CRITICAL GROWTH QUASILINEAR ELLIPTIC PROBLEMS WITH SHIFTING SUBCRITICAL PERTURBATION*

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Abstract. We consider a family of quasilinear elliptic problems on bounded domains with perturbed critical growth term; we prove the existence of nontrivial solutions when the perturbation depends on a parameter which forces the solutions to have a suitable behavior in order to interact with it. We also study the behavior of the solutions for varying parameter, and we show that concentration phenomena appear as the parameter tends to infinity.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary, let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denote the p-Laplacian operator $(1 < p < n)$, and let $p^* = \frac{np}{n-p}$ be the critical Sobolev exponent of the embedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$; for all $k > 0$ we consider the problem

$$(P_k) \quad \begin{cases} -\Delta_p u = g(x)[(u-k)^+]^{q-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where g is a positive function, $s^+ = \max\{s, 0\}$ and $p \le q < p^*$. We also consider the limit problem

$$(P_\infty) \quad \begin{cases} -\Delta_p u = u^{p^*-1} & \text{in } \Omega \\ u \ge 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

The study of critical growth problems started with the case $p = 2$ in the celebrated paper by Brezis-Nirenberg [7] where one can also find several

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applications in physics and geometry. By now, it is well-known that the existence of nontrivial solutions of (P_∞) depends on the geometry of the set Ω; in particular, if Ω is star-shaped (that is, there exists a point $\bar{x} \in \Omega$ such that the line segment $[x, \bar{x}]$ is contained in Ω whenever $x \in \Omega$), the generalized Pohožaev identity of Pucci-Serrin [28] (see also [22]) implies that (P_∞) admits only the trivial solution in the space $W_0^{1,p}$. Therefore, in order to find nontrivial solutions of (P_∞), one needs to modify the geometry of Ω or to perturb the critical term u^{p^*-1}: it is still an open problem to understand if, for a given star-shaped domain Ω, there is some explicit correspondence between the perturbations of u^{p^*-1} and the perturbations of the domain Ω.

As far as we are aware, the perturbation of the domain has only been treated in the case $p = 2$: first, existence results for (P_∞) were obtained for domains Ω having nontrivial topology [4, 9] and later for contractible domains [10, 12] together with multiplicity results [26, 27]. This shows that the geometry (and not only the topology) of the domain is responsible for the existence of nontrivial solutions of (P_∞), at least in the case $p = 2$. On the other hand, the critical term has been perturbed either by replacing it with subcritical terms $u^{p^*-1-\varepsilon}$ as in [3, 17, 24, 30] (and by letting $\varepsilon \to 0$) or by adding some subcritical perturbation $f(x, u)$ as in [2, 7, 16, 18, 20, 21, 22, 31, 32, 33, 34].

Hoping to shed some light on the previous problems, in this paper we perturb (P_∞) in a different fashion: we add the subcritical term $f(x, s) = g(x)(s^+)^{q-1}$, and we study the behavior of the solutions when it "shifts", that is, the behavior of the solutions of (P_k) for varing k. We restrict our attention to a particular class of solutions which we name *mountain-pass solutions* (mp-solutions) (see Definition 1 below); under suitable assumptions on f we first prove the existence of at least an mp-solution of (P_k) and its continuous dependence with respect to k. Since (P_∞) does not admit an mp-solution (see Theorem 1 below) this shows a striking difference between (P_k) and (P_∞) *independent of the geometry* of the domain Ω; therefore, it is of some interest to study the behavior of mp-solutions u_k as $k \to \infty$. By means of the concentration-compactness principle [25], in Theorem 3 we prove that $|\nabla u_k|^p$ and $u_k^{p^*}$ converge in the sense of measures to the same positive measure, a Dirac mass having support at a point $x_0 \in \bar{\Omega}$. Moreover, any mp-solution u_k has to interact with the perturbation f; this allows us to characterize the blow-up of the L^∞ and $W^{1,\infty}$ norms as $k \to \infty$. Since the problems we consider are not autonomous, under further assumptions on f we may force the concentration point x_0 to belong to a suitable subset of $\bar{\Omega}$.

2. Notation and results

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with smooth boundary $\partial\Omega$; we define the *interior diameter* of Ω by $R_\Omega = \sup_{x \in \Omega}\{d(x, \partial\Omega)\}$; in other words, R_Ω denotes the supremum of the radii of the open balls contained in Ω. We denote by $\|\cdot\|$ the $W_0^{1,p}$-norm and by $\|\cdot\|_s$ the L^s-norm ($s \in [1, \infty]$). We also recall that there exists a positive constant S (the best Sobolev constant of the embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, see [25, 35]), independent of Ω, such that

$$\|u\|^p \geq S\|u\|_{p^*}^p \qquad \forall u \in W_0^{1,p}(\Omega) ; \tag{1}$$

this inequality is strict unless $\Omega = \mathbb{R}^n$ (which is not our case).

