Critical growth problems for polyharmonic operators

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Abstract

We prove that critical growth problems for polyharmonic operators admit nontrivial solutions for a wide class of lower order perturbations of the critical term. The results highlight the phenomenon of bifurcation of the critical dimensions discovered by Pucci-Serrin [21]; moreover, we show that another bifurcation seems to appear for "nonresonant" dimensions.

1 Introduction

In a recent paper [11] an orthogonalizing technique has been developed for the study of critical growth problems in semilinear elliptic equations of second order; such technique is based on variational methods: to assure that the considered minimax levels are at suitable energy values, certain approximating functions having disjoint support with the Sobolev concentrating functions are constructed. We call Sobolev concentrating functions some truncations of the positive radial entire functions which achieve the best constant in Sobolev inequalities, see [24]; in the celebrated paper by Brezis-Nirenberg [5] such functions have been found responsible for the lack of compactness of the problem, see also [7].

In this paper we show that this orthogonalizing technique also applies to higher order semilinear elliptic equations: we consider the problem

$$\begin{cases} (-\Delta)^K u = g(x,u) + |u|^{K_*-2}u & \text{in } \Omega \\ D^k u = 0 \quad \text{on } \partial\Omega & k = 0, ..., K-1 \end{cases} \tag{1.1}$$

where $K \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ ($n \geq 2K+1$) is an open bounded domain with smooth boundary, $K_* = \frac{2n}{n-2K}$ is the critical Sobolev exponent (the largest p for which the imbedding $H_K := H_0^K(\Omega) \subset L^p(\Omega)$ is continuous) and $g(\cdot, s)$ has subcritical growth at infinity (i.e. $\lim_{|s|\to\infty} \frac{g(\cdot,s)}{|s|^{K_*-1}} = 0$); with our notation $1_* = 2^* = \frac{2n}{n-2}$. Problem (1.1) has been studied by many authors even in the case $K > 1$, see e.g. [4, 9, 12, 13, 19, 21]. We endow the Hilbert space H_K with the scalar product

$$(u,v) = \begin{cases} \int_\Omega (\Delta^M u)(\Delta^M v) & \text{if} \quad K = 2M \\ \int_\Omega (\nabla \Delta^M u)(\nabla \Delta^M v) & \text{if} \quad K = 2M+1 \end{cases}$$

and we denote by $\|\cdot\|_K$ the corresponding norm; $|\cdot|_p$ denotes the $L^p(\Omega)$-norm, $p \in [1,\infty]$. Let $\Sigma_K := \{\lambda_j^K\}_{j\in\mathbf{N}}$ denote the spectrum of $(-\Delta)^K$ relative to the homogeneous Dirichlet problem in Ω; it is well-known that $\Sigma_K \subset \mathrm{I\!R}^+$ and that $\lambda_j^K \to +\infty$ as $j \to \infty$: in the sequel we will omit the index K on λ_j. Define the functional $J : H_K \to \mathrm{I\!R}$ by

$$J(u) = \frac{1}{2}\|u\|_K^2 - \int_\Omega G(x,u)dx - \frac{1}{K_*}|u|_{K_*}^{K_*} ,$$

where $G(x,s) = \int_0^s g(x,t)dt$; if g is continuous then $J \in C^1(H_K, \mathrm{I\!R})$ and the critical points of J correspond to solutions of equation (1.1). Since the imbedding $H_K \subset L^{K_*}(\Omega)$ is not compact the functional J does not satisfy the PS condition: in [5] it is shown that the corresponding functional only satisfies the PS condition at certain energy levels. The failure of the PS condition is not only technical; by using a generalized Pohožaev identity, Pucci-Serrin [20] proved non-existence results: in particular they show that if $\lambda < 0$ and Ω is star-shaped then the problem

$$\begin{cases} (-\Delta)^K u = \lambda u + |u|^{K_*-2}u & \text{in } \Omega \\ D^k u = 0 \qquad \text{on } \partial\Omega & k = 0, ..., K-1 \end{cases} \tag{1.2}$$

only admits the trivial solution $u \equiv 0$.

Under minimal assumptions on the lower order term g we will prove that (1.1) admits nontrivial solutions: the proofs are obtained by slight modifications of those in [2, 7, 11]; however, our results highlight some basic facts.

First, they bring further evidence to the "strange" behaviour of the critical dimensions. One of the basic problems concerning (1.2) is to determine the smallest constant $\lambda^* \geq 0$ for which the relation $\lambda \in (\lambda^*, \lambda_1)$ implies the existence of a radial positive solution of (1.2) when Ω is the unit ball of $\mathrm{I\!R}^n$. Pucci-Serrin [21] call *critical dimensions* the dimensions for which $\lambda^* > 0$ and conjecture, for a given $K \geq 1$, that the critical dimensions are $n = 2K+1, ..., 4K-1$: this conjecture was already proved for $K = 1$ (see [5]), next it has been proved for $K = 2$ (see [9, 21]), subsequently for $K = 3, 4$ (see [12]) and finally it is "almost" proved for all $K \geq 1$ (see [13]). We will prove that if $K \geq 1$ and $n \geq 4K$ then (1.2) admits a nontrivial solution for all $\lambda \in (0, \lambda_1)$ independently of the geometry of Ω; we recall that, as noticed by Grunau [13], one cannot expect the existence of positive solutions of (1.2) in general domains: the domain Ω needs to have a positive Green function. We also deal with a more general subcritical term $g(x,u)$: it turns out that the critical dimensions are exactly the dimensions for which an additional assumption on the lower order term g is needed in the nonresonant case for (1.1), see (2.4) below; in other words, for these dimensions, we cannot prove that the bifurcation branch of nontrivial solutions of (1.2) arising from an eigenvalue λ_{j+1} can be extended to all $\lambda \in (\lambda_j, \lambda_{j+1})$.

