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Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Part III

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Abstract

We study the asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters. Thanks to an additional (fixed) parameter, we show that two different critical exponents play a crucial role in the asymptotic analysis, giving an explanation of the phenomena discovered in Gazzola et al. (Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear) and Gazzola and Serrin (Ann. Inst. H. Poincaré Anal. Non Linéaire 19 (2002) 477).

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1. Introduction

Let $\Delta_m u = \text{div}(|\nabla u|^{m-2}\nabla u)$ denote the degenerate m-Laplace operator and let

$$m^* = \frac{nm}{n-m}$$

be the critical Sobolev exponent for the embedding $D^{1,m}(\mathbb{R}^n) \subset L^{m^*}(\mathbb{R}^n)$. In this paper, we study the asymptotic behavior of ground states of the quasilinear elliptic

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equations

$$-\Delta_m u = -\delta u^{q-1} + u^{p-1} \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $\delta > 0$, $n > m > 1$ and $1 < q < p < m^*$. Here and in the sequel, by a ground state we mean a $C^1(\mathbb{R}^n)$ nonnegative nontrivial radial distribution solution of (1.1) vanishing at infinity.

We know from [6,12] that (1.1) admits a unique ground state for all δ, p, q in the given range. On the contrary, if either $p = m^*$ and $\delta > 0$ or $\delta = 0$ and $p \in (q, m^*)$ then (1.1) admits no ground states, see [9,10]. Finally, if both $\delta = 0$ and $p = m^*$, then (1.1) becomes

$$-\Delta_m u = u^{m^*-1} \quad \text{in } \mathbb{R}^n \tag{1.2}$$

and (1.2) admits the one-parameter family of positive ground states (see [14]) given by

$$U_d(x) = d[1 + D(d^{\frac{m}{n-m}}|x|)^{\frac{m}{m-1}}]^{-\frac{n-m}{m}} \quad (d > 0) \tag{1.3}$$

with $D = D_{m,n} = (m-1)/(n-m)n^{\frac{1}{m-1}}$ and $U_d(0) = d$.

Our purpose is to study the behavior of ground states of (1.1) in these limiting situations, namely when $p \uparrow m^*$ and/or $\delta \downarrow 0$. Note that the case $q = m$ is also somehow a limit case since if $q < m$ then the ground state of (1.1) has compact support, whereas if $q \geqslant m$ it remains positive on $\mathbb{R}^n$, see [3]. And precisely in the case $q = m$, this behavior has been determined in [4,5] where a new phenomenon was highlighted: an unexpected "discontinuous" dependence of the behavior on the parameters m and n was found. In order to better understand this phenomenon, we introduce here the additional free parameter q. And indeed, our results shed some light on this strange behavior and we may attempt some explanations. The new parameter q allows us to interpret the above-mentioned discontinuous dependence in terms of two critical exponents. We will show that in the description of the asymptotic behavior of ground states of (1.1) a crucial role is played by the two numbers

$$m_R = \frac{n(m-1)}{n-m}, \quad m_* = \frac{m(n-1)}{n-m}$$

which satisfy

$$1 < m_R < m_* < m^*.$$

Note that $m_R = m_* - 1$ and that $m_R - m$ has the same sign as $m^2 - n$.

It is well-known that the best Sobolev constant S in the inequality for the embedding $D^{1,m}(\Omega) \subset L^{m^*}(\Omega)$ is independent of the domain Ω and that it is not attained if $\Omega \neq \mathbb{R}^n$. In fact, if Ω is bounded more can be said, a so-called remainder term appears. In [1], it is shown that for any bounded domain $\Omega \subset \mathbb{R}^n$ and any

$1 \leqslant q < m_R$ there exists an optimal constant $C = C(\Omega, q) > 0$ such that

$$||\nabla u||^m_{L^m(\Omega)} \geqslant S||u||^m_{L^{m^*}(\Omega)} + C||u||^m_{L^q(\Omega)}$$

and $C(\Omega, q) \to 0$ as $q \to m_R^-$. For this reason, we call m_R the *critical remainder exponent*.

The number m_* is called Serrin's exponent, see [11]. It is shown independently in [8,13] that the inequality $-\Delta_m u \geqslant u^{p-1}$ (where $p > 1$) admits a nonnegative nontrivial solution if and only if $p > m_*$.

In Theorem 1, we show that Serrin's exponent m_* is also the borderline between existence and nonexistence for the "coercive" problem $\Delta_m W = W^{q-1}$. More precisely, we prove that this equation admits a (unique) nonnegative radial solution on $\mathbb{R}^n \setminus \{0\}$ which blows up at the origin like the fundamental solution if and only if $q < m_*$. The nonexistence statement for $q \geqslant m_*$ is a consequence of removable singularities [15].

Then, we start our asymptotic analysis by maintaining $\delta > 0$ fixed and letting $p \uparrow m^*$. In Theorems 2 and 3 we show that the ground state u of (1.1) converges to a Dirac measure having mass at the origin and that $u(0)$ blows up with different rates when $q > m_R$, $q = m_R$ and $q < m_R$. This fact is strictly related to the L^q summability of the functions U_d in (1.3) which fails precisely if $q \leqslant m_R$. As already mentioned, if $q < m$ then the ground state of (1.1) has compact support (a ball); in Theorem 4, we show that the radius of the ball tends to 0 as $p \uparrow m^*$ and we give the precise rate of its extinction. Once more, the critical exponents m_R and m_* play a major role. In Theorem 5, we rescale in a suitable fashion the ground state u and we show that the rescaled function converges to the solution W of the problem $\Delta_m W = W^{q-1}$ previously determined in Theorem 1: since W is nontrivial only if $q < m_*$, this gives a further different behavior of the ground state according to the sign of $q - m_*$.

Our asymptotic analysis is continued by maintaining p fixed and letting $\delta \downarrow 0$. In Theorem 6, we prove that in such a case $u \to 0$ uniformly in $\mathbb{R}^n$ and we determine the precise rate of convergence; moreover, when u is compactly supported (i.e. $q < m$) we show that the radius of the ball supporting it diverges to infinity. This means that the ground state spreads out as $\delta \downarrow 0$. Since this behavior is somehow opposite to the concentration phenomenon obtained when $p \uparrow m^*$ it is natural to inquire what happens when *both* $\delta \downarrow 0$ and $m^* - p \downarrow 0$ (this justifies the title of the paper). In Theorem 7, we show that if this occurs at a suitable "equilibrium behavior" $\delta = \delta(m^* - p)$ then the ground state does not concentrate nor spread out, it converges uniformly in $\mathbb{R}^n$ to one of the functions U_d in (1.3). The rate of this equilibrium behavior depends on the sign of $q - m_R$.

Some of our statements are obtained by adapting the proofs in [4,5] while some others (as Theorem 1, Theorem 4, Theorem 5 and the second part of Theorem 6) are based on new ideas. Furthermore, we emphasize once more that our study for general q gives a complete picture of the phenomenon thanks to the critical exponents m_R and m_*.

2. Main results

Throughout the paper, we define $r=|x|$. Thanks to the rescaling,

$$v(x)=\delta^{-\frac{1}{p-q}}u\left(\frac{x}{\delta^{\frac{p-m}{m(p-q)}}}\right)$$

we have that u is a ground state of (1.1) if and only if v is a ground state of the equation

$$-\Delta_m v=-v^{q-1}+v^{p-1}\quad \text{in } \mathbb{R}^n. \tag{2.1}$$

Therefore, when $\delta>0$ is fixed, we may restrict our attention to (2.1).

Consider first the auxiliary problem obtained by deleting the largest power term in (2.1):

$$\Delta_m W=W^{q-1}\quad \text{in } \mathbb{R}^n\backslash\{0\} \tag{2.2}$$

supplemented with the "boundary" conditions

$$\lim_{x\to 0}|x|^{\frac{n-m}{m-1}}W(x)=A_{m,n}, \tag{2.3}$$

$$\lim_{|x|\to\infty}W(x)=0, \tag{2.4}$$

where $A_{m,n}=D^{-\frac{n-m}{m}}$ and $D=D_{m,n}$ is defined in (1.3). We have

Theorem 1. *Let* $n>m>1$.

(i) *If* $q\geqslant m_*$ *then* (2.2)–(2.3) *has no solution.*

(ii) *If* $m<q<m_*$ *then* (2.2)–(2.3) *admits a unique nonnegative radial solution* W_q. *Moreover,* $W_q(r)>0$ *in* $(0,\infty)$ *and* $W_q(r)=O(r^{-\frac{m}{q-m}})$ as $r\to\infty$.

(iii) *If* $1<q\leqslant m$ *then* (2.2)–(2.4) *admits a unique nonnegative radial solution* W_q. *Moreover, if* $q=m$ *then* $W_q(r)>0$ *on* $(0,\infty)$ *and there exists* $\nu>0$ *such that* $W_q(r)=O(e^{-\nu r})$ *as* $r\to\infty$, *while if* $q<m$ *then* W_q *has compact support.*

The nonexistence result for $q\geqslant m_*$ is essentially due to Vázquez–Veron [15]. On the other hand, statements (ii) and (iii) in Theorem 1 require a fairly complicated proof, involving new ideas which may be of some interest also independently of our context. Clearly, Theorem 1 is true also if $A_{m,n}$ in (2.3) is replaced by any other positive constant, see the rescaling (4.33).

Note that when $q<m_R$, Theorem 1 states that the following constant is well-defined:

$$I_{m,n,q}=\int_0^\infty r^{n-1}W_q^q(r)\,\mathrm{d}r. \tag{2.5}$$

Also recall that the beta function $B(\cdot,\cdot)$ is defined by

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,\mathrm{d}t \quad a,b>0.$$

We then introduce the following constants:

$$\beta_{m,n,q} = \frac{m^*}{q}(m^*-q)\,\frac{B(\frac{n(m-1)}{m},\frac{n-m}{m}(q-m_R))}{B(\frac{n(m-1)}{m},\frac{n}{m})} \quad (q>m_R), \tag{2.6}$$

$$\mu_{m,n} = \frac{nm^2}{(n-m)^2(m-1)}\,\frac{1}{B(\frac{n(m-1)}{m},\frac{n}{m})}, \tag{2.7}$$

$$\gamma_{m,n,q} = \frac{m^*}{q}(m^*-q)\,\frac{m}{m-1}\,D^{\frac{n(m-1)}{m}}\,\frac{I_{m,n,q}}{B(\frac{n(m-1)}{m},\frac{n}{m})} \quad (q<m_R), \tag{2.8}$$

where $D = D_{m,n}$ is defined in (1.3).

With these constants we describe the asymptotic behavior of the solution u of (2.1) at the origin when $p \to m^*$:

Theorem 2. *For all* $1<q<p<m^*$, *let* u *be the unique ground state of* (2.1). *Then, writing* $\varepsilon = m^* - p$, *we have*

$$\lim_{\varepsilon\to 0} \varepsilon[u(0)]^{m^*-q} = \beta_{m,n,q} \quad \text{if } q>m_R,$$
$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{|\ln\varepsilon|}[u(0)]^{\frac{n}{n-m}} = \mu_{m,n} \quad \text{if } q=m_R,$$
$$\lim_{\varepsilon\to 0} \varepsilon[u(0)]^{\frac{m^*-q}{m_*-q}} = \gamma_{m,n,q} \quad \text{if } q<m_R, \tag{2.9}$$

where the constants $\beta_{m,n,q}$, $\mu_{m,n}$ *and* $\gamma_{m,n,q}$ *are defined in* (2.6), (2.7) *and* (2.8), *respectively.*

Note that at the "turning point" $q = m_R$ we have $\frac{m^*-q}{m_*-q} = m^* - q = \frac{n}{n-m}$. Moreover, $\beta_{m,n,q} \to \infty$ as $q \downarrow m_R$ and $\gamma_{m,n,q} \to \infty$ as $q \uparrow m_R$.

Theorem 3 asserts that u concentrates at $x = 0$. We state this fact in more details as

Theorem 3. *For all* $1<q<p<m^*$, *let* u *be the unique ground state of* (2.1). *Then, writing* $\varepsilon = m^* - p$, *there exist* $v_{m,n}>0$ *and* $C_{m,n}>0$ *depending only on* m,n *such that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u^{m^*}(x)\,dx = v_{m,n}, \quad \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} |\nabla u(x)|^m\,dx = v_{m,n},$$

$$\lim_{\varepsilon\to 0}\,[u(0)u^{m-1}(x)]\leqslant C_{m,n}|x|^{m-n}\quad \forall x\neq 0,$$

$$\lim_{\varepsilon\to 0}\,[u(0)|\nabla u(x)|^{m-1}]\leqslant \left(\frac{n-m}{m-1}\right)^{m-1} C_{m,n}|x|^{1-n}\quad \forall x\neq 0.$$

These facts imply that

$$u^{m^*}\to \nu_{m,n}\delta_0 \quad \textit{and} \quad |\nabla u|^m\to \nu_{m,n}\delta_0 \quad \textit{as } \varepsilon\to 0$$

in the sense of distributions; here δ_0 *is the Dirac measure concentrated at* $x=0$.

If $q<m$, then the unique ground state of (2.1) has compact support; in the next statement, we give an asymptotic estimate of its support as $p\to m^*$: note that $m<m_R$ if and only if $n<m^2$.

