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Positivity preserving property for a class of biharmonic elliptic problems

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Abstract

The lack of a general maximum principle for biharmonic equations suggests to study under which boundary conditions the positivity preserving property holds. We show that this property holds in general domains for suitable linear combinations of Dirichlet and Navier boundary conditions. The spectrum of this operator exhibits some unexpected features: radial data may generate nonradial solutions. These boundary conditions are also of some interest in semilinear equations, since they enable us to give explicit radial singular solutions to fourth order Gelfand-type problems.

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1. Introduction

When dealing with fourth order elliptic equations of the form

$$\Delta^2 u = \phi \quad \text{in } \Omega, \tag{1}$$

where Ω is a smooth bounded domain of $\mathbf{R}^n$ ($n \geqslant 2$) and $\phi \in L^2(\Omega)$, one must face the lack of a general maximum or comparison principle (positivity preserving property). More precisely, it

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is not yet clear under which assumptions the following implication holds:

$$\phi \geqslant 0 \quad \text{in } \Omega \quad \Longrightarrow \quad u \geqslant 0 \quad \text{in } \Omega. \tag{2}$$

It is well known that for higher order problems, and more generally for elliptic systems, the lack of the maximum principle strongly depends on the kind of boundary conditions imposed on the solutions and, for systems, on some kind of quasi-monotonicity of the problem under consideration [20]. For problems of the form (1), the choice of the boundary conditions is usually related to the physical problem described by the equation, see [25].

In the case of *Navier* boundary conditions, i.e., $u = \Delta u = 0$ on $\partial\Omega$, implication (2) holds. In this case, the positivity property is not sensitive to the geometric or topological properties of the domain Ω. In fact, *smoothness* of the domain is here crucial since it has recently been shown by Nazarov and Sweers [21] that for planar domains with an interior corner, (2) may not hold.

On the other hand, if we consider *Dirichlet* boundary conditions $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, (2) is guaranteed only in some special domains. About one century ago, Boggio [5] and Hadamard [16] believed, by physical intuition, that (2) would hold in convex domains. Their conjecture was also supported by the validity of (2) in balls, see [6]. Only much later this conjecture was disproved: for example, it was shown that (2) does not hold for certain ellipses [13,23] and for squares [9]. On the contrary, more recent results show that (2) holds for domains close (in a suitable sense) to the planar disk, see [15,22]. Furthermore, it has been proved in [12] that (2) may also hold for some nonconvex domains. See [11,14] for up to date results on this problem.

In this paper we study the positivity problem for (1) trying to give at least an initial answer to the following question:

which boundary conditions guarantee that (2) *holds in a general domain* Ω?

Of course, it is not a simple matter to give a complete answer to this question but, as we shall see during the course, boundary conditions of particular interest seem to be

$$u = \Delta u - d\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $d \in \mathbf{R}$. These conditions are in some sense intermediate between Dirichlet conditions (corresponding to $d = -\infty$) and Navier conditions (corresponding to $d = 0$).

We show that (1), combined with (3) for suitable values of d, enjoys of the positivity preserving property. We also study the "boundary spectrum" of (1)–(3), namely the values of d for which there exist nontrivial solutions when $\phi \equiv 0$. This spectrum has some unexpected features. For instance, when Ω is the unit ball and ϕ is radial, one may find nonradial solutions to (1)–(3), see Section 3.

Even more involved appear *semilinear* elliptic problems of the form:

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - d\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\lambda > 0$ and $f \in C^2(\mathbf{R}_+, \mathbf{R}_+)$ is an increasing and strictly convex function with $f(0) = 1$ so that $u \equiv 0$ is not a solution of (4). For (4) we must face all the typical problems associated with the corresponding second order equation

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega \tag{5}$$

such as nonuniqueness of solutions, nonexistence, breaking of symmetry, lack of compactness related to critical growth of the function f, as well as the already mentioned lack of a general comparison principle. We show that when d is in the range which ensures (2), the results available for (5) may be extended to (4).

A further argument in favor of the boundary conditions (3) is the possibility of writing explicit singular solutions of (4) in the unit ball $\mathbf{B}$. We consider just the simplest cases where

$$f(s) = e^s \quad \text{or} \quad f(s) = (1+s)^p \quad \text{with } p > \frac{n}{n-4}. \tag{6}$$

It is well known [8,17–19] that if $\Omega = \mathbf{B}$, then the corresponding second order equation (5) under Dirichlet boundary conditions admits explicit singular solutions for suitable values of λ. For nonlinearities as in (6) the equation in (4) still admits explicit singular solutions in $\mathbf{B}$ but they do not satisfy neither the Dirichlet nor the Navier boundary conditions, see [2,4]. For this reason, these singular solutions were called *ghost solutions* in [4]. In this work, for the nonlinearities in (6), we find d (depending on the dimension n and on f) such that these singular solutions satisfy the boundary conditions in (4) and the value of d lies precisely in the positivity preserving range.

This paper is organized as follows: in next section we state our main results. In Section 3 we consider the positivity preserving property of the linear problem (1)–(3) together with the naturally associated boundary eigenvalue problem. In the same section we state some questions that remain to be solved and that we think are of particular interest for the continuation of this work. We also give some hints on how to proceed for general polyharmonic semilinear problems. Finally, in Sections 4–10 we prove our results.

2. Main results

For a smooth bounded domain $\Omega \subset \mathbf{R}^n$ ($n \geqslant 2$), we define $\mathcal{H}(\Omega) := [H^2 \cap H_0^1(\Omega)] \setminus H_0^2(\Omega)$ and

$$\sigma := \inf_{u \in \mathcal{H}(\Omega)} \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2}. \tag{7}$$

The minimization problem (7) is related to the following linear equation

$$\begin{cases} \Delta^2 u = \phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - d\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $\phi \in L^2(\Omega)$ and $d \in \mathbf{R}$. By *solution* of (8) we mean a weak solution, namely $u \in H^2 \cap H_0^1(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} = \int_\Omega \phi v \quad \text{for all } v \in H^2 \cap H_0^1(\Omega).$$

Let us mention that although most of the functions we consider are not continuous, throughout the paper we omit writing "a.e." in Ω or on $\partial\Omega$ (in the former case with respect to the Lebesgue measure, in the latter with respect to the $(n-1)$-dimensional Hausdorff measure).

In view of the characterization (7), it is not difficult to show that, for $d < \sigma$, the map

$$u \mapsto \left(\int_\Omega |\Delta u|^2 - d \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 \right)^{1/2} =: \|u\|_d$$

is a norm on $H^2 \cap H_0^1(\Omega)$ that we shall denote by $\|\cdot\|_d$.

Our first results concern problems (7) and (8).

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ $(n \geqslant 2)$ be a smooth connected bounded domain, let σ be as in* (7) *and let $d < \sigma$. Then*:

(i) *For any $\phi \in L^2(\Omega)$, problem* (8) *admits a unique solution $u \in H^4(\Omega)$. Moreover* (8) *is satisfied pointwise in Ω. In particular, if $\phi \in H^k(\Omega)$ for some $k \geqslant 1$, then $u \in H^{k+4}(\Omega)$.*
(ii) *If $d \geqslant 0$ and $\phi \geqslant 0$ $(\phi \not\equiv 0)$ in Ω, then the solution u of* (8) *satisfies $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.*
(iii) *The infimum in* (7) *is achieved and, up to a multiplicative constant, the minimizer $\bar u$ for* (7) *is unique, strictly positive in Ω, satisfies $\frac{\partial \bar u}{\partial \nu} < 0$ on $\partial\Omega$ and it solves* (8) *when $d = \sigma$ and $\phi \equiv 0$. Furthermore, $\bar u \in C^\infty(\Omega)$ and, up to the boundary, $\bar u$ is as smooth as the boundary permits.*

In Theorem 1 the upper bound $d < \sigma$ is sharp. Indeed, if we take $d = \sigma$, then Theorem 1(iii) and Fredholm alternative tell us that problem (8) lacks either existence or uniqueness, depending on ϕ. On the other hand, we do not know if the positivity preserving property (2) in Theorem 1 continues to hold for $d < 0$. Let us mention that it does hold in the 1-dimensional case:

Theorem 2. *Let $\delta > 0$ and let $u \in C^4(-1,1) \cap C^2[-1,1]$ be such that $u \not\equiv 0$ and*

(i) $u^{\mathrm{iv}} \geqslant 0$ *in* $(-1,1)$;
(ii) $u(\pm 1) = 0$;
(iii) $u''(1) + \delta u'(1) = u''(-1) - \delta u'(-1) = 0$.

