Lambda\int_{\mathbf{B}} \tau(x)w^{p'-1}z\,dx \quad \text{for every } z \in L^{p'}(\mathbf{B}),$$

which implies that $\mathcal{G}(\tau w) = \Lambda w^{p'-1}$ almost everywhere in $\mathbf{B}$. Therefore, setting $u = w^{p'-1} = \dfrac{1}{\Lambda}\mathcal{G}(\tau w) \in \mathcal{H}^m$, we obtain by (4.1)

$$\Lambda\|u\|_m^2 = \Lambda\langle u,u\rangle_m = \langle \mathcal{G}(\tau w), w^{p'-1}\rangle_m = \int_{\mathbf{B}} \tau(x)w^{p'}\,dx = \int_{\mathbf{B}} \tau(x)u^p\,dx = 1,$$

so that

$$S \leqq \frac{\|u\|_m^2}{|\tau^{1/p}u|_p^2} = \frac{1}{\Lambda}. \tag{4.5}$$

Now (i) is a consequence of (4.4) and (4.5). But then the first inequality in (4.4) must be an equality, and (ii) follows. Similarly, the first inequality in (4.5) must be an equality, and (iii) follows. □

Next, we consider the set $\mathscr{H}$ of all closed half-spaces in $\mathbb{R}^n$ such that $0 \in \partial H$. For $H \in \mathscr{H}$, we let $\sigma_H : \mathbb{R}^n \to \mathbb{R}^n$ denote the reflection at the boundary ∂H of H. For simplicity, we also put $\bar{x} = \sigma_H(x)$ for $x \in \mathbb{R}^N$ when the underlying half space H is understood. For a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, we define the *polarization* v_H of v relative to H by

$$v_H(x) = \begin{cases} \max\{v(x), v(\bar{x})\}, & x \in H, \\ \min\{v(x), v(\bar{x})\}, & x \in \mathbb{R}^n \backslash H. \end{cases} \tag{4.6}$$

We note the following simple and useful property.

Lemma 13. *Let $w : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and let $H \in \mathscr{H}$. Then, for a.e. $x \in \mathbb{R}^n$ we have $w_H(\bar{x}) - w(\bar{x}) = w(x) - w_H(x)$.*

Proof. By definition we have $w_H(x) + w_H(\bar{x}) = w(x) + w(\bar{x})$ for a.e. $x \in H$, which proves the statement. □

We also need the following property of the Green function G.

Lemma 14. *Let $H \in \mathscr{H}$, and let $x, y \in H$, $x \neq y$. Then*

$$G(x, y) = G(\bar{x}, \bar{y}) \geqq G(x, \bar{y}) = G(\bar{x}, y) \tag{4.7}$$

and the inequality is strict if $x, y \in \operatorname{int}(\mathbf{B} \cap H)$.

Proof. By continuity, it suffices to consider $x, y \in \operatorname{int}(\mathbf{B} \cap H)$. Consider the squared distance function defined in (2.1), the θ function defined in (2.2) and observe that

$$d(x, y) = d(\bar{x}, \bar{y}) < d(x, \bar{y}) = d(\bar{x}, y) \quad \text{for } x, y \in \operatorname{int}(H) \tag{4.8}$$

and

$$\theta(x, y) = \theta(x, \bar{y}) = \theta(\bar{x}, y) = \theta(\bar{x}, \bar{y}), \tag{4.9}$$

so the equalities in (4.7) follow directly from the representation (2.3). Moreover, Lemma 2, (4.8) and (4.9) imply that $H\big(d(x, y), \theta(x, y)\big) > H\big(d(x, \bar{y}), \theta(x, y)\big)$. Hence $G(x, y) > G(x, \bar{y})$ by (2.3), as claimed. □

In the following, for every function $w : \mathbf{B} \to \mathbb{R}$ we let w also denote the corresponding trivial extension ($w \equiv 0$ outside of $\mathbf{B}$) to $\mathbb{R}^n$. Lemmas 13 and 14 enable us to compare double "convolutions" of functions with the corresponding double "convolutions" of their polarizations.

Lemma 15. *Let $w \in L^{p'}(\mathbf{B})$, and let $H \in \mathscr{H}$. Then:*

$$\int_{\mathbf{B}\times\mathbf{B}} G(x,y)\tau(x)\tau(y)w(x)w(y)\,dx\,dy \leqq \int_{\mathbf{B}\times\mathbf{B}} G(x,y)\tau(x)\tau(y)w_H(x)w_H(y)\,dx\,dy, \tag{4.10}$$

where equality holds if and only if $w = w_H$ *or* $w \circ \sigma_H = w_H$ *a.e. in* $\mathbf{B}$.