Theorem 4. *For all* $1<q<m<p<m^*$ *let* u *be the unique ground state of* (2.1). *Then, writing* $\varepsilon = m^*-p$ *and* $\operatorname{supp}(u)=\overline{B_\rho(0)}$ *we have* $\rho\to 0$ *as* $\varepsilon\to 0$. *To be more precise we have the following estimates*:

$$C_1\leqslant \liminf_{\varepsilon\to 0}\ \rho\,\varepsilon^{\frac{q-m}{(m^*-q)(m_*-q)(n-m)}}\leqslant \limsup_{\varepsilon\to 0}\ \rho\,\varepsilon^{\frac{q-m}{(m^*-q)(m_*-q)(n-m)}}<+\infty \quad \textit{if } q>m_R,$$

$$C_2\leqslant \liminf_{\varepsilon\to 0}\ \rho\left(\frac{|\ln\varepsilon|}{\varepsilon}\right)^{\frac{n-m^2}{n(n-m)}}\leqslant \limsup_{\varepsilon\to 0}\ \rho\left(\frac{|\ln\varepsilon|}{\varepsilon}\right)^{\frac{n-m^2}{n(n-m)}}<+\infty \quad \textit{if } q=m_R,$$

$$C_3\leqslant \liminf_{\varepsilon\to 0}\ \rho\,\varepsilon^{\frac{q-m}{(m^*-q)(n-m)}}\leqslant \limsup_{\varepsilon\to 0}\ \rho\,\varepsilon^{\frac{q-m}{(m^*-q)(n-m)}}<+\infty \quad \textit{if } q<m_R,$$

where $C_1, C_2, C_3>0$ *are constants depending only on* m, n, q.

Our next result gives the asymptotic behavior of the ground state u of (2.1) "outside the origin" when $p\to m^*$. We introduce the constant

$$\kappa_q := \frac{1}{(n-m)(m_*-q)} \quad \forall q<m_* \tag{2.10}$$

and we state

Theorem 5. *For all* $1<q<p<m^*$ *let* u *be the unique ground state of problem* (2.1). *Then, writing* $\alpha=u(0)$ *and* $\varepsilon=m^*-p$, *we have*

(i) *If* $1<q<m_*$, *then*

$$\lim_{\varepsilon\to 0}\ \alpha^{m\kappa_q}u(\alpha^{(q-m)\kappa_q}r)=W_q(r)\quad \forall r>0, \tag{2.11}$$

where W_q *is the unique nonnegative radial solution of* (2.2)–(2.4), *see Theorem* 1.

(ii) *If* $m_* \leqslant q < m^*$, *then*

$$\lim_{\varepsilon\to 0} \alpha^{m\kappa} u(\alpha^{(q-m)\kappa} r) = 0 \quad \forall r>0 \ \forall \kappa \in \mathbb{R}. \tag{2.12}$$

Now we change notations: we denote by v the unique ground state of (2.1) and set

$$\beta = v(0). \tag{2.13}$$

By the already mentioned existence and uniqueness results for (2.1), β is a well-defined function of the four parameters m, n, p and q. If $q<m$ then the support of v is a closed ball centered at the origin (see [3] and Proposition 2 below), so that we can put

$$\text{supp}\,(v) = \overline{B_R(0)}. \tag{2.14}$$

Then the following result holds

Theorem 6. *For all* $\delta>0$, *let* u *be the unique ground state of* (1.1) *with* $1<q<p<m^*$, *let* v *be the unique ground state of* (2.1) *and let* β *be as in* (2.13). *Then* $u(0) = \delta^{\frac{1}{p-q}}\beta$ *and for any* p *fixed and* $x\neq 0$ *we have*

$$u(x) = u(0) - \frac{m-1}{m}\left(\frac{\beta^{p-1}-\beta^{q-1}}{n}\right)^{\frac{1}{m-1}} \delta^{\frac{p-1}{(p-q)(m-1)}} |x|^{\frac{m}{m-1}} + o(\delta^{\frac{p-1}{(p-q)(m-1)}}) \quad \textit{as } \delta\to 0. \tag{2.15}$$

Moreover, if $q<m$, *then*

$$\text{supp}\,(u) = \overline{B_\rho(0)}, \tag{2.16}$$

where $\rho = R\delta^{-\frac{p-m}{m(p-q)}}$ *and* R *as in* (2.14).

Remark 1. In some cases (e.g. if $q\leqslant m$ or m^*-p is small enough) by arguing as in [5, Theorem 1] we see that if $\ell = \frac{n(p-m)}{m}$, there exists a positive constant $\alpha_{m,n,p,q}$ independent of δ such that

$$\int_{\mathbb{R}^n} u^\ell(x)\,dx = \alpha_{m,n,p,q} \quad \forall \delta>0.$$

This gives an idea of the way the convergence $u\to 0$ occurs.

In our last statement we determine an equilibrium behavior in such a way that $u(0)$ remains bounded away from 0 and infinity when both $p\to m^*$ and $\delta\to 0$.

Theorem 7. *Let $d>0$ and for all $1<q<p<m^*$, let u be the unique ground state of problem* (1.1). *Let $\beta_{m,n,q}$, $\mu_{m,n}$ and $\gamma_{m,n,q}$ be as in* (2.6), (2.7) and (2.8), *respectively. Then, writing $\varepsilon = m^* - p$ and taking*

$$\delta = \delta(\varepsilon) = \begin{cases} d^{m^*-q}\ \beta_{m,n,q}^{-1}\ \varepsilon & \text{if } q>m_R, \\ d^{m^*-q}\mu_{m,n}^{-1}\dfrac{\varepsilon}{|\ln \varepsilon|} & \text{if } q=m_R, \\ d^{m^*-q}\ \gamma_{m,n,q}^{q-m_*}\ \varepsilon^{m_*-q} & \text{if } q<m_R, \end{cases}$$

we have

$$u(0)\to d, \quad u\to U_d \qquad \text{as } \varepsilon\to 0$$

uniformly on $\mathbb{R}^n$, where U_d is the function defined in (1.3).

3. Preliminary results

In radial coordinates, Eq. (2.1) becomes

$$\begin{cases} (|u'|^{m-2}u')' + \dfrac{n-1}{r}|u'|^{m-2}u' - u^{q-1} + u^{p-1} = 0, \\ u'(0) = 0, \\ \lim\limits_{r\to\infty} u(r) = 0. \end{cases} \tag{3.1}$$

We first recall a known result:

Proposition 1. *For all $1<q<p<m^*$, problem* (3.1) *admits a unique solution u. Such a solution obeys the following Pohožaev-type identity*

$$\int_0^\infty \left[\left(1-\frac{m^*}{q}\right)u^q(r) + \left(\frac{m^*}{p}-1\right)u^p(r)\right]r^{n-1}\,dr = 0. \tag{3.2}$$

Proof. Existence is proved in [6], see also [2]. Uniqueness is proved in [12]. The Pohožaev-type identity is proved in [9], see also Proposition 3 in [5]. □

We now describe the asymptotic behavior at infinity of the solution of (3.1). In the following statement we collect a number of known results. Only (iv) seems to be new: it improves (iii) when $q<m_*$ and it plays an important role in what follows.

Proposition 2. *Assume that $1<q<p<m^*$.*

(i) *If $q<m$, then the unique solution of* (3.1) *has compact support in* $[0,\infty)$.

(ii) *If* $q=m$, *then the unique solution* u *of* (3.1) *satisfies* $u>0$ *and*

$$u(r)\leqslant \mu e^{-\nu r},\quad |u'(r)|\leqslant \mu e^{-\nu r},\quad |u''(r)|\leqslant \mu e^{-\nu r}\qquad \forall r\geqslant 0 \tag{3.3}$$

for some constants $\mu,\nu>0$.

(iii) *If* $q>m$, *then the unique solution* u *of* (3.1) *satisfies* $u>0$ *and*

$$r^{\frac{m(n-1)}{q(m-1)}}u(r)\to 0\quad \text{as } r\to\infty. \tag{3.4}$$

(iv) *If* $m<q<m_*$, *then the unique solution of* (3.1) *also satisfies*

$$\exists C,R>0,\quad u(r)\leqslant Cr^{-\frac{m}{q-m}}\quad \forall r\geqslant R \tag{3.5}$$

and

$$r^{n-1}|u'(r)|^{m-1}\to 0\quad \text{as } r\to\infty. \tag{3.6}$$

Proof. Part (i) is an immediate consequence of Proposition 1.3.1 in [3]. For the proof of part (ii) see Theorem 8 in [5]. Part (iii) can be obtained using the limits at p. 184 in [9]. It remains to prove part (iv). To the solution u of (3.1) we associate the energy function

$$E(r)\coloneqq \frac{m-1}{m}|u'(r)|^m-\frac{1}{q}u^q(r)+\frac{1}{p}u^p(r)\quad r>0 \tag{3.7}$$

which satisfies $E(r)>0$ for all $r\geqslant 0$, see Proposition 2 in [5]. Since $u(r)\to 0$ as $r\to\infty$, for any $\rho>0$ there exists $\bar{R}>0$ such that $u(r)\leqslant\rho$ for any $r\geqslant\bar{R}$. Choose ρ so that $\frac{s^q}{q}-\frac{s^p}{p}\geqslant\frac{s^q}{2q}$ for all $0<s\leqslant\rho$; then, by positivity of E, we have

$$\frac{m-1}{m}|u'(r)|^m>\frac{1}{q}u^q(r)-\frac{1}{p}u^p(r)\geqslant\frac{1}{2q}u^q(r)\quad\forall r\geqslant\bar{R}.$$

Hence, recalling that $u'<0$ in view of [6, Theorem 1], we obtain

$$-\frac{u'(r)}{u^{\frac{q}{m}}(r)}\geqslant\left(\frac{m}{2q(m-1)}\right)^{\frac{1}{m}}\quad\forall r\geqslant\bar{R}.$$

Integrating this inequality over the interval $(\bar{R},r)$, we obtain

$$\left(\frac{q}{m}-1\right)^{-1}\left(u^{1-\frac{q}{m}}(r)-u^{1-\frac{q}{m}}(\bar{R})\right)\geqslant\left(\frac{m}{2q(m-1)}\right)^{\frac{1}{m}}(r-\bar{R})$$

and hence there exist constants $C>0$ and $R>\bar{R}$ such that

$$u(r)\leqslant \left[\left(\frac{m}{2q(m-1)}\right)^{\frac{1}{m}}\left(\frac{q}{m}-1\right)(r-\bar{R})+u^{1-\frac{q}{m}}(\bar{R})\right]^{-\frac{m}{q-m}}\leqslant Cr^{-\frac{m}{q-m}}\quad \forall r\geqslant R.$$

which proves (3.5). In order to prove (3.6), recall that by [9, Lemma 5.1] the limit exists. Suppose for contradiction that

$$\lim_{r\to\infty} r^{n-1}|u'(r)|^{m-1}>0.$$

Then, by using de l'Hopital rule and the fact that $q<m_*$, we infer

$$\lim_{r\to\infty}\frac{u(r)}{r^{-\frac{m}{q-m}}}=\frac{q-m}{m}\lim_{r\to\infty} r^{\frac{q}{q-m}}|u'(r)|=+\infty$$

which contradicts (3.5). $\square$

Note that, taking into account (3.6) and integrating (3.1) (in divergence form) over $[0,\infty)$ yields

$$\int_0^\infty r^{n-1}u^{q-1}(r)\,dr=\int_0^\infty r^{n-1}u^{p-1}(r)\,dr. \tag{3.8}$$

In the remaining part of this section, we follow closely the approach in [5]. We just briefly recall the basic points. From now on we denote

$$\varepsilon:=m^*-p.$$

Using Proposition 1, we see that if u is the unique solution of (3.1), then

$$\alpha:=u(0)>\left(\frac{(m^*-q)(m^*-\varepsilon)}{\varepsilon q}\right)^{\frac{1}{p-q}}$$

since otherwise the left-hand side of (3.2) would be strictly negative. Clearly, the previous inequality implies that

$$\lim_{\varepsilon\to 0}\alpha=+\infty \tag{3.9}$$

and

$$\omega:=\varepsilon\alpha^{p-q}\geqslant\frac{(m^*-q)(m^*-\varepsilon)}{q}\quad \forall\varepsilon\in(0,m^*-q). \tag{3.10}$$

Let u be the unique solution of (3.1) and consider the function $y=y(r)$ defined by

$$y(r)=\frac{1}{\alpha}u(\alpha^{-\frac{p-m}{m}}r) \tag{3.11}$$

so that y is the unique solution of the problem

$$\begin{cases} (|y'|^{m-2}y')' + \dfrac{n-1}{r}|y'|^{m-2}y' - \eta y^{q-1} + y^{p-1} = 0 \quad (r>0), \\ y(0) = 1, y'(0) = 0 \end{cases} \tag{3.12}$$

with $\eta = \alpha^{q-p}$. By (3.9), we immediately deduce that $\eta \to 0$ as $\varepsilon \to 0$.