Second, they show that a bifurcation of dimensions also seems to appear in the *limiting dimension* corresponding to $n = 4$ when $K = 1$. In a remarkable paper [6], it has been proved that if $K = 1$ and $n \geq 4$ then (1.2) admits nontrivial solutions for all $\lambda > 0$; however, as was pointed out to us by H. Brezis, the arguments of [6] do not apply in the case $n = 4$ and $\lambda \in \Sigma_1$: therefore, we can say that $n = 4$ is a limiting nonresonant dimension for (1.2) when $K = 1$, see also [11] where a different approach is used. In the general case $K \geq 1$ we encounter the same difficulty when $n \in [4K, (2+2\sqrt{2})K)$, see Corollary 2.2 below; using the above terminology, for these nonresonant dimensions we prove that the bifurcation branch of nontrivial solutions of (1.2) arising from λ_{j+1} can be extended to all $\lambda \in (\lambda_j, \lambda_{j+1})$ but we do not know if it reaches $\lambda = \lambda_j$. Theorem 2.6 below states that in higher dimensions, that is $n > (2+\sqrt{2})K$, the bifurcation branch "crosses" λ_j. Although the different behaviour of certain dimensions seems to be strictly related with the tools involved in our proofs, we believe that some hidden reason might exist: to formulate a conjecture we also extend the result of [7] to the case $K \geq 2$.

Finally, our results show that the orthogonalizing method introduced in [11] may be used to solve more general semilinear elliptic problems at critical growth having a linking variational structure (in this case $\lambda_1 \leq \liminf_{s\to 0} \frac{2G(x,s)}{s^2} < \infty$); we believe that with minor changes the same arguments apply to elliptic problems with variable coefficients such as the Yamabe [25] generalized problem

$$-\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) = g(x,u) + |u|^{4/(n-2)}u \ .$$

2 Results and remarks

We assume that g is subcritical in the following sense:

$$\begin{aligned}
&g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function such that} \\
&\forall \varepsilon > 0 \quad \exists a_\varepsilon \in L^{\frac{2n}{n+2K}} \ , \quad |g(x,s)| \leq a_\varepsilon(x) + \varepsilon |s|^{\frac{n+2K}{n-2K}} \quad \forall s \in \mathbb{R} \ , \quad \text{for a.e. } x \in \Omega \ ;
\end{aligned} \tag{2.1}$$

moreover, if $G(x,s) := \int_0^s g(x,t)dt$, the perturbation $g(x,u)$ may change sign, provided that

$$G(x,s) \geq 0 \qquad \text{for a.e. } x \in \Omega \ , \quad \forall s \in \mathbb{R} \ . \tag{2.2}$$

In fact, assumption (2.2) can be weakened in "high" dimensions so that also the primitive G is allowed to change sign, see [11]: here, we will not go deeply into this analysis.

We first consider the case where the functional J has a mountain-pass geometrical structure [1]: some non-existence results for (1.1) can be found in [13, 21] while for existence results concerning (1.2) in the case $K = 2$ we refer to [9]. We will assume that there exist $\delta > 0$ and $\sigma > 0$ such that

$$G(x,s) \leq \frac{1}{2}(\lambda_1 - \sigma)s^2 \qquad \text{for a.e. } x \in \Omega \ , \quad \forall |s| \leq \delta \ . \tag{2.3}$$

In the cases $n = 2K+1, ..., 4K-1$ we need a growth condition at infinity, that is,

$$\begin{aligned}
&\exists \Omega_0 \subset \Omega, \ \Omega_0 \text{ open, nonempty, such that} \\
&\lim_{s\to+\infty} \frac{G(x,s)}{s^{4K/(n-2K)}} = +\infty \quad \text{uniformly w.r.t. } x \in \Omega_0 \ ;
\end{aligned} \tag{2.4}$$

if in Ω_0 the term G is a pure power, that is $G(x,s)=s^p$, then the above assumption requires for $g(s)$ a growth strictly greater than $s^{(6K-n)/(n-2K)}$: this is precisely condition (A_1) in [19] for the case $K=2$. Compare also this result with the non-existence result of Theorem 4 in [21].

For all the other dimensions (namely, $n\ge 4K$) we will assume that

$$\begin{aligned}&\exists\Omega_0\subset\Omega,\ \Omega_0 \text{ open, nonempty,}\quad \exists b>a>0\ ,\ \exists\mu>0\quad\text{such that}\\ &G(x,s)\ge\mu\quad\text{for a.e. } x\in\Omega_0\ ,\ \forall s\in[a,b]\ .\end{aligned}\tag{2.5}$$

With the above assumptions we will prove the following

Theorem 2.1 *For $n\ge 4K+1$ assume (2.1)-(2.3) (2.5), for $n=4K$ also assume that μ in (2.5) is large enough, for $n=2K+1,...,4K-1$ assume (2.1)-(2.4); then (1.1) admits a nontrivial solution.*

Besides being the "boundary" between the critical and what we call nonresonant dimensions, the dimension $4K$ is very particular as it involves logarithms in the standard estimates relative to the Sobolev concentrating functions, see (3.16) below: in the above statement it is required that the constant μ is large enough and this assumption is necessary as shown by Theorem 2.3 in [5]; note also that by reasoning as in [2] one could instead require that there exist $a,\mu>0$ such that $G(x,s)\ge\mu s^2$ for all $s\in[0,a]$ and for a.e. $x\in\Omega_0$.