Consider now the subcritical perturbation $f(x,s) = g(x)(s^+)^{q-1}$; our assumptions on q vary according to whether $n \geq p^2$ or $p < n < p^2$; this distinction is not purely technical since the latter case corresponds to the so-called critical dimensions of Pucci-Serrin [29] (see [14]) and the fundamental solution of the operator $-\Delta_p$ has some local summability properties only in this case [23]; we also refer to [15, 19] for a link with Sobolev inequalities. We assume that

$$q \in [p, p^*) \quad \text{if } n \geq p^2 > 1 , \qquad q \in (\alpha_{np}, p^*) \quad \text{if } 1 < p < n < p^2 , \tag{2}$$

where $\alpha_{np} = \frac{p(np+p-2n)}{(p-1)(n-p)}$ (note that $\alpha_{np} < p^*$). We assume that g satisfies

(G) $\begin{cases} g \in C(\Omega), \quad g(x) \geq 0 \text{ in } \Omega, \ g \not\equiv 0, \quad g \in L^{\frac{np}{np+(p-n)q}}(\Omega); \\ \text{if } q = p \text{ in (2), then } \frac{np}{np+(p-n)q} = \frac{n}{p} \text{ and we assume that } \|g\|_{\frac{n}{p}} < S. \end{cases}$

For all $k \in (0, \infty]$, the particular solutions of (P_k) we consider are obtained as suitable critical points of the functional $J_k : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_k(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{q}\int_\Omega g(x)[(u-k)^+]^q - \frac{1}{p^*}\int_\Omega |u|^{p^*} \qquad \text{if } k \in (0, \infty)$$

$$J_\infty(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*} ;$$

more precisely, we are interested in mountain-pass critical points, which we are going to characterize. Consider the cone of positive functions

$$\mathcal{C} := \{u \in W_0^{1,p}; \ u(x) \geq 0 \text{ for almost every } x \in \Omega\} ;$$

since the leading term of the functional J_∞ is negative at infinity, the set $M := \{u \in \mathcal{C} : J_\infty(u) < 0\}$ is nonempty; take any $v \in M$, and consider the class

$$\Gamma_v := \{\gamma \in C([0,1]; \mathcal{C}) : \gamma(0) = 0, \ \gamma(1) = v\} \tag{3}$$

and the infmax value

$$\alpha_k := \inf_{\gamma \in \Gamma_v} \max_{t \in [0,1]} J_k(\gamma(t)) \ . \tag{4}$$

Let $k \in (0, \infty)$; then since $J_k(u) \leq J_\infty(u)$ for all $u \in W_0^{1,p}$, we have $J_k(v) < 0$ for all $v \in M$ and α_k is a mountain-pass level for J_k; see [1]. Note also that the set M is connected in $W_0^{1,p}$; therefore, α_k *does not* depend on the choice of v and the following definition makes sense:

Definition 1. For all $k \in (0, \infty]$ let α_k be as in (4); we say that a solution $u_k \in W_0^{1,p}$ of (P_k) is an *mp-solution* if $J_k(u_k) = \alpha_k$.

Our starting point is a nonexistence result; as a straightforward consequence of an inequality in [17], we have

Theorem 1. *There exist no mp-solutions of* (P_∞).

On the other hand, we have existence and stability of the mp-solutions of (P_k) for varying k:

Theorem 2. *Assume* (2) *and* (G)*. Then, for all* $k \in (0, \infty)$ *there exists an mp-solution* $u_k \in W_0^{1,p} \cap C^{1,\alpha}(\bar{\Omega})$ *of* (P_k) *which satisfies* $\frac{\partial u_k}{\partial n} < 0$ *on* $\partial \Omega$; *moreover, there exist* $x_1^k, x_2^k \in \Omega$ *such that* $u_k(x_1^k) > k$ *and* $|\nabla u_k(x_2^k)| > \frac{k}{R_\Omega}$. *Finally, if* $\{k_m\} \subset (0, \infty)$ *is a sequence such that* $k_m \to k \in (0, \infty)$ *and* $\{u_{k_m}\}$ *denotes a sequence of corresponding mp-solutions of* (P_{k_m})*, then there exists an mp-solution* u_k *of* (P_k) *such that* $u_{k_m} \to u_k$ *in the* $W_0^{1,p}$*-norm topology, up to a subsequence.*

We prove Theorems 1 and 2 in Section 4. Theorems 1 and 2 highlight a qualitative change of behavior between (P_k) for $k \in (0, \infty)$ and (P_∞); therefore, we study the behavior of the solutions as the subcritical perturbation shifts to infinity. The following result holds:

Theorem 3. *Assume* (2)*,* (G) *and, for all* $k \in (0, \infty)$*, let* u_k *be an mp-solution of* (P_k)*. If* $k \to +\infty$*, then*

    (i) $u_k \rightharpoonup 0$ *in* $W_0^{1,p}$;

    (ii) $u_k \to 0$ *in the* $W_0^{1,s}$*-norm topology for all* $s \in [1, p)$.

Moreover, there exist $x_0 \in \bar{\Omega}$ *and a subsequence* $\{u_{k_m}\} \subset \{u_k\}$ *such that (as* $k_m \to \infty$)*:*

    (iii) $|\nabla u_{k_m}|^p \rightharpoonup^* S^{n/p} \delta_{x_0}$ *in the weak*-measure topology;*

    (iv) $u_{k_m}^{p^*} \rightharpoonup^* S^{n/p} \delta_{x_0}$ *in the weak*-measure topology.*

Finally, if $x_1^{k_m}, x_2^{k_m}$ *denote any of the points found in Theorem 2, then we have* $x_1^{k_m} \to x_0$ *and* $x_2^{k_m} \to x_0$ *as* $k_m \to \infty$.

Theorem 2 states that for all $k \in (0, \infty)$ the sets

$$\Omega(u_k) := \{x \in \Omega : u_k(x) > k\} \ \text{ and } \ \Omega(\nabla u_k) := \{x \in \Omega : |\nabla u_k(x)| > \frac{k}{R_\Omega}\} \tag{5}$$

are nonempty. Moreover, by Theorem 3 we know that as $k \to \infty$, $\Omega(u_k)$ and $\Omega(\nabla u_k)$ collapse to a single point $x_0 \in \bar{\Omega}$.

