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# A PARTIALLY HINGED RECTANGULAR PLATE AS A MODEL FOR SUSPENSION BRIDGES

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Abstract. A plate model describing the statics and dynamics of a suspension bridge is suggested. A partially hinged plate subject to nonlinear restoring hangers is considered. The whole theory from linear problems, through nonlinear stationary equations, ending with the full hyperbolic evolution equation is studied. This paper aims to be the starting point for more refined models.

## 1. Introduction.

Due to the videos available on the web [34], the Tacoma Narrows Bridge collapse is certainly the most impressive failure of the history of bridges. But, unfortunately, it is not an isolated event, many other bridges collapsed in the past, see [3, 15]. According to [14], around 400 recorded bridges failed for several different reasons and the ones who failed after year 2000 are more than 70. Strong aerodynamic instability is manifested, in particular, in suspension bridges which usually have fairly long spans. Hence reliable mathematical models appear necessary for a precise description of the instability and of the structural behavior of suspension bridges under the action of dead and live loads.

On one hand, realistic models appear too complicated to give helpful hints when making plans. On the other hand, simplified models do not describe with sufficient accuracy the complex behavior of actual bridges. We refer to [10] for a survey of some existing models.

A one-dimensional simply supported beam suspended by hangers was suggested as a model for suspension bridges in [19, 27, 28]. It is assumed that when the hangers are stretched there is a restoring force which is proportional to the amount of stretching but when the beam moves in the opposite direction, the hangers slacken and there is no restoring force exerted on it. If $u = u(x,t)$ denotes the vertical displacement of the beam (of length $L$) in the downward direction, the following fourth order nonlinear equation is derived

$$u_{tt} + u_{xxxx} + \gamma u^+ = f(x,t)\,, \qquad x \in (0,L)\,, \quad t > 0\,, \tag{1}$$

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where $u^+ = \max\{u, 0\}$, $\gamma u^+$ represents the force due to the hangers, and $f$ represents the forcing term acting on the bridge, including its own weight per unit length. For time periodic $f$, McKenna-Walter [27] prove the existence of multiple periodic solutions of (1). Moreover, in [28] they normalize (1) by taking $\gamma = 1$ and $f \equiv 1$: then by seeking traveling waves $u(x,t) = 1 + w(x - ct)$ they end up with the ODE

$$w''''(s) + kw''(s) + \psi(w(s)) = 0 \qquad (s \in \mathbb{R},\ k = c^2)\,,$$

where

$$\psi(w) = [w+1]^+ - 1\,, \tag{2}$$

a term which takes into account both the restoring force due to the hangers and external forces including gravity.

Soon after the Tacoma Narrows Bridge collapse [32, 34], three engineers were assigned to investigate and report to the Public Works Administration. The Report [4] considers *...the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*, see [32, p.63]. But if one views the bridge as a beam as in (1), there is no way to highlight torsional oscillations. A model suggested by McKenna [25] considers the cross section of the bridge as a rod, free to rotate about its center which behaves as a forced oscillator subject to the forces exerted by the two lateral hangers. After normalization, the force is taken again as in (2). In order to smoothen the force by maintaining the asymptotically linear behavior at 0, McKenna-Tuama [26] also consider $\psi(w) = c(e^{aw} - 1)$ for some $a, c > 0$. Then, after adding some damping and forcing, [25, 26] were able to numerically replicate in a cross section the sudden transition from standard and expected vertical oscillations to destructive and unexpected torsional oscillations. More recently, Arioli-Gazzola [5] reconsidered this model and studied its isolated version (energy conservation) with nonlinear restoring forces due to the hangers: they were able to display a sudden appearance of torsional oscillations. This phenomenon was explained using the stability of a fixed point of a suitable Poincaré map. The full bridge was then modeled in [5] by considering a finite number of parallel rods linked to the two nearest neighbors rods with attractive linear forces representing resistance to longitudinal and torsional stretching; this discretization of a suspension bridge is justified by the positive distance between hangers. The sudden appearance of torsional oscillations was highlighted also within the multiple rods model.