From Theorem 2.1 we immediately deduce the following

Corollary 2.1 *If $n\ge 4K$ then (1.2) admits a nontrivial solution for all $\lambda\in(0,\lambda_1)$.*

For the (conjectured) critical dimensions $n=2K+1,...,4K-1$ we can only prove the existence of a nontrivial solution for values of λ "near" to λ_1: let S_K denote the best constant of the imbedding $H_K\subset L^{K_*}(\Omega)$ (see [24]) and define the number

$$\Lambda_K^\Omega:=S_K|\Omega|^{-2K/n}\ ;\tag{2.6}$$

we will prove

Theorem 2.2 *Let $n=2K+1,...,4K-1$; then for all $\lambda\in(\lambda_1-\Lambda_K^\Omega,\lambda_1)$ (1.2) admits a nontrivial solution.*

In particular, Theorem 2.2 establishes the existence of a constant $\lambda^*=\lambda^*(n,K,\Omega)\in[0,\lambda_1-\Lambda_K^\Omega]$ such that (1.2) admits a nontrivial solution whenever $\lambda\in(\lambda^*,\lambda_1)$. The constant Λ_2^Ω appears in the statement of Theorem 1.5 in [9]; when $\Omega=B$ (unit ball of $\mathbb{R}^n$), the same constant Λ_K^B has been determined in [13] and a lower bound for $\lambda^*(2K+1,K,B)$ is given in Theorem 2 in [21].

Next we consider a more general case of non-resonance near the origin, i.e. the case where J has a linking structure [22]: we assume that there exist $j \geq 1$, $\delta > 0$ and $\sigma > 0$ such that

$$\begin{cases} \frac{1}{2}(\lambda_j + \sigma)s^2 \leq G(x,s) \leq \frac{1}{2}(\lambda_{j+1} - \sigma)s^2 & \text{for a.e. } x \in \Omega \ , \quad \forall |s| \leq \delta \\ G(x,s) \geq \frac{1}{2}(\lambda_j + \sigma)s^2 - \frac{1}{K_*}|s|^{K_*} & \text{for a.e. } x \in \Omega \ , \quad \forall s \neq 0 \ . \end{cases} \tag{2.7}$$

Then, we will prove

Theorem 2.3 *For $n \geq 4K$ assume (2.1) (2.2) (2.7), for $n = 2K+1, ..., 4K-1$ assume also (2.4); then (1.1) admits a nontrivial solution.*

In the case of resonance near the origin we assume that there exist $\delta, \sigma > 0$ and $\mu \in (0, 1/K_*)$ such that

$$\begin{cases} \frac{1}{2}\lambda_j s^2 \leq G(x,s) \leq \frac{1}{2}(\lambda_{j+1} - \sigma)s^2 & \text{for a.e. } x \in \Omega \ , \quad \forall |s| \leq \delta \\ G(x,s) \geq \frac{1}{2}\lambda_j s^2 - (\frac{1}{K_*} - \mu)|s|^{K_*} & \text{for a.e. } x \in \Omega \ , \quad \forall s \in \mathbb{R} \ ; \end{cases} \tag{2.8}$$

moreover, we require that

$$\begin{array}{l} \exists \Omega_0 \subset \Omega, \ \Omega_0 \text{ open, nonempty, such that} \\ \lim\limits_{s \to +\infty} \dfrac{G(x,s)}{s^{8Kn/(n^2-4K^2)}} = +\infty \quad \text{uniformly w.r.t. } x \in \Omega_0 \ . \end{array} \tag{2.9}$$

We will prove the following

Theorem 2.4 *Let $n \geq 2K+1$ and assume (2.1), (2.2), (2.8), (2.9); then (1.1) admits a nontrivial solution.*

To study the phenomenon of "bifurcation of the dimensions" we restrict our attention to the simpler problem (1.2): let $[\cdot]$ denote the entire part function and let $n_K := [(2 + 2\sqrt{2})K]$; if $G(x,s) = \frac{1}{2}\lambda s^2$ condition (2.9) is satisfied for $n \geq n_K + 1$, therefore, from Theorems 2.1, 2.3 and 2.4 we obtain

Corollary 2.2 *If $n \geq n_K + 1$ then equation (1.2) admits a nontrivial solution $\forall \lambda > 0$.*
If $n = 4K, ..., n_K$ then equation (1.2) admits a nontrivial solution $\forall \lambda > 0$ such that $\lambda \notin \Sigma_K$.