Then, $u > 0$ in $(-1,1)$.

In the case $\Omega = \mathbf{B}$ (the unit ball), we may compute explicitly σ (defined in (7)) and determine the minimizer.

Theorem 3. *Let $\mathbf{B} \subset \mathbf{R}^n$ $(n \geqslant 2)$ be the unit ball centered at zero. Then,*

$$\int_{\mathbf{B}} |\Delta u|^2 \geqslant n \int_{\partial \mathbf{B}} \left| \frac{\partial u}{\partial \nu} \right|^2 \quad \textit{for all } u \in H^2 \cap H_0^1(\mathbf{B}). \tag{9}$$

Moreover, the constant n is optimal and equality holds in (9) *if and only if $u(x) = \alpha(1 - |x|^2)$ for some $\alpha \in \mathbf{R}$.*

For the linear problem, our last result concerns the first eigenvalue λ_1 of the operator Δ^2 under the boundary conditions in (8), namely

$$\lambda_1 := \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 - d \int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^2}{\int_\Omega u^2}. \tag{10}$$

We prove:

Theorem 4. *Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded connected domain, let σ be as in* (7) *and let $0 \leqslant d < \sigma$. Then the first eigenfunction ϕ_1 (corresponding to λ_1 in* (10)) *is strictly of one sign in Ω.*

We now turn to the nonlinear problem (4). For simplicity, we restrict here to space dimensions $n \geqslant 5$. So, let $\Omega \subset \mathbf{R}^n$ $(n \geqslant 5)$ be a smooth bounded domain and fix $q > \frac{n}{4}$ such that $q \geqslant 2$. Then, we set:

$$X_d(\Omega) := \left\{ v \in W^{4,q}(\Omega);\ v = \Delta v - d \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

Following [2] we consider solutions of (4) according to

Definition 5. We say that $u \in L^2(\Omega)$ is a solution of (4) if $f(u) \in L^1(\Omega)$ and

$$\int_\Omega u \Delta^2 v = \lambda \int_\Omega f(u) v \quad \text{for all } v \in X_d(\Omega).$$

Moreover, if $u \in L^\infty(\Omega)$ we say that u is *regular* while if $u \notin L^\infty(\Omega)$ we say that u is *singular*. Finally, we say that a solution u_λ of (4) is *minimal* if $u_\lambda \leqslant u$ in Ω, for any other possible solution u of (4).

For any

$$f \in C^2(\mathbf{R}_+, \mathbf{R}_+) \text{ is increasing and strictly convex with } f(0) = 1 \tag{11}$$

we set $\alpha_f := \max\{\alpha > 0\colon\ f(s) \geqslant \alpha s \text{ for all } s \geqslant 0\} > 0$. Theorem 1 shows that the positivity preserving property holds for (8), provided that $d < \sigma$. This fact enables us to prove the following theorem.

Theorem 6. *Assume that f satisfies* (11) *and let $0 \leqslant d < \sigma$. Then, there exists λ^* with*

$$0 < \lambda^* \leqslant \frac{\lambda_1}{\alpha_f},$$

such that (4) *admits a positive minimal solution if $0 < \lambda < \lambda^*$ and no solutions if $\lambda > \lambda^*$. Moreover, the minimal solutions are pointwise increasing with respect to λ. Finally, if the nonlinearity f satisfies the additional condition*

$$\liminf_{s \to +\infty} \frac{f'(s)s}{f(s)} > 1 \tag{12}$$

and for every $0 < \lambda < \lambda^*$ the minimal solution is regular, then (4) *admits at least a positive solution also for $\lambda = \lambda^*$.*

At this point one should determine assumptions on f which ensure that minimal solutions are regular. The next two statements give a partial answer to this question.

Theorem 7. *Assume that f satisfies* (11) *and let $0 \leqslant d < \sigma$. If f also satisfies*

$$s \mapsto \log f(s) \quad \textit{is nonconstant increasing and convex}, \tag{13}$$

then, for every $0 < \lambda < \lambda^*$, the minimal solution of (4) *is regular.*

Theorem 8. *Assume that $0 \leqslant d < \sigma$ and $1 < p \leqslant \frac{n+4}{n-4}$. Then, the solutions of* (4) *with $f(s) = (1+s)^p$ are regular. Moreover, if $1 < p < \frac{n+4}{n-4}$,* (4) *admits at least two positive solutions if $0 < \lambda < \lambda^*$ and a unique solution if $\lambda = \lambda^*$.*

Our last statement concerns problem (4) in the case where f satisfies (6) and $\Omega = \mathbf{B}$. A crucial role will be played by the functions $h(s) := \frac{f'(s)}{f''(s)}$ and $g(s) := (h(s))^2(f'(s) - 1)$ and by the two values $h(0)$ and $g(0)$. Let us remark that

$$\begin{aligned} f(s) = e^s \quad &\Longrightarrow \quad h(0) = 1, \quad g(0) = 0, \quad \text{and} \\ f(s) = (1+s)^p \quad &\Longrightarrow \quad h(0) = g(0) = \frac{1}{p-1}. \end{aligned} \tag{14}$$

Setting

$$\lambda_\sigma := 8h(0)\big(2g(0)+1\big)\big(n-2-4g(0)\big)\big(n-4-4g(0)\big), \tag{15}$$

we have

$$\begin{aligned} f(s) = e^s \quad &\Longrightarrow \quad \lambda_\sigma = 8(n-2)(n-4), \quad \text{and} \\ f(s) = (1+s)^p \quad &\Longrightarrow \quad \lambda_\sigma = \frac{8}{(p-1)^4}\big[(n-2)(n-4)(p-1)^3 \\ &\qquad + 2\big(n^2 - 10n + 20\big)(p-1)^2 - 16(n-4)(p-1) + 32\big]. \end{aligned}$$

Remark 9. Condition $p > \frac{n}{n-4}$ in (6) ensures that $\lambda_\sigma > 0$ also in the power case. Let us mention that if one also allows $1 < p \leqslant \frac{n}{n-4}$ then, at least formally, one has $\lambda_\sigma = 0$ if $p = \frac{n+4-2i}{n-2i}$ for some $i = 1, 2$. In this case, in view of Theorem 10 below, the singular solution becomes $v_\sigma(x) := |x|^{2i-n} - 1$ which is, up to the addition of a constant, the fundamental solution of $(-\Delta)^i$. This explains why $\Delta^2 v_\sigma = 0$.