The nonlinear behavior of suspension bridges is by now well established, see e.g. [7, 10, 17, 31]. After replacing the nonlinear term $\psi(w)$ in (2) by a fairly general superlinear term $h(w)$, traveling waves of (1) display self-excited oscillations, see [6, 12, 13]: the solution may blow up in finite time with wide oscillations. So, a reliable model for suspension bridges should be nonlinear and it should have enough degrees of freedom to display torsional oscillations. In this respect, Lazer-McKenna [20, Problem 11] suggest to study the following equation

$$\Delta^2 u + c^2 \Delta u + h(u) = 0 \qquad \text{in } \ \mathbb{R}^n \tag{3}$$

where $h$ is "like" $\psi$ in (2). The purpose of the present paper is to set up the full theory for (3) in a bounded domain (representing the roadway) and to study the corresponding evolution problem similar to (1).

A long narrow rectangular thin plate hinged at two opposite edges and free on the remaining two edges well describes the roadway of a suspension bridge which, at the short edges, is supported by the ground. Let $L$ denote its length and $2\ell$ denote its width; a realistic assumption is that $2\ell \cong \frac{L}{100}$. For simplicity, we take $L = \pi$ so that, in the sequel,

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 \ .$$

Our purpose is to provide a reliable model and to study the corresponding Euler-Lagrange equations. Since several energies are involved, we reach this task in several steps. We first recall the derivation of the bending elastic energy of a deflected plate, according to the Kirchhoff-Love [16, 22] theory. Then we consider the action of both dead and live loads described by some forcing term $f$; the equilibrium position of the plate $u$ is then the minimum of a convex energy functional and is the unique solution of

$$\Delta^2 u = f(x, y) \qquad \text{in } \Omega \tag{4}$$

under suitable boundary conditions. We set up the correct variational formulation of (4) (Theorem 3.1) and when $f$ depends only on the longitude, $f = f(x)$, we are able to determine the explicit form of $u$ by separating variables (Theorem 3.2). In order to analyze the oscillating modes of the bridge, we also consider the eigenvalue problem

$$\Delta^2 w = \lambda w \qquad \text{in } \Omega \tag{5}$$

where $\lambda$ is the eigenvalue and $w = w(x, y)$ is the eigenfunction. We characterize in detail the spectrum and the corresponding eigenfunctions (Theorem 3.4). The eigenvalues exhibit some weakness on the long edges and manifest a tendency to display a torsional component, see Figure 3.

Then we introduce into the model the elastic restoring force due to the hangers which is confined in a proper subset $\omega$ of $\Omega$ such as two small rectangles close to the horizontal edges, see Figure 1. The restoring force $h = h(x, y, u)$ is superlinear


FIGURE 1. The plate $\Omega$ and its subset $\omega$ (dark grey) where the hangers act.

with respect to $u$, which yields a superquadratic potential energy $\int_\omega H(x, y, u)$. A particular form of $h$ is suggested to describe the precise behavior of hangers, see (15) below. The equilibrium position is then given by the unique solution of

$$\Delta^2 u + h(x, y, u) = f(x, y) \qquad \text{in } \Omega \ . \tag{6}$$

Finally, if the force $f$ is variable in time, so is the the equilibrium position and also the kinetic energy of the structure comes into the energy balance. This leads to the fourth order wave equation

$$u_{tt} + \Delta^2 u + h(x, y, u) = f(x, y, t) \qquad \text{in } \Omega \times (0, T) \tag{7}$$

where $(0, T)$ is an interval of time. Well-posedness of an initial-boundary-value-problem is shown in Theorem 3.6. Our future target is to reproduce within our plate model the same oscillating behavior visible at the Tacoma Bridge [34]. This paper should be considered as a first necessary step in order to reach more challenging results.

This paper is organized as follows. In Section 2 we describe the physical model and we derive the PDE's which have to be solved. In Section 3 we state our main results: existence, uniqueness, and qualitative behavior of the solutions of the PDE's. The remaining sections of the paper are devoted to the proofs of these results.