By using an idea of [7], for all dimensions we estimate the left neighborhood of λ_{j+1} to which the bifurcation branch of nontrivial solutions of (1.2) extends, and we generalize the statement of Theorem 2.2:

Theorem 2.5 *Let $n \geq 2K+1$, Λ_K^Ω as in (2.6) and let M_{j+1} be the multiplicity of λ_{j+1}; if $\lambda \in (\lambda_{j+1} - \Lambda_K^\Omega, \lambda_{j+1})$ then (1.2) admits at least M_{j+1} (pairs of) nontrivial solutions.*

Finally, we prove that if $n \geq n_K + 1$ then the bifurcation branch starting from the eigenvalue $\lambda_{j+1} \in \Sigma_K$, $j \geq 1$, can be extended up to a left neighborhood of λ_j:

Theorem 2.6 *Let $n \geq n_K + 1$; then, $\forall \lambda_j \in \Sigma_K$, there exists $\delta_j > 0$ such that if $\lambda \in (\lambda_j - \delta_j, \lambda_j)$ equation (1.2) admits at least M_j+1 (pairs of) nontrivial solutions (here M_j denotes the multiplicity of λ_j).*

Some remarks are now in order. In spite of the above results, we believe that the bifurcation branch *always* crosses λ_j ($j \geq 1$) even if in some cases this may happen at high energy levels: this fact is suggested (when $K = 1$) by the results in domains having some symmetries [8, 10]; variational methods as used in [5, 6, 7, 11] may not work because the PS condition does not hold for high energies, see Lemma 3.1 below. Theorem 2.5 states, in particular, that if $\lambda_{j+1} - \lambda_j < \Lambda_K^\Omega$ then (1.2) admits a solution for $\lambda = \lambda_j$; however, if $\lambda_{j+1} - \lambda_j \geq \Lambda_K^\Omega$ and if $n \leq n_K$ then (1.2), with $\lambda = \lambda_j$, may not have solutions at energy below the compactness threshold: we conjecture that there exist some j for which the bifurcation branches of nontrivial solutions of (1.2) behave as in Figure 1 below. Let

$$J_\lambda(u) := \frac{1}{2}\|u\|_K^2 - \frac{\lambda}{2}|u|_2^2 - \frac{1}{K_*}|u|_{K_*}^{K_*}$$

then I_λ denotes the infimum of the nontrivial critical levels of J_λ

$$I_\lambda := \inf\{J_\lambda(u);\ u \neq 0,\ J'_\lambda(u) = 0\}\ :$$

as $J'_\lambda(u)[u] = 0$ implies $J_\lambda(u) = \frac{K}{n}|u|_{K_*}^{K_*}$, we know that $I_\lambda \geq 0$. In other words, I_λ is a measure of the distance (the L^{K^*}-norm) between the origin and the "nearest" nontrivial critical point of J_λ: such distance tends to 0 at resonance, i.e. $\lim_{\lambda \to \lambda_{j+1}^-} I_\lambda = 0$, $\forall j \in \mathbb{N}$. In Figure 1 the branch (I) corresponds to the (conjectured) critical dimensions ($n = 2K+1, ..., 4K-1$), the branch (II) corresponds to the (conjectured) limiting dimensions ($n = 4K, ..., n_K$) and the branch (III) corresponds to higher dimensions ($n \geq n_K + 1$).

Figure 1

Struwe ([23], Chapter III, Remark following Theorem 2.6) states that the result in [7] implies that for $\lambda \to \infty$ (and $n \geq 3$) the number of solutions of equation (1.2) (when $K = 1$) tends to infinity, arguing that by Weyl's formula the number of eigenvalues in $(\lambda, \lambda + \Lambda_1^\Omega)$ tends to infinity. As was pointed out by B. Ruf, this is not the case: consider the cube $C := (0, \pi)^3$, which has Dirichlet eigenvalues of the form $\lambda_j = k_1^2 + k_2^2 + k_3^2$ $(k_1, k_2, k_3 \in \mathbb{N})$, i.e. $\min_j(\lambda_{j+1} - \lambda_j) = 1$. A numerical calculation gives $\Lambda_1^C = S_1/\pi^2 = 3(4\pi)^{-2/3} \approx 0.555$: hence, if $\lambda \in (\lambda_j, \lambda_{j+1} - \Lambda_1^C)$ the theorem in [7] yields no solution at all and the situation of Figure 1 may occur for infinitely many eigenvalues λ_j. On the other hand, it is well-known [15, 16, 17, 18] that small perturbations of the domain may or may not preserve multiple eigenvalues: in domains having only simple eigenvalues the remark of Struwe may apply; the existence, the finiteness or the infiniteness of the number of eigenvalues for which the behaviour of Figure 1 holds are then strictly related to the geometry of the domain Ω and this number seems to be unstable with respect to perturbations of the domain.