Furthermore, if we choose

$$d := n - 2 - 4g(0), \tag{16}$$

we have $0 < d < n$, so that, in view of Theorems 1 and 3, the problems are well-posed in $\mathbf{B}$ and the positivity preserving property holds. Let us rewrite (4) with the choice of f as in (6), of $\Omega = \mathbf{B}$ and d as in (16). When $f(u) = e^u$ we obtain

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \\ \Delta u - (n-2)\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{17}$$

while if $f(u) = (1+u)^p$ we have

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \\ \Delta u - \frac{(n-2)p-(n+2)}{p-1}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{18}$$

We can now state our result on singular solutions:

Theorem 10. *Let $\lambda = \lambda_\sigma$. Then, the function $u_\sigma(x) := -4\log|x|$ (respectively $v_\sigma(x) := |x|^{-\frac{4}{p-1}} - 1$) is a singular radial solution of problem* (17) (*respectively* (18) *with $p > \frac{n}{n-4}$*) *in* $\mathbf{B}$.

When $f(s) = (1+s)^p$, the singular solution v_σ found in Theorem 10 is a solution of (18), according to Definition 5, only if $p > \frac{8+n}{n}$. Indeed, if $p \leqslant \frac{8+n}{n}$ then $v_\sigma \notin L^2(\mathbf{B})$, see Remark 27 at the end of the paper (notice that $\frac{n}{n-4} \geqslant \frac{8+n}{n} \Leftrightarrow n \leqslant 8$).

3. Further results and open problems

3.1. A boundary eigenvalue problem

In view of Theorem 1(iii), minimizers of (7) solve the linear problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - d\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

when $d = \sigma$. Problem (19) may be seen as a boundary eigenvalue problem. Theorem 1(iii) states that the least eigenvalue is $d = \sigma$, that it is simple and that the corresponding eigenfunction is of one sign. A natural question is then

what about other eigenvalues?

In the one-dimensional case (19) reads

$$u^{\mathrm{iv}} = 0 \quad \text{in } (-1, 1), \quad u(\pm 1) = u''(-1) + du'(-1) = u''(1) - du'(1) = 0. \tag{20}$$

Some computations lead to the following statement:

Proposition 11. *Problem* (20) *admits a nontrivial solution if and only if $d \in \{1, 3\}$. When $d = 1$ an eigenfunction is given by $u(x) = 1 - x^2$, whereas if $d = 3$ we have $u(x) = x^3 - x$.*

Therefore, besides the first (radial) eigenvalue $d = 1$, there exists just another (nonradial) eigenvalue $d = 3$. Then, a second natural question is

if $n \geqslant 2$, is it possible to describe the spectrum for any bounded domain $\Omega \subset \mathbf{R}^n$?

When Ω is the unit ball, the following holds:

Proposition 12. *If $\Omega = \mathbf{B}$, problem* (19) *admits a unique radial eigenvalue given by $d = n$.*

Proof. One solves the corresponding ordinary differential equation $(r^{n-1}(\Delta u)')' = 0$ and finds $u(r) = \alpha + \beta r^2 + \gamma r^{4-n} + \delta r^{2-n}$. Then $\delta = 0$, since otherwise $u \notin H^1(\mathbf{B})$. Moreover, $\gamma = 0$ because $|x|^{4-n}$ is the fundamental solution which satisfies $\Delta^2 |x|^{4-n} = \delta_0$ (the Dirac mass at the origin). Finally, forcing u to satisfy the boundary conditions in (19) yields $d = n$ and $u(r) = c(1 - r^2)$ for any $c \in \mathbf{R}$. □

Then, a further question is:

when $\Omega = \mathbf{B}$, can we characterize nonradial eigenvalues for (19)?

Inspired by the one-dimensional case, we see that $d = n + 2$ is an eigenvalue with corresponding (independent) eigenfunctions $u_i(x) = (1 - |x|^2)x_i$ for $i = 1, \dots, n$. By continuing the pattern, we find that $d = n + 4$ is an eigenvalue with corresponding eigenfunctions $v_i(x) = (1 - |x|^2) \times (x_1^2 - x_i^2)$ for $i = 2, \dots, n$. Summarizing, we have:

Proposition 13. *When $\Omega = \mathbf{B}$, $d = n + 2$ is a nonradial eigenvalue having* (*at least*) *multiplicity n and $d = n + 4$ is a nonradial eigenvalue having* (*at least*) *multiplicity $n - 1$.*

Our final questions then read:

are the multiplicities in Proposition 13 *exact*? *What about multiplicities of other eigenvalues*?

3.2. The positivity preserving property for the linear problem

We consider a simple example which shows how positivity is lost in (8) as d reaches σ and in which way it is asymptotically restored when $d \to \infty$. We consider the problem

$$\begin{cases} \Delta^2 u = 1 & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \\ \Delta u - d\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbf{B}, \end{cases} \tag{21}$$

where $\mathbf{B}$ is the unit ball in $\mathbf{R}^3$. Some computations show that (21) admits a unique radial solution for all $d \neq 3$ which is given by

$$u_d(x) = \frac{|x|^4}{120} - \frac{5-d}{60(3-d)}|x|^2 + \frac{7-d}{120(3-d)}.$$

It is readily seen that $u_d > 0$ in $\mathbf{B}$ whenever $d < 3$. Moreover, $u_d < 0$ for all $d \in (3, 7)$. For $d = 7$, $u_d(0) = 0$ but u_d remains strictly negative elsewhere. If $d > 7$ then u_d is sign-changing. The normal derivative on the boundary is $u_d'(1) = \frac{1}{15(d-3)}$ which is positive precisely for $d > 3$. Finally, as $d \to \infty$ we have $u_d(x) \to \frac{(1-|x|^2)^2}{120}$ which is positive.

Let us also mention that, in view of Proposition 13, when $d = 5$ or $d = 7$, problem (21) admits infinitely many nonradial solutions: it suffices to add to u_d any of the eigenfunctions u_i or v_i found in the previous subsection.

More generally, if we replace "1" in (21) with "$|x|^k$" (for some $k \geqslant 0$) then the solutions become

$$\frac{|x|^{k+4}}{(k+2)(k+3)(k+4)(k+5)} - \frac{(k+5-d)|x|^2}{2(k+2)(k+3)(k+5)(3-d)} + \frac{k^2+9k+14-d(k+2)}{2(k+2)(k+3)(k+4)(k+5)(3-d)}.$$

These functions exhibit the same behavior as u_d for varying d.

3.3. Boundary conditions for polyharmonic problems

We explain here how one should proceed in order to determine boundary conditions satisfied by both the explicit singular solutions u_σ and v_σ for the polyharmonic problem

$$\begin{cases} (-\Delta)^m u = \lambda f(u) & \text{in } \mathbf{B}, \\ u = 0 & \text{on } \partial\mathbf{B}, \\ \frac{\partial^{i+1} u}{\partial \nu^{i+1}} + a_i \frac{\partial^i u}{\partial \nu^i} = 0 & \text{on } \partial\mathbf{B},\ i = 1, \dots, (m-1), \end{cases} \tag{22}$$

where the a_i's are constants depending on f (which is of the kind (6)). We consider here boundary conditions in the radial setting. In this framework, when $m = 2$, the boundary conditions in (17) and (18) become

$$\frac{\partial^2 u}{\partial \nu^2} + a\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathbf{B}, \tag{23}$$

with $a = 1 + 4g(0)$. For general $m \geqslant 1$, the singular solutions are given by $u_\sigma(x) := -2m \log|x|$ (exponential nonlinearity) and $v_\sigma(x) := |x|^{-\frac{2m}{p-1}} - 1$ (power nonlinearity), see Theorem 24 below. We fix some integer q such that $m \leqslant q \leqslant 2m-1$ and we introduce the column vectors in $\mathbf{R}^{q+1}$:

$$U_\sigma = \big[u_\sigma(1), u'_\sigma(1), u''_\sigma(1), \ldots, u^{(q)}_\sigma(1)\big]^{\mathrm{T}} \quad \text{and} \quad V_\sigma = \big[v_\sigma(1), v'_\sigma(1), v''_\sigma(1), \ldots, v^{(q)}_\sigma(1)\big]^{\mathrm{T}}. \tag{24}$$

Then, to find boundary conditions satisfied by both v_σ and u_σ corresponds to determining a $m \times (q+1)$ matrix $M = [b_{ij}]$ having rank m and such that

$$MU_\sigma = 0 \quad \text{and} \quad MV_\sigma = 0.$$

Notice that the restriction $q \geqslant m$ is necessary. Indeed, if we take $q = m-1$ we obtain a square matrix M. If M is singular it gives rise to an underdetermined boundary value problem, whereas if M is invertible it gives rise to Dirichlet boundary conditions which we already know must be avoided.