## 2. **The physical model.**

2.1. **A linear model for a partially hinged plate.** The bending energy of the plate $\Omega$ involves curvatures of the surface. Let $\kappa_1$ and $\kappa_2$ denote the principal curvatures of the graph of a smooth function $u$ representing the vertical displacement of the plate in the downward direction, then a simple model for the bending energy of the deformed plate $\Omega$ is

$$\mathbb{E}_B(u)=\frac{E\,d^3}{12(1-\sigma^2)}\int_\Omega\left(\frac{\kappa_1^2}{2}+\frac{\kappa_2^2}{2}+\sigma\kappa_1\kappa_2\right)dxdy \tag{8}$$

where $d$ denotes the thickness of the plate, $\sigma$ the Poisson ratio defined by $\sigma=\frac{\lambda}{2(\lambda+\mu)}$ and $E$ the Young modulus defined by $E=2\mu(1+\sigma)$, with the so-called Lamé constants $\lambda,\mu$ that depend on the material. For physical reasons it holds that $\mu>0$ and usually $\lambda>0$ so that

$$0<\sigma<\frac{1}{2}. \tag{9}$$

Moreover, it always holds true that $\sigma>-1$ although some exotic materials have a negative Poisson ratio, see [18]. For metals the value of $\sigma$ lies around 0.3, see [22, p.105], while for concrete $0.1<\sigma<0.2$.

For small deformations the terms in (8) are taken as approximations being purely quadratic with respect to the second order derivatives of $u$. More precisely, for small deformations $u$, one has

$$(\kappa_1+\kappa_2)^2\approx(\Delta u)^2\ ,\quad \kappa_1\kappa_2\approx\det(D^2u)=u_{xx}u_{yy}-u_{xy}^2\ ,$$

and therefore

$$\frac{\kappa_1^2}{2}+\frac{\kappa_2^2}{2}+\sigma\kappa_1\kappa_2\approx\frac{1}{2}(\Delta u)^2+(\sigma-1)\det(D^2u).$$

Then, if $f$ denotes the external vertical load acting on the plate $\Omega$ and if $u$ is the corresponding (small) deflection of the plate in the vertical direction, by (8) we have that the total energy $\mathbb{E}_T$ of the plate becomes

$$\begin{aligned}\mathbb{E}_T(u)&=\mathbb{E}_B(u)-\int_\Omega fu\,dxdy \qquad (10)\\ &=\frac{E\,d^3}{12(1-\sigma^2)}\int_\Omega\left(\frac{1}{2}\,(\Delta u)^2+(\sigma-1)\det(D^2u)\right)dxdy-\int_\Omega fu\,dxdy.\end{aligned}$$

By replacing the load $f$ with $\frac{Ed^3}{12(1-\sigma^2)}f$ and up to a constant multiplier, the energy $\mathbb{E}_T$ may be written as

$$\mathbb{E}_T(u)=\int_\Omega\left(\frac{1}{2}(\Delta u)^2+(1-\sigma)(u_{xy}^2-u_{xx}u_{yy})-fu\right)dxdy\,. \tag{11}$$

Note that for $\sigma>-1$ the quadratic part of the functional (11) is positive. This variational formulation appears in [8], while a discussion for a boundary value problem for a thin elastic plate in a somehow old fashioned notation is made by Kirchhoff [16], see also [11, Section 1.1.2] for more details and references.

The unique minimizer $u$ of $\mathbb{E}_T$, satisfies the Euler-Lagrange equation (4). We now turn to the boundary conditions to be associated to (4). We seek the ones representing the physical situation of a plate modeling a bridge. Due to the connection with the ground, the plate $\Omega$ is assumed to be hinged on its vertical edges and hence

$$u(0,y)=u_{xx}(0,y)=u(\pi,y)=u_{xx}(\pi,y)=0 \qquad \forall y\in(-\ell,\ell)\ . \tag{12}$$