Proof. Setting

$$A(g,h) := \int_{\mathbf{B}\times\mathbf{B}} G(x,y)\tau(x)\tau(y)g(x)h(y)\,dx\,dy \quad \text{for } g,h \in L^{p'}(\mathbf{B})$$

and using Lemmas 13 and 14 we find

$$\begin{aligned}
A(w_H,w_H) - A(w_H,w) &= \int_{\mathbf{B}\times\mathbf{B}} G(x,y)\tau(x)\tau(y)w_H(x)[w_H(y)-w(y)]\,dx\,dy \\
&= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} + \int_{\mathbf{B}\cap H\times(\mathbf{B}\setminus H)} + \int_{(\mathbf{B}\setminus H)\times\mathbf{B}\cap H} + \int_{(\mathbf{B}\setminus H)\times(\mathbf{B}\setminus H)} \dots\,dx\,dy \\
&= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} \big[w_H(x)\big(G(x,y)-G(x,\bar{y})\big) + w_H(\bar{x})\big(G(\bar{x},y)-G(\bar{x},\bar{y})\big)\big] \\
&\quad \times \tau(x)\tau(y)[w_H(y)-w(y)]\,dx\,dy \\
&= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} [G(x,y)-G(x,\bar{y})]\tau(x)\tau(y) \\
&\quad \times [w_H(x)-w_H(\bar{x})][w_H(y)-w(y)]\,dx\,dy.
\end{aligned} \tag{4.11}$$

Again by Lemmas 13 and 14 and with the same decomposition of the domain of integration we find

$$\begin{aligned}
A(w_H,w) - A(w,w) &= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} [G(x,y)-G(\bar{x},y)]\tau(x)\tau(y)[w_H(x)-w(x)] \\
&\quad \times [w(y)-w(\bar{y})]\,dx\,dy \\
&= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} [G(x,y)-G(x,\bar{y})]\tau(x)\tau(y)[w_H(y)-w(y)] \\
&\quad \times [w(x)-w(\bar{x})]\,dx\,dy.
\end{aligned} \tag{4.12}$$

Combining (4.11) and (4.12), we obtain

$$\begin{aligned}
A(w_H,w_H) - A(w,w) &= \int_{\mathbf{B}\cap H\times\mathbf{B}\cap H} [G(x,y)-G(x,\bar{y})]\tau(x)\tau(y)[w_H(y)-w(y)] \\
&\quad \times [w_H(x)-w_H(\bar{x})+w(x)-w(\bar{x})]\,dx\,dy \geqq 0,
\end{aligned}$$

since $G(x,y) - G(x,\bar{y}) \geqq 0$ by Lemma 14,

$$w_H(y) - w(y) \geqq 0 \quad \text{and} \quad w_H(x) - w_H(\bar{x}) + w(x) - w(\bar{x}) \geqq 0 \quad \text{for } x \in H.$$

Hence (4.10) follows. Moreover, putting $U_1 := \{x \in \mathbf{B} \cap H : w(x) > w(\bar{x})\}$ and $U_2 := \{y \in \mathbf{B} \cap H : w_H(y) > w(y)\}$, we find that

$$\begin{aligned} &A(w_H, w_H) - A(w, w) \\ &\quad \geqq \int_{U_1 \times U_2} [G(x, y) - G(x, \bar{y})]\tau(x)\tau(y)[w_H(y) - w(y)] \\ &\quad\quad \times [w_H(x) - w_H(\bar{x}) + w(x) - w(\bar{x})]\, dx\, dy, \end{aligned}$$

and the right-hand side is positive if and only if $U_1 \times U_2$ has positive Lebesgue measure. Hence we conclude that equality holds in (4.10) if and only if $|U_1| = 0$ or $|U_2| = 0$, i.e., if and only if $w \circ \sigma_H = w_H$ or $w = w_H$ a.e. in $\mathbf{B}$. □

The next step is a comparison statement for minimizers of (1.6):

Lemma 16. *Let u be a (positive) minimizer for* (1.6)*. Then for each $H \in \mathscr{H}$ one of the following is true*:

(i) $u > u \circ \sigma_H$ *in* $\mathbf{B} \cap \operatorname{int}(H)$.

