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CRITICAL EXPONENTS WHICH RELATE EMBEDDING INEQUALITIES WITH QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. We show that three numbers which are critical for suitable embedding inequalities are also critical for existence results for some m-Laplace quasilinear elliptic problems with polynomial reaction term.

1. **Introduction.** In this paper, we show that three critical exponents for suitable embedding inequalities are related to existence/nonexistence of positive solutions to some second order quasilinear elliptic problems. The idea of writing this paper comes from the striking fact that the very same exponents appear in several different (and apparently unlinked) contexts. Our purpose is precisely to link different kinds of results and to attempt an explanation for these links. To this end, we also suggest a number of open problems.

We consider equations of "polynomial m-Laplace" type such as

$$-\Delta_m u = u^{p-1} \qquad \text{or} \qquad -\Delta_m u = -u^{q-1} + u^{p-1} \tag{1}$$

where $1 < q < p$, $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$ and the equations are considered either in the whole $\mathbb{R}^n$ or in a bounded domain $\Omega \subset \mathbb{R}^n$ combined with suitable boundary conditions. Here and in the sequel, we assume that

$$1 < m < n \ .$$

Throughout the paper, by *solution* of (1) we mean a $W^{1,m}$ weak solution which, according to well-known regularity results [8, 36], belongs to $C^{1,\alpha}_{\rm loc}$ whenever $p \le m^*$ (the Sobolev critical exponent).

2. **Sobolev exponent.**

2.1. **Sobolev inequalities.** The critical Sobolev exponent arises from the following question in functional analysis. Let $\Omega \subseteq \mathbb{R}^n$ be an open domain, which is the largest $q \ge m$ such that the embedding $W^{1,m}(\Omega) \subset L^q(\Omega)$ is continuous? It is well-known [1] that the answer is $q = m^*$ where

$$m^* = \frac{nm}{n-m} \ .$$

It is also well-known that the best Sobolev constant

$$S \ = \inf_{u \in D^{1,m}(\Omega)\setminus\{0\}} \left(\frac{\|\nabla u\|_{L^m(\Omega)}}{\|u\|_{L^{m^*}(\Omega)}} \right)^m \tag{2}$$

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does not depend on the domain Ω and that the infimum on the right hand side of (2) is achieved only if $\Omega \equiv \mathbb{R}^n$. In this case, it is achieved by the one-parameter family of functions (see [35])

$$U_d(x) = d\left[1 + D\left(d^{\frac{m}{n-m}}|x|\right)^{\frac{m}{m-1}}\right]^{-\frac{n-m}{m}} \qquad (d > 0), \tag{3}$$

where $d = U_d(0) = \|U_d\|_\infty$ and $D = D_{m,n} = \frac{m-1}{(n-m)n^{1/(m-1)}}$. Clearly, for any $\alpha > 0$ and any $x_0 \in \mathbb{R}^n$, the function $\alpha U_d(x - x_0)$ is also a positive minimizer for (2). In fact, the functions of the family (3) solve the equation

$$-\Delta_m u = u^{m^*-1} \qquad \text{in } \mathbb{R}^n\ . \tag{4}$$

By combining the results in [4, 20] one has that if $m = 2$, then the functions in (3) are the only positive solutions of (4) having maximum at the origin $x = 0$.

Open Problem 1. Show that the functions in (3) are the only positive solutions (having maximum at the origin) of (4) also when $m \neq 2$. Partial results are obtained in [7]. □

2.2. **Entire solutions.** Let us first consider what Ni-Serrin [25] call the *normal case*. We study the existence of *ground states*, namely nonnegative nontrivial radially symmetric and vanishing at infinity solutions of the equation

$$-\Delta_m u = -u^{q-1} + u^{p-1} \qquad \text{in } \mathbb{R}^n\ . \tag{5}$$

Radial symmetry of smooth *positive* solutions of (5) has been widely studied. First, it has been proved in [20] in the case $m = 2$ under an additional assumption on the decay at infinity of the solution; subsequently, this assumption was removed in [32]. This statement was extended to the case $1 < m < 2$ in [6]. When $m > 2$ the situation is more delicate and radial symmetry of positive solutions of (5) is known only under the additional assumption that the solution admits a unique critical point, see [32].

On the other hand, positive solutions to (5) exist only if $q \geq m$ while if $1 < q < m$ ground states have compact support (a ball). In such case, radially symmetric solutions about different centers and with disjoint supports may be "sticked" together in order to obtain multibump solutions, see again [32] and references therein. Hence, it is readily seen that radial symmetry of nonnegative solutions may fail when $1 < q < m$.