Consider the function

$$z(r) = (1 + (1-\eta)^{\frac{1}{m-1}} D r^{\frac{m}{m-1}})^{-\frac{n-m}{m}} \quad \forall r \geqslant 0, \tag{3.13}$$

where D is the constant defined in (1.3). Then z solves the equation

$$(|z'|^{m-2}z')' + \frac{n-1}{r}|z'|^{m-2}z' + (1-\eta)z^{m^*-1} = 0.$$

Moreover, if $A_{m,n} = D^{-\frac{n-m}{m}}$, then z satisfies

$$z(r) \sim \frac{A_{m,n}}{(1-\alpha^{q-p})^{\frac{n-m}{m(m-1)}}} r^{-\frac{n-m}{m-1}} \quad \text{as } r \to \infty. \tag{3.14}$$

In the spirit of [7], we establish an important comparison result:

Lemma 1. *Let y be the unique solution of* (3.12) *and let z be as in* (3.13), *then*

$$y(r) < z(r) \quad \forall r > 0.$$

Proof. It follows closely the proof of [5, Lemma 1]. One has just to be careful when dealing with compact support solutions, namely in the case $q < m$. □

In the sequel, we sometimes consider the functions $y = y(r)$ and $z = z(r)$ to be defined on $\mathbb{R}^n$, that is, $y = y(x)$ and $z = z(x)$. In particular, the function y solves the partial differential equation

$$-\Delta_m y = -\eta y^{q-1} + y^{p-1}, \quad \eta = \alpha^{q-p}. \tag{3.15}$$

We introduce the two constants (depending on ε):

$$C_1 = C_1(\varepsilon) = \left(\frac{\varepsilon}{m^* - q}\right)^{\frac{\varepsilon}{p-q}}, \quad C_2 = C_2(\varepsilon) = \left(\frac{C_1}{1-\eta}\right)^{\frac{1}{m-1}}. \tag{3.16}$$

Arguing as in [5] we establish:

Lemma 2. *Let u be the solution of* (3.1), *y as in* (3.11), *z as in* (3.13), *then*:

(i) $s^{p-1} - \eta s^{q-1} \leqslant C_1 \alpha^{\varepsilon} s^{m^*-1}$ *for all $s > 0$ and* $\lim_{\varepsilon \to 0} C_1 = 1$

(ii) $y(r) > C_2\alpha^{\frac{\varepsilon}{m-1}}z(r) - (C_2\alpha^{\frac{\varepsilon}{m-1}} - 1)$ *for all* $r > 0$, $\lim_{\varepsilon\to 0} C_2 = 1$ *and* $C_2\alpha^{\frac{\varepsilon}{m-1}} > 1$.
(iii) *For all* $r > 0$ *we have*

$$0 < z\left(\alpha^{\frac{p-m}{m}} r\right) - \frac{1}{\alpha}u(r) < c\varepsilon|\ln\varepsilon|, \quad \lim_{\varepsilon\to 0}\ [\alpha^{\frac{1}{m-1}}|u'(r)|] \leqslant \frac{n-m}{m-1}A_{m,n}\, r^{-\frac{n-1}{m-1}}. \tag{3.17}$$

(iv) *There exist* $c_2 > c_1 > 0$ (*depending only on* m, n, q) *such that*

$$c_1\omega \int_{\mathbb{R}^n} y^p(x)\,dx \leqslant \int_{\mathbb{R}^n} y^q(x)\,dx \leqslant c_2\omega \int_{\mathbb{R}^n} y^p(x)\,dx.$$

(v) *There exists* $C > 0$ (*depending only on* m, n) *such that*

$$\int_{\mathbb{R}^n} y^p(x)\,dx \geqslant (C\alpha^\varepsilon)^{-\frac{n-m}{m}} \quad \textit{and} \quad \int_{\mathbb{R}^n} |\nabla y(x)|^m\,dx \geqslant (C\alpha^\varepsilon)^{-\frac{n-m}{m}}.$$

Proof. (i) It follows after some computations of differential calculus.
(ii) It follows by using (i), see [5, Lemma 2].
(iii) The proof of the first of (3.17) can be obtained by Lemma 1, the rescaling (3.11) and following the same lines used to obtain (49) and (59) in [5]. The second estimate in (3.17) can be obtained in the same way as in the proof of Theorem 3 in [5].
(iv) See [5, Lemma 3] and also Proposition 1.
(v) It follows by using (i), see [5, Lemma 4]; here C is the best Sobolev constant for the inequality of the embedding $D^{1,m} \subset L^{m^*}$. □

We now distinguish two cases according to whether the function z defined in (3.13) satisfies $z \in L^q(\mathbb{R}^n)$ or not. Since $z(x) \approx |x|^{-(n-m)/(m-1)}$ as $|x| \to \infty$, the first case occurs when $q > m_R$.

3.1. The case $q > m_R$

This case is somehow simpler: we establish

Lemma 3. *Let* $q > m_R$, *then*

$$\lim_{\varepsilon\to 0} \alpha^\varepsilon = 1 \tag{3.18}$$

and there exists $K > 0$ (*depending only on* n, m, q) *such that* $\omega = \varepsilon\alpha^{p-q} \leqslant K$ *for all sufficiently small* ε.

Proof. As $q > m_R$ (i.e. $z \in L^q$), by Lemma 1 we get a uniform upper bound for $||y||_q$, provided ε is sufficiently small. This, together with Lemma 2(iv) and (v), yields

$$\varepsilon\,\alpha^{m^*-q-\varepsilon\frac{n}{m}} \leqslant C. \tag{3.19}$$

This inequality proves (3.18), see also [5, Lemma 5] for the details.

In turn, using (3.19) and (3.18) we obtain

$$\omega = \varepsilon\alpha^{m^*-q-\varepsilon} = \varepsilon\alpha^{m^*-q-\varepsilon\frac{n}{m}}\,\alpha^{\varepsilon\frac{n-m}{m}} \leqslant K$$

and the proof is complete, see also [5, Lemma 6]. □

3.2. *The case $q \leqslant m_R$*

In this case, $z \notin L^q$ and the situation is more difficult. To compensate the nonsummability of z, we will consider an exponent $\ell = \ell(\varepsilon)$ larger than m_R and convergent to it when $\varepsilon \to 0$. The next statement is the extension of [5, Lemma 7] to our context:

Lemma 4. *Suppose that $q \leqslant m_R$, then there exists $K_1 = K_1(m, n, q) > 0$, such that*

$$\varepsilon\alpha^{\frac{m^*-q}{m_*-q}} \leqslant K_1 |\ln \varepsilon|^{\frac{1}{m_*-q}}.$$

Proof. Let $\ell = \ell(\varepsilon) > m_R$ to be determined later. Then, by (3.13), $z \in L^\ell(\mathbb{R}^n)$. Hence, by Lemma 1, there exists $\hat{d} = \hat{d}(\ell) > 0$ such that

$$\int_{\mathbb{R}^n} y^\ell(x)\,dx \leqslant \hat{d} < \infty; \tag{3.20}$$

moreover, after some calculations one sees that

$$\hat{d}(\ell) = O\left(\frac{1}{\ell - m_R}\right) \quad \text{as } \ell \downarrow m_R. \tag{3.21}$$

By Lemma 2(iv) and (v), we have

$$\int_{\mathbb{R}^n} y^q(x)\,dx \geqslant C\omega\alpha^{-\varepsilon\frac{n-m}{m}}. \tag{3.22}$$

On the other hand, $y = y(r)$ solves the ordinary differential equation

$$(r^{n-1}|y'(r)|^{m-1})' = r^{n-1}(-\eta y^{q-1}(r) + y^{p-1}(r)) \quad (r > 0).$$

If we integrate it over $(0, \infty)$, then the left-hand side vanishes: indeed, the boundary term obviously vanishes at $r = 0$ while it vanishes at infinity because y has compact support (if $q < m$) or because of (3.6) and (3.11) (if $q \geqslant m$). Therefore, returning to cartesian coordinates and recalling Lemma 1, we find

$$\int_{\mathbb{R}^n} y^{q-1}(x)\,dx = \alpha^{p-q} \int_{\mathbb{R}^n} y^{p-1}(x)\,dx \leqslant C\alpha^{p-q}; \tag{3.23}$$

here we used the fact that $z \in L^{p-1}(\mathbb{R}^n)$ if $\varepsilon < m/(n-m)$.

As $q \leqslant m_R < \ell$, we have $\vartheta := (\ell - q + 1)^{-1} \in (0,1)$; then, using Hölder inequality, we obtain

$$\int_{\mathbb{R}^n} y^q(x)\,dx \leqslant \left(\int_{\mathbb{R}^n} y^{q-1}(x)\,dx \right)^{1-\vartheta} \left(\int_{\mathbb{R}^n} y^{\ell}(x)\,dx \right)^{\vartheta}. \tag{3.24}$$

Now we choose

$$\ell = \frac{m-1}{1-|\ln \varepsilon|^{-1}} \left(\frac{n}{n-m} - \frac{1}{|\ln \varepsilon|} \right);$$

with this choice, $\ell > m_R$ whenever $|\ln \varepsilon| > 1$. Then, after some calculations we have

$$\vartheta = (\ell - q + 1)^{-1} = \frac{1}{m_* - q} + O\left(\frac{1}{|\ln \varepsilon|} \right) \quad \text{as } \varepsilon \to 0 \tag{3.25}$$

and

$$\frac{1}{\ell - m_R} = \frac{n-m}{m(m-1)} (|\ln \varepsilon| - 1). \tag{3.26}$$

Combining (3.20)–(3.24) and (3.26) yields

$$\varepsilon \alpha^{(p-q)\vartheta - \varepsilon \frac{n-m}{m}} \leqslant c |\ln \varepsilon|^{\vartheta} \leqslant c |\ln \varepsilon|^{\frac{1}{m_* - q}}$$

and therefore, by (3.25), we have

$$\varepsilon \alpha^{\frac{m^* - q}{m_* - q} + O(\frac{1}{|\ln \varepsilon|})} \leqslant c |\ln \varepsilon|^{\frac{1}{m_* - q}}.$$

Moreover, for ε small enough, we have

$$\alpha \leqslant C \left(\frac{|\ln \varepsilon|^{\frac{1}{m_* - q}}}{\varepsilon} \right)^{2\frac{m_* - q}{m^* - q}}$$

and hence $\alpha^{|O(\frac{1}{|\ln \varepsilon|})|}$ is bounded and this completes the proof of the lemma. □

We seek a more precise estimate on the function $C_3 = C_3(\varepsilon)$. By Lemma 4 and the fact that $\alpha > 1$, (3.18) follows again, namely

$$\lim_{\varepsilon \to 0} \alpha^{\varepsilon} = 1 \quad \forall q \in (1, m^*); \tag{3.27}$$

hence, (5.1) still holds and $C_3 \to 1$ as $\varepsilon \to 0$. After an easy calculation, one finds that the function $C_1 = C_1(\varepsilon)$ defined in (3.16), satisfies $C_1 \leqslant 1 + c\varepsilon|\ln \varepsilon|$ for some constant $c > 0$ depending only on m, n, q. Moreover by (3.10) we have $\eta < c\varepsilon$ and hence also the function $C_2 = C_2(\varepsilon)$ defined in (3.16) satisfies $C_2 \leqslant 1 + c\varepsilon|\ln \varepsilon|$. Finally,

$$1 < C_3 = C_2 \alpha^{\frac{\varepsilon}{m-1}} \leqslant 1 + c\varepsilon|\ln \varepsilon|. \tag{3.28}$$

for ε small enough. Let R be the unique value of r such that $z(r) = v\varepsilon|\ln \varepsilon|$ where $v > 0$ is a sufficiently large constant, see below. By Lemma 2(ii) and (3.28), we have

$$y(r) > C_3 z(r) - (C_3 - 1)\frac{z(r)}{z(R)} > \left(1 - \frac{C_3 - 1}{v\varepsilon|\ln \varepsilon|}\right) z(r) \geqslant \left(1 - \frac{c}{v}\right) z(r) \quad \forall r \in [0, R].$$

Then, fixing v large enough, we infer

$$y(r) \geqslant \tfrac{1}{2} z(r) \quad \forall r \in [0, R]. \tag{3.29}$$

The next lemma shows a different behavior of the parameter $\omega = \varepsilon\alpha^{p-q}$ when compared to the case $q > m_R$ where ω remains bounded as $\varepsilon \to 0$.

Lemma 5. *Assume $q \leqslant m_R$. Then there exists $K_2 = K_2(m, n, q) > 0$ such that for ε small enough*

$$\varepsilon \alpha^{\frac{m^* - q}{m_* - q}} \geqslant K_2 |\ln \varepsilon|^{-\frac{m_R - q}{m_* - q}} \quad \textit{if} \;\; q < m_R \tag{3.30}$$

and

$$\varepsilon \alpha^{\frac{n}{n-m}} \geqslant K_2 |\ln \varepsilon| \quad \textit{if} \;\; q = m_R. \tag{3.31}$$

Proof. By (3.29), one can repeat the proof of [5, Lemma 8] with some minor modifications. □

4. Proof of Theorem 1

The proof is delicate, covering a number of pages. Here we sketch the main steps and we refer to the subsection below for the details.

Statement (i) follows from Theorem 1.1 in [15].

When $1 < q < m_*$, the existence of a nonnegative radial solution W_q of (2.2)–(2.4) is stated in Proposition 3. The uniqueness of the solution W_q is established by Proposition 4; the fact that the boundary condition (2.4) is not needed in the statement when $q > m$ is shown in Lemma 6. The compact support statement for W_q when $1 < q < m$ and the positivity of W_q when $q \geqslant m$ are obtained in Lemma 7.

Finally, the decay conditions at infinity for W_q are obtained in Lemma 8 (case $q>m$) and in [4] (case $q=m$).