In some cases we may be more precise on the concentration point x_0; assume that

$$\text{supp } g := \Omega_g \subset \Omega \ , \tag{6}$$

where supp g denotes the (compact) support of the function g. Then, the concentration point x_0 belongs to Ω_g:

Theorem 4. *Assume* (2)*,* (G) *and* (6)*; for all* k*, let* u_k *be an mp-solution of* (P_k)*. Then, for all* $k \in (0, \infty)$ *there exists* $x_1^k \in \Omega_g$ *such that* $u_k(x_1^k) > k$*; moreover, if* x_0 *and* $\{u_{k_m}\}$ *are as in Theorem 3, then for any such* $x_1^{k_m}$ *we have* $x_1^{k_m} \to x_0$ *as* $k_m \to \infty$*; in particular,* $x_0 \in \Omega_g$*.*

The above result does not come unexpectedly; indeed, in the autonomous case (and $p = 2$) the solutions of some perturbed critical growth problems have the concentration point (for vanishing perturbation) in a critical point of the regular part of the Green function relative to $-\Delta$; see [31, 32]. For nonautonomous perturbations the situation is different [34], and the concentration point depends on the behavior of the perturbation itself.

The proofs of Theorems 3 and 4 are quoted in Section 5.

3. PRELIMINARY LEMMAS

Again we denote $f(x, s) = g(x)(s^+)^{q-1}$ and $F(x, s) = \frac{1}{q} g(x)(s^+)^q$. We first establish a technical result:

Lemma 1. *Let* $\{k_m\} \subset (0, \infty)$ *be a sequence such that* $k_m \to k \in (0, +\infty]$*; let* $\{u_m\} \subset W_0^{1,p}$ *be a sequence such that* $u_m \rightharpoonup u$ *for some* $u \in W_0^{1,p}$*. Then*
(i) *if* $k < \infty$*, then*

$$f(x, u_m - k_m) \to f(x, u - k) \text{ in } L^{\frac{np}{np-n+p}}, \ \ F(x, u_m - k_m) \to F(x, u - k) \text{ in } L^1;$$

(ii) *if* $k = \infty$*, then*

$$f(x, u_m - k_m) \to 0 \quad \text{in } L^{\frac{np}{np-n+p}}, \qquad F(x, u_m - k_m) \to 0 \quad \text{in } L^1 \ .$$

Proof. Since $\|(u_m - k_m)^+\| \leq \|u_m\|$, there exists $\bar{u} \in W_0^{1,p}$ such that $(u_m - k_m)^+ \rightharpoonup \bar{u}$, up to a subsequence; by pointwise convergence we infer that $\bar{u} = (u - k)^+$ if $k < \infty$ and $\bar{u} = 0$ if $k = \infty$; hence,

$$(u_m - k_m)^+ \rightharpoonup (u - k)^+ \quad \text{if } k < \infty \qquad (u_m - k_m)^+ \rightharpoonup 0 \quad \text{if } k = \infty \ .$$

By repeating the above arguments for any subsequence of $\{u_m\}$, we infer that the previous convergences hold on the whole sequence. The result follows by taking into account (2) (namely that f has subcritical growth) and by arguing as in Theorem 2.2.7 in [8]. $\qquad\qquad\square$

Let $I \in C^1(W_0^{1,p}, \mathbb{R})$; we recall that a sequence $\{u^m\} \subset W_0^{1,p}$ is called a Palais-Smale sequence (PS for brevity) for I at level c if $I(u^m) \to c$ and $I'(u^m) \to 0$ in $W^{-1,p'}$, $p' = \frac{p}{p-1}$. The functional I is said to satisfy the PS condition at level c, if every PS sequence at level c is precompact.

Lemma 2. *Let* $k \in (0, \infty)$ *and let* $c \in (0, \frac{S^{n/p}}{n})$; *then the functional* J_k *satisfies the PS condition at level* c.

Proof. Let $\{u^m\}$ be a PS sequence at level $c \in (0, \frac{S^{n/p}}{n})$ for J_k; by Lemma 1 in [2] we infer that there exists $u \in W_0^{1,p} \setminus \{0\}$ such that $u^m \rightharpoonup u$ (up to a subsequence) and $J'_k(u) = 0$.

By Theorem 2.1 in [5] we infer that $\nabla u^m(x) \to \nabla u(x)$ for almost every $x \in \Omega$; therefore, by Theorem 1 in [6] we get

$$\|u^m\|^p - \|u\|^p = \|u^m - u\|^p + o(1) \qquad \text{as } m \to \infty . \tag{7}$$

Since $\{u^m\}$ is a (bounded) PS sequence and $J'_k(u) = 0$, we obtain

$$o(1) = J'_k(u^m)[u^m - u] - J'_k(u)[u^m - u]$$
$$= \int_\Omega \big(|\nabla u^m|^{p-2}\nabla u^m - |\nabla u|^{p-2}\nabla u\big)(\nabla u^m - \nabla u) - \int_\Omega f(x, u^m - k)(u^m - u)$$
$$+ \int_\Omega f(x, u - k)(u^m - u) - \int_\Omega |u^m|^{p^*-1}(u^m - u) + \int_\Omega |u|^{p^*-1}(u^m - u);$$

hence, by (1), by Lemma 1 (with $k_m \equiv k$), by (7) and again by the Brezis-Lieb lemma [6] we get

$$o(1) = \|u^m - u\|^p - \|u^m - u\|_{p^*}^{p^*} \geq \|u^m - u\|^p (1 - S^{-p^*/p}\|u^m - u\|^{p^*-p}). \tag{8}$$