However, all the just recalled results suggest to restrict our attention to radially symmetric solutions. In this situation, we have

Theorem 1. *Assume that* $1 < q < p$.

(*i*) *If* $p < m^*$ *then* (5) *admits a unique ground state.*

(*ii*) *If* $p \geq m^*$ *then* (5) *admits no ground states.*

References. When $p < m^*$, see [19] (existence) and [31] (uniqueness). When $p \geq m^*$, nonexistence follows from a generalized Pohožaev identity [25]. □

Next we turn to what Ni-Serrin [26] call the *anomalous case*:

$$-\Delta_m u = u^{p-1} \qquad \text{in } \mathbb{R}^n\ . \tag{6}$$

In this case we have the following result:

Theorem 2. *Assume that* $p > 1$.

(*i*) *If* $p < m^*$ *then the unique bounded solution of* (6) *is* $u \equiv 0$.

(*ii*) *If* $p \geq m^*$ *then* (6) *admits infinitely many ground state.*

References. For statement (i), see [33] and previous work in [21] for $m = 2$. Statement (ii) is a particular case of [26, Theorem 6.4]; multiplicity is obtained thanks to rescaling. □

2.3. **Solutions in bounded domains.** We consider the problem

$$\begin{cases} -\Delta_m u = u^{p-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{7}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and $p > m$. Again, Sobolev exponent is the "borderline" between existence and nonexistence but this time it is the other way around when compared to Theorem 2:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and let $p > m > 1$.*
 (i) *If $p < m^*$ then* (7) *has at least a positive solution.*
 (ii) *If $p \geq m^*$ and Ω is star-shaped, then* (7) *has no positive solutions.*

References. For the existence of a positive solution when $p < m^*$, see [13] where one also finds references for previous work in the semilinear case $m = 2$. The nonexistence statement for $p \geq m^*$ follows from Pohožaev identity [29] when $m = 2$ and from its extension by Pucci-Serrin [30] when $m \neq 2$; in the latter case the C^2 regularity of solutions is not the correct framework and a subsequent paper by Guedda-Veron [22] clarifies this point. □

In statement (ii) the assumption that Ω is star-shaped is crucial. Indeed, there are well-known examples of contractible domains for which (7) admits a positive solution when $m = 2$ and $p \geq 2^*$; we refer to [27, 28] for general results and a fairly complete list of further references.

Open Problem 2. Extend the existence results in [27, 28] to general $m \neq 2$. Even if we believe the very same statements to be true, this is *not* a simple exercise, it is a difficult problem strictly related to Open Problem 1. □

Therefore, Theorem 3 seems to emphasize a different behavior for (7) when the parameter p is above/below the Sobolev exponent m^* only when combined with suitable geometric assumptions on the domain. In fact, independently of the geometry of the (bounded) domain Ω, there is one big difference between the subcritical case $p < m^*$ and the critical case $p = m^*$. If $1 < p \leq m^*$, solutions of (7) may be sought as critical points of the smooth functional

$$J(u) = \frac{1}{m}\int_\Omega |\nabla u|^m - \frac{1}{m^*}\int_\Omega |u|^{m^*} \qquad u \in W_0^{1,m}(\Omega)\ .$$

We may then consider the *Nehari manifold* relative to J

$$\mathcal{N} = \{u \in W_0^{1,m}(\Omega),\ \langle J'(u), u\rangle = 0,\ u \neq 0\}\ .$$

Finally, we say that a solution u of (7) is a *least energy solution* if $J(u) = \inf_{\mathcal{N}} J$. Then, we have

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be any open bounded domain and let $p > m > 1$.*
 (i) *If $p < m^*$ then* (7) *admits a positive least energy solution.*
 (ii) *If $p = m^*$ then* (7) *admits no least energy solutions.*

References. Statement (i) is essentially proved in [13]; when $p < m^*$ the functional J satisfies the Palais-Smale condition and therefore it admits a mountain-pass critical point which is a least energy solution. It may be chosen nonnegative since $J(|u|) = J(u)$. Finally, it is positive by the maximum principle, see [37]. Statement (ii) is just Proposition 2 in [14]. □

3. **Trace exponent.**

3.1. **Trace inequalities.** In order to maintain some analogies with Sobolev exponent, we start again from a problem in functional analysis. Let $\Omega \subset \mathbb{R}^n$ be an open domain with (nonempty) smooth boundary $\partial\Omega$ and let $\gamma : W^{1,m}(\Omega) \mapsto L^q(\partial\Omega)$ be the trace operator. Which is the largest $q \geq m$ for which the trace operator is continuous? It is known [1, Theorem 5.22] that $q = m_T$ where

$$m_T = \frac{m(n-1)}{n-m} \ .$$

In other words, we have the following result:

$$\exists C = C_\Omega > 0 \ , \qquad \|\gamma u\|_{L^{m_T}(\partial\Omega)} \leq C\|u\|_{W^{1,m}(\Omega)} \qquad \forall u \in W^{1,m}(\Omega)$$

and m_T is the largest exponent possible.