3 Proof of the results

Let e_i be an L^2 normalized eigenvector relative to $\lambda_i \in \Sigma_K$, let $H^- := \mathrm{span}\{e_i;\ i \leq j\}$, let $H^+ := (H^-)^\perp$ and let $P_j : H \to H^-$ denote the orthogonal projection. Let Ω_0 be as in (2.4) and (2.9) (we may assume that $0 \in \Omega_0 \subset \Omega$); for $m \in \mathbb{N}$ large enough (so that $B_{1/m} \subset \Omega_0$), define

$$\zeta_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m} \\ Q_K(m|x|) & \text{if } x \in B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } x \in \Omega \setminus B_{2/m}\ , \end{cases}$$

where Q_K is the polynomial of degree $2K-1$ for which $\zeta_m \in C^{K-1}(\Omega)$ (for instance, $Q_2(m|x|) = -2m^3|x|^3 + 9m^2|x|^2 - 12m|x| + 5$). Clearly, $|D^k\zeta_m|_\infty \leq cm^k$ for $k = 0, ..., K$: then, if we let $e_i^m := \zeta_m e_i$ we have $e_i^m \to e_i$ in H_K (as $m \to \infty$) and, by reasoning as in [11], there exists $c_j > 0$ such that for large enough m we have

$$\max_{\{u \in H_m^-;\ \int u^2 = 1\}} \|u\|_K^2 \leq \lambda_j + c_j m^{2K-n} \tag{3.1}$$

where $H_m^- := \mathrm{span}\{e_i^m;\ i \leq j\}$; moreover, for large enough m we also have

$$P_j H_m^- = H^- \qquad \text{and} \qquad H_m^- \oplus H^+ = H\ . \tag{3.2}$$

Note that if (2.1), (2.2) and either (2.7) or (2.8) hold then

$$\exists \alpha, \rho > 0 \quad \text{such that} \quad J(u) \geq \alpha \qquad \forall u \in \partial B_\rho \cap H^+\ . \tag{3.3}$$

Next, we introduce the family of functions

$$U_\varepsilon^K(x) := c_n \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(n-2K)/2} \qquad (\varepsilon > 0)\ , \tag{3.4}$$

which solve the problem $(-\Delta)^K u = |u|^{K_*-2}u$ in $\mathbb{R}^n$ and which satisfy $\|U_\varepsilon^K\|_K^2 = |U_\varepsilon^K|_{K_*}^{K_*} = S_K^{n/2K}$ for all $\varepsilon > 0$, see [14, 24]; in fact,

$$c_n = c(n,K) = \left\{ \prod_{i=1-K}^{K} (n-2i) \right\}^{(n-2K)/4K}$$

but we omit the index K. Take a positive cut-off function $\eta \in C_c^\infty(B_{1/m})$ such that $\eta \equiv 1$ in $B_{1/2m}$, $\eta \leq 1$ in $B_{1/m}$ and $|D^k\eta|_\infty \leq cm^k$ for $k = 0, ..., K$; consider the sequence of Sobolev concentrating functions $u_\varepsilon^K(x) := \eta(x)U_\varepsilon^K(x)$: by reasoning as in [13] (see also [5, 11, 23]) we obtain

$$\begin{gathered}(\varepsilon \to 0) \Rightarrow \left(\|u_\varepsilon^K\|_K^2 \leq S_K^{n/2K} + c\varepsilon^{n-2K},\ |u_\varepsilon^K|_{K_*}^{K_*} \geq S_K^{n/2K} - c\varepsilon^n \right) \\ (m \to \infty,\ \varepsilon = \varepsilon_m = o(\tfrac{1}{m})) \Rightarrow \left(\|u_m^K\|_K^2 \leq S_K^{n/2K} + c(\varepsilon m)^{n-2K},\ |u_m^K|_{K_*}^{K_*} \geq S_K^{n/2K} - c(\varepsilon m)^n \right)\end{gathered} \tag{3.5}$$

where $u_m^K = u_{\varepsilon_m}^K$. For all $\varepsilon > 0$, $m \in \mathbb{N}$ and $v \in H_m^- \oplus \mathbb{R}^+\{u_\varepsilon^K\}$ there exist $r \geq 0$ and $w \in H_m^-$ such that $v = w + ru_\varepsilon^K$ and

$$\mathrm{supp}(u_\varepsilon^K) \cap \mathrm{supp}(w) = \emptyset\ : \tag{3.6}$$

hence, $J(v) = J(w) + J(ru_\varepsilon^K)$ with $J(w) \leq \max_{u\in H_m^-} J(u) =: \omega_m \to 0$ (as $m \to \infty$); furthermore, by (2.1) and (3.5) we get

$$J(ru_\varepsilon^K) \leq r^2\|u_\varepsilon^K\|_K^2 - r^{K_*}(|u_\varepsilon^K|_{K_*}^{K_*} + o(1)) \leq S_K^{n/2K}(r^2 - r^{K_*}) + o(r^{K_*}) \quad \text{as } r \to \infty$$

which becomes negative if $r = R > \rho$ (ρ as in (3.3)) and R is large enough. By (3.6) we can choose R so large that if $Q_m^\varepsilon := [(B_R \cap H_m^-) \oplus [0,R]\{u_\varepsilon^K\}]$ then $\max_{v\in\partial Q_m^\varepsilon} J(v) \leq \omega_m \to 0$ as $m \to \infty$; note that

$$\forall m, \varepsilon > 0 \qquad Q_m^\varepsilon \subset B_{2R}\ . \tag{3.7}$$

By (3.2), $\partial B_\rho \cap H^+$ and ∂Q_m^ε link and by standard minimax methods [22] we obtain a PS sequence, i.e. a sequence $\{u_m\} \subset H_K$ such that $J(u_m) \to c$ (for a certain $c \in \mathbb{R}$) and $J'(u_m) \to 0$ in H^{-K}. We do not concern ourselves here with the relative compactness of PS sequences: with our assumptions on the lower order term g we cannot prove that the critical levels of J are positive and we cannot find a "range of compactness" as in [5]. However, by reasoning as in [11] one can prove