The inequality in (8) implies that either $\|u^m - u\| \to 0$ or $o(1) \geq 1 - S^{-p^*/p}\|u^m - u\|^{p^*-p}$; therefore, we are done if we prove that the latter may not occur. For the sake of contradiction, assume that

$$\|u^m - u\|^p \geq S^{n/p} + o(1) ; \tag{9}$$

then, by Lemma 1, (7), the Brezis-Lieb lemma and the first equality in (8) we get

$$J_k(u^m) = J_k(u^m - u) + J_k(u) + o(1) = \frac{1}{n}\|u^m - u\|^p + J_k(u) + o(1) . \tag{10}$$

Finally, note that (2) and (G) imply that

$$\forall u \in W_0^{1,p} , \quad J_k'(u) = 0 \Longrightarrow J_k(u) \geq 0 ; \tag{11}$$

this follows by taking into account that $J_k'(u)[u] = 0$ and by inserting this in the expression of $J_k(u)$. Then, by (9), (10) and (11) we get

$$\liminf_{m \to \infty} J_k(u^m) \geq \frac{1}{n} S^{n/p},$$

which contradicts the assumption $c < \frac{S^{n/p}}{n}$; therefore, (9) does not hold and $\|u^m - u\| \to 0$. $\qquad\square$

Fix any $v \in M$; we wish now to estimate α_k as defined in (4):

Lemma 3. *For all $k \in (0, \infty)$ we have $\alpha_k < \frac{S^{n/p}}{n}$.*

Proof. By (G) there exist $K > 0$ and a nonempty open set $\omega \subset \Omega$ such that

$$g(x) \geq K \qquad \forall x \in \omega ; \tag{12}$$

without loss of generality we may assume that ω contains the origin O; let $B(\rho)$ be the ball centered at O of radius $\rho > 0$ sufficiently small so that $\bar{B}(\rho) \subset \omega$, and take a positive cut-off function $\eta \in C_c^\infty(\omega)$ such that $\eta \equiv 1$ in $B(\rho)$, $\eta \leq 1$ in ω. We argue as in [2, 7, 16] and consider the one-parameter family of entire functions

$$u_\varepsilon^*(x) := \frac{c_n \varepsilon^{\frac{n-p}{p(p-1)}}}{\left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} \qquad (\varepsilon > 0) ,$$

for which the equality in (1) holds. For all $\varepsilon > 0$ let $u_\varepsilon(x) := \eta(x) u_\varepsilon^*(x)$, and we claim that if ε is small enough, then

$$\max_{t \geq 0} J(t u_\varepsilon) < \frac{1}{n} S^{n/p} . \tag{13}$$

For the sake of contradiction, assume that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$J(t_\varepsilon u_\varepsilon) = \max_{t \geq 0} J(t u_\varepsilon) \geq \frac{1}{n} S^{n/p} . \tag{14}$$

If (14) holds, it is proved in [2] (see also [16, 20]) that as $\varepsilon \to 0$ we have $t_\varepsilon \to 1$ and

$$\frac{\|t_\varepsilon u_\varepsilon\|^p}{p} - \frac{\|t_\varepsilon u_\varepsilon\|_{p^*}^{p^*}}{p^*} \leq \frac{S^{n/p}}{n} + O\left(\varepsilon^{(n-p)/(p-1)}\right) . \tag{15}$$

Next, we need an estimate of $\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^q$; we have to distinguish three different cases.

The case $n > p^2$. For all $\varepsilon < \rho$ sufficiently small and for all $x \in B(\varepsilon)$ we have $t_\varepsilon u_\varepsilon(x) \geq c\varepsilon^{-(n-p)/p} \geq 2k$, and therefore by (12) we get

$$\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^q \geq c \int_{B(\varepsilon)} \varepsilon^{-q(n-p)/p} = c\varepsilon^n \varepsilon^{-q(n-p)/p} \ .$$

As $n > p^2$, by (2) we have $n - \frac{q(n-p)}{p} \leq p < \frac{n-p}{p-1}$; then, if we set $\delta = \frac{n-p}{p-1} - n + \frac{q(n-p)}{p}$, we have $\delta > 0$, and for ε small enough we have

$$\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^q \geq c\varepsilon^{-\delta} \varepsilon^{(n-p)/(p-1)} \ ;$$

this, together with (15), yields

$$J(t_\varepsilon u_\varepsilon) \leq \frac{S^{n/p}}{n} + \left(O(1) - c\varepsilon^{-\delta}\right)\varepsilon^{(n-p)/(p-1)} < \frac{S^{n/p}}{n}$$

for sufficiently small ε. Therefore, (14) cannot be true and (13) holds.

The case $n = p^2$. In this case we have $\frac{n-p}{p-1} = p$, and so we need to prove that $\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^q$ goes to 0 more slowly than ε^p as $\varepsilon \to 0$. If $q > p$ we may argue as for the case $n > p^2$; therefore, we consider only the case $q = p$. By a simple calculation, for all $k > 0$ there exist $C_k > 0$ such that, for $\varepsilon > 0$ small enough the following implication holds:

$$|x| < C_k \varepsilon^{1/p} \quad \Longrightarrow \quad t_\varepsilon u_\varepsilon(x) - k \geq \frac{1}{2} t_\varepsilon u_\varepsilon^*(x) \ ;$$

therefore, by (12) we get

$$\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^p \geq c \int_{B(C_k \varepsilon^{1/p})} [t_\varepsilon u_\varepsilon^*(x)]^p \geq c \int_{B(C_k \varepsilon^{1/p}) \setminus B(C_k \varepsilon)} [u_\varepsilon^*(x)]^p$$

$$\geq c\varepsilon^p \int_{B(C_k \varepsilon^{1/p}) \setminus B(C_k \varepsilon)} \frac{1}{|x|^{p^2}} \geq c\varepsilon^p \int_{C_k \varepsilon}^{C_k \varepsilon^{1/p}} \frac{1}{r} \geq c\varepsilon^p \log|\varepsilon| \ .$$

This estimate, together with (15), yields $J(t_\varepsilon u_\varepsilon) < \frac{S^p}{n}$ for sufficiently small ε and proves (13).