4.1. Asymptotic behavior of the solutions

If W is a nonnegative radial solution of (2.2), then $W=W(r)$ satisfies

$$(r^{n-1}|W'(r)|^{m-2}W'(r))' = r^{n-1}W^{q-1}(r) \quad (r>0). \tag{4.1}$$

Moreover, the boundary conditions (2.3), (2.4) become

$$\lim_{r\to 0} r^{\frac{n-m}{m-1}}W(r) = A_{m,n}, \quad \lim_{r\to\infty} W(r) = 0. \tag{4.2}$$

We first show that the second condition in (4.2) is automatically satisfied when $q>m$: note that the assumption $q>m$ is needed only in the proof of Step 3.

Lemma 6. *Let $n>m>1$ and $m<q<m_*$. Then any nonnegative radial solution W of* (2.2) *satisfying* (2.3) *necessarily satisfies* (2.4).

Proof. We argue directly with Eq. (4.1).

Step 1. We show that $\lim_{r\to\infty} W(r)$ exists.

It suffices to show that W is ultimately monotone. If not, then W has a local minimum at some $R>0$ and therefore $R^{n-1}|W'(R)|^{m-2}W'(R)=0$. Eq. (4.1) shows that the map $r\mapsto r^{n-1}|W'(r)|^{m-2}W'(r)$ is nondecreasing; hence, $W'(r)\geqslant 0$ for all $r\geqslant R$, giving a contradiction.

Step 2. We show that $\lim_{r\to\infty} W(r)\notin(0,\infty)$.

For contradiction, assume that $\lim_{r\to\infty} W(r)=C\in(0,\infty)$. Then, by (4.1) we infer

$$\lim_{r\to\infty} r^{n-1}|W'(r)|^{m-2}W'(r) = +\infty.$$

Therefore, we may use the de l'Hopital rule and (4.1) to obtain

$$\lim_{r\to\infty}\frac{|W'(r)|^{m-2}W'(r)}{r} = \lim_{r\to\infty}\frac{r^{n-1}|W'(r)|^{m-2}W'(r)}{r^n} = \lim_{r\to\infty}\frac{r^{n-1}W^{q-1}(r)}{nr^{n-1}} = \frac{C^{q-1}}{n}>0.$$

This implies that $\lim_{r\to\infty} W'(r)=+\infty$, contradiction.

Step 3. We show that $\lim_{r\to\infty} W(r)\neq\infty$.

Consider the "standard" energy function

$$E(r) = \frac{m-1}{m}|W'(r)|^m - \frac{1}{q}W^q(r). \tag{4.3}$$

Then, by using (4.1) one sees that

$$E'(r) = -\frac{n-1}{r}|W'(r)|^m \leqslant 0. \tag{4.4}$$

For contradiction, assume that $\lim_{r\to\infty} W(r) = \infty$. Then, by the argument in Step 1 we see that there exists a unique critical point R of W which is a global minimum. In such a point, we obviously have $E(R)\leqslant 0$ and by (4.4) we deduce that

$$\frac{m-1}{m}|W'(r)|^m \leqslant \frac{1}{q} W^q(r) \quad \forall r\geqslant R.$$

Then, since $q>m$, we infer that

$$\frac{n-1}{r}|W'(r)|^{m-1} = o(W^{q-1}(r)) \quad \text{as } r\to\infty.$$

And this, together with (4.1), implies that

$$\lim_{r\to\infty} \frac{(m-1)|W'(r)|^{m-2}W''(r)}{W^{q-1}(r)} = 1. \tag{4.5}$$

We claim that (4.5) yields

$$\lim_{r\to\infty} W'(r) = +\infty. \tag{4.6}$$

Indeed, if $m\geqslant 2$, this follows at once from (4.5). If $1<m<2$, (4.5) shows that W'' is ultimately positive so that $\lim W'$ exists; it cannot be 0 because $W'(r)>0$ for $r\geqslant R$ and therefore (4.5) implies (4.6).

Thanks to (4.6), we may apply de l'Hopital rule which, combined with (4.5), gives

$$\lim_{r\to\infty} \frac{\frac{m-1}{m}|W'(r)|^m}{\frac{1}{q}W^q(r)} = \lim_{r\to\infty} \frac{(m-1)|W'(r)|^{m-2}W''(r)}{W^{q-1}(r)} = 1.$$

This shows that there exists $C>0$ such that for sufficiently large r, say $r\geqslant \bar{R}$, we have

$$\frac{W'(r)}{W^{q/m}(r)} \geqslant C \quad \forall r\geqslant \bar{R}.$$

Integrating this inequality over $[\bar{R}, r]$ gives

$$-\frac{m}{q-m}[W^{1-q/m}(r) - W^{1-q/m}(\bar{R})] \geqslant C(r-\bar{R})$$

and (recall $q>m$) the contradiction follows by letting $r\to\infty$. $\square$

Even if the function W is singular at $r=0$, the proof of Lemma 7 below follows the same lines as in [3] with some minor changes; therefore, we just refer to the corresponding statement in [3].

Lemma 7. *Let W be a nonnegative solution of* (4.1)–(4.2); *then*

(i) *For any* $r_2 > r_1 > 0$ *the following identity holds*

$$\frac{m-1}{m}|W'(r_2)|^m - \frac{m-1}{m}|W'(r_1)|^m + (n-1)\int_{r_1}^{r_2}\frac{|W'(r)|^m}{r}\,dr$$
$$= \frac{W^q(r_2) - W^q(r_1)}{q}. \tag{4.7}$$

(ii) *We have* $W'(r) < 0$ *for any* $r > 0$ *such that* $W(r) > 0$.
(iii) *If* $W(r) > 0$ *for all* $r > 0$ *then* $W'(r) \to 0$ *as* $r \to \infty$.
(iv) *If* $q < m$, *then there exists* $R > 0$ *such that* $W(r) = 0$ *for all* $r \geqslant R$.
(v) *If* $q \geqslant m$, *then* $W(r) > 0$ *for all* $r > 0$.

Proof. (i) See [3, Lemma 1.1.2].
(ii) From (i) and arguing as in [3, Lemma 1.2.4] one gets $W'(r) \leqslant 0$ for any $r > 0$; the *strict* inequality follows from the form of (4.1).
(iii) See [3, Lemma 1.2.1].
(iv) Argue as in [3, Proposition 1.3.1] by using (i)–(iii).
(v) Argue as in [3, Proposition 1.3.2] by using (i)–(iii). □

Finally, we determine the asymptotic behavior at infinity of the solutions of (4.1):

Lemma 8. *Suppose that* $m < q < m_*$. *If* W *is a nonnegative solution of* (4.1)–(4.2) *then*

$$W(r) = O\left(r^{-\frac{m}{q-m}}\right) \quad \text{as } r \to \infty \quad \text{and} \quad \lim_{r\to\infty} r^{n-1}|W'(r)|^{m-1} = 0.$$

Proof. Consider again the energy function E defined in (4.3). By (4.2), (4.4) and Lemma 7 we infer that $E(r) > 0$ for all $r > 0$. The proof is now similar to that of Proposition 2(iv). □

4.2. Existence

The results in this section are inspired to [4] but the proofs are tedious and slightly different from [4] because the exponents involved depend on q. For this reason, we briefly sketch the proofs.

Assume $q < m_*$, let κ_q be as in (2.10) and let u be the unique solution of (3.1). Throughout this section we consider the functions

$$w(r) = \alpha^{m\kappa_q} u(\alpha^{(q-m)\kappa_q} r) \quad (\alpha = u(0)) \tag{4.8}$$

and

$$v(r) = r^{\frac{n-m}{m-1}} w(r). \tag{4.9}$$

Then, the function $w = w(r)$ satisfies

$$(r^{n-1}|w'(r)|^{m-2}w'(r))' = r^{n-1}w^{q-1}(r)(1 - u^{p-q}(\alpha^{(q-m)\kappa_q}r)). \tag{4.10}$$

Our first purpose is to prove that the family of functions $v = v_\varepsilon$ defined in (4.9) converges as $\varepsilon \to 0$:

Lemma 9. *There exists a function* $V \in \mathrm{Lip}_{\mathrm{loc}}(0, \infty)$ *such that* $0 \leqslant V(r) \leqslant 2A_{m,n}$ *for all* $r > 0$ *and* (*up to a subsequence*)

$$\lim_{\varepsilon \to 0} v(r) = V(r)$$

pointwise in $(0, \infty)$ *and uniformly on compact sets of* $(0, \infty)$.

Proof. We first claim that if ε is small enough, then

$$0 < v(r) \leqslant 2A_{m,n}, \quad |v'(r)| \leqslant 2A_{m,n}\frac{n-m}{m-1}\frac{1}{r} \qquad \forall r > 0. \tag{4.11}$$

By Lemma 1 and the substitutions (4.8)–(4.9) we have for all $r > 0$,

$$\begin{aligned} v(r) &= \alpha^{m\kappa_q} r^{\frac{n-m}{m-1}} u(\alpha^{(q-m)\kappa_q} r) < \alpha^{m\kappa_q+1} r^{\frac{n-m}{m-1}} z(\alpha^{\frac{p-m}{m}+(q-m)\kappa_q} r) \\ &< \alpha^{m\kappa_q+1} r^{\frac{n-m}{m-1}} \frac{A_{m,n}}{(1-\alpha^{q-p})^{\frac{n-m}{m(m-1)}}} (\alpha^{\frac{p-m}{m}+(q-m)\kappa_q} r)^{-\frac{n-m}{m-1}} \\ &= \alpha^{\varepsilon\frac{n-m}{m(m-1)}} \frac{A_{m,n}}{(1-\alpha^{q-p})^{\frac{n-m}{m(m-1)}}} \end{aligned} \tag{4.12}$$

and the first part of (4.11) follows by (3.9) and (3.27). A similar argument, combined with (3.17), gives for all $r > 0$:

$$\begin{aligned} |v'(r)| &\leqslant \alpha^{m\kappa_q} \left[\alpha^{(q-m)\kappa_q} r^{\frac{n-m}{m-1}} |u'(\alpha^{(q-m)\kappa_q} r)| + \frac{n-m}{m-1} r^{\frac{n-2m+1}{m-1}} u(\alpha^{(q-m)\kappa_q} r) \right] \\ &\leqslant 2A_{m,n}\frac{n-m}{m-1}\frac{1}{r} \end{aligned}$$

which completes the proof of (4.11).

The statement follows at once from (4.11) and the Ascoli–Arzelà Theorem. □

Here and in the sequel we put

$$\rho_\varepsilon = \alpha^{-\kappa_q(m-1)(m^*-q)}\varepsilon^{-\theta}, \quad 0 < \theta < \min\left\{(m-1)\kappa_q, \frac{m-1}{n-m}\right\}.$$

With this choice of ρ_ε, the following lemma holds.

Lemma 10. *We have* $\lim_{\varepsilon\to 0}\rho_\varepsilon = 0$ *and* $\lim_{\varepsilon\to 0} v(\rho_\varepsilon) = A_{m,n}$.

Proof. By (3.10) and Lemma 3, we know that $\omega\in[m^* - q, K]$ for small enough ε, namely

$$\alpha\approx\varepsilon^{-\frac{1}{m^*-q}} \quad \text{as } \varepsilon\to 0 \text{ if } q>m_R$$

whence by Lemmas 4 and 5,

$$\alpha\approx\varepsilon^{-\frac{m_*-q}{m^*-q}} \quad \text{as } \varepsilon\to 0 \text{ if } q\leqslant m_R,$$

up to a logarithmic term. Hence, by definition of ρ_ε, it follows that $\rho_\varepsilon\to 0$ as $\varepsilon\to 0$. To prove the second part of the lemma, we observe that, by (3.27) and (4.12)

$$\lim_{\varepsilon\to 0}\sup v(\rho_\varepsilon)\leqslant A_{m,n}. \tag{4.13}$$

By (3.17), (4.8) and (4.9), for all $r>0$ we have

$$v(r) > r^{\frac{n-m}{m-1}}\alpha^{m\kappa_q+1} z(\alpha^{\frac{p-m}{m}+(q-m)\kappa_q} r) - c r^{\frac{n-m}{m-1}}\alpha^{m\kappa_q+1}\varepsilon|\ln\varepsilon|. \tag{4.14}$$

Furthermore, it follows from the definition of ρ_ε and κ_q, that

$$\rho_\varepsilon^{\frac{n-m}{m-1}}\alpha^{m\kappa_q+1}\varepsilon|\ln\varepsilon|\to 0 \quad \text{as } \varepsilon\to 0 \tag{4.15}$$

and

$$\alpha^{\frac{p-m}{m}+(q-m)\kappa_q}\rho_\varepsilon\to\infty \quad \text{as } \varepsilon\to 0. \tag{4.16}$$

By (3.14) and (4.16), one can easily see that

$$z(\alpha^{\frac{p-m}{m}+(q-m)\kappa_q}\rho_\varepsilon)\sim\frac{A_{m,n}}{(1-\alpha^{q-p})^{\frac{n-m}{m(m-1)}}}(\alpha^{\frac{p-m}{m}+(q-m)\kappa_q}\rho_\varepsilon)^{-\frac{n-m}{m-1}} \quad \text{as } \varepsilon\to 0$$

and hence

$$\alpha^{m\kappa_q+1}\rho_\varepsilon^{\frac{n-m}{m-1}} z(\alpha^{\frac{p-m}{m}+(q-m)\kappa_q}\rho_\varepsilon)\to A_{m,n} \quad \text{as } \varepsilon\to 0. \tag{4.17}$$

Thus, inserting (4.15) and (4.17) into (4.14) one has

$$\lim_{\varepsilon\to 0}\inf v(\rho_\varepsilon)\geqslant A_{m,n}$$

and this, with (4.13), completes the proof of the lemma. $\square$

Next, we prove an integral identity:

Lemma 11. *The function v defined in* (4.9) *solves the following integral identity*

$$v(r) = \frac{m-1}{n-m} a^{\frac{1}{m-1}} - b r^{\frac{n-m}{m-1}} + a^{\frac{1}{m-1}} r^{\frac{n-m}{m-1}} \int_r^1 g\left(\frac{\int_{\rho_\varepsilon}^t s^{n-1} w^{q-1}(s)(1-u^{p-q}(\alpha^{(q-m)\kappa_q} s))\, ds}{a}\right) t^{-\frac{n-1}{m-1}}\, dt, \tag{4.18}$$

where $a = a_\varepsilon = \rho_\varepsilon^{n-1} |w'(\rho_\varepsilon)|^{m-1}$, $b = \frac{m-1}{n-m} a^{\frac{1}{m-1}} - v(1)$ *and* $g(\zeta) = (1-\zeta)^{\frac{1}{m-1}} - 1$ *for all* $\zeta < 1$.