The deflection of the fully hinged rectangular plate $\Omega$ (that is $u=u_{\nu\nu}=0$ on $\partial\Omega$) under the action of a distributed load has been solved by Navier [30] in 1823, see also [23, Section 2.1]. The general problem of a load on the rectangular plate $\Omega$ with two opposite hinged edges was considered by Lévy [21], Zanaboni [35], and Nadai [29], see also [23, Section 2.2] for the analysis of different kinds of boundary conditions on the remaining two edges $y=\pm\ell$. In the plate $\Omega$, representing the roadway of a suspension bridge, the horizontal edges $y=\pm\ell$ are free and the boundary conditions there become (see e.g. [33, (2.40)])

$$u_{yy}(x,\pm\ell)+\sigma u_{xx}(x,\pm\ell)=0\ ,\quad u_{yyy}(x,\pm\ell)+(2-\sigma)u_{xxy}(x,\pm\ell)=0 \quad \forall x\in(0,\pi)\,. \tag{13}$$

In Section 3.1 we show how these boundary conditions arise. Note that free boundaries yield small stretching energy for the plate; this is the reason why we take $c=0$ in (3).

Summarizing, the whole set of boundary conditions for a rectangular plate $\Omega=(0,\pi)\times(-\ell,\ell)$ modeling a suspension bridge is (12)-(13) and the boundary value problem reads

$$\begin{cases}\Delta^2u=f & \text{in } \Omega\\ u(0,y)=u_{xx}(0,y)=u(\pi,y)=u_{xx}(\pi,y)=0 & \text{for } y\in(-\ell,\ell)\\ u_{yy}(x,\pm\ell)+\sigma u_{xx}(x,\pm\ell)=0 & \text{for } x\in(0,\pi)\\ u_{yyy}(x,\pm\ell)+(2-\sigma)u_{xxy}(x,\pm\ell)=0 & \text{for } x\in(0,\pi)\ .\end{cases} \tag{14}$$

2.2. **A nonlinear model for a dynamic suspension bridge.** Assume that the bridge is suspended by hangers whose action is concentrated in the union of two thin strips parallel to the two horizontal edges of the plate $\Omega$, i.e. in a set of the type $\omega:=(0,\pi)\times[(-\ell,-\ell+\varepsilon)\cup(\ell-\varepsilon,\ell)]$ with $\varepsilon>0$ small.

In order to describe the action of the hangers we introduce a continuous function $g:\mathbb{R}\to\mathbb{R}$ satisfying

$$g\in C^1(0,+\infty),\quad g(s)=0 \text{ for any } s\le 0,\quad g'(0^+)>0,\quad g'(s)\ge 0 \text{ for any } s>0\,.$$

Then, the restoring force due to the hangers takes the form

$$h(x,y,u)=\Upsilon(y)g(u+\gamma x(\pi-x)) \tag{15}$$

where $\Upsilon$ is the characteristic function of $(-\ell,-\ell+\varepsilon)\cup(\ell-\varepsilon,\ell)$ and $\gamma>0$. This choice of $h$ is motivated by the fact that the action of the hangers is larger around the central part of the bridge $x=\pi/2$, than on its sides $x=0$ and $x=\pi$ where the bridge is supported. This parabolic behavior is a consequence of the prestressing procedure and appears quite visible in certain bridges such as the Deer Isle Bridge, see Figure 2.

FIGURE 2. The Deer Isle Bridge.

More generally we may consider a force $h$ satisfying the following assumptions:

$$h:\Omega\times\mathbb{R}\to\mathbb{R}\quad\text{ is a Carathéodory function,}\tag{16}$$

$$s\mapsto h(\cdot,\cdot,s)\quad\text{is nondecreasing in }\mathbb{R}\,,\quad\exists\overline{s}\in\mathbb{R}\,,\ h(\cdot,\cdot,\overline{s})=0\,,\tag{17}$$

and $h$ is locally Lipschitzian with respect to $s$, i.e.

$$L_I:=\sup_{(x,y)\in\Omega,\,s_1,s_2\in I,\,s_1\neq s_2}\left|\frac{h(x,y,s_1)-h(x,y,s_2)}{s_1-s_2}\right|<+\infty\tag{18}$$

for any bounded interval $I\subset\mathbb{R}$.