The case $p < n < p^2$. By arguing as in the case $n > p^2$ we get again for all $\varepsilon < \rho$ sufficiently small

$$\int_\Omega g(x)[(t_\varepsilon u_\varepsilon - k)^+]^q \geq c\varepsilon^n \varepsilon^{-q(n-p)/p} \ .$$

Since $q > \alpha_{np}$, we have again $n - \frac{q(n-p)}{p} < \frac{n-p}{p-1}$ and (13) follows.

Conclusion: for all $1 < p < n$, if $\varepsilon > 0$ is small enough (13) holds and proves that $\alpha_k < \frac{S^{n/p}}{n}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

With the previous lemmas we may guarantee the existence of mp-solutions of (P_k); we also prove that the mp-solutions are "low energy solutions":

Lemma 4. *For all $k \in (0, \infty)$ there exists at least an mp-solution u_k of (P_k) satisfying $\frac{\partial u_k}{\partial n} < 0$ on $\partial\Omega$; furthermore, any mp-solution u_k of (P_k) satisfies $J_k(u_k) < \frac{S^{n/p}}{n}$ and $u_k \in W_0^{1,p} \cap C^{1,\alpha}(\bar{\Omega})$.*

Proof. The variational characterization of α_k as in (4) ensures the existence of a PS sequence $\{u^m\}$ for J_k of mountain-pass type at level α_k (see [1]); since $J_k(|u|) \leq J_k(u)$ for all $u \in W_0^{1,p}$, we may assume that $\{u^m\} \subset \mathcal{C}$. Lemma 3 ensures that $\alpha_k < \frac{S^{n/p}}{n}$, and Lemma 2 implies that the PS condition holds for $\{u^m\}$; the existence of an mp-solution $u_k \in \mathcal{C}$ of (P_k) then follows.

For the local regularity of u_k we refer to [11, 37]; the proof that u_k has Hölder-continuous derivatives at the boundary is given in [36]; see also Proposition A.5 in [13]. Since u_k solves (P_k), by (2) and (G) we infer that $-\Delta_p u_k \geq 0$ in Ω so that by Theorem 5 in [38] we infer that $u_k > 0$ in Ω and $\frac{\partial u_k}{\partial n} < 0$ on $\partial\Omega$. $\square$

Now we prove that all the mp-solutions are contained in a suitable ball of the space $W_0^{1,p}$:

Lemma 5. *There exists a constant $\Lambda > 0$ such that for all $k > 0$ and for all u_k being an mp-solution of (P_k) we have $\|u_k\| \leq \Lambda$.*

Proof. By Lemma 3, for all mp-solution u_k of (P_k) we have

$$\frac{1}{p}\|u_k\|^p - \frac{1}{q}\int_\Omega g(x)[(u_k - k)^+]^q - \frac{1}{p^*}\|u_k\|_{p^*}^{p^*} < \frac{S^{n/p}}{n}, \qquad (16)$$

and, since $J_k'(u_k)[u_k] = 0$, we have

$$\|u_k\|^p = \int_\Omega g(x)[(u_k - k)^+]^{q-1}u_k + \|u_k\|_{p^*}^{p^*}. \qquad (17)$$

Assume for the sake of contradiction that there exists a sequence $\{u_{k_m}^m\}$ of mp-solutions of (P_{k_m}) (the sequence k_m may also be constant!) such that $\|u_{k_m}^m\| \to \infty$ as $m \to \infty$; then by (16) we also have $\|u_{k_m}^m\|_{p^*} \to \infty$; moreover,

$$\int_\Omega g(x)[(u_{k_m}^m - k_m)^+]^{q-1}u_{k_m}^m = o(\|u_{k_m}^m\|_{p^*}^{p^*}) \qquad \text{and} \qquad (18)$$

$$\int_\Omega g(x)[(u_{k_m}^m - k_m)^+]^q = o(\|u_{k_m}^m\|_{p^*}^{p^*});$$

indeed, by (2), (G) and Hölder's inequality we have

$$\int_\Omega g(x)[(u_{k_m}^m - k_m)^+]^{q-1}u_{k_m}^m \leq \|g\|_{\frac{np}{np+(p-n)q}}\|u_{k_m}^m\|_{p^*}^q$$

and a similar result holds for $\int_\Omega g(x)[(u_{k_m}^m - k_m)^+]^q$.

By (16) and (18), as $m \to \infty$ we get $\|u_{k_m}^m\|^p \leq \frac{p}{p^*}\|u_{k_m}^m\|_{p^*}^{p^*}(1 + o(1)) \asymp \frac{p}{p^*}\|u_{k_m}^m\|_{p^*}^{p^*}$, while by (17) and (18) we get $\|u_{k_m}^m\|^p \asymp \|u_{k_m}^m\|_{p^*}^{p^*}$; the contradiction is achieved and the lemma is proved. $\square$

We conclude this section by proving that the (weak) limit of mp-solutions is also a solution:

Lemma 6. *Let $\{k_m\} \subset (0, \infty)$ be a sequence such that $k_m \to k \in (0, +\infty]$; for all m let u_{k_m} be an mp-solution of (P_{k_m}). Then, there exists $u \in W_0^{1,p}$ (eventually trivial) which solves (P_k) and such that $u_{k_m} \rightharpoonup u$, up to a subsequence.*