Clearly, with this lower bound for q the simplest choice is $q = m$. Moreover, in order to have a vanishing trace for u_σ and v_σ we set

$$b_{11} = 1, \qquad b_{i1} = 0 \quad \text{for all } i = 2, \ldots, m, \qquad b_{1j} = 0 \quad \text{for all } j = 2, \ldots, m+1. \tag{25}$$

In order to simplify further, one should choose most of the remaining b_{ij}'s equal to 0. We decide to set $b_{ij} = 0$ for $j \neq i, i+1$. Therefore, we are, finally, led to consider a matrix of the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & a_1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & a_2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & a_{m-2} & 1 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & a_{m-1} & 1 \end{pmatrix}$$

and we choose the a_i's as in (45) in order to fulfill the boundary conditions. Clearly, different choices of q and of the b_{ij}'s can be made.

We conclude this section, by showing that elliptic regularity may be applied to (22).

Proposition 14. *Let $\Omega \subset \mathbf{R}^n$ be a smooth domain, not necessarily bounded. For any choice of the constants a_i's, the boundary conditions in* (22) *satisfy the complementing condition.*

Proof. We follow the notation of [1, pp. 625–633]. For all $\xi \in \mathbf{R}^n$, the principal part of the characteristic polynomial associated to the operator $(-\Delta)^m$ is $L'(\xi) = |\xi|^{2m}$. If ν denotes the outward unit normal to $\partial\Omega$ and ξ any vector parallel to $\partial\Omega$, we get that $\tau = i|\xi|$ is the unique root with positive imaginary part of $L'(\xi + \tau\nu)$, as polynomial in τ, and it has multiplicity m. Hence,

$$M^+(\xi, \tau) = \big(\tau - i|\xi|\big)^m = \sum_{j=0}^{m} \gamma_j \tau^j, \quad \text{where } \gamma_j = \binom{m}{j}\big(-i|\xi|\big)^{m-j}.$$

We associate to the boundary conditions in (22) the polynomials

$$B_i'(\xi+\tau\nu)=\begin{cases}1 & \text{if } i=0,\\ \tau^{i+2} & \text{if } i=1,\dots,m-1.\end{cases}$$

To prove that the complementing condition holds one has to check that, for every $\xi\neq 0$, the $B_j'(\xi+\tau\nu)$ are linearly independent modulo $M^+(\xi,\tau)$. This follows by observing that:

$$B_i'(\xi+\tau\nu)=\begin{cases}1 \bmod [M^+(\xi,\tau)] & \text{if } i=0,\\ \tau^{i+2} \bmod [M^+(\xi,\tau)] & \text{if } i=1,\dots,m-3,\\ -\sum_{j=0}^{m-1}\gamma_j\tau^j \bmod [M^+(\xi,\tau)] & \text{if } i=m-2,\\ \bigl(\gamma_{m-1}\sum_{j=0}^{m-1}\gamma_j\tau^j-\sum_{j=1}^{m-1}\gamma_{j-1}\tau^j\bigr) \bmod [M^+(\xi,\tau)] & \text{if } i=m-1\end{cases}$$

which, for every $\xi\neq 0$, are linearly independent as polynomials in τ. $\square$

4. Proof of Theorem 1

We first prove that the complementing condition is satisfied.

Lemma 15. *The boundary conditions in* (8) *satisfy the complementing condition.*

Proof. We follow again the notation of [1, pp. 625–633]. We have $M^+(\xi,\tau)=(\tau-i|\xi|)^2=\tau^2-2i|\xi|\tau-|\xi|^2$, while the polynomials associated to the boundary conditions are $B_1'(\xi+\tau\nu)=1$ and $B_2'(\xi+\tau\nu)=\tau^2+|\xi|^2$. The complementing condition holds if we show that, for every $\xi\neq 0$, $B_1'(\xi+\tau\nu)$ and $B_2'(\xi+\tau\nu)$ are linearly independent modulo $M^+(\xi,\tau)$. This follows by observing that

$$B_1'(\xi+\tau\nu)=1 \mod \bigl[M^+(\xi,\tau)\bigr],\qquad B_2'(\xi+\tau\nu)=2i|\xi|\tau+2|\xi|^2 \mod \bigl[M^+(\xi,\tau)\bigr]$$

which, for every $\xi\neq 0$, are linearly independent as polynomials in τ. $\square$

Proof of Theorem 1(i). On the space $H^2\cap H_0^1(\Omega)$ we define the bilinear form

$$A(u,v):=\int_\Omega \Delta u\Delta v-d\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{\partial u}{\partial\nu}.$$

This form is clearly continuous and symmetric. Moreover, by definition of σ, we have

$$A(u,u)=\int_\Omega|\Delta u|^2-d\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2\geqslant\frac{\sigma-d}{\sigma}\int_\Omega|\Delta u|^2\quad\text{for all } u\in H^2\cap H_0^1(\Omega)$$

so that the form A is coercive. Since $H^2 \cap H_0^1(\Omega) \subset L^2(\Omega)$, we have $L^2(\Omega) \subset [H^2 \cap H_0^1(\Omega)]'$ (the dual space). Hence, by applying the Lax–Milgram Theorem, we infer that for any $\phi \in L^2(\Omega)$, the problem

$$\int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} = \int_\Omega \phi v \quad \text{for all } v \in H^2 \cap H_0^1(\Omega) \tag{26}$$

admits a unique solution $u \in H^2 \cap H_0^1(\Omega)$. Assume for a moment that $u \in H^4(\Omega)$, then taking $v \in C_c^\infty(\Omega)$ in (26) and integrating by parts we get

$$\int_\Omega \Delta^2 u v = \int_\Omega \phi v \quad \text{for all } v \in C_c^\infty(\Omega)$$

which proves that

$$\Delta^2 u = \phi \quad \text{in } \Omega. \tag{27}$$

Let $v \in H^2 \cap H_0^1(\Omega)$. Integrating by parts in (26) and using (27) gives

$$\int_{\partial\Omega} \left(\Delta u - d\frac{\partial u}{\partial \nu}\right)\frac{\partial v}{\partial \nu} = 0 \quad \text{for all } v \in H^2 \cap H_0^1(\Omega)$$

which, by arbitrariness of v, shows that $\Delta u - d\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Hence, the second boundary condition is satisfied under the additional assumption that $u \in H^4(\Omega)$. Since the complementing condition is satisfied in view of Lemma 15, regularity theory applies (see [1]) so that we indeed have $u \in H^4(\Omega)$ and the proof is complete. □

An alternative proof of existence and uniqueness may be obtained by exploiting the strict convexity of the functional

$$I(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{d}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 - \int_\Omega \phi u, \quad u \in H^2 \cap H_0^1(\Omega). \tag{28}$$

The solution of (8) is the unique minimizer of I.

To conclude we still have to show that if $\phi \in H^k(\Omega)$ for some $k \geqslant 1$, then $u \in H^{k+4}(\Omega)$ but, in view of Lemma 15, this follows directly by elliptic regularity.