(ii) $u < u \circ \sigma_H$ *in* $\mathbf{B} \cap \operatorname{int}(H)$.

(iii) $u \equiv u \circ \sigma_H$ *in* $\mathbf{B}$.

Proof. By Lemma 11 we may assume that u is positive, and we may normalize u such that $\int_{\mathbf{B}} \tau(x) u^p\, dx = 1$. By Proposition 12, $w = u^{p-1}$ is a maximizer for (4.3). Since τ is a radial function, a straightforward computation (see e.g. [21]) shows that $|\tau^{1/p'} w_H|_{p'} = |\tau^{1/p'} w|_{p'}$. Hence Lemma 15 implies that $w = w_H$ or $w \circ \sigma_H = w_H$ a.e. in $\mathbf{B}$. Since u is continuous, we conclude that

$$u \geqq u \circ \sigma_H \quad \text{in } \mathbf{B} \cap H \tag{4.13}$$

or

$$u \leqq u \circ \sigma_H \quad \text{in } \mathbf{B} \cap H. \tag{4.14}$$

We first consider (4.13), and we suppose in addition that $u(x_0) > u(\sigma_H(x_0))$ for some $x_0 \in H$. Then, by continuity, $u > u \circ \sigma_H$ in a subset of $\mathbf{B} \cap H$ of positive measure. Using Lemma 14 we estimate for every $x \in \mathbf{B} \cap \operatorname{int}(H)$

$$\begin{aligned} u(x) - u(\bar{x}) &= \int_{\mathbf{B}} [G(x, y) - G(\bar{x}, y)]\tau(y) u^{p-1}(y)\, dy \\ &= \int_{\mathbf{B} \cap H} \big([G(x, y) - G(\bar{x}, y)]\tau(y) u^{p-1}(y) - [G(x, \bar{y}) - G(\bar{x}, \bar{y})]\tau(y) u^{p-1}(\bar{y})\big)\, dy \\ &= \int_{\mathbf{B} \cap H} [G(x, y) - G(\bar{x}, y)]\tau(y)[u^{p-1}(y) - u^{p-1}(\bar{y})]\, dy > 0. \end{aligned}$$

Hence we obtain (i). Similarly, assuming (4.14) and $u(x_0) < u(\sigma_H(x_0))$ for some $x_0 \in \mathbf{B} \cap H$, we obtain (ii). It follows that one of the cases (i)–(iii) occurs, as claimed. □

For any unit vector $e \in \mathbb{R}^n$ we now define

$$\mathscr{H}(e) := \{H \in \mathscr{H} : e \in \mathrm{int}(H)\}.$$

We will prove Theorem 2 with the help of the following characterization.

Lemma 17. *Let $e \in \mathbb{R}^n$ be a unit vector. A continuous function $v : \mathbb{R}^n \to \mathbb{R}$ with compact support is foliated Schwarz symmetric with respect to e if and only if $v = v_H$ for every $H \in \mathscr{H}(e)$.*

Proof. For nonnegative v this follows immediately from [21], Lemma 2.6. But, as noted subsequently in [4], Lemma 2.4, there is no need to assume nonnegativity. Alternatively, the characterization also follows from [6], Lemma 4.2. □

We may now complete the *proof of Theorem* 2. Let u be a positive minimizer for (1.6). Take $x_0 \in \mathbf{B}\backslash\{0\}$ with $u(x_0) = \max\{u(x) x \in \mathbf{B}, |x| = |x_0|\}$, and put $e = \dfrac{x_0}{|x_0|}$. Then Lemma 16 implies that

$$u = u_H \quad \text{for every } H \in \mathscr{H}(e).$$

Hence u is foliated Schwarz symmetric with respect to e by Lemma 17. Therefore we can write $u = u(r, \theta)$, where $r = |x|$ and $\theta = \arccos \dfrac{x}{|x|} \cdot e$. It remains to prove that

$$\text{(4.15)} \qquad \text{either } u \text{ is radial, or } u(r,\theta) \text{ is strictly decreasing in } \theta \in (0,\pi) \text{ for } 0 < r \leqq 1.$$