Proof. By Lemma 5, the sequence $\{u_{k_m}\}$ is bounded in $W_0^{1,p}$; therefore, there exists $u \in W_0^{1,p}$ such that $u_{k_m} \rightharpoonup u$, up to a subsequence. For all m and all $v \in W_0^{1,p}$ we have

$$\int_\Omega |\nabla u_{k_m}|^{p-2}\nabla u_{k_m}\nabla v = \int_\Omega f(x, u_{k_m} - k_m)v + \int_\Omega u_{k_m}^{p^*-1}v \ ;$$

by Lemma 1 and by letting $k_m \to k$ we obtain that u solves (P_k). $\square$

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. By (15) and the variational characterization of α_∞ we infer $\alpha_\infty \leq \frac{1}{n}S^{n/p}$; then by Proposition 2 in [17], any nontrivial solution u to (P_∞) satisfies $J_\infty(u) > \alpha_\infty$ and Theorem 1 follows. $\square$

Remark. By arguing as in the proof of Lemma 9 below, one could prove that $\alpha_\infty = \frac{1}{n}S^{n/p}$. $\square$

In order to prove Theorem 2 we need two lemmas:

Lemma 7. *There exists $\beta > 0$ (independent of k) such that for all $k \in (0, \infty)$, if u_k denotes an mp-solution of (P_k), then the segment $\gamma_k = [O, \beta\frac{u_k}{\|u_k\|}]$ satisfies $\max_{u \in \gamma_k} J_k(u) = \alpha_k$, where α_k is defined in (4).*

Proof. We first claim that

$$\exists \ \underline{\alpha} > 0 \quad \text{such that} \quad \alpha_k \geq \underline{\alpha} \qquad \forall k \in (0, \infty) \ . \tag{19}$$

Indeed, by (1), (G) and Hölder's inequality, there exist three constants $c_1, c_2, c_3 > 0$ such that

$$\text{if } q = p \qquad J_k(u) \geq c_1\|u\|^p - c_2\|u\|^{p^*}$$
$$\text{if } q > p \qquad J_k(u) \geq c_1\|u\|^p - c_2\|u\|^q - c_3\|u\|^{p^*}.$$

In both cases there exist $\rho > 0$ and $\underline{\alpha} > 0$ such that $J_k(u) \geq \underline{\alpha}$ if $\|u\| = \rho$ and $J_k(u) > 0$ if $0 < \|u\| \leq \rho$; this proves that the $W_0^{1,p}$-sphere of radius ρ separates O and M, and hence any path $\gamma \in \Gamma$ satisfies $\max_{t \in [0,1]} J_k(\gamma(t)) \geq \underline{\alpha}$ and (19) follows by (4).

Next, note that (17) and (19) yield

$$\underline{\alpha} \leq J_k(u_k) \leq \frac{1}{p}\|u_k\|^p = \frac{1}{p}\int_\Omega g(x)[(u_k - k)^+]^{q-1}u_k + \frac{1}{p}\|u_k\|_{p^*}^{p^*}$$

$$\leq c\|u_k\|_{p^*}^q + \frac{1}{p}\|u_k\|_{p^*}^{p^*} ,$$

and therefore

$$\exists \delta > 0 \quad \text{such that} \quad \|u_k\|_{p^*} \geq \delta \tag{20}$$

for all $k \in (0, \infty)$ and all u_k an mp-solution of (P_k).

Finally, we prove that for all $k \in (0, \infty)$ we have

$$\alpha_k = \max_{t \geq 0} J_k(t u_k) \tag{21}$$

where α_k is as in (4) and u_k is any mp-solution of (P_k). This proof is a slight generalization of that of Lemma 4.1 in [39], and we just quickly outline it. Consider the function $\phi_k(t) = J_k(t u_k)$; since (17) holds, we have

$$\phi_k'(t) = (t^{p-1} - t^{p^*-1})\|u_k\|_{p^*}^{p^*} + \int_\Omega g(x)u_k(t^{p-1}[(u_k - k)^+]^{q-1} - [(tu_k - k)^+]^{q-1})$$

so that $\phi_k(t)$ is increasing for $t \in (0, 1)$ and decreasing for $t \in (1, \infty)$; this proves (21).

To conclude, let Λ be as in Lemma 5 and let δ be as in (20); also define

$$T := \max\left\{\left(\frac{n}{n-p}\frac{\Lambda^p}{\delta^{p^*}}\right)^{(n-p)/p^2}, 1\right\}.$$

Then we have

$$J_k(Tu_k) \leq J_\infty(Tu_k) = \frac{T^p}{p}\|u_k\|^p - \frac{T^{p^*}}{p^*}\|u_k\|_{p^*}^{p^*} \leq \frac{T^p}{p}\Lambda^p - \frac{T^{p^*}}{p^*}\delta^{p^*} \leq 0,$$

the latter inequality being a direct consequence of the definition of T. Now take $\beta = \Lambda T$; since the function ϕ_k defined above is decreasing on $(1, \infty)$ we get

$$J_k\left(\beta\frac{u_k}{\|u_k\|}\right) = J_k\left(\frac{\Lambda T}{\|u_k\|}u_k\right) \leq J_k(Tu_k) \leq 0,$$

and the lemma is proved thanks to (21). $\qquad\square$

Lemma 8. *Let $\{k_m\} \subset (0, \infty)$ be a sequence such that $k_m \to k < \infty$; then $\alpha_{k_m} \to \alpha_k$. Moreover, for all compact intervals $I \subset (0, \infty)$, there exists $\delta_I > 0$ such that if $k \in I$, then $\alpha_k \leq \frac{S^{n/p}}{n} - \delta_I$.*