Proof. After integration of (4.10) over the interval (ρ_ε, r) one has

$$w'(r) = -r^{-\frac{n-1}{m-1}} \left(a - \int_{\rho_\varepsilon}^r s^{n-1} w^{q-1}(s)(1-u^{p-q}(\alpha^{(q-m)\kappa_q} s))\, ds\right)^{\frac{1}{m-1}};$$

integrating the latter over $(r, 1)$ and using (4.9), we obtain

$$v(r) = r^{\frac{n-m}{m-1}} v(1) + r^{\frac{n-m}{m-1}} \int_r^1 \left(a - \int_{\rho_\varepsilon}^t s^{n-1} w^{q-1}(s)(1-u^{p-q}(\alpha^{(q-m)\kappa_q} s))\, ds\right)^{\frac{1}{m-1}} t^{-\frac{n-1}{m-1}}\, dt$$

and this, by definition of a, b and g, completes the proof of the lemma. □

Next we prove that the functions $a = a_\varepsilon$ and $b = b_\varepsilon$ are uniformly bounded when $\varepsilon \to 0$.

Lemma 12. *There exist* $C_1, C_2, C_3 > 0$ *such that, for* ε *small enough*,

$$C_1 < a_\varepsilon < C_2 \quad \text{and} \quad |b_\varepsilon| < C_3.$$

Proof. Recalling (3.6) and (4.8), the integration of (4.10) over $(\rho_\varepsilon, \infty)$ yields

$$a = \int_{\rho_\varepsilon}^\infty s^{n-1} w^{q-1}(s)(1-u^{p-q}(\alpha^{(q-m)\kappa_q} s))\, ds \leqslant \int_0^\infty s^{n-1} w^{q-1}(s)\, ds.$$

Therefore, with some changes of variables and by using (4.8), (3.8), (3.11), (3.27) and Lemma 1 we obtain that $a = a_\varepsilon$ is upper bounded for small ε.

Then, taking $r = 1$ in (4.18) and using (4.11), we have

$$0 \leqslant v(1) = \frac{m-1}{n-m} a^{\frac{1}{m-1}} - b \leqslant 2A_{m,n}$$

which proves that $|b_\varepsilon|$ is bounded.

It remains to prove that a is bounded away from 0. To this end, we first claim that

$$\int_{\rho_\varepsilon}^{1} s^{n-1} w^{q-1}(s) u^{p-q}(\alpha^{(q-m)\kappa_q} s)\, ds = o(1) \quad \text{as } \varepsilon \to 0. \tag{4.19}$$

To show this, we use (4.8), (4.9) and (4.11) to obtain

$$\begin{aligned} \int_{\rho_\varepsilon}^{1} s^{n-1} w^{q-1}(s) u^{p-q}(\alpha^{(q-m)\kappa_q} s)\, ds &= \alpha^{-m\kappa_q(p-q)} \int_{\rho_\varepsilon}^{1} s^{\frac{1-2m}{m-1}+\varepsilon\frac{n-m}{m-1}} v^{p-1}(s)\, ds \\ &\leqslant \alpha^{-m\kappa_q(p-q)} (2A_{m,n})^{p-1} \int_{\rho_\varepsilon}^{1} s^{\frac{1-2m}{m-1}+\varepsilon\frac{n-m}{m-1}}\, ds \end{aligned}$$

and (4.19) follows by recalling (3.27) and the definitions of ρ_ε and θ.

Consider again $g(\zeta) = (1-\zeta)^{\frac{1}{m-1}} - 1$ (for $\zeta < 1$) and let

$$\xi(t) = \xi_\varepsilon(t) = \int_{\rho_\varepsilon}^{t} s^{n-1} w^{q-1}(s)(1 - u^{p-q}(\alpha^{(q-m)\kappa_q} s))\, ds.$$

By [4, Lemma 3] and (4.19), one sees that if $\xi(t) < 0$, then for all $t \in (\rho_\varepsilon, 1]$ we have

$$a^{\frac{1}{m-1}} g\left(\frac{\xi(t)}{a}\right) \leqslant C\left[a^{\frac{1}{m-1}} + \left(\int_{\rho_\varepsilon}^{t} s^{n-1} w^{q-1}(s) u^{p-q}(\alpha^{(q-m)\kappa_q} s)\, ds\right)^{\frac{1}{m-1}}\right] \leqslant C a^{\frac{1}{m-1}} + o(1)$$

as $\varepsilon \to 0$; whence, if $\xi(t) > 0$, then obviously $a^{\frac{1}{m-1}} g(\frac{\xi(t)}{a}) < 0$. Therefore, after multiplication by $t^{-\frac{n-1}{m-1}}$ and integration over $(\rho_\varepsilon, 1)$ we get

$$a^{\frac{1}{m-1}} \rho_\varepsilon^{\frac{n-m}{m-1}} \int_{\rho_\varepsilon}^{1} g\left(\frac{\xi(t)}{a}\right) t^{-\frac{n-1}{m-1}}\, dt \leqslant C a^{\frac{1}{m-1}} + o(1) \quad \text{as } \varepsilon \to 0.$$

Then, putting $r = \rho_\varepsilon$ in (4.18), we have

$$v(\rho_\varepsilon) \leqslant \frac{m-1}{n-m} a^{\frac{1}{m-1}} + a^{\frac{1}{m-1}} \rho_\varepsilon^{\frac{n-m}{m-1}} \int_{\rho_\varepsilon}^{1} g\left(\frac{\xi(t)}{a}\right) t^{-\frac{n-1}{m-1}}\, dt \leqslant C a^{\frac{1}{m-1}} + o(1) \quad \text{as } \varepsilon \to 0.$$

Therefore, by Lemma 10 and letting $\varepsilon \to 0$, we obtain

$$\liminf_{\varepsilon \to 0} a^{\frac{1}{m-1}} \geqslant C A_{m,n}$$

which completes the proof of the lemma. □

Thanks to Lemma 12, we may prove an important convergence result.

Lemma 13. *The following limit exists and*

$$\lim_{\varepsilon\to 0} a^{\frac{1}{m-1}}\rho_\varepsilon^{\frac{n-m}{m-1}}\int_{\rho_\varepsilon}^{1} g\left(\frac{\xi(t)}{a}\right)t^{-\frac{n-1}{m-1}}\,dt = 0,$$

where ξ *and* g *are the functions defined in the proof of Lemma* 12.

Proof. First note that, by (4.19), $\xi(t)\geqslant -o(1)$ as $\varepsilon\to 0$. Moreover, for all $t\in(\rho_\varepsilon,1]$

$$\xi(t)\leqslant \int_{\rho_\varepsilon}^{t} s^{\frac{nm-2m+1-q(n-m)}{m-1}}v^{q-1}(s)\,ds\leqslant (2A_{m,n})^{q-1}(m-1)\kappa_q\, t^{\frac{1}{(m-1)\kappa_q}},$$

where in the last inequality we used (4.11). By Lemma 12 and using again [4, Lemma 3], thanks to the previous lines we infer

$$-Ct^{\frac{1}{(m-1)\kappa_q}}\leqslant g\left(\frac{\xi(t)}{a}\right)\leqslant o(1)\quad (C>0). \tag{4.20}$$

In turn, by (4.20) and letting $\varepsilon\to 0$ we obtain

$$\left|a^{\frac{1}{m-1}}\rho_\varepsilon^{\frac{n-m}{m-1}}\int_{\rho_\varepsilon}^{1} g\left(\frac{\xi(t)}{a}\right)t^{-\frac{n-1}{m-1}}\,dt\right|\leqslant a^{\frac{1}{m-1}}\rho_\varepsilon^{\frac{n-m}{m-1}}\int_{\rho_\varepsilon}^{1}\left[Ct^{\frac{1}{(m-1)\kappa_q}}+o(1)\right]t^{-\frac{n-1}{m-1}}dt = o(1)$$

since $\rho_\varepsilon\to 0$ (by Lemma 10), a is bounded (by Lemma 12) and $q<m_*$. □

Inserting $r=\rho_\varepsilon$ into (4.18), by Lemmas 10, 12 and 13 we have

$$\lim_{\varepsilon\to 0}\frac{m-1}{n-m}a^{\frac{1}{m-1}} = A_{m,n}. \tag{4.21}$$

Fix $r>0$ and let $\varepsilon\to 0$ through an appropriate subsequence, so that $v(r)\to V(r)$ (see Lemma 9) and $b\to B$ (see Lemma 12). Thus, by (4.18), (4.19), (4.21) and Lebesgue Theorem, the function V solves the integral equation

$$V(r) = A_{m,n} - Br^{\frac{n-m}{m-1}} + A_{m,n}\frac{n-m}{m-1}R(r)\quad \forall r>0, \tag{4.22}$$

where

$$R(r) = r^{\frac{n-m}{m-1}}\int_r^1 g\left(\left(\frac{n-m}{m-1}A_{m,n}\right)^{1-m}\int_0^t s^{\frac{nm-2m+1-q(n-m)}{m-1}}V^{q-1}(s)\,ds\right)t^{-\frac{n-1}{m-1}}\,dt\quad \forall r>0.$$

Once more, in the application of Lebesgue Theorem, we used the important restriction $q<m_*$.

Note that the function $R(r)$ defined above satisfies

$$\lim_{r\to 0} R(r) = 0; \tag{4.23}$$

indeed, by invoking again [4, Lemma 3], we infer

$$|R(r)| \leqslant r^{\frac{n-m}{m-1}} \int_r^1 C t^{\frac{1}{(m-1)\kappa_q}} t^{-\frac{n-1}{m-1}} dt = o(1) \quad \text{as } r \to 0,$$

where we used the assumption $q < m_*$.

Hence, by (4.22) and (4.23), it follows that

$$\lim_{r \to 0} V(r) = A_{m,n} > 0. \tag{4.24}$$

This allows us to say that V is a nontrivial function, giving sense to the convergence result of Lemma 9. Put

$$W(r) = r^{-\frac{n-m}{m-1}} V(r) \quad \forall r > 0. \tag{4.25}$$

We finally establish

Proposition 3. *Assume that* $1 < q < m_*$; *then, there exists at least a nonnegative radial solution* W_q *of* (2.2)–(2.4).

Proof. Reversing the steps used to derive the integral identity (4.18), one has that the function W defined in (4.25) solves the ordinary differential equation (4.1) and hence (2.2).

The condition (2.3) follows at once from (4.24) and (4.25).

Finally, V is bounded by Lemma 9; by (4.25), this implies that W also satisfies (2.4).

Therefore, the radial function $W = W(r)$ defined in (4.25) solves (2.2)–(2.4) $\square$

4.3. Uniqueness

In this section, we assume that $q < m_*$ and we prove that the solution of problem (4.1)–(4.2) is unique. To this end, we first give a comparison result:

Lemma 14. *Assume* $q < m$ *and let* W_1 *and* W_2 *be two different nonnegative solutions of* (4.1)–(4.2) *having respective supports* B_{R_1} *and* B_{R_2} *with* $R_1 \leqslant R_2$; *then, up to switching* W_1 *and* W_2 *in the case* $R_1 = R_2$, *we have*

$$W_1(r) < W_2(r), \quad |W_1'(r)| < |W_2'(r)| \quad \forall r < R_2. \tag{4.26}$$

Assume $q \geqslant m$ *and let* W_1 *and* W_2 *be two different positive solutions of* (4.1)–(4.2); *then, up to switching* W_1 *and* W_2, *we have*

$$W_1(r) < W_2(r), \quad |W_1'(r)| < |W_2'(r)| \quad \forall r > 0. \tag{4.27}$$

Proof. Assume $q<m$. Suppose first that $R_1<R_2$. It is clear that (4.26) is satisfied on $[R_1, R_2)$. Let $\underline{R}\in[0, R_1)$ be the infimum value of r for which (4.26) holds; then, we have

$$|W_1'(r)|<|W_2'(r)| \quad \forall r\in(\underline{R}, R_1) \quad \text{and} \quad 0 = W_1(R_1)<W_2(R_1). \tag{4.28}$$

Suppose for contradiction that $\underline{R}>0$, then $W_1(\underline{R})\leqslant W_2(\underline{R})$ and by (4.7) and (4.28), we have

$$\begin{aligned}\frac{m-1}{m}|W_1'(\underline{R})|^m &= (n-1)\int_{\underline{R}}^{R_1}\frac{|W_1'(t)|^m}{t}\,dt+\frac{W_1^q(\underline{R})}{q}\\ &< (n-1)\int_{\underline{R}}^{R_1}\frac{|W_2'(t)|^m}{t}\,\mathrm{d}t+\frac{W_2^q(\underline{R})}{q}\\ &< (n-1)\int_{\underline{R}}^{R_2}\frac{|W_2'(t)|^m}{t}\,\mathrm{d}t+\frac{W_2^q(\underline{R})}{q}=\frac{m-1}{m}|W_2'(\underline{R})|^m\end{aligned}$$

that is,

$$|W_1'(\underline{R})|<|W_2'(\underline{R})|. \tag{4.29}$$

By (4.28), we can easily see that $W_1(\underline{R})<W_2(\underline{R})$ and this, together with (4.29), contradicts the definition of $\underline{R}$. Hence, $\underline{R}=0$ and the proof is so complete in the case $R_1<R_2$.