Proof of Theorem 1(ii). Assume that $d \geqslant 0$ and $\phi \geqslant 0$ ($\phi \not\equiv 0$). Statement (ii) is a straightforward consequence of the following lemma. □

Lemma 16. *Assume that $d \geqslant 0$. Then for all $u \in H^2 \cap H_0^1(\Omega) \setminus \{0\}$ there exists $w \in H^2 \cap H_0^1(\Omega)$ such that*:

$$\text{(j)} \quad w > 0 \quad \text{in } \Omega, \qquad \text{(jj)} \quad \frac{\partial w}{\partial \nu} < 0 \quad \text{on } \partial\Omega, \qquad \text{(jjj)} \quad I(w) \leqslant I(u).$$

Proof. Given $u \in H^2 \cap H_0^1(\Omega) \setminus \{0\}$, let $w \in H^2 \cap H_0^1(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta w = |\Delta u| & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

By the maximum principle for superharmonic functions, (j) and (jj) follow.

Moreover, both $w \pm u$ are superharmonic in Ω and vanish on $\partial\Omega$. This proves that

$$|u| \leqslant w \quad \text{in } \Omega, \qquad \left|\frac{\partial u}{\partial \nu}\right| \leqslant \left|\frac{\partial w}{\partial \nu}\right| \quad \text{on } \partial\Omega.$$

In turn, these inequalities (and $-\Delta w = |\Delta u|$) prove (jjj). □

Proof of Theorem 1(iii). For all $u \in \mathcal{H}(\Omega)$ let

$$\sigma(u) := \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^2}.$$

Then, we may rewrite (7) as

$$\sigma := \inf_{u \in \mathcal{H}(\Omega)} \sigma(u). \tag{30}$$

In view of the compact embedding $H^2(\Omega) \subset H^1(\partial\Omega)$, the infimum in (30) is achieved.

To show that the minimizers are strictly positive, we observe that the functions w, defined in the proof of Lemma 16, clearly belong to $\mathcal{H}(\Omega)$. Then it is easy to get, for $\sigma(u)$, an equivalent version of Lemma 16 and conclude.

Let us now show that the minimizer $\bar{u}$ for (30) solves the Euler equation and it is a smooth function. Consider the functional I in (28) with $\phi \equiv 0$ and $d = \sigma$. Then, by definition of σ, we have $I(u) \geqslant 0$ for all $u \in H^2 \cap H_0^1(\Omega)$. Since $I(\bar{u}) = 0$, $\bar{u}$ is a minimizer for I and it solves the Euler equation as shown in the above proof of (i). Finally, interior smoothness of $\bar{u}$ is a consequence of elliptic regularity, whereas smoothness up to the boundary follows from Lemma 15, see [1].

In order to conclude the proof of statement (iii) we still have to show that the minimizer $\bar{u}$ is unique. By contradiction; let $v \in H^2 \cap H_0^1(\Omega)$ be another minimizer. We have already shown that $v > 0$ in Ω. For every $c \in \mathbf{R}$, define $v_c := v + c\bar{u}$. Exploiting the fact that both v and $\bar{u}$ solve (8) when $d = \sigma$ and $\phi \equiv 0$, it is easy to verify that also v_c is a minimizer. But, unless v is a multiple of $\bar{u}$, there exists some c such that v_c changes sign in Ω. This leads to a contradiction and completes the proof. □

5. Proof of Theorem 2

We proceed in four steps.

Step 1. We show that u'' changes sign in $[-1, 1]$.

For contradiction, assume first that $u'' \geqslant 0$ in $[-1, 1]$. Then, u is convex and has a global minimum at some $x_m \in (-1, 1)$. Moreover, $u'(-1) < u'(x_m) = 0$. Since $u''(-1) \geqslant 0$, this contradicts (iii). A similar argument (changing all the signs!) enables us to rule out the possibility that $u'' \leqslant 0$ in $[-1, 1]$.

Step 2. We show that u'' cannot have only one zero in $(-1, 1)$.

For contradiction, assume that there exists $z \in (-1, 1)$ such that $u''(z) = 0$, $u'' < 0$ in $(-1, z)$ and $u'' > 0$ in $(z, 1)$. Then, since $u'(-1) = \delta^{-1}u''(-1) \leqslant 0$ in view of (iii), u is strictly negative in a right neighborhood of $x = -1$, it has a flex point at $x = z$ and a local minimum at some $x_m \in (z, 1)$. Moreover, since $u'' > 0$ in $(z, 1)$, we also have $u'(1) > u'(x_m) = 0$. Since $u''(1) \geqslant 0$, this contradicts (iii). Reversing all signs and arguing from $x = 1$ backwards enables us to show that there exists no $z \in (-1, 1)$ such that $u''(z) = 0$, $u'' > 0$ in $(-1, z)$ and $u'' < 0$ in $(z, 1)$.

Step 3. We show that u'' cannot have infinitely many zeros in $[-1, 1]$. Since assumption (i) tells us that u'' is a convex function, it may have infinitely many zeros only if $u'' \equiv 0$ on some interval $I \subset [-1, 1]$. The case $I = [-1, 1]$ is excluded by the assumption $u \not\equiv 0$. If $I = [-1, z]$ for some $z \in (-1, 1)$, by convexity of u'' we obtain $u''(1) > 0$ and $u'(1) > 0$ which contradicts (iii). If $I = [z, 1]$ for some $z \in (-1, 1)$, by convexity of u'' we obtain $u''(-1) > 0$ and $u'(-1) < 0$ which contradicts (iii). Similarly, we can exclude the case where $I = [z_1, z_2]$ for some $-1 < z_1 < z_2 < 1$.

Step 4. Conclusion. By combining the statements of steps 1–3 with assumption (i) (convexity of u''), we deduce that there exist $-1 < z_1 < z_2 < 1$ such that $u''(z_1) = u''(z_2) = 0$, $u'' > 0$ on $[-1, z_1) \cup (z_2, 1]$, $u'' \leqslant 0$ on (z_1, z_2). The same arguments used in step 2, show that u is strictly positive in a right neighborhood of $x = -1$, it has a flex point at $x = z_1$ and a local maximum at some $x_M \in (z_1, z_2)$. For contradiction, assume that $u(\bar{x}) = 0$ for some $\bar{x} \in (x_M, 1)$. Then, by (ii), u admits a local minimum at some $x_m \in [\bar{x}, 1)$ so that necessarily $z_2 \in (x_M, x_m)$. Hence, $u'' > 0$ in $[x_m, 1]$ and, in turn, $u'(1) > u'(x_m) = 0$. Since also $u''(1) > 0$, this contradicts (iii).

Remark 17. The above proof establishes that under the assumptions of Theorem 2 the function u is positive and it has exactly two flex points.