Lemma 3.1 *Assume (2.1) and let $\{u_m\} \subset H_K$ be a PS sequence for J; then there exists $u \in H_K$ such that $u_m \rightharpoonup u$ up to a subsequence and $J'(u) = 0$. Moreover, if $J(u_m) \to c$ with $c \in (0, \frac{K}{n}S_K^{n/2K})$ then $u \not\equiv 0$ and u is a nontrivial solution of (1.1); finally, if $\frac{1}{2}g(x,s) - G(x,s) + \frac{K}{n}|s|^{K_*} \geq 0$ for all $s \in \mathbb{R}$ and for a.e. $x \in \Omega$ then such a sequence is precompact.*

Hence, Theorems 2.3 and 2.4 follow if the above linking arguments yield a PS sequence for J at a level strictly lower than $\frac{K}{n}S_K^{n/2K}$. Let $\Gamma := \{h \in C(\bar{Q}_m^\varepsilon, H);\ h(u) = u,\ \forall u \in \partial Q_m^\varepsilon\}$: we obtain a PS sequence for J at level

$$c = \inf_{h\in\Gamma} \max_{u\in Q_m^\varepsilon} J(h(u))\ ;$$

as the identity $Id \in \Gamma$, Theorems 2.3 and 2.4 follow if we prove that for ε small enough, there results

$$\max_{u\in Q_m^\varepsilon} J(u) < \frac{K}{n} S_K^{n/2K} \ . \tag{3.8}$$

Proof of Theorem 2.3. Assume (2.1)-(2.7) if $n \geq 4K$ and (2.1)-(2.4) if $n = 2K+1, ..., 4K-1$ and take m large enough so that (σ as in (2.7), c_j as in (3.1))

$$\sigma > c_j m^{2K-n} \ . \tag{3.9}$$

We claim that (3.8) holds; if not, as the set $\{u \in Q_m^\varepsilon;\ J(u) \geq 0\}$ is compact, for all $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that, for $v_\varepsilon := w_\varepsilon + t_\varepsilon u_\varepsilon^K$, we have

$$\frac{1}{2}\|v_\varepsilon\|_K^2 - \int_\Omega G(x, v_\varepsilon) - \frac{1}{K_*}|v_\varepsilon|_{K_*}^{K_*} \geq \frac{K}{n} S_K^{n/2K} \ . \tag{3.10}$$

By (3.7), the sequences $\{t_\varepsilon\} \subset \mathbb{R}^+$ and $\{w_\varepsilon\} \subset H_m^-$ are bounded: up to subsequences we have $t_\varepsilon \to t_0 \geq 0$. By (2.7), (3.1) and (3.9) we get

$$J(w_\varepsilon) \leq 0 \tag{3.11}$$

and since G has subcritical growth at infinity, we have $\lim_{\varepsilon\to 0} \int_\Omega G(x, t_\varepsilon u_\varepsilon^K) = 0$; therefore, by (3.5) and (3.6) we obtain

$$J(v_\varepsilon) \leq J(t_\varepsilon u_\varepsilon^K) \leq S_K^{n/2K} \left(\frac{t_0^2}{2} - \frac{t_0^{K_*}}{K_*} \right) + o(1) \ . \tag{3.12}$$

Moreover, $\frac{x^2}{2} - \frac{x^{K_*}}{K_*} < \frac{K}{n}$ $\forall x \geq 0,\ x \neq 1$ and if $t_0 \neq 1$ then (3.12) contradicts (3.10); hence, if (3.10) holds

$$\lim_{\varepsilon\to 0} t_\varepsilon = 1 \ . \tag{3.13}$$

By using (3.5), as $\varepsilon \to 0$, we obtain

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon^K\|_K^2 - \frac{1}{K_*}|t_\varepsilon u_\varepsilon^K|_{K_*}^{K_*} \leq \frac{K}{n} S_K^{n/2K} + \frac{1}{2}\left(t_\varepsilon^2 - 1 - \frac{n-2K}{n}(t_\varepsilon^{K_*} - 1) \right) S_K^{n/2K} + c\varepsilon^{n-2K} \ ;$$

since $\max_{x\geq 0}\{x^2 - 1 - \frac{n-2K}{n}(x^{K_*} - 1)\} = 0$, as $\varepsilon \to 0$ we have

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon^K\|_K^2 - \frac{1}{K_*}|t_\varepsilon u_\varepsilon^K|_{K_*}^{K_*} \leq \frac{K}{n} S_K^{n/2K} + c\varepsilon^{n-2K} \ . \tag{3.14}$$

If $n = 2K+1, ..., 4K-1$, for ε small enough we have $B_\varepsilon \subset B_{1/2m} \subset \Omega_0$, then by (2.2) and (3.4) we get

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^K) \geq \int_{B_\varepsilon} G\left(x, c_n t_\varepsilon \frac{\varepsilon^{(n-2K)/2}}{[\varepsilon^2 + |x|^2]^{(n-2K)/2}} \right) \ ;$$

by (2.4) there exist $\bar{s} > 0$ and an increasing function $\varphi = \varphi(s)$ with $\lim_{s\to+\infty} \varphi(s) = +\infty$ such that

$$\forall s \geq \bar{s} \qquad G(x, s) \geq \varphi(s) s^{4K/(n-2K)} \qquad \text{for a.e. } x \in \Omega_0 \ . \tag{3.15}$$