Suppose now $R_1 = R_2$. We claim that there exists at most one value $R\in(0, R_1)$ such that $W_1(R) = W_2(R)$; more precisely, up to switching W_1 and W_2, we show that

$$\begin{aligned}&\exists R\in(0, R_1), \quad W_1(R) = W_2(R) \Rightarrow W_1(r)<W_2(r),\\ &|W_1'(r)|<|W_2'(r)| \quad \forall r<R.\end{aligned} \tag{4.30}$$

For one such R, by uniqueness for the Cauchy problem, we have $|W_1'(R)|<|W_2'(R)|$. Then, there exists a left neighborhood $(\underline{R}, R)$ of R such that

$$W_1(r)<W_2(r) \quad \text{and} \quad |W_1'(r)|<|W_2'(r)| \qquad \forall r\in(\underline{R}, R); \tag{4.31}$$

we take $\underline{R}$ as the infimum value for which (4.31) holds. Suppose for contradiction that $\underline{R}>0$. Then, by (4.7) and (4.31), we have

$$\begin{aligned}\frac{m-1}{m}|W_1'(\underline{R})|^m &= \frac{m-1}{m}|W_1'(R)|^m+(n-1)\int_{\underline{R}}^{R}\frac{|W_1'(t)|^m}{t}\,dt+\frac{W_1^q(\underline{R})-W_1^q(R)}{q}\\ &< \frac{m-1}{m}|W_2'(R)|^m+(n-1)\int_{\underline{R}}^{R}\frac{|W_2'(t)|^m}{t}\,dt+\frac{W_2^q(\underline{R})-W_2^q(R)}{q}\\ &= \frac{m-1}{m}|W_2'(\underline{R})|^m.\end{aligned}$$

Moreover, (4.31) and the fact that $W_1(R) = W_2(R)$ yield $W_1(\underline{R}) < W_2(\underline{R})$ which contradicts the definition of $\underline{R}$. Therefore, $\underline{R} = 0$ and (4.30) follows.

Since (4.30) states that there exists at most one value $R \in (0, R_1)$ such that $W_1(R) = W_2(R)$, we can suppose, up to switching W_1 and W_2, that

$$\exists R \in [0, R_1) \quad \text{such that } W_1(r) < W_2(r) \quad \forall r \in (R, R_1).$$

Let R be the infimum of such values and assume for contradiction that $R > 0$. Then, for any $r \in [R, R_1)$ integrate the two equations (4.1) for W_1 and W_2 over the interval $[r, R_1]$ to obtain

$$\begin{aligned} r^{n-1}|W_1'(r)|^{m-1} &= \int_r^{R_1} t^{n-1} W_1^{q-1}(t)\, dt \\ &< \int_r^{R_1} t^{n-1} W_2^{q-1}(t)\, dt = r^{n-1}|W_2'(r)|^{m-1} \end{aligned} \tag{4.32}$$

and hence $|W_1'(r)| < |W_2'(r)|$ for any $r \in [R, R_1)$. Moreover, since $W_1(R_1) = W_2(R_1)$, then $W_1(R) < W_2(R)$ which shows that $R = 0$. This completes the proof in the case $q < m$.

Assume now $q \geqslant m$. Arguing as above, we may prove again (4.30) with $R_1 = \infty$. Since by Lemma 8, $r^{n-1}|W_i'(r)|^{m-1} \to 0$ as $r \to \infty$ for $i = 1, 2$, then we also obtain (4.32) which completes the proof as in the case $q < m$. $\square$

If W solves Eq. (4.1), then another solution of (4.1) is given by

$$W_\lambda(r) = \frac{1}{\lambda} W(\lambda^{\frac{m-q}{m}} r) \quad \forall \lambda > 0. \tag{4.33}$$

Moreover, by (4.2), the rescaled function W_λ satisfies

$$r^{\frac{n-m}{m-1}} W_\lambda(r) \to \frac{A_{m,n}}{\lambda^{\frac{(n-m)(m_*-q)}{m(m-1)}}} \quad \text{as } r \to 0, \tag{4.34}$$

and if $q < m$,

$$R_\lambda = R\lambda^{-\frac{m-q}{m}}, \tag{4.35}$$

where R and R_λ are the radii of the supports of W and W_λ. We prove the uniqueness result in the case $q < m_*$:

Proposition 4. *Suppose that* $1 < q < m_*$. *Let* W_1 *and* W_2 *be two solutions of problem* (4.1)–(4.2). *Then* $W_1 \equiv W_2$.

Proof. Assume $q < m$. Let R_1, R_2 be the radii of the supports of W_1 and W_2, respectively. We first show that

$$R_1 = R_2. \tag{4.36}$$

Suppose for contradiction that $R_1 < R_2$. Let $W_{1,\lambda}$ be the rescaled function of W_1 according to (4.33); by (4.35), there exists $\lambda < 1$ such that $R_{1,\lambda} = R_1 \lambda^{-\frac{m-q}{m}} < R_2$. Then, by Lemma 14, we obtain

$$W_{1,\lambda}(r) < W_2(r) \quad \forall r \in (0, R_2). \tag{4.37}$$

By (4.2), (4.34) and the fact that $q < m_*$, we have

$$\lim_{r\to 0} r^{\frac{n-m}{m-1}} W_2(r) = A_{m,n}, \quad \lim_{r\to 0} r^{\frac{n-m}{m-1}} W_{1,\lambda}(r) = \frac{A_{m,n}}{\lambda^{\frac{(n-m)(m_*-q)}{m(m-1)}}} > A_{m,n}$$

which contradicts (4.37). Hence, (4.36) holds. Then, by Lemma 14 we obtain

$$\begin{aligned} &\forall \lambda < 1 \quad W_{1,\lambda}(r) > W_2(r) \quad \forall r \in (0, R_2), \\ &\forall \lambda > 1 \quad W_{1,\lambda}(r) < W_2(r) \quad \forall r \in (0, R_2). \end{aligned} \tag{4.38}$$

Finally, since $W_{1,\lambda} \to W_1$ pointwise on $(0, \infty)$ as $\lambda \to 1$, by (4.38) we deduce that $W_1 \equiv W_2$.

Assume now $m \leqslant q < m_*$ and consider again the rescaled function $W_{1,\lambda}$; if $\lambda > 1$, then by (4.34) and Lemma 14, we obtain

$$W_{1,\lambda}(r) < W_2(r) \quad \forall r > 0.$$

The conclusion is now similar to the case $q < m$. $\square$

5. Proof of Theorem 2

5.1. *The case $q > m_R$*

As a direct consequence of (3.11) and (3.17), we get

$$0 < z(r) - y(r) < c\varepsilon |\ln \varepsilon| \quad \forall r > 0. \tag{5.1}$$

Since $\eta \to 0$ as $\varepsilon \to 0$, then z converges pointwise for any $r \geqslant 0$ to the function

$$z_0(r) = (1 + D r^{\frac{m}{m-1}})^{-\frac{n-m}{m}} \quad \text{as } \varepsilon \to 0;$$

hence, by (5.1), also the function y converges pointwise to z_0 as $\varepsilon \to 0$. Moreover, by (3.10) and Lemma 3, $\omega \in [m^* - q, K]$ (for small enough ε) so that $\omega = \omega(\varepsilon)$ converges, up to a subsequence, to some limit $\omega_0 \in [m^* - q, K]$ (in fact we will show that ω_0 is the limit of ω as $\varepsilon \to 0$ in the continuum). Recalling Lemma 1, that $z \in L^q$ and taking into account that $y \leqslant 1$ and $q < p$, we may then apply Lebesgue Theorem to the Pohožaev identity (3.2) relative to Eq. (3.12) and obtain

$$\int_0^\infty z_0^q(r)\, r^{n-1}\, dr = \omega_0 \frac{q}{m^*(m^*-q)} \int_0^\infty z_0^{m^*}(r) r^{n-1}\, dr. \tag{5.2}$$

After a change of variables we obtain

$$\int_0^\infty z_0^q(r) r^{n-1}\, dr = \frac{m-1}{m} D^{-\frac{m-1}{m}n} B\left(\frac{n(m-1)}{m}, \frac{q(n-m)-n(m-1)}{m}\right) \tag{5.3}$$

and

$$\int_0^\infty z_0^{m^*}(r) r^{n-1}\, dr = \frac{m-1}{m} D^{-\frac{n(m-1)}{m}} B\left(\frac{n(m-1)}{m}, \frac{n}{m}\right). \tag{5.4}$$

Inserting (5.3) and (5.4) into (5.2) we obtain

$$\omega_0 = \frac{m^*}{q}(m^*-q) \frac{B\left(\frac{n(m-1)}{m}, \frac{(q-m_R)(n-m)}{m}\right)}{B\left(\frac{n(m-1)}{m}, \frac{n}{m}\right)}$$

which proves (2.9) in the case $q > m_R$.

5.2. The case $q < m_R$

Let u be the solution of (3.1); after the substitution (4.8) we obtain

$$\int_0^\infty r^{n-1} u^q(r)\, dr = \alpha^{\frac{q-m^*}{m_*-q}} \int_0^\infty r^{n-1} w^q(r)\, dr.$$

By (4.9), Lemma 9 and (4.25) we know that $w(r) \to W(r)$ for any $r > 0$; in order to apply Lebesgue Theorem we need a uniform L^1 majorization of the last integrand. We first estimate its behavior in a neighborhood of infinity (we do not consider the case $q = m$ because it has already been studied in [4,5]).

Lemma 15. *Assume $q \leqslant m_R$.*

If $1 < q < m$, then there exists $R > 0$ such that supp $(w) \subset B_R(0)$ *for ε small enough.*

If $q > m$, then there exist $C, R > 0$ (independent of ε) such that $w(r) \leqslant C r^{-\frac{m}{q-m}}$ for all $r \geqslant R$.

Proof. If $1 < q < m$, the statement follows from Lemma 17 below.

So, assume that $q > m$. By (4.8) and (4.10) we know that w satisfies

$$(|w'|^{m-2} w')' + \frac{n-1}{r} |w'|^{m-2} w' - w^{q-1} + \alpha^{-m\kappa_q(p-q)} w^{p-1} = 0.$$

Consider the corresponding energy function

$$E(r) = \frac{m-1}{m}|w'(r)|^m - \frac{1}{q}w^q(r) + \frac{\alpha^{-m\kappa_q(p-q)}}{p}w^p(r)$$

which satisfies $E(r)>0$ for all r, see Proposition 2 for the details. By (3.9), there exist $C, \rho > 0$, independent of ε, such that

$$\frac{1}{q}s^q - \frac{\alpha^{-m\kappa_q(p-q)}}{p}s^p > Cs^q \quad \forall s \leqslant \rho, \ \forall \varepsilon < \bar{\varepsilon}$$

for $\bar{\varepsilon} > 0$ small enough. By (4.9) and (4.11) we know that

$$w(r) \leqslant 2A_{m,n} r^{-\frac{n-m}{m-1}} \quad \forall r > 0, \ \forall \varepsilon \text{ small enough.}$$

Therefore, there exists $R>0$ independent of ε such that

$$w(r) \leqslant \rho \quad \forall r \geqslant R, \ \forall \varepsilon \text{ small enough.}$$

Hence, by positivity of E we obtain

$$\frac{m-1}{m}|w'(r)|^m > Cw^q(r) \quad \forall r \geqslant R, \ \forall \varepsilon \text{ small enough}$$

and the statement follows as in Proposition 2(iv). □

Concerning the behavior at the origin, by (4.11) we have

$$r^{n-1}w^q(r) = r^{n-1-q\frac{n-m}{m-1}}v^q(r) \leqslant (2A_{m,n})^q \, r^{n-1-q\frac{n-m}{m-1}} \quad \forall r > 0. \tag{5.5}$$

Note that the function on the right-hand side of (5.5) is integrable in a neighborhood of the origin since $q < m_R$. This, together with Lemma 15, enables us to apply Lebesgue Theorem and to infer that $r^{n-1}W^q(r) \in L^1(0,\infty)$ and (see (2.5))

$$\int_0^\infty r^{n-1}w^q(r)\,dr \to \int_0^\infty r^{n-1}W^q(r)\,dr = I_{m,n,q}. \tag{5.6}$$

On the other hand, by (3.11) and the convergence $y \to z_0$, we also have

$$\int_0^\infty r^{n-1}u^p(r)\,dr \to \int_0^\infty r^{n-1}z_0^{m^*}(r)\,dr = \frac{m-1}{m}D^{-\frac{n(m-1)}{m}}B\left(\frac{n(m-1)}{m}, \frac{n}{m}\right). \tag{5.7}$$

Inserting (5.6) and (5.7) into (3.2) proves (2.9) in the case $q < m_R$.