We follow the argument in [13], p. 204. We already know that no half-space $H \subset \mathscr{H}_e$ satisfies property (ii) of Lemma 16. Moreover, if property (i) of this lemma holds for all half-spaces $H \subset \mathscr{H}_e$, then $u(r,\theta)$ is strictly decreasing in $\theta \in (0,\pi)$ for $0 < r \leqq 1$. It remains to consider the case where property (iii) of Lemma 16 holds for some $H_0 \subset \mathscr{H}_e$. Let $0 < \theta_0 < \pi/2$ be the angle formed by e and the hyperplane ∂H_0. Let $e_0 = \sigma_{H_0}(e)$. Then $\arccos(e_0 \cdot e) = 2\theta_0$. Moreover, (iii) implies that $u(re_0) = u(re)$ for $0 \leqq r \leqq 1$. Since u is nonincreasing in the angle $\theta \in (0,\pi)$, we conclude that $u(r,\theta) = u(r,0)$ for all $\theta \leqq 2\theta_0$. From Lemma 16 we then deduce that (iii) holds for all $H \subset \mathscr{H}_e$ for which the angle between e and H is less than $2\theta_0$. Then, by the same argument as before, $u(r,\theta) = u(r,0)$ for all $\theta \leqq \min\{4\theta_0, \pi\}$. Arguing successively, in a finite number of steps we obtain $u(r,\theta) = u(r,0)$ for all $\theta \leqq \pi$. This shows that u is radial. We therefore conclude (4.15), as claimed.

5. Nonradial minimizers

In this section we prove Theorem 3. To this end, we introduce the subspace of radial functions $\mathscr{H}_r^m := \{u \in \mathscr{H}^m;\ u \text{ radially symmetric}\}$ and consider the related minimization problem:

$$(5.1) \qquad S_r(m,p,\alpha) := \inf_{u \in \mathscr{H}_r^m \setminus \{0\}} \frac{\|u\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |u|^p\,dx\right)^{2/p}}.$$

Now Theorem 3 can be rephrased in the following way.

Theorem 4. *For any $\alpha > 0$ there exists $p(\alpha) \in (2, 2^*)$ such that $S_r(m,p,\alpha) > S(m,p,\alpha)$ for $p \in [p(\alpha), 2^*]$. Hence, for $p \in [p(\alpha), 2^*)$, no minimizer of* (1.8) *is radially symmetric.*

For the proof we need two lemmas which deal with the *limit case* where $p = 2^*$.

Lemma 18. *Let $S_* := S_{2^*}^m$ be as in* (1.4)*. Then, $S(m, 2^*, \alpha) = S_*$, and the infimum in* (1.8) *for $p = 2^*$ is not attained.*

Proof. Since $|x|^\alpha < 1$ in $\mathbf{B}$, we have $S(m, 2^*, \alpha) \geqq S_*$. Let $\{u_k\} \subset \mathscr{C}_0^\infty(\mathbf{B})$ be a minimizing sequence for S_*. Fix $y \in \mathbf{B}$ with $|y| > 1/2$ and consider

$$v_k \in \mathscr{C}_0^\infty(\mathbf{B}), \quad v_k(x) := \begin{cases} u_k\left(\dfrac{x-y}{1-|y|}\right) & \text{if } x \in B_{1-|y|}(y), \\ 0 & \text{if } x \in \mathbf{B} \setminus B_{1-|y|}(y). \end{cases}$$

We then have $\|v_k\|_m^2 = (1-|y|)^{n-2m} \|u_k\|_m^2$ and

$$\begin{aligned} \int_{\mathbf{B}} |x|^\alpha |v_k(x)|^{2^*}\,dx &= (1-|y|)^n \int_{\mathbf{B}} \big|z(1-|y|) + y\big|^\alpha |u_k(z)|^{2^*}\,dz \\ &\geqq (1-|y|)^n (2|y|-1)^\alpha \int_{\mathbf{B}} |u_k(z)|^{2^*}\,dz, \end{aligned}$$

hence

$$S(m, 2^*, \alpha) \leqq \frac{\|v_k\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |v_k|^{2^*}\,dx\right)^{2/2^*}} \leqq (2|y|-1)^{-2\alpha/2^*} \frac{\|u_k\|_m^2}{|u_k|_{2^*}^2} = (2|y|-1)^{-2\alpha/2^*} \big(S_* + o(1)\big).$$

Consequently, $S_* \leqq S(m, 2^*, \alpha) \leqq (2|y|-1)^{-2\alpha/2^*} S_*$ for any $|y| > 1/2$. Since y can be chosen arbitrarily close to $\partial \mathbf{B}$, we conclude that $S(m, 2^*, \alpha) = S_*$.