Next, note that if ε is small enough we have (recall that (3.13) holds)

$$c_n t_\varepsilon \frac{\varepsilon^{(n-2K)/2}}{[\varepsilon^2+|x|^2]^{(n-2K)/2}} > \bar{s} \ , \quad \forall x \in B_\varepsilon \ ;$$

hence, by (3.15),

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^K) \geq c_n \int_{B_\varepsilon} \varphi(c\varepsilon^{(2K-n)/2})\varepsilon^{-2K} \geq c\varphi(c\varepsilon^{(2K-n)/2})\varepsilon^{n-2K} \ .$$

Consider now the case $n \geq 4K$: if ε is small enough, as in [11] one finds $c_1 > 0$ such that $G(x, t_\varepsilon u_\varepsilon^K) \geq c[U_\varepsilon^K(x)]^2$ if $|x| \in (c_1\sqrt{\varepsilon}, 1/2m)$; therefore,

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^K) \geq c\varepsilon^{n-2K} \int_{c_1\sqrt{\varepsilon}}^{1/2m} r^{4K-n-1}dr \geq c\varepsilon^{n-2K} \begin{cases} \varepsilon^{(4K-n)/2} & \text{if } n \geq 4K+1 \\ |\log\varepsilon| & \text{if } n = 4K \ . \end{cases} \tag{3.16}$$

Therefore, for all $n \geq 2K+1$ there exists a function $\tau = \tau(\varepsilon)$ such that $\lim_{\varepsilon\to 0} \tau(\varepsilon) = +\infty$ and such that for ε small enough we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^K) \geq \tau(\varepsilon) \cdot \varepsilon^{n-2K} \ ; \tag{3.17}$$

hence, by (3.6), (3.11) and (3.14), if we take ε small enough we get

$$J(v_\varepsilon) \leq \frac{1}{2}\|t_\varepsilon u_\varepsilon^K\|_K^2 - \int_\Omega G(x, t_\varepsilon u_\varepsilon^K) - \frac{1}{K_*}|t_\varepsilon u_\varepsilon^K|_{K_*}^{K_*} \leq \frac{K}{n}S_K^{n/2K} + \varepsilon^{n-2K}(c - \tau(\varepsilon)) < \frac{K}{n}S_K^{n/2K}$$

which contradicts (3.10), and (3.8) follows. □

Proof of Theorem 2.4. Assume (2.1), (2.2), (2.8), (2.9); here, we need to take into account the dependence on m: we let $m \to \infty$ and we choose

$$\varepsilon_m = m^{-(n+2K)/2K} \tag{3.18}$$

so that $\varepsilon_m = o(1/m)$ and (3.5) applies. We claim that (3.8) holds; by contradiction assume that for all m large enough there exists $v_m \in Q_m$ (here $Q_m = Q_m^\varepsilon$ and we denote u_m^K, w_m, v_m instead of $u_\varepsilon^K, w_\varepsilon, v_\varepsilon$) such that

$$\frac{1}{2}\|v_m\|_K^2 - \int_\Omega G(x, v_m) - \frac{1}{K_*}|v_m|_{K_*}^{K_*} \geq \frac{K}{n}S_K^{n/2K} \ . \tag{3.19}$$

Let $v_m = w_m + t_m u_m^K$: if (3.19) holds, then by (3.7) the sequences $\{t_m\}$ and $\{w_m\}$ satisfy again

$$t_m \geq c > 0 \qquad \text{and} \qquad \|w_m\|_K \leq c \ . \tag{3.20}$$

By (2.9) there exists an increasing function τ such that $\lim_{x\to+\infty} \tau(x) = +\infty$ satisfying $G(x,s) \geq \tau(s) \cdot s^{8Kn/(n^2-4K^2)}$ for a.e. $x \in \Omega_0$ and for all $s \geq 0$; therefore, by (3.18) and (3.20)

$$\begin{aligned} \int_\Omega G(x, t_m u_m^K) &\geq c\int_0^{\varepsilon_m} \left(\frac{\varepsilon_m^{(n-2K)/2}}{(\varepsilon_m^2+r^2)^{(n-2K)/2}}\right)^{8Kn/(n^2-4K^2)} \cdot \tau\left(c\frac{\varepsilon_m^{(n-2K)/2}}{(\varepsilon_m^2+r^2)^{(n-2K)/2}}\right) \cdot r^{n-1}dr \\ &\geq c(\varepsilon_m^{(2K-n)/2})^{8Kn/(n^2-4K^2)} \cdot \tau(c\varepsilon_m^{(2K-n)/2}) \cdot \varepsilon_m^n \\ &\geq cm^{n(2K-n)/2K} \cdot \phi(m) \end{aligned}$$

with $\phi(m) \to \infty$ as $m \to \infty$. Hence, by (3.5), (3.20) and by replacing (3.18) in (3.14) we obtain

$$J(t_m u_m^K) \le \frac{K}{n} S_K^{n/2K} - c m^{n(2K-n)/2K} \phi(m) \ . \tag{3.21}$$