The force $h$ admits a potential energy given by $\int_\Omega H(x,y,u)\,dxdy$ where we put $H(x,y,s):=\int_{\overline{s}}^s h(x,y,\tau)d\tau$ for any $s\in\mathbb{R}$. The total static energy of the bridge is obtained by adding this potential energy to the elastic energy of the plate (11):

$$\mathbb{E}_T(u)=\int_\Omega\left(\frac{1}{2}(\Delta u)^2+(1-\sigma)(u_{xy}^2-u_{xx}u_{yy})+H(x,y,u)-fu\right)dxdy\,.\tag{19}$$

The Euler-Lagrange equation is obtained by minimizing this convex functional:

$$\begin{cases}\Delta^2u+h(x,y,u)=f & \text{in }\Omega\\ u(0,y)=u_{xx}(0,y)=u(\pi,y)=u_{xx}(\pi,y)=0 & \text{for }y\in(-\ell,\ell)\\ u_{yy}(x,\pm\ell)+\sigma u_{xx}(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\\ u_{yyy}(x,\pm\ell)+(2-\sigma)u_{xxy}(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\ .\end{cases}\tag{20}$$

Finally, assume that the external force also depends on time, $f=f(x,y,t)$. If $m$ denotes the mass of the plate, then the corresponding deformation $u$ has a kinetic energy given by the integral

$$\frac{m}{2|\Omega|}\int_\Omega u_t^2\,dxdy\ .$$

By the time scaling $t\mapsto\sqrt{m|\Omega|^{-1}}t$, we can set $m|\Omega|^{-1}=1$. This term should be added to the nonlinear static energy (19):

$$\mathcal{E}_u(t):=\int_\Omega\frac{1}{2}u_t^2\,dxdy+\int_\Omega\left(\frac{1}{2}(\Delta u)^2+(1-\sigma)(u_{xy}^2-u_{xx}u_{yy})+H(x,y,u)-fu\right)dxdy\,.\tag{21}$$

This represents the total energy of a nonlinear dynamic bridge. As for the action, one has to take the difference between kinetic energy and potential energy and integrate on an interval $[0,T]$:

$$\mathcal{A}(u) := \int_0^T \left[\int_\Omega \frac{1}{2}u_t^2\,dxdy\right] dt - \int_0^T \left[\int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) + H(x,y,u) - fu\right) dxdy\right] dt\,.$$

The equation of the motion of the bridge is obtained by taking critical points of the functional $\mathcal{A}$:

$$u_{tt} + \Delta^2 u + h(x,y,u) = f \qquad \text{in } \Omega\times(0,T)\,.$$

Due to internal friction, we add a damping term and obtain

$$\begin{cases} u_{tt}+\delta u_t+\Delta^2 u+h(x,y,u)=f & \text{in } \Omega\times(0,T)\\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)=u_{xx}(\pi,y,t)=0 & \text{for } (y,t)\in(-\ell,\ell)\times(0,T)\\ u_{yy}(x,\pm\ell,t)+\sigma u_{xx}(x,\pm\ell,t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,T)\\ u_{yyy}(x,\pm\ell,t)+(2-\sigma)u_{xxy}(x,\pm\ell,t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,T)\\ u(x,y,0)=u_0(x,y)\,,\quad u_t(x,y,0)=u_1(x,y) & \text{for } (x,y)\in\Omega \end{cases} \tag{22}$$

where $\delta$ is a positive constant. Notice that this equation also arises in different contexts, see e.g. [9, equation (17)], and is sometimes called the Swift-Hohenberg equation.

3. **Main results.** Our first purpose is to minimize the energy functional $\mathbb{E}_T$, defined in (11), on the space

$$H^2_*(\Omega) := \Big\{w\in H^2(\Omega);\ w=0 \text{ on } \{0,\pi\}\times(-\ell,\ell)\Big\}\,.$$

We also define

$$\mathcal{H}(\Omega) := \text{ the dual space of } H^2_*(\Omega)$$

and we denote by $\langle\cdot,\cdot\rangle$ the corresponding duality. Since we are in the plane, $H^2_*(\Omega)\subset C^0(\overline{\Omega})$ so that the condition on $\{0,\pi\}\times(-\ell,\ell)$ introduced in the definition of $H^2_*(\Omega)$ is satisfied pointwise and

$$L^p(\Omega)\subset\mathcal{H}(\Omega) \qquad \forall 1\le p\le\infty\,. \tag{23}$$

If $f\in L^1(\Omega)$ then the functional $\mathbb{E}_T$ is well-defined in $H^2_*(\Omega)$, while if $f\in\mathcal{H}(\Omega)$ we need to replace $\int_\Omega fu$ with $\langle f,u\rangle$ although we will not mention this in the sequel. The first somehow standard statement is the connection between minimizers of the energy function $\mathbb{E}_T$ and solutions of (14). It shows that the variational setting is correct and it allows to derive the boundary conditions.