5.3. *The case $q = m_R$*

When $q = m_R$, the limit (5.7) still holds. The problem is the behavior of the first term in (3.2); indeed, the right-hand side in (5.5) becomes Cr^{-1} and is no longer integrable at the origin. Nevertheless, by splitting the integral into two parts, we obtain

Lemma 16. *Let* $q = m_R$ *and let* w *be as in* (4.8). *Then*

$$\int_0^\infty r^{n-1} w^{m_R}(r)\, dr = \frac{n(m-1)}{(n-m)^2} D^{-\frac{n(m-1)}{m}} \ln\alpha + O(\ln\ln\alpha) \quad \text{as } \alpha \to \infty.$$

Proof. Note first that since $q = m_R$, by Lemmas 4 and 5, we get

$$\frac{\ln\alpha}{|\ln\varepsilon|} \to \frac{n-m}{n} \quad \text{as } \varepsilon \to 0. \tag{5.8}$$

We split the integral at the value

$$R_0 = R_0(\varepsilon) = |\ln\varepsilon|^{-\frac{2(m-1)}{n-m}}; \tag{5.9}$$

the statement of the lemma follows if we show the two estimates

$$\int_{R_0}^\infty r^{n-1} w^{m_R}(r)\, dr = O(\ln\ln\alpha) \quad \text{as } \alpha \to \infty, \tag{5.10}$$

$$\int_0^{R_0} r^{n-1} w^{m_R}(r)\, dr = \frac{n(m-1)}{(n-m)^2} D^{-\frac{n(m-1)}{m}} \ln\alpha + O(\ln\ln\alpha) \quad \text{as } \alpha \to \infty. \tag{5.11}$$

When $q = m_R = m$ (i.e. $n = m^2$), these estimates are already known, see [4].

Consider first the case $q = m_R > m$. Let C, R as in Lemma 15, then for small ε we have

$$\int_R^\infty r^{n-1} w^{m_R}(r)\, dr \leqslant C^{m_R} \int_R^\infty r^{-\frac{n(n-m)}{m^2-n}-1}\, dr < c \tag{5.12}$$

(recall $m^2 > n$). Since $R_0 \to 0$ as $\varepsilon \to 0$, we may suppose that $R_0 < R$; then, (4.9) and (4.11) yield

$$\int_{R_0}^R r^{n-1} w^{m_R}(r)\, dr \leqslant (2A_{m,n})^{m_R} \int_{R_0}^R r^{-1}\, dr = (2A_{m,n})^{m_R} \ln\frac{R}{R_0} = c\ln|\ln\varepsilon| + O(1).$$

This, together with (5.12) and (5.8) proves (5.10) in the case $q > m$.

Suppose now $q = m_R < m$. Let R be as in Lemma 15. Then, in the same way as in the case $q > m$, we obtain

$$\int_{R_0}^\infty r^{n-1} w^{m_R}(r)\, dr = \int_{R_0}^R r^{n-1} w^{m_R}(r)\, dr \leqslant (2A_{m,n})^{m_R} \frac{2(m-1)}{n-m} \ln|\ln\varepsilon| + O(1)$$

and (5.10) also holds in the case $q < m$.

In order to prove (5.11) we put

$$\varphi(r) = \frac{1}{[1 + D\alpha^{\frac{p-m}{m-1}} r^{\frac{m}{m-1}}]^{\frac{n-m}{m}}}$$

and we show that there exist $c_1, c_2 > 0$ such that (recall $\eta = \alpha^{q-p}$)

$$w^{m_R}(r) \leqslant \alpha^{\frac{n^2(m-1)}{(n-m)^2}} \{\varphi^{m_R}(\alpha^{\frac{m^2-n}{(n-m)^2}} r) + c_1 \eta \varphi^{m_R - 1}(\alpha^{\frac{m^2-n}{(n-m)^2}} r)\} \tag{5.13}$$

and

$$w^{m_R}(r) \geqslant \alpha^{\frac{n^2(m-1)}{(n-m)^2}} \{\varphi^{m_R}(\alpha^{\frac{m^2-n}{(n-m)^2}} r) - c_2 \varepsilon |\ln \varepsilon| \varphi^{m_R - 1}(\alpha^{\frac{m^2-n}{(n-m)^2}} r)\} \quad \forall r \geqslant 0. \tag{5.14}$$

Indeed, by (3.17) we have $u(r) \leqslant \alpha(\varphi(r) + C\eta)$ for all r and for some $C > 0$; then taking the m_Rth power and after the substitution (4.8), the upper bound (5.13) is obtained. On the other hand, by (3.17) we also obtain

$$u(r) \geqslant \alpha(\varphi(r) - C\varepsilon |\ln \varepsilon|) \quad \forall r \geqslant 0 \tag{5.15}$$

for some $C > 0$. Take $c_2 = C$; if the right-hand side of (5.14) is negative, there is nothing to prove. If it is positive, then taking the m_Rth power of (5.15) and after the substitution (4.8), the lower bound (5.14) is obtained, eventually by choosing a larger value for c_2.

Put

$$\int_0^{R_0} r^{n-1} w^{m_R}(r)\, dr = I + J$$

with the principal part I defined by

$$I = \alpha^{\frac{n^2(m-1)}{(n-m)^2}} \int_0^{R_0} r^{n-1} \varphi^{m_R}(\alpha^{\frac{m^2-n}{(n-m)^2}} r)\, dr.$$

We first estimate I. After the substitution

$$t = D\alpha^{\frac{nm}{(n-m)^2} - \frac{\varepsilon}{m-1}} r^{\frac{m}{m-1}}, \quad T = D\alpha^{\frac{nm}{(n-m)^2} - \frac{\varepsilon}{m-1}} R_0^{\frac{m}{m-1}} \tag{5.16}$$

we obtain

$$I = \frac{m-1}{m} D^{-\frac{n(m-1)}{m}} \alpha^{\frac{\varepsilon n}{m}} \int_0^T \frac{t^{\frac{n(m-1)}{m} - 1}}{(1+t)^{\frac{n(m-1)}{m}}}\, dt;$$

since $T \to \infty$ as $\alpha \to \infty$ and (3.27) holds, we have

$$I = \frac{m-1}{m} D^{-\frac{n(m-1)}{m}} \ln T + O(1) \quad \text{as } \alpha \to \infty$$

so that, by (5.9) and (5.16), we find

$$I = \frac{n(m-1)}{(n-m)^2} D^{-\frac{n(m-1)}{m}} \ln \alpha + O(\ln \ln \alpha).$$

It remains to estimate the remainder term

$$J = \int_0^{R_0} r^{n-1} [w^{m_R}(r) - \alpha^{\frac{n^2(m-1)}{(n-m)^2}} \varphi^{m_R}(\alpha^{\frac{m^2-n}{(n-m)^2}} r)]\, dr;$$

to this purpose we will use the upper and lower bounds (5.13) and (5.14). Let us define

$$J_0 = \alpha^{\frac{n^2(m-1)}{(n-m)^2}} \int_0^{R_0} r^{n-1} \varphi^{m_R-1}(\alpha^{\frac{m^2-n}{(n-m)^2}} r)\, dr, \tag{5.17}$$

then, with the change of variables (5.16), we have

$$J_0 = \frac{m-1}{m} D^{-\frac{n(m-1)}{m}} \alpha^{\frac{\varepsilon n}{m}} \int_0^T \frac{t^{\frac{n(m-1)}{m}-1}}{(1+t)^{\frac{nm-2n+m}{m}}}\, dt.$$

Since there exist $c_3, c_4 > 0$ such that

$$\frac{t^{\frac{n(m-1)}{m}-1}}{(1+t)^{\frac{nm-2n+m}{m}}} \leqslant \begin{cases} c_3 t^{\frac{n(m-1)}{m}-1} & \text{if } t \leqslant 1, \\ c_4 t^{\frac{n-2m}{m}} & \text{if } t > 1, \end{cases}$$

then $J_0 = O(T^{\frac{n-m}{m}})$ as $\alpha \to \infty$. Therefore, by (5.13) and (5.14) we obtain

$$-c\varepsilon |\ln \varepsilon| T^{\frac{n-m}{m}} \leqslant -c_2 \varepsilon |\ln \varepsilon| J_0 \leqslant J \leqslant c_1 \eta J_0 \leqslant c \eta T^{\frac{n-m}{m}}. \tag{5.18}$$

Moreover, by (5.9) and (5.16) we have $\eta T^{\frac{n-m}{m}} = o(1)$ and $\varepsilon |\ln \varepsilon| T^{\frac{n-m}{m}} = O(1)$ as $\alpha \to \infty$; for the second estimate we also used Lemma 4. Inserting these asymptotics into (5.18) we have $J = O(1)$ as $\alpha \to \infty$, so that (5.11) holds. □

We can now complete the proof of Theorem 2. By (4.8), (5.8) and Lemma 16 we obtain as $\varepsilon \to 0$:

$$\int_0^\infty r^{n-1} u^{m_R}(r)\, dr = \alpha^{-\frac{n}{n-m}} \int_0^\infty r^{n-1} w^{m_R}(r)\, dr \approx \frac{m-1}{n-m} D^{-\frac{n(m-1)}{m}} \alpha^{-\frac{n}{n-m}} |\ln \varepsilon|. \tag{5.19}$$

Finally, inserting (5.7) and (5.19) into (3.2) proves (2.9) in the case $q = m_R$.

6. Proof of Theorem 3

The proof of this result is essentially given in [5] and hence we omit it. We refer in particular to Section 5.3 and 6 in [5].

7. Proof of Theorem 4

As $q<m$, the ground state u of (2.1) is compact supported. Let $w = w_\varepsilon$ be as in (4.8), let R_ε be such that $\overline{B_{R_\varepsilon}(0)} = \text{supp}\,(w_\varepsilon)$ and let W be as in (4.25). By Proposition 3, W solves (2.2)–(2.4); moreover by Lemma 7, W has bounded support since $q<m$. We can state the following

Lemma 17. *We have*

$$R \leqslant \liminf_{\varepsilon\to 0} R_\varepsilon \leqslant \limsup_{\varepsilon\to 0} R_\varepsilon < +\infty, \tag{7.1}$$

where $R>0$ is the radius of the support of the function W defined in (4.25).

Proof. We first show that $R \leqslant \liminf_{\varepsilon\to 0} R_\varepsilon$. By definition of R we have

$$W(R-\lambda)>0 \quad \forall \lambda \in (0,R)$$

and hence by the pointwise convergence $w_\varepsilon \to W$ we deduce that

$$w_\varepsilon(R-\lambda)>0$$

for all ε small enough. Then we have

$$R_\varepsilon > R-\lambda \quad \forall \lambda \in (0,R),\ \forall \varepsilon \text{ small enough}$$

and the first inequality in (7.1) follows.

It remains to prove that R_ε is uniformly bounded from above when $\varepsilon \to 0$.

Suppose that there exists $\bar{\varepsilon}>0$ such that

$$R_{\bar{\varepsilon}} > R. \tag{7.2}$$

If such $\bar{\varepsilon}$ does not exist, the last inequality in (7.1) follows readily.

If such $\bar{\varepsilon}$ exists, the proof will be complete once we show that there exists $\underline{\varepsilon} \in (0,\bar{\varepsilon})$ such that

$$R_\varepsilon \leqslant R_{\bar{\varepsilon}} \quad \forall \varepsilon \in (0,\underline{\varepsilon}). \tag{7.3}$$

Assume for contradiction that there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that

$$\varepsilon_k \to 0 \quad \text{as } k\to\infty \quad \text{and} \quad R_{\varepsilon_k} > R_{\bar{\varepsilon}}. \tag{7.4}$$

Since $w_{\varepsilon_k}(R) \to W(R) = 0$ as $k \to \infty$ and $w_{\bar{\varepsilon}}(R) > 0$ (recall $R_{\bar{\varepsilon}} > R$) there exists $\bar{k} \in \mathbb{N}$ such that

$$w_{\varepsilon_k}(R) < w_{\bar{\varepsilon}}(R) \quad \forall k \geqslant \bar{k}. \tag{7.5}$$

For $k \geqslant \bar{k}$ let

$$R_1 = R_1(k) = \max\{r \in (R, R_{\bar{\varepsilon}}); w_{\varepsilon_k}(r) = w_{\bar{\varepsilon}}(r)\}; \tag{7.6}$$

R_1 is well-defined by (7.4) and (7.5).

Consider the functions

$$f_{\varepsilon_k}(s) = -s^{q-1} + \alpha_k^{-m\kappa_q(p_k-q)} s^{p_k-1} \quad \forall s \geqslant 0$$

and

$$f_{\bar{\varepsilon}}(s) = -s^{q-1} + \alpha^{-m\kappa_q(\bar{p}-q)} s^{\bar{p}-1} \quad \forall s \geqslant 0,$$

with $\alpha_k = u_{\varepsilon_k}(0)$, $\alpha = u_{\bar{\varepsilon}}(0)$, $p_k = m^* - \varepsilon_k$, $\bar{p} = m^* - \bar{\varepsilon}$.