Proof. Let $R > 0$ and take any $u \in B_R = \{u \in W_0^{1,p} : \|u\| \leq R\}$; by Lagrange's Theorem for almost every $x \in \Omega$ there exists $\sigma_m(x) \in (k_m, k)$ (or the converse if $k < k_m$) such that

$$|g(x)[(u(x)-k_m)^+]^q - g(x)[(u(x)-k)^+]^q| = q|k_m-k|g(x)\cdot[(u(x)-\sigma_m(x))^+]^{q-1}$$

for almost every $x \in \Omega$. Hence, by Hölder's inequality we get (for some $C(R) > 0$)

$$\left| \frac{1}{q} \int_\Omega \left(g(x)[(u(x)-k_m)^+]^q - g(x)[(u(x)-k)^+]^q \right) \right| \leq C(R)|k_m - k|,$$

which proves that $J_{k_m} \to J_k$ uniformly on bounded subsets of $W_0^{1,p}$. Consider now the set $\Delta = \{\gamma_j : j = k_1, k_2, \dots \text{ or } j = k\}$, where γ_j is the segment relative to (P_j) introduced in Lemma 7; then $\Delta \subset B_\beta$ (the $W_0^{1,p}$-ball of radius β), and by uniform convergence on B_β we get

$$\alpha_{k_m} = \inf_{\gamma \in \Delta} \max_{u \in \gamma} J_{k_m}(u) \to \inf_{\gamma \in \Delta} \max_{u \in \gamma} J_k(u) = \alpha_k .$$

Now let $I \subset (0, \infty)$ be a compact interval; we just saw that the map $k \mapsto \alpha_k$ is continuous, and therefore it attains a maximum $\overline{\alpha}$ on I; by Lemma 4, we have $\alpha_k < \frac{S^{n/p}}{n}$ for all $k \in I$, and therefore $\overline{\alpha} = \frac{S^{n/p}}{n} - \delta_I$ for some $\delta_I > 0$.□

Proof of Theorem 2. Let $k \in (0, \infty)$; the existence of an mp-solution u_k to (P_k) is proved in Lemma 4; we now prove the existence of x_1^k and x_2^k as in the statement of Theorem 2.

Let u_k be any mp-solution of (P_k), and assume for the sake of contradiction that $\|u_k\|_\infty \leq k$; then $(u_k - k)^+ \equiv 0$ so that u_k solves (P_∞) which, by Theorem 1, has no mp-solutions. The contradiction is achieved, and hence $\|u_k\|_\infty > k$; by Lemma 4, u_k is continuous, and therefore the set $\Omega(u_k)$ defined in (5) is not empty; so, we may take $x_1^k \in \Omega(u_k)$.

Let $y \in \partial\Omega$ satisfy $d(x_1^k, y) = d(x_1^k, \partial\Omega)$. Since $u_k \in C^1(\bar{\Omega})$ by Lemma 4, and since $d(x_1^k, y) \leq R_\Omega$ and $u_k(y) = 0$, there exists a subsegment Σ of positive one-dimensional measure of the segment (x_1^k, y) where $|\nabla u_k(x)| > \frac{k}{R_\Omega}$ for all $x \in \Sigma$; take any $x_2^k \in \Sigma$.

In order to complete the proof of Theorem 2 we need to strengthen the statement of Lemma 6; the weak convergence has to be tight convergence and u has to be an mp-solution.

So, take the same assumptions of Lemma 6 with $k < \infty$; then by Lemma 1 we have

$$J_k(u_{k_m}) - J_{k_m}(u_{k_m}) = \frac{1}{q} \int_\Omega g(x) \left([(u_{k_m} - k_m)^+]^q - [(u_{k_m} - k)^+]^q \right) \to 0$$

by Lemma 8; this implies $J_k(u_{k_m}) \to c < \frac{S^{n/p}}{n}$, up to a subsequence. On the other hand, Lemma 1 also yields

$$\sup_{\|v\|=1} \left| J'_k(u_{k_m})[v] - J'_{k_m}(u_{k_m})[v] \right|$$

$$= \sup_{\|v\|=1} \left| \int_\Omega g(x) \left([(u_{k_m} - k_m)^+]^{q-1} - [(u_{k_m} - k)^+]^{q-1} \right) v \right| \to 0;$$

since $J'_{k_m}(u_{k_m}) = 0$, this proves that $J'_k(u_{k_m}) \to 0$ in $W^{-1,p'}$. Therefore, $\{u_{k_m}\}$ is a PS sequence at level $c < \frac{S^{n/p}}{n}$ for the functional J_k and $u_{k_m} \to u$ in the norm topology of $W_0^{1,p}$ by Lemma 2.

Taking into account that $u_{k_m} \to u$, by Lemmas 1 and 8, we get

$$J_k(u) = \lim_{m\to\infty} J_{k_m}(u_{k_m}) = \lim_{m\to\infty} \alpha_{k_m} = \alpha_k$$

so that u is an mp-solution of (P_k). $\qquad\square$

Remark. It is clear from the proof that one could take the whole sequence $\{u_{k_m}\}$ (instead of a subsequence) provided one had uniqueness results.