Now suppose by contradiction that $S(m, 2^*, \alpha)$ is attained at some $u \in \mathscr{H}^m$. Then

$$S_* \leqq \frac{\|u\|_m^2}{|u|_{2^*}^2} < \frac{\|u\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |u|^{2^*}\,dx\right)^{2/2^*}} = S(m, 2^*, \alpha),$$

contrary to the equality we just proved. Hence $S(m, 2^*, \alpha)$ is not attained, as claimed. □

Lemma 19. *The infimum $S_r(m, 2^*, \alpha)$ in* (5.1) *is attained.*

Proof. Let $\{u_k\} \subset \mathscr{H}_r^m$ be a minimizing sequence for (5.1), normalized such that $\|u_k\|_m = 1$ for all k. Up to a subsequence, we may assume that $u_k \to u$ a.e. and $u_k \rightharpoonup u$ weakly in $\mathscr{H}_r^m$ for some $u \in \mathscr{H}_r^m$ such that $\|u\|_m \leqq 1$. We claim that

$$\int_{\mathbf{B}} |x|^\alpha |u_k|^{2^*}\,dx \to \int_{\mathbf{B}} |x|^\alpha |u|^{2^*}\,dx \quad \text{as } k \to \infty. \tag{5.2}$$

Indeed, if $r \in (0,1)$, then by boundedness of $\|u_k - u\|_m$ and by [23], Radial Lemma 1, we know that there exists $C(r) > 0$ such that $|u_k(x) - u(x)| \leqq C(r)$ for all $x \in \mathbf{B}\backslash B_r(0)$. Then, we may apply Lebesgue Theorem to obtain $\|u_k - u\|_{L^{2^*}(\mathbf{B}\backslash B_r(0))} \to 0$. Hence,

$$\begin{aligned}\left|\int_{\mathbf{B}} |x|^\alpha (|u_k|^{2^*} - |u|^{2^*})\,dx\right| &\leqq \int_{B_r(0)} |x|^\alpha (|u_k|^{2^*} + |u|^{2^*})\,dx + \int_{\mathbf{B}\backslash B_r(0)} |x|^\alpha \big||u_k|^{2^*} - |u|^{2^*}\big|\,dx \\ &\leqq r^\alpha \int_{B_r(0)} (|u_k|^{2^*} + |u|^{2^*})\,dx + o(1) \\ &\leqq r^\alpha S_*^{2^*/2}(\|u_k\|_m^{2^*} + \|u\|_m^{2^*}) + o(1) \leqq 2r^\alpha S_*^{2^*/2} + o(1).\end{aligned}$$

By arbitrariness of r, we obtain (5.2). Consequently, we have

$$\frac{\|u\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |u|^{2^*}\,dx\right)^{2/2^*}} \leqq \liminf_{k\to\infty} \frac{\|u_k\|_m^2}{\left(\int_{\mathbf{B}} |x|^\alpha |u_k|^{2^*}\,dx\right)^{2/2^*}} = S_r(m, 2^*, \alpha),$$

and hence u is a minimizer for (5.1). □

We may now complete the *proof of Theorem* 4. Lemmas 18 and 19 immediately imply that $S_r(m, 2^*, \alpha) > S(m, 2^*, \alpha)$. Since $p \mapsto S_r(m, p, \alpha)$ is continuous and $p \mapsto S(m, p, \alpha)$ is upper semicontinuous as $p \to 2^*$, there exists $p(\alpha) \in (2, 2^*)$ such that $S_r(m, p, \alpha) > S(m, p, \alpha)$ for all $p \in [p(\alpha), 2^*]$.

Note added in proof. After the paper was accepted, we learned that the moving plane method has already been applied to some integral equations in papers of Chang and Yang (Math. Res. Lett. **4** (1997), 91–102), Y. Y. Li (J. Eur. Math. Soc. **6** (2004), 153–180), Birkner, López-Mimbela and Wakolbinger (Ann. Inst. H. Poincaré Anal. Non Lin. **22** (2005), 83–97) and Chen, Li and Ou (Comm. Pure Appl. Math. **59** (2006), 330–343). Our method has common points with some of these papers but also contains new features. In particular, it deals with very general nonlinearities and completely reduces the problem to Green function inequalities.

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Dipartimento di Matematica, Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: elvise.berchio@unito.it

Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 33, 20133 Milano, Italy
e-mail: filippo.gazzola@polimi.it

Mathematisches Institut, Universität Giessen, Arndtstr. 2, 35392 Giessen, Germany
e-mail: tobias.weth@math.uni-giessen.de

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