Then, by (4.8) we see that w_{ε_k} and $w_{\bar{\varepsilon}}$ solve, respectively, the equations

$$\begin{aligned} (r^{n-1}|w'_{\varepsilon_k}(r)|^{m-1})' &= r^{n-1} f_{\varepsilon_k}(w_{\varepsilon_k}(r)), \\ (r^{n-1}|w'_{\bar{\varepsilon}}(r)|^{m-1})' &= r^{n-1} f_{\bar{\varepsilon}}(w_{\bar{\varepsilon}}(r)). \end{aligned} \tag{7.7}$$

Integrating the two equations in (7.7) on the interval $[R_1, R_{\varepsilon_k}]$, using the fact that $w'_{\varepsilon_k}(R_{\varepsilon_k}) = w'_{\bar{\varepsilon}}(R_{\varepsilon_k}) = 0$, we obtain after subtraction

$$R_1^{n-1}|w'_{\bar{\varepsilon}}(R_1)|^{m-1} - R_1^{n-1}|w'_{\varepsilon_k}(R_1)|^{m-1} = \int_{R_1}^{R_{\varepsilon_k}} r^{n-1}[f_{\varepsilon_k}(w_{\varepsilon_k}(r)) - f_{\bar{\varepsilon}}(w_{\bar{\varepsilon}}(r))]\, dr. \tag{7.8}$$

Note that by definition of R_1, $|w'_{\bar{\varepsilon}}(R_1)| \geqslant |w'_{\varepsilon_k}(R_1)|$ and hence by (7.8) we have

$$\int_{R_1}^{R_{\varepsilon_k}} r^{n-1}[f_{\varepsilon_k}(w_{\varepsilon_k}(r)) - f_{\bar{\varepsilon}}(w_{\bar{\varepsilon}}(r))]\, dr \geqslant 0. \tag{7.9}$$

Since $w_{\varepsilon_k}(R) \to 0$ as $k \to \infty$ and $w'_{\varepsilon_k}(r) < 0$ for any $r \in (0, R_{\varepsilon_k})$, for any $\sigma > 0$ and sufficiently large $\bar{k}$ we have

$$w_{\varepsilon_k}(R_1) < \sigma \quad \forall k \geqslant \bar{k}. \tag{7.10}$$

We fix $\sigma < (\frac{q-1}{\bar{p}-1})^{\frac{1}{\bar{p}-q}} \alpha^{m\kappa_q}$ since $(\frac{q-1}{\bar{p}-1})^{\frac{1}{\bar{p}-q}} \alpha^{m\kappa_q}$ is the unique positive minimum point of the function $f_{\bar{\varepsilon}}$. With this choice of σ fix $\bar{k}$ as in (7.10); in this way by (7.4), (7.6), (7.10) and the fact that w_{ε_k} is decreasing, we have

$$w_{\bar{\varepsilon}}(r) \leqslant w_{\varepsilon_k}(r) \leqslant w_{\varepsilon_k}(R_1) < \left(\frac{q-1}{\bar{p}-1}\right)^{\frac{1}{\bar{p}-q}} \alpha^{m\kappa_q} \quad \forall r \in [R_1, R_{\varepsilon_k}], \ \forall k \geqslant \bar{k}. \tag{7.11}$$

By elementary calculus, after another suitable restriction on $\bar{k}$, we have

$$f_{\varepsilon_k}(s) < f_{\bar{\varepsilon}}(s) \quad \forall s \in \left(0, \left(\frac{q-1}{\bar{p}-1}\right)^{\frac{1}{\bar{p}-q}} \alpha^{m\kappa_q}\right), \ \forall k \geqslant \bar{k}; \tag{7.12}$$

in particular, since $f_{\bar{\varepsilon}}$ is decreasing on the interval $(0, (\frac{q-1}{\bar{p}-1})^{\frac{1}{\bar{p}-q}} \alpha^{m\kappa_q})$, by (7.11) and (7.12), we have

$$f_{\varepsilon_k}(w_{\varepsilon_k}(r)) < f_{\bar{\varepsilon}}(w_{\varepsilon_k}(r)) \leqslant f_{\bar{\varepsilon}}(w_{\bar{\varepsilon}}(r)) \quad \forall r \in (R_1, R_{\varepsilon_k})$$

and this contradicts (7.9) after integration over (R_1, R_{ε_k}). □

Thanks to Lemma 17 and the rescaling (4.8) we can complete the proof of Theorem 4; indeed, let ρ be as in Theorem 4, then we have

$$\rho = \rho_\varepsilon = \alpha^{(q-m)\kappa_q} R_\varepsilon$$

and letting $\varepsilon \to 0$ we obtain

$$R \leqslant \liminf_{\varepsilon \to 0} \ \rho \alpha^{(m-q)\kappa_q} \leqslant \limsup_{\varepsilon \to 0} \ \rho \alpha^{(m-q)\kappa_q} < +\infty;$$

the proof of Theorem 4 can be obtained after a calculation which uses the estimates on α of Theorem 2 in the three cases $q > m_R$, $q = m_R$ and $q < m_R$.

Remark 2. We believe that all the limits in Lemma 17 exist and are equal to R, the radius of the support of W; in such case, the limits in Theorem 4 also exist. However, this result would require the continuous dependence of $u(0)$ and R_ε on ε, which seems a hard matter.

8. Proof of Theorem 5

8.1. The case $q < m_*$

It follows at once from (4.8), (4.9), Lemma 9, (4.25) and Proposition 3.

8.2. The case $q = m_*$

Let κ be any real number, then by (3.17) and thanks to the fact that $q = m_*$ is equivalent to $\frac{m}{q-m} = \frac{n-m}{m-1}$, we have

$$\lim_{\varepsilon\to 0} \alpha^{\frac{m}{q-m}\kappa} u(\alpha^\kappa r) = \lim_{\varepsilon\to 0} \alpha^{\frac{n-m}{m-1}\kappa} u(\alpha^\kappa r) \leqslant \lim_{\varepsilon\to 0} \frac{\alpha^{\frac{\varepsilon(n-m)}{m(m-1)}} \alpha^{-\frac{1}{m-1}}}{(1-\eta)^{\frac{n-m}{m(m-1)}} D^{\frac{n-m}{m}} r^{\frac{n-m}{m-1}}} = 0 \quad \forall r > 0$$

by the convergence $\eta \to 0$ and since both (3.9) and(3.27) hold. This implies (2.12) with $\kappa(q-m)$ in place of κ (note that $q = m_* > m$).

8.3. The case $m_* < q < m^*$

In this case, with an abuse of notation we still let κ_q be as in (2.10). Note that $\kappa_q < -\frac{1}{m}$.

Let κ be an arbitrary real number; we will treat the two cases $\kappa < -\frac{1}{m}$ and $\kappa > \kappa_q$ separately. If $\kappa < -\frac{1}{m}$, then by (3.17), we have

$$\alpha^{m\kappa} u(\alpha^{(q-m)\kappa} r) < \alpha^{1+m\kappa} \to 0 \quad \text{as } \varepsilon \to 0 \ \forall r > 0$$

which proves (2.12). If $\kappa > \kappa_q$, by (3.17) we have

$$\alpha^{m\kappa} u(\alpha^{(q-m)\kappa} r) < \frac{\alpha^{1+m\kappa}}{C\alpha^{\frac{m}{m-1}} [\alpha^{(q-m)\kappa} r]^{\frac{n-m}{m-1}}} \leqslant \frac{C}{r^{\frac{n-m}{m-1}}} \alpha^{\frac{\kappa-\kappa_q}{(m-1)\kappa_q}} \to 0 \quad \text{as } \varepsilon \to 0 \ \forall r > 0$$

and (2.12) follows again.

9. Proof of Theorem 6

Let $u = u(r)$ be the unique ground state of (1.1). Let

$$v(r) = \delta^{-\frac{1}{p-q}} u\left(\frac{r}{\delta^{\frac{p-m}{m(p-q)}}}\right); \tag{9.1}$$

then v is the unique ground state of Eq. (2.1). By (2.13) and (9.1) we have $u(0) = \delta^{\frac{1}{p-q}}\beta$ and the first part of Theorem 6 follows. Since $v = v(r)$ solves the ordinary differential equation

$$(r^{n-1}|v'(r)|^{m-1})' = r^{n-1}(-v^{q-1}(r) + v^{p-1}(r)),$$

after integration over $[0, r]$ we obtain

$$\begin{aligned}|v'(r)|^{m-1} &= \frac{1}{r^{n-1}} \int_0^r t^{n-1}(-v^{q-1}(t) + v^{p-1}(t))\, dt \\ &= \frac{1}{r^{n-1}} \int_0^r t^{n-1}(-\beta^{q-1} + \beta^{p-1} + o(1))\, dt \\ &= \frac{r}{n}(-\beta^{q-1} + \beta^{p-1} + o(1)) \quad \text{as } r \to 0.\end{aligned}$$

Taking the $1/(m-1)$ power and integrating from 0 to r we have

$$v(r) = \beta - \frac{m-1}{m}\left(\frac{\beta^{p-1} - \beta^{q-1}}{n}\right)^{\frac{1}{m-1}} r^{\frac{m}{m-1}} + o(r^{\frac{m}{m-1}}) \quad \text{as } r \to 0.$$

This, together with (9.1), gives (2.15).

If $q < m$, the estimate (2.16) on the support of u is a straightforward consequence of (9.1).

10. Proof of Theorem 7

Let $u(x; \delta)$ be the unique ground state of (1.1) where $\delta > 0$. Then, thanks to the estimates of Theorem 2 and the rescaling (9.1) we obtain the following estimates for $u(0; \delta)$:

$$u(0; \delta) \sim \begin{cases} \delta^{\frac{1}{p-q}} (\beta_{m,n,q} \varepsilon^{-1})^{\frac{1}{m^*-q}} & \text{if } q > m_R \\ \delta^{\frac{1}{p-q}} \left(\mu_{m,n} \frac{|\ln \varepsilon|}{\varepsilon}\right)^{\frac{n-m}{n}} & \text{if } q = m_R \\ \delta^{\frac{1}{p-q}} (\gamma_{m,n,q} \varepsilon^{-1})^{\frac{m_*-q}{m^*-q}} & \text{if } q < m_R \end{cases} \quad \text{as } \varepsilon \to 0;$$

hence, if $\delta = \delta(\varepsilon)$ as in the statement of Theorem 7 then

$$u(0; \delta(\varepsilon)) \to d \quad \text{as } \varepsilon \to 0. \tag{10.1}$$

By (3.17) and (9.1), we have

$$0 < u(0; \delta(\varepsilon)) \cdot z(u(0; \delta(\varepsilon))^{\frac{p-m}{m}} x) - u(x; \delta(\varepsilon)) < c\, u(0; \delta(\varepsilon)) \cdot \varepsilon |\ln \varepsilon|; \tag{10.2}$$

moreover, by (10.1) and the fact that $\eta \to 0$ as $\varepsilon \to 0$, we infer

$$u(0; \delta(\varepsilon)) z(u(0; \delta(\varepsilon))^{\frac{p-m}{m}} x) \to d[1 + D(d^{\frac{m}{n-m}} |x|)^{\frac{m}{m-1}}]^{-\frac{n-m}{m}} \equiv U_d(x).$$

This, together with (10.2), yields $u(\cdot; \delta(\varepsilon)) \to U_d$ uniformly on $\mathbb{R}^n$ as $\varepsilon \to 0$.

References

[1] H. Egnell, F. Pacella, M. Tricarico, Some remarks on Sobolev inequalities, Nonlinear Anal. TMA 13 (1989) 671–681.
[2] A. Ferrero, F. Gazzola, On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations, Differential and Integral Equations, to appear.
[3] B. Franchi, E. Lanconelli, J. Serrin, Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, Adv. Math. 118 (1996) 177–243.
[4] F. Gazzola, L.A. Peletier, P. Pucci, J. Serrin, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Part II, Ann. Inst. H. Poincaré Anal. Non Lineaire, to appear.
[5] F. Gazzola, J. Serrin, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, Ann. Inst. H. Poincaré Anal. Non Lineaire 19 (2002) 477–504.
[6] F. Gazzola, J. Serrin, M. Tang, Existence of ground states and free boundary problems for quasilinear elliptic operators, Adv. Differential Equations 5 (2000) 1–30.
[7] M.C. Knaap, L.A. Peletier, Quasilinear elliptic equations with nearly critical growth, Comm. Partial Differential Equations 14 (1989) 1351–1383.
[8] E. Mitidieri, S. Pohožaev, Apriori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Proc. Steklov Inst. Math. 234 (2001) 1–362 (translated from Russian).
[9] W.M. Ni, J. Serrin, Non existence theorems for quasilinear partial differential equations, Rend. Circ. Mat. Palermo Series II 8 (Centenary Supplement) (1985) 171–185.
[10] W.M. Ni, J. Serrin, Existence and non existence theorems for ground states of quasilinear partial differential equations, the anomalous case, Accad. Naz. dei Lincei, Atti dei Convegni 77 (1986) 231–257.
[11] J. Serrin, Isolated singularities of solutions of quasilinear equations, Acta Math. 113 (1965) 219–241.
[12] J. Serrin, M. Tang, Uniqueness of ground states for quasilinear elliptic equations, Indiana Univ. Math. J. 49 (2000) 897–923.
[13] J. Serrin, H. Zou, Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, Acta Math. 189 (2002) 79–142.
[14] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[15] J.L. Vázquez, L. Veron, Removable singularities of some strongly nonlinear elliptic equations, Manuscripta Math. 33 (1980) 129–144.