3.2. **The Emden-Fowler inversion.** One of the most common tools used to tackle radially symmetric solutions of quasilinear equations of the kind

$$-\Delta_m u = f(u) \tag{8}$$

is the so-called Emden-Fowler inversion, see e.g. [23]. Writing (8) in radial coordinates ($|x| = r$, $u = u(r)$) and performing the inversion

$$t = \left(\frac{n-m}{m-1}\,\frac{1}{r}\right)^{(n-m)/(m-1)} \ , \qquad y(t) = u(r) \ , \tag{9}$$

the equation (8) becomes the ode

$$\left(|y'|^{m-2}y'\right)' \ + \ t^{-m_T} f(y) \ = \ 0 \tag{10}$$

and the trace exponent m_T appears.

Even if at a first glance this looks very artificial, it has a deep meaning. Indeed, the change of variables (9) is the most natural one as it switches the origin with infinity and it "deletes" the first derivative term so that flex points of the solutions of (10) coincide with the zeros of f. Moreover, as $t \to 0$ (i.e. $|x| \to \infty$ in (8)), the differential operator behaves like the reaction term $f(u)$ times a singularity with the trace exponent. Therefore, this singularity seems to be intrinsic of the operator Δ_m.

3.3. **Inequalities and equations in unbounded domains.** We first consider weak solutions of the quasilinear elliptic problem

$$\begin{cases} -\Delta_m u \geq u^{p-1} & \text{in } \mathbb{R}^n \\ u \geq 0 \ , \quad u \not\equiv 0 & \text{in } \mathbb{R}^n \ . \end{cases} \tag{11}$$

By weak solution u of (11) we mean

$$\int_{\mathbb{R}^n} |\nabla u|^{m-2}\nabla u \nabla\phi \geq \int_{\mathbb{R}^n} u^{p-1}\phi \qquad \forall\phi \in C_c^\infty(\mathbb{R}^n) \ , \quad \phi \geq 0 \ .$$

The first result which highlights the importance of the trace exponent is

Theorem 5. *Assume that $p > 1$.*

(i) If $p \leq m_T$, then (11) has no weak solutions $u \in W^{1,m}_{\rm loc}(\mathbb{R}^n)$.

(ii) If $p > m_T$ then (11) has infinitely many solutions.

References. Statement (i) has been obtained independently and with different proofs in [24, Theorem 12.1] and in [33, Theorem I]. For statement (ii), it is shown in [24, p.53] that the functions $u(x) = \varepsilon(1+|x|^{m/(m-1)})^{(1-m)/(p-m)}$ are entire solutions of (11) provided ε is sufficiently small. □

We also have a somehow "dual" statement. Consider the equation

$$\Delta_m u = u^{q-1} \qquad \text{in } \mathbb{R}^n \setminus \{0\} \tag{12}$$

together with the "boundary conditions"

$$\lim_{x\to 0} |x|^{(n-m)/(m-1)} u(x) = D^{(m-n)/m} \qquad \lim_{|x|\to\infty} u(x) = 0 \ , \tag{13}$$

where $D = D_{m,n}$ is as in (3). Clearly, (13) implies that u has at $x = 0$ the same singularity of the fundamental solution of $-\Delta_m u = \delta_0$. We have

Theorem 6. *Assume that* $q > 1$.