6. Proof of Theorem 3

For $\Omega = \mathbf{B}$, take a minimizer of (30), namely a function $u \in \mathcal{H}(\mathbf{B})$ such that

$$\int_{\mathbf{B}} |\Delta u|^2 = \sigma \int_{\partial\mathbf{B}} \left|\frac{\partial u}{\partial \nu}\right|^2. \tag{31}$$

In view of Theorem 1, we may assume $u > 0$ and

$$\int_{\partial\mathbf{B}} \frac{\partial u}{\partial \nu} < 0. \tag{32}$$

For all $\varepsilon > 0$ let $v_\varepsilon := u + \varepsilon(|x|^2 - 1)$. Then,

$$\Delta v_\varepsilon = \Delta u + 2n\varepsilon \quad \text{in } \mathbf{B}, \qquad \frac{\partial v_\varepsilon}{\partial \nu} = \frac{\partial u}{\partial \nu} + 2\varepsilon \quad \text{on } \partial\mathbf{B}.$$

Therefore,

$$\int_{\mathbf{B}} |\Delta v_\varepsilon|^2 = \int_{\mathbf{B}} |\Delta u|^2 + 4n\varepsilon \int_{\mathbf{B}} \Delta u + 4n\omega_n\varepsilon^2, \qquad \int_{\partial\mathbf{B}} \left|\frac{\partial v_\varepsilon}{\partial \nu}\right|^2 = \int_{\partial\mathbf{B}} \left|\frac{\partial u}{\partial \nu}\right|^2 + 4\varepsilon \int_{\partial\mathbf{B}} \frac{\partial u}{\partial \nu} + 4\omega_n\varepsilon^2,$$

where ω_n denotes the $(n-1)$-dimensional Hausdorff measure of $\partial\mathbf{B}$. Hence, recalling (31) and using the divergence theorem, we get

$$I_\varepsilon := \int_{\mathbf{B}} |\Delta v_\varepsilon|^2 - \sigma \int_{\partial\mathbf{B}} \left|\frac{\partial v_\varepsilon}{\partial\nu}\right|^2 = 4\varepsilon(n-\sigma)\int_{\partial\mathbf{B}} \frac{\partial u}{\partial\nu} + 4\omega_n(n-\sigma)\varepsilon^2. \tag{33}$$

If $\sigma < n$, in view of (32), the previous equation gives $I_\varepsilon < 0$ for ε sufficiently small, which contradicts the definition of σ in (30). Hence, $\sigma \geqslant n$. In order to prove the converse inequality, consider the function $\bar{u}(x) := 1 - |x|^2$. It is readily seen that $\sigma(\bar{u}) = n$. This proves that $\sigma = n$ and that $\bar{u}$ is a minimizer for (30).

Finally, from Theorem 1, it follows that nontrivial multiples of $\bar{u}$ are the only minimizers for (30).

7. Proof of Theorem 4

Let λ_1 be defined as in (10), from the compactness of the embedding $H^2 \cap H_0^1(\Omega) \subset L^2(\Omega)$ we get that the infimum in (10) is achieved so the minimizer ϕ_1 exists. To show that $\phi_1 > 0$ in Ω, we define

$$\Sigma(u) := \frac{\int_\Omega |\Delta u|^2 - d\int_{\partial\Omega} \left|\frac{\partial u}{\partial\nu}\right|^2}{\int_\Omega u^2}.$$

Then it is not difficult to show that for $\Sigma(u)$ a similar version of Lemma 16 holds, from which the strict positivity of ϕ_1 follows.

8. Proof of Theorems 6 and 7

8.1. Preliminary lemmas

The proofs of Theorems 6 and 7 require some technical results. Firstly, we prove a weak form of (2).

Lemma 18. *Let $0 \leqslant d < \sigma$ and assume that $u \in L^2(\Omega)$ satisfies*

$$\int_\Omega u\Delta^2 v \geqslant 0 \quad \textit{for all } v \in X_d(\Omega),\ v \geqslant 0 \textit{ in } \Omega. \tag{34}$$

Then, $u \geqslant 0$ in Ω . Moreover, one has either $u \equiv 0$ or $u > 0$ in Ω.

Proof. If $d = 0$, we have Navier boundary conditions and the result is well known (it follows by applying twice the maximum principle for $-\Delta$). So, let us assume that $d > 0$. Take $\phi \in C_c^\infty(\Omega)$, $\phi \geqslant 0$ in Ω and let v be the classical solution of

$$\begin{cases} \Delta^2 v = \phi & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \Delta v - d\frac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 1, v exists, belongs to $X_d(\Omega)$ and $v \geqslant 0$ in Ω. Hence we may use v in (34) as a test function:

$$\int_\Omega u\Delta^2 v = \int_\Omega u\phi \geqslant 0$$

for all $\phi \in C_c^\infty(\Omega)$, $\phi \geqslant 0$ in Ω. Thus $u \geqslant 0$ in Ω.

Let us now show that one has either $u \equiv 0$ or $u > 0$ in Ω. Assume that u is not strictly positive in Ω and let χ be the characteristic function of the set $\{x \in \Omega\colon\ u(x) = 0\}$. Clearly $\chi \geqslant 0$ in Ω and $\chi > 0$ on a subset of positive measure. Let v_0 be the unique solution of

$$\begin{cases} \Delta^2 v_0 = \chi & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \\ \Delta v_0 - d\frac{\partial v_0}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 1, $v_0 > 0$ in Ω and $\frac{\partial v_0}{\partial \nu} < 0$ on $\partial\Omega$, so $\Delta v_0 < 0$ on $\partial\Omega$ (recall that $d > 0$). The proof of the lemma may now be completed by arguing as in [2, Lemma 1] with obvious changes. □

Next, we state a result concerning with the existence and uniqueness of weak solutions.

Lemma 19. *Let $0 \leqslant d < \sigma$. For all $f \in L^1(\Omega)$ such that $f \geqslant 0$ in Ω there exists a unique $u \in L^1(\Omega)$ such that $u \geqslant 0$ in Ω and*

$$\int_\Omega u\Delta^2 v = \int_\Omega f v \quad \text{for all } v \in C^4(\overline{\Omega}) \cap X_d(\Omega). \tag{35}$$

Moreover, there exists $C > 0$ such that $\|u\|_1 \leqslant C\|f\|_1$.

Proof. It may be obtained by arguing as in the proof of [7, Lemma 1] with two minor modifications. First, the standard positivity preserving property for $-\Delta$ used there should here be replaced by Lemma 18. Second, the existence and uniqueness result for the "truncated problems" here follows from Theorem 1. □

Finally, we establish a weak form of the super-subsolution method.

Lemma 20. *Let $0 \leqslant d < \sigma$ and $\lambda > 0$. Assume that there exists $\bar{u} \in L^2(\Omega)$ such that $\bar{u} \geqslant 0$ in Ω, $f(\bar{u}) \in L^1(\Omega)$ and*

$$\int_\Omega \bar{u}\Delta^2 v \geqslant \lambda \int_\Omega f(\bar{u}) v \quad \text{for all } v \in X_d(\Omega),\ v \geqslant 0 \text{ in } \Omega.$$

Then there exists a solution u of (4) *such that $0 \leqslant u \leqslant \bar{u}$ in Ω.*

Proof. Let $u_0 = \bar{u}$ and, for all $m \geqslant 1$, define inductively the function u_{m+1} as the unique solution of

$$\int_\Omega u_{m+1}\Delta^2 v = \lambda \int_\Omega f(u_m)v \quad \text{for all } v \in X_d(\Omega),\ v \geqslant 0 \text{ in } \Omega. \tag{36}$$

By Lemmas 18 and 19 we infer the existence of such functions and that $0 \leqslant u_{m+1} \leqslant u_m$ in Ω, $u_m \in L^2(\Omega)$ and $f(u_m) \in L^1(\Omega)$. Hence, there exists $u \in L^2(\Omega)$, $u \geqslant 0$ such that $u_{m+1} \to u$ pointwise in Ω and $f(u) \in L^1(\Omega)$. Finally, by Lebesgue's theorem, we may pass to the limit in (36) so that u solves (4) in the sense of Definition 5. □

8.2. Proof of Theorem 6

Put

$$\Lambda := \{\lambda \geqslant 0\colon \text{ (4) admits a nonnegative solution}\} \quad \text{and} \quad \lambda^* := \sup \Lambda.$$

Clearly $0 \in \Lambda$, so $\Lambda \neq \emptyset$. On the other hand, by a standard application of the implicit function theorem, we deduce that $\lambda^* > 0$. Moreover, by Lemmas 18 and 20, we infer that for every $\lambda \in \Lambda$ minimal solutions exist. In particular, the family of minimal solutions, parametrized with respect to λ, is increasing (so that Λ is an interval).