5. Proof of Theorems 3 and 4

We first prove that the energy of the mp-solutions tends to the critical threshold as the perturbation shifts to infinity:

Lemma 9. *For all $k \in (0, \infty)$ let u_k be an mp-solution of (P_k). Then $\lim_{k\to\infty} J_k(u_k) = \frac{1}{n} S^{n/p}$.*

Proof. By Lemma 6, up to a subsequence, we have $u_k \rightharpoonup u$ for some $u \in W_0^{1,p}$ solving (P_∞); therefore, by Lemma 1 we know that $\int_\Omega g(x)[(u_k - k)^+]^q \to 0$; hence, by (1) we obtain

$$J_k(u_k) = \frac{1}{p}\|u_k\|^p - \frac{1}{p^*}\|u_k\|_{p^*}^{p^*} + o(1) \geq \frac{1}{p}\|u_k\|^p - \frac{1}{p^* S^{p^*/p}}\|u_k\|^{p^*} + o(1) .$$

Define $\phi(t) = \frac{1}{p}t^p - \frac{1}{p^* S^{p^*/p}}t^{p^*}$; then, ϕ attains its maximum on $\mathbb{R}^+$ when $t = S^{n/p^2}$ and $\phi(S^{n/p^2}) = \frac{1}{n}S^{n/p}$. Since the $W_0^{1,p}$-sphere of radius S^{n/p^2} separates O and M, by the variational characterization of α_k we get

$$J_k(u_k) = \alpha_k \geq \frac{1}{n}S^{n/p} + o(1) .$$

The converse inequality follows by applying Lemma 3 for all k.

Finally, by repeating the above steps on any weakly convergent subsequence of $\{u_k\}$ we obtain the result on the whole sequence. $\qquad\square$

In particular, the previous lemma establishes that we do not have the alternative of Section 5 in [17]. To complete the proof of Theorem 3 we need the following result:

Lemma 10. *For all $k \in (0, \infty)$ let u_k be an mp-solution of (P_k); then as $k \to \infty$, $u_k \rightharpoonup 0$ in $W_0^{1,p}$ and $u_k \to 0$ in the $W_0^{1,q}$-norm topology for all $q \in [1, p)$. Moreover, there exist $x_0 \in \bar{\Omega}$ and a subsequence $\{u_{k_m}\} \subset \{u_k\}$ such that (as $k_m \to \infty$)*

(i) $|\nabla u_{k_m}|^p \rightharpoonup^* S^{n/p} \delta_{x_0}$ in the weak*-measure topology;

(ii) $u_{k_m}^{p^*} \rightharpoonup^* S^{n/p} \delta_{x_0}$ in the weak*-measure topology.

Proof. Let $u \in W_0^{1,p}$ be the solution of (P_∞) found in Lemma 6 so that $u_k \rightharpoonup u$, up to a subsequence. Since u_k solves (P_k), we have $J_k'(u_k)[u_k] = 0$, which, substituted into the expression of J_k, yields

$$J_k(u_k) = \frac{1}{n}\|u_k\|^p - \frac{1}{q}\int_\Omega g(x)[(u_k - k)^+]^q + \frac{1}{p^*}\int_\Omega g(x)[(u_k - k)^+]^{q-1}u_k \ ;$$

therefore, by Lemmas 1 and 9 we infer that $\|u_k\|^p \to S^{n/p}$. Then, since $J_\infty'(u)[u] = 0$, by the lower semicontinuity of the $W_0^{1,p}$-norm with respect to weak convergence we have

$$J_\infty(u) = \frac{1}{n}\|u\|^p \le \liminf_{k \to \infty} \frac{1}{n}\|u_k\|^p = \frac{1}{n}S^{n/p} \ ;$$

hence, by Proposition 2 in [17] we infer $u \equiv 0$. By repeating the above steps on any weakly convergent subsequence of $\{u_k\}$, we infer that $u_k \rightharpoonup 0$ in $W_0^{1,p}$ on the whole sequence.

Let $a(x, s, \zeta) = |\zeta|^{p-2}\zeta$ and $g_k(x) = g(x)[(u_k(x) - k)^+]^{q-1} + u_k^{p^*-1}(x)$ so that g_k is bounded in $W^{-1,p'} \cap L^1$; then, the convergence in the $W_0^{1,q}$-norm follows for all $q \in [1, p)$ by Theorem 2.1 in [5].

Finally, as $u_k \rightharpoonup 0$, by Lemma I.1 in [25], $|\nabla u_{k_m}|^p$ and $u_{k_m}^{p^*}$ converge in the weak*-measure topology to two bounded, positive measures which are proportional to δ_{x_0} for some $x_0 \in \bar{\Omega}$; in order to evaluate the mass of these δ_{x_0} we take into account Lemma 9 and proceed as in Lemma 3 and Theorem 6 in [17]; then we infer (i) and (ii). $\qquad\square$

Proof of Theorem 3. By Lemma 10, we may select a subsequence $\{u_{k_m}\}$ such that both the sequences $\{x_1^{k_m}\}$ and $\{x_2^{k_m}\}$ converge to the same point $x_0 \in \bar{\Omega}$, and the proof of Theorem 3 is complete. $\qquad\square$

Proof of Theorem 4. Similarly, to prove Theorem 4, let u_k be any mp-solution of (P_k) and assume that $\Omega(u_k) \cap \Omega_g = \emptyset$; then, we get $g(x)[(u_k -$

$k)^+]^{q-1} \equiv 0$, which gives a contradiction in view of Theorem 1. So, for all k there exists $x_1^k \in \Omega(u_k) \cap \Omega_g$, and the limit x_0 of the subsequence $\{x_1^{k_m}\}$ belongs to Ω_g. $\qquad\square$

Remark. If Ω is star-shaped then (P_∞) admits only the trivial solution; hence, in such a case, it is clear from the above proofs that Theorems 3 and 4 hold for any bounded sequence of solutions $\{u_k\}$ of (P_k) and not only for a sequence of mp-solutions.

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