(i) *If* $q < m_T$, *then* (12)-(13) *has a unique nonnegative radial solution.*

(ii) *If* $q \geq m_T$ *then* (12)-(13) *has no solution.*

References. Statement (i) when $q = m$ is proved in [17] while for general $q < m_T$ we refer to [11]. If $q < m$ the solution has compact support while if $q \geq m$ it is positive [11]; moreover, if $q = m$ it has exponential decay [18]. Statement (ii) is a straightforward consequence of [38, Theorem 1.1]. □

3.4. **A doubly critical Neumann problem.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the Neumann problem

$$\begin{cases} -\Delta_m u + u^{m-1} = u^{m^*-1} - \alpha u^{q-1} & \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{14}$$

where $\alpha > 0$ and $m < q < m^*$. Clearly, (14) admits a constant solution which may or may not be an "interesting" solution according to the following definition. As in Section 2.3 we consider the related action functional

$$J(u) = \frac{1}{m}\|\nabla u\|^m_{L^m(\Omega)} + \frac{1}{m}\|u\|^m_{L^m(\Omega)} + \frac{\alpha}{q}\|u\|^q_{L^q(\Omega)} - \frac{1}{m^*}\|u\|^{m^*}_{L^{m^*}(\Omega)} \ ,$$

the associated Nehari manifold

$$\mathcal{N} = \{u \in W^{1,m}(\Omega),\ \langle J'(u), u\rangle = 0,\ u \neq 0\} \ ,$$

and we say that a solution u of (14) is a least energy solution if $J(u) = \inf_{\mathcal{N}} J$. Then, we have

Theorem 7. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded, let* $\alpha > 0$ *and let* $m < q \leq m_T$.

(i) *If* $q < m_T$, *then* (14) *admits a least energy solution for all* $\alpha > 0$.

(ii) *If* $m = 2$, $n \geq 5$, $q = 2_T$, *then there exists* $\alpha_0 = \alpha_0(\Omega) > 0$ *such that if* $\alpha < \alpha_0$ *then* (14) *admits a least energy solution, while if* $\alpha > \alpha_0$ *then* (14) *admits no least energy solutions.*

References. Statement (i) is proved in [39, Theorem 4.3] (case $m = 2$) and [40] ($m > 1$). Statement (ii) is proved in [5]. In case (ii), the "limit" situation $\alpha = \alpha_0$ is discussed in [5]. □

Open Problem 3. Remove the restriction on the dimension $n \geq 5$ in statement (ii). It is known that existence results for *semilinear* critical growth problems are different in dimension $n = 3$ [3], $n = 4$ [16], and $n \geq 5$. □

Open Problem 4. Prove Theorem 7 (*ii*) in the quasilinear case $m \neq 2$. This statement seems difficult to obtain in this case; indeed, when $m = 2$ one takes advantage of the fact that $2(2_T - 2) = 2^* - 2$, see [5]. It seems that if $m \neq 2$ a technical obstruction arises [34]. Moreover, one could also study the case $m_T < q < m^*$; we conjecture that in such case least energy solutions do not exist for any $\alpha > 0$. □

4. Critical remainder exponent.

4.1. Sobolev inequalities with remainder terms.

As already mentioned in Section 2.1, the best Sobolev constant S for the embedding $D^{1,m}(\Omega) \subset L^{m^*}(\Omega)$ is independent of the domain $\Omega \subset \mathbb{R}^n$ and it is *not attained* whenever Ω is bounded. Then, it is natural to inquire if for some $q \geq 1$ there exists a constant $C = C(\Omega, q) > 0$ such that

$$\|\nabla u\|^m_{L^m(\Omega)} \geq S\|u\|^m_{L^{m^*}(\Omega)} + C\|u\|^m_{L^q(\Omega)} \qquad \forall u \in W^{1,m}_0(\Omega) \ .$$

And, if affirmative, which is the largest exponent q such that $C(\Omega, q) > 0$?

Let us introduce a third (smaller) critical exponent, namely

$$m_R = \frac{n(m-1)}{n-m} = m_T - 1 \ ;$$

then, the answer is given by the following

Theorem 8. *Let $n > m > 2 - \frac{1}{n}$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Then, for all $q \in [1, m_R)$ there exists $C_q > 0$ such that*

$$\|\nabla u\|^m_{L^m(\Omega)} \geq S\|u\|^m_{L^{m^*}(\Omega)} + C_q\|u\|^m_{L^q(\Omega)} \qquad \forall u \in W^{1,m}_0(\Omega) \ .$$

Moreover, $\lim_{q \to m_R} \overline{C_q} = 0$, *where*

$$\overline{C_q} \ = \ \inf_{\|u\|_q = 1} \left(\|\nabla u\|^m_m - S\|u\|^m_{m^*}\right) \ .$$

References. See [3] for a proof based on the Pohožaev identity in the case $m = 2$. A similar proof for general $m > 1$ may be found in [9], see also [15]. A direct proof yielding a stronger result which involves weak norms in the case $m = 2$ is given in [2]. □