Now we establish that λ^* is finite. To this end, let $\lambda \in \Lambda$ and let u be the corresponding positive solution of (4). By definition of α_f it follows that $f(u) \geqslant \alpha_f u$ in Ω ($f(u) \not\equiv \alpha_f u$). Then, by using the characterization of λ_1 and ϕ_1 given in Theorem 4 and by Definition 5, we obtain

$$\lambda_1 \int_\Omega u\phi_1 = \int_\Omega u\Delta^2\phi_1 = \lambda \int_\Omega f(u)\phi_1 > \lambda\alpha_f \int_\Omega u\phi_1.$$

This yields

$$\lambda^* \leqslant \frac{\lambda_1}{\alpha_f}.$$

To conclude the proof we still have to study the case $\lambda = \lambda^*$. First we need the following statement concerning the stability of the minimal regular solution.

Lemma 21. *Let $\lambda \in (0, \lambda^*)$ and suppose that the corresponding minimal solution u_λ of* (4) *is regular. Let μ_1 and ψ_1 denote, respectively, the least eigenvalue and the corresponding eigenfunction of the linearized operator $\Delta^2 - \lambda f'(u_\lambda)$ under the boundary conditions in* (4)*. Then $\mu_1 > 0$ and ψ_1 is strictly of one sign in Ω.*

Proof. Consider the variational characterization of μ_1:

$$\mu_1(\lambda) = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 - d\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 - \lambda \int_\Omega f'(u_\lambda)u^2}{\int_\Omega u^2}.$$

From the compactness of the embedding $H^2 \cap H_0^1(\Omega) \subset L^2(\Omega)$ and the regularity of u_λ, we deduce that the infimum is achieved. Hence, the minimizer ψ_1 exists. To show that $\psi_1 > 0$ in Ω, we fix $\lambda \in (0, \lambda^*)$ and we define the functional

$$\Gamma(u) := \frac{\int_\Omega |\Delta u|^2 - d\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 - \lambda \int_\Omega f'(u_\lambda)u^2}{\int_\Omega u^2}.$$

One can then apply the arguments of the proof of Lemma 16 to $\Gamma(u)$ to obtain $\psi_1 > 0$ in Ω. Finally, to show that $\mu_1 > 0$ one can follow the proof of [2, Proposition 4], see also [4, Proposition 3.5]. □

Next, we note that by Lemma 20 the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing for all $x \in \Omega$ so that we may define

$$u^*(x) := \lim_{\lambda \to \lambda^*} u_\lambda(x), \quad x \in \Omega. \tag{37}$$

The following statement completes the proof of Theorem 6.

Lemma 22. *Assume that the minimal solution u_λ of* (4) *is regular for all $\lambda \in (0, \lambda^*)$ and f satisfies condition* (12). *Then, the function u^* defined in* (37) *satisfies $u^* \in H^2 \cap H_0^1(\Omega)$ and solves* (4) *for $\lambda = \lambda^*$.*

Proof. Let u_λ be the minimal solution of (4), then

$$\int_\Omega u_\lambda \Delta^2 v = \lambda \int_\Omega f(u_\lambda)v \quad \text{for all } v \in X_d(\Omega), \tag{38}$$

so that by Lemma 21, after an integration by parts, we get

$$\lambda \int_\Omega f'(u_\lambda)u_\lambda^2 \leqslant \int_\Omega |\Delta u_\lambda|^2 - d\int_{\partial\Omega} \left|\frac{\partial u_\lambda}{\partial \nu}\right|^2 = \int_\Omega u_\lambda \Delta^2 u_\lambda = \lambda \int_\Omega f(u_\lambda)u_\lambda. \tag{39}$$

From (12) it follows that there exist $\varepsilon > 0$ and $C > 0$ such that $(1+\varepsilon)f(s)s \leqslant f'(s)s^2 + C$ for all $s \geqslant 0$. This fact, combined with (39), yields the existence of $C_1 > 0$ such that

$$\int_\Omega f(u_\lambda)u_\lambda < C_1.$$

Therefore

$$\|u_\lambda\|_d^2 = \int_\Omega |\Delta u_\lambda|^2 - d\int_{\partial\Omega} \left|\frac{\partial u_\lambda}{\partial \nu}\right|^2 = \lambda \int_\Omega f(u_\lambda)u_\lambda < \lambda^* C_1.$$

Letting $\lambda \to \lambda^*$, in the above inequality we deduce that, up to a subsequence,

$$u_\lambda \rightharpoonup u^* \quad \text{in } H^2 \cap H_0^1(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

And this allows us to pass to the limit in (38). Therefore, u^* solves (4) for $\lambda = \lambda^*$. $\square$

8.3. Proof of Theorem 7

Assume that (13) holds and set $\varphi(s) := \log f(s)$. Then, (4) may be rewritten as

$$\begin{cases} \Delta^2 u = \lambda e^{\varphi(u)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - d\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{40}$$

where $\varphi \in C^2(\mathbf{R}_+)$, is increasing and convex. Of course, problem (4) and problem (40) have the same solutions. The regularity of the minimal solution of (40) may be obtained as in the proof of [4, Lemma 5.2] (see also previous work [2]), where problem (40) is studied in the case $d = 0$.

9. Proof of Theorem 8

We begin with the regularity result:

Proposition 23. *Assume that $1 < p \leqslant \frac{n+4}{n-4}$ and let $u \in H^2 \cap H_0^1(\Omega)$ be a solution of* (4) *with $f(u) = (1+u)^p$. Then u is regular.*

Proof. If we show that $u \in L^q(\Omega)$ for every $q < \infty$, the statement follows by elliptic regularity.

Thanks to [4, (6.1)], we have that for every $\varepsilon > 0$ there exist $q_\varepsilon \in L^{\frac{n}{4}}(\Omega)$ and $F_\varepsilon \in L^\infty(\Omega)$ such that

$$\big(1 + u(x)\big)^p = q_\varepsilon(x)u(x) + F_\varepsilon(x) \quad \text{and} \quad \|q_\varepsilon\|_{\frac{n}{4}} < \varepsilon. \tag{41}$$

So, for every $\varepsilon > 0$, the equation in (4) can be rewritten as

$$\Delta^2 u = \lambda\big(q_\varepsilon(x)u(x) + F_\varepsilon(x)\big) \quad \text{in } \Omega.$$

In order to complete our argument we may follow steps 2 and 3 in [24]. The only difference between [24] and this context being the Green function used there; here we have to consider the Green function of the operator Δ^2 with the boundary conditions in (4) whose existence follows from Theorem 1(i). $\square$

Assume now that $1 < p < \frac{n+4}{n-4}$ and consider the action functional associated to problem (4) with $f(s) = (1+s)^p$:

$$J(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{d}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 - \frac{\lambda}{p+1}\int_\Omega |1+u|^{p+1}. \tag{42}$$

In view of Theorem 6, we know that there exists $\lambda^* > 0$ such that, for every $\lambda < \lambda^*$, we have at least a solution of (4), the minimal one. Moreover, by Lemma 21, the least eigenvalue of the linearized operator $\Delta^2 - \lambda p(1+u_\lambda)^{p-1}$ is strictly positive. This enables us to argue as in the proof of [10, Theorem 2.1] with few changes; therefore, the existence of a mountain pass critical point v_λ for J follows. In order to show that it is positive, we need to recall its variational characterization in [3]. Take $\bar{u} \in H^2 \cap H_0^1(\Omega)$ such that

$$J(\bar{u}) < J(u_\lambda), \qquad -\Delta\bar{u} > 0 \quad \text{in } \Omega.$$

In particular, this implies $\bar{u} > 0$ in Ω. Let $\Gamma := \{\gamma \in C^0([0,1], H^2 \cap H_0^1(\Omega));\ \gamma(0) = u_\lambda, \gamma(1) = \bar{u}\}$. Then,

$$c := \min_{\gamma\in\Gamma} \max_{t\in[0,1]} J\big(\gamma(t)\big)$$

is the mountain pass level of J corresponding to the solution v_λ. Since the Palais–Smale condition holds, there exists an "optimal path" $\bar{\gamma} \in \Gamma$ such that $\max_{t\in[0,1]} J(\bar{\gamma}(t)) = c$.