To estimate $J(w_m)$ note that by (2.8) and (3.1) we get (for large m)

$$J(w_m) \le \frac{1}{2}\|w_m\|_K^2 - \frac{\lambda_j}{2}|w_m|_2^2 - \mu |w_m|_{K_*}^{K_*} \le \frac{c_j}{2}|w_m|_2^2 \cdot m^{2K-n} - c|w_m|_2^{K_*} \ ;$$

since (3.20) holds and since $\max_{x\ge 0}\{\frac{1}{2}c_j m^{2K-n} \cdot x^2 - c \cdot x^{2n/(n-2K)}\} = cm^{n(2K-n)/2K}$, if m is large enough we have

$$J(w_m) \le c m^{n(2K-n)/2K} \ . \tag{3.22}$$

By (3.6), (3.21) and (3.22) we finally obtain

$$J(v_m) = J(t_m u_m) + J(w_m) \le \frac{K}{n} S_K^{n/2K} - c m^{n(2K-n)/2K} \left(\phi(m) - 1\right) < \frac{K}{n} S_K^{n/2K} \ , \tag{3.23}$$

for m sufficiently large: this contradicts (3.19) and completes the proof of Theorem 2.4. □

Proof of Theorem 2.5. Let $V := \text{span}\{e_i;\ i \le j+1\}$ and $H^+ := \overline{\text{span}\{e_i;\ i \ge j+1\}}$; then $\dim V - \text{codim} H^+ = M_{j+1}$. Reasoning as in Lemma 2.4 in [7] and by the assumption on λ we obtain

$$J(u) \le \frac{1}{2}(\lambda_{j+1} - \lambda)|\Omega|^{2K/n}|u|_{K_*}^2 - \frac{1}{K_*}|u|_{K_*}^{K_*} < \frac{K}{n} S_K^{n/2K} \qquad \forall u \in V \ ;$$

moreover, (3.3) holds. The result follows by applying Theorem 2.4 in [3]. □

Proof of Theorem 2.6. Let $j \in \mathbb{N}$ and choose ε_m as in (3.18): as $g(x,s) = \lambda_j s$ satisfies (2.1), (2.2), (2.8), (2.9), by (3.23) we infer that there exists a function $\delta(m)$ with $\lim_{m\to\infty} \delta(m) = 0$ and $\bar{m} \in \mathbb{N}$ such that

$$\delta(m) > 0 \quad \text{and} \quad \max_{u \in Q_m} \left(\frac{1}{2}\|u\|_K^2 - \frac{\lambda_j}{2}|u|_2^2 - \frac{n-2K}{2n}|u|_{K_*}^{K_*} \right) < \frac{K}{n} S_K^{n/2K} - \delta(m) \qquad \forall m \ge \bar{m} \ ;$$

therefore, there exists $\tilde{m} \ge \bar{m}$ such that $\sup_{m \ge \bar{m}} \delta(m) = \delta(\tilde{m}) =: \beta > 0$. As the set $B := \{u \in Q_{\tilde{m}};\ J(u) \ge 0\}$ is compact, $\gamma := \sup_{u\in B}|u|_2 < \infty$. Take $\delta_j = \beta/\gamma$, let $\lambda \in (\lambda_j - \delta_j, \lambda_j)$ and consider the functional

$$J(u) = \frac{1}{2}\|u\|_K^2 - \frac{\lambda}{2}|u|_2^2 - \frac{n-2K}{2n}|u|_{K_*}^{K_*} \ .$$

If we set $H^+ := \overline{\text{span}\{e_i;\ i \ge j\}}$ then (3.3) holds; furthermore,

$$\max_{u \in Q_{\tilde{m}}} J(u) < \max_{u \in Q_{\tilde{m}}} J(u) + \delta_j \gamma < \frac{K}{n} S_K^{n/2K} \ :$$

since $\dim(H_m^- \oplus \mathbb{R}\{u_\varepsilon^K\}) - \text{codim} H^+ = M_j + 1$, to conclude it suffices to apply Theorem 2.4 in [3] with $b = \frac{K}{n} S_K^{n/2K}$. □

Proof of Theorem 2.1. By standard minimax arguments [1] we can prove that the functional J admits a PS sequence of mountain-pass type; next, we need to prove that the PS sequence for J is below the level $\frac{K}{n}S_K^{n/2K}$: to this end we prove that

$$\max_{t\geq 0} J(tu_\varepsilon^K) < \frac{K}{n}S_K^{n/2K} \tag{3.24}$$

for ε small enough. If not, there exists $t_\varepsilon > 0$ such that $J(t_\varepsilon u_\varepsilon^K) \geq \frac{K}{n}S_K^{n/2K}$ for all ε: we can prove again that (3.13) and (3.14) hold. If $n = 2K+1, ..., 4K-1$ then (3.15) holds and we get again (3.17): this implies (3.24) as in the proof of Theorem 2.3. If $n \geq 4K$, by reasoning as in [2] we get

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^K) \geq c\mu\varepsilon^{n/2} \ .$$

If $n \geq 4K+1$ then $\frac{n}{2} < 2n-4K$, we obtain again (3.17) and we can conclude as above. If $n = 4K$ we need to choose μ large enough so that $O(\varepsilon^{2K}) < c\mu\varepsilon^{2K}$ where $O(\varepsilon^{2K})$ comes from (3.14); then, we obtain (3.24) for small ε. □

Proof of Theorem 2.2. This can be obtained as for Theorem 2.5 with V being the line $V = \{te_1 : t \in \mathbb{R}\}$; see also [2]. □

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