For this reason we call the number m_R the *critical remainder exponent.*

Open Problem 5. Prove the result in [2] in the $W^{1,m}$ setting, namely that for any bounded domain $\Omega \subset \mathbb{R}^n$ there exists $C > 0$ such that

$$\|\nabla u\|^m_{L^m(\Omega)} \geq S\|u\|^m_{L^{m^*}(\Omega)} + C\|u\|^m_{L^{m_R}_w(\Omega)} \qquad \forall u \in W^{1,m}_0(\Omega) \ .$$

When repeating the proof in [2] a "wrong" weak norm appears, see [15]. □

Open Problem 6. Remove the assumption $m > 2 - \frac{1}{n}$ in Theorem 8. We believe that it holds true for all $n > m > 1$. □

The critical remainder exponent m_R has also a different characterization in terms of summability of the minimizers for the Sobolev ratio (2):

Theorem 9. *Consider the functions U_d of the family* (3) *and let $q \geq 1$. Then $U_d \in L^q(\mathbb{R}^n)$ if and only if $q > m_R$.*

References. This is just a simple calculation, see [11]. □

4.2. **Asymptotic behavior of ground states.** A particular interesting behavior is exhibited by the L^∞ norm of ground states to the following equation as p approaches (from below) the Sobolev exponent m^*:

$$-\Delta_m u = -u^{q-1} + u^{p-1} \qquad \text{in } \mathbb{R}^n \ . \tag{15}$$

By Theorem 1, (15) admits a unique ground state for all $1 < q < p < m^*$ while it admits no ground states if $1 < q < p = m^*$. Therefore, it is of some interest to study the case where p approaches the Sobolev exponent m^*.

We quote a result from [11]; as far as we are aware it is the only statement involving all the three critical exponents:

Theorem 10. *For all $1 < q < p < m^*$ let u be the unique ground state of* (15). *Then there exist constants $\alpha_{m,n,q}, \beta_{m,n}, \gamma_{m,n,q} > 0$ such that*

$$\lim_{p\to m^*} (m^* - p)[u(0)]^{m^*-q} = \alpha_{m,n,q} \qquad \text{if } q > m_R$$

$$\lim_{p\to m^*} \frac{m^* - p}{|\log(m^* - p)|}[u(0)]^{\frac{n}{n-m}} = \beta_{m,n} \qquad \text{if } q = m_R$$

$$\lim_{p\to m^*} (m^* - p)[u(0)]^{\frac{m^*-q}{m_T-q}} = \gamma_{m,n,q} \qquad \text{if } q < m_R \ .$$

References. See Theorem 2 in [11] and previous work in the "linear" case $q = m$ in [17, 18]. The constants $\alpha_{m,n,q}, \beta_{m,n}, \gamma_{m,n,q}$ are explicitly determined in [11] and it is shown that

$$\lim_{q\downarrow m_R} \alpha_{m,n,q} = +\infty \ , \qquad \lim_{q\uparrow m_R} \gamma_{m,n,q} = +\infty \ .$$

Note that $\frac{m^*-q}{m_T-q} = m^* - q = \frac{n}{n-m}$ whenever $q = m_R$ and this gives "continuity of the exponents" in the above statement. □

5. **Concluding remarks.** All the above listed results show that the three exponents m^*, m_T and m_R which are critical for embedding inequalities are also critical for existence results for quasilinear equations of the kinds (1). In (1) polynomial reaction terms are considered and this fact seems to be the key ingredient to introduce L^p spaces in the discussion and, subsequently, explanations in terms of embedding inequalities. It is shown in [10] that when more general reaction terms are considered, a precise definition of "subcritical" seems not available. However, it is by now clear that there exists a strict link between embedding inequalities and quasilinear elliptic problems with power-like reaction terms. What is missing is a full explanation of this fact. On one hand some embedding inequalities may be obtained as consequence of suitable minimization problems for constrained functionals; if the minimum exists, then we have a solution for the Euler-Lagrange equation. On the other hand, is there a general way to deduce results for equations directly from embedding inequalities and vice-versa? If affirmative, which is the role of the domain (bounded/unbounded), of the boundary conditions (Dirichlet/Neumann)? And which exponent explains which phenomena?

The subsequent steps of the preliminary remarks contained in this paper could be to try to tackle all these problems from a different point of view. Before wondering about existence/nonexistence results for (8) one should try to understand *a priori* which are the critical exponents and which embedding inequalities are responsible of the answer. To this end, a promising model problem is considered in [12]: it involves an anisotropic quasilinear elliptic operator which fails to possess some of the typical features of elliptic operators such as regularity theory, maximum principles, homogeneous eigenvalue problems.

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