Lemma 16 defines a continuous map $\Psi : H^2 \cap H_0^1(\Omega) \to H^2 \cap H_0^1(\Omega)$ such that for any $u \in H^2 \cap H_0^1(\Omega)$ we have $\Psi(u) = w$ with w solution of (29). Therefore, we have

$$|\Delta u| = \big|\Delta\Psi(u)\big|, \qquad |1+u| \leqslant 1 + \Psi(u) \quad \text{in } \Omega, \qquad \left|\frac{\partial u}{\partial\nu}\right| \leqslant \left|\frac{\partial\Psi(u)}{\partial\nu}\right| \quad \text{on } \partial\Omega. \tag{43}$$

Note that u_λ and $\bar{u}$ are fixed points for Ψ so that $\Psi \circ \bar{\gamma} \in \Gamma$. Moreover, by (43) we infer that $J(\Psi(\bar{\gamma}(t))) \leqslant J(\bar{\gamma}(t))$ for all $t \in [0,1]$. Hence, by definition of c, we have $\max_{t\in[0,1]} J(\Psi(\bar{\gamma}(t))) = c$ so that also $\Psi \circ \bar{\gamma}$ is an optimal path and the point which achieves $\max_{t\in[0,1]} J(\Psi(\bar{\gamma}(t)))$ is a positive mountain pass solution of (4).

Finally, to complete the proof of Theorem 8 we have to show that the extremal solution u^*, which exists by Lemma 22, is unique. To this end, recall that, from Proposition 23, u^* is a classical solution. Therefore, it suffices to argue as in the proof of [8, Lemma 2.6].

10. Proof of Theorem 10

In this section, we prove Theorem 10 in its polyharmonic version. We refer to problem (22) and we first emphasize that the boundary conditions are well-defined. Indeed, if we assume that $u \in H^m(\mathbf{B})$, then $\frac{\partial^i u}{\partial\nu^j} \in H^{m-i-\frac{1}{2}}(\partial\mathbf{B})$. However, since for every $i = 2, \ldots, (m-1)$ there exists b_i such that $\frac{\partial^{i+2}u}{\partial\nu^{i+2}} = b_i \frac{\partial u}{\partial\nu}$, we have that $\frac{\partial^{i+2}u}{\partial\nu^{i+2}} \in H^{m-\frac{3}{2}}(\partial\mathbf{B})$.

The singular parameter λ_σ defined in (15) for the biharmonic operator is here defined by

$$\lambda_\sigma := 2^m h(0) \prod_{i=0}^{m-1} \big(mg(0) + i\big) \prod_{i=1}^{m} \big(n - 2i - 2mg(0)\big). \tag{44}$$

Therefore, we have

$$f(s) = e^s \quad \Longrightarrow \quad \lambda_\sigma = 2^m m! \prod_{i=1}^{m} (n-2i),$$

$$f(s) = (1+s)^p \quad \Longrightarrow \quad \lambda_\sigma = 2^m \prod_{i=0}^{m-1} \left(\frac{m}{p-1} + i\right) \prod_{i=1}^{m} \left(n - 2i - \frac{2m}{p-1}\right).$$

Finally, we choose

$$a_i := 2mg(0) + i, \quad i = 1, \ldots, (m-1). \tag{45}$$

Then, the polyharmonic version of Theorem 10 reads:

Theorem 24. *Let $\lambda = \lambda_\sigma$ be as in* (44) *and let the a_i's be as in* (45)*. Then, the function $u_\sigma(x) := -2m \log|x|$ (respectively $v_\sigma(x) := |x|^{-\frac{2m}{p-1}} - 1$) is a singular radial solution of problem* (22) *with $f(u) = e^u$ (respectively $f(u) = (1+u)^p$ with $p > \frac{n}{n-2m}$).*

We start by showing that u_σ and v_σ satisfy the corresponding equations.

Lemma 25. *Let $\lambda = \lambda_\sigma$ be as in* (44)*, then $u_\sigma(x) := -2m \log|x|$ satisfies $(-\Delta)^m u = \lambda e^u$ in* **B**.

Proof. For $\alpha > 0$, we define $\tilde{u}(r) := -\alpha \log r$. Using the radial expression of the Laplacian and arguing by induction on m we obtain

$$(-\Delta)^m \tilde{u} = \alpha 2^{m-1}(m-1)! \prod_{i=1}^{m} (n-2i) r^{-2m}.$$

The statement then follows by taking $\alpha = 2m$. $\square$

Lemma 26. *Let $\lambda = \lambda_\sigma$ be as in* (44) *and let $p > \frac{n}{n-2m}$. Then $v_\sigma(x) := |x|^{-\frac{2m}{p-1}} - 1$ satisfies $(-\Delta)^m u = \lambda(1+u)^p$ in* **B**.

Proof. For $\alpha > 0$, we define $\tilde{v}(r) := r^{-\frac{\alpha}{p-1}} - 1$. Arguing by induction on m we obtain

$$(-\Delta)^m \tilde{v} = 2^m \prod_{i=0}^{m-1} \left(\frac{\alpha}{2(p-1)} + i \right) \prod_{i=1}^{m} \left(n - 2i - \frac{\alpha}{p-1} \right) r^{-(\frac{\alpha}{p-1} + 2m)}.$$

The statement then follows by taking $\alpha = 2m$. $\square$

By induction one can also show that for all $k = 1, 2, \ldots$ we have

$$u_\sigma^{(k)}(r) = (-1)^k 2m(k-1)!\, r^{-k} \quad \text{and} \quad v_\sigma^{(k)}(r) = (-1)^k \prod_{i=0}^{k-1} \left(\frac{2m}{p-1} + i \right) r^{-(\frac{2m}{p-1} + k)}. \tag{46}$$

We may now prove that v_σ and u_σ satisfy the boundary conditions in (22). To this end, we observe that the right-hand sides of (46) with $r = 1$, also read as

$$(-1)^k h(0) 2m \prod_{i=1}^{k-1} \big(2mg(0) + i \big),$$

where $h(0)$ and $g(0)$ are defined in (14). Hence,

$$u_\sigma^{(k+1)}(1) = (-1)^{k+1} h(0) 2m \prod_{i=1}^{k} \bigl(2mg(0) + i\bigr) = -\bigl(2mg(0) + k\bigr) u_\sigma^{(k)}(1)$$

and similarly for v_σ. This shows that the boundary conditions are satisfied. In view of Lemmas 25 and 26, the proof of Theorem 24 is so complete.

Remark 27. Notice that, $v_\sigma \in L^2(\mathbf{B})$ if and only if $p > \frac{4m+n}{n}$. Indeed,

$$\int_{\mathbf{B}} |v_\sigma|^2 \, dx = C_n \int_0^1 \bigl(r^{-\frac{2m}{p-1}} - 1\bigr)^2 r^{n-1} \, dr,$$

which is finite for $p > \frac{4m+n}{n}$. Similarly, using (46) we see that $v_\sigma \in H^m(\mathbf{B})$ if and only if $p > \frac{n+2m}{n-2m}$.

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