**Theorem 3.1.** *Assume (9) and let $f\in\mathcal{H}(\Omega)$. Then there exists a unique $u\in H^2_*(\Omega)$ such that*

$$\int_\Omega [\Delta u\Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\,dxdy = \langle f,v\rangle \qquad \forall v\in H^2_*(\Omega)\,; \tag{24}$$

*moreover, $u$ is the minimum point of the convex functional $\mathbb{E}_T$. Finally, if $f\in L^2(\Omega)$ then $u\in H^4(\Omega)$, and if $u\in C^4(\overline{\Omega})$ then $u$ is a classical solution of (14).*

Since we have in mind a long narrow rectangle, that is $\ell\ll\pi$, it is reasonable to assume that the forcing term does not depend on $y$. So, we now assume that

$$f=f(x)\,,\qquad f\in L^2(0,\pi). \tag{25}$$

In this case, we may solve (14) following [23, Section 2.2] although the boundary conditions (13) require some additional effort. A similar procedure can be used also for some forcing terms depending on $y$ such as $e^{\pm y}f(x)$ or $yf(x)$, see [23]. We extend the source $f$ as an odd $2\pi$-periodic function over $\mathbb{R}$ and we expand it in Fourier series

$$f(x)=\sum_{m=1}^{+\infty}\beta_m\sin(mx)\ ,\qquad \beta_m=\frac{2}{\pi}\int_0^\pi f(x)\sin(mx)\,dx\ ,\tag{26}$$

so that $\{\beta_m\}\in\ell^2$ and the series converges in $L^2(0,\pi)$ to $f$. Then we define the constants

$$A=A(m,\ell):=\frac{\sigma}{1-\sigma}\ \frac{\beta_m}{m^4}\ \frac{(1+\sigma)\sinh(m\ell)-(1-\sigma)m\ell\cosh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell)-(1-\sigma)m\ell}\ ,\tag{27}$$

$$B=B(m,\ell):=\sigma\ \frac{\beta_m}{m^4}\ \frac{\sinh(m\ell)}{(3+\sigma)\sinh(m\ell)\cosh(m\ell)-(1-\sigma)m\ell}\ ,\tag{28}$$

and we prove

**Theorem 3.2.** *Assume (9) and that $f$ satisfies (25)-(26). Then the unique solution of (14) is given by*

$$u(x,y)=\sum_{m=1}^{+\infty}\left[\frac{\beta_m}{m^4}+A\cosh(my)+Bmy\sinh(my)\right]\sin(mx)$$

*where the constants $A$ and $B$ are defined in (27) and (28).*

When $\ell\to 0$, the plate $\Omega$ tends to become a one dimensional beam of length $\pi$. We wish to analyze the behavior of the solution and of the energy in this limit situation. To this end, we re-introduce the constants appearing in (10) that were normalized in (11). Let $f\in L^2(\Omega)$ be as in (25) and let $u^\ell$ be a solution of the problem

$$\begin{cases}\frac{E\,d^3}{12(1-\sigma^2)}\Delta^2u^\ell=f & \text{in }\Omega\\ u^\ell(0,y)=u^\ell_{xx}(0,y)=u^\ell(\pi,y)=u^\ell_{xx}(\pi,y)=0 & \text{for }y\in(-\ell,\ell)\\ u^\ell_{yy}(x,\pm\ell)+\sigma u^\ell_{xx}(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\\ u^\ell_{yyy}(x,\pm\ell)+(2-\sigma)u^\ell_{xxy}(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\end{cases}\tag{29}$$

whose total energy is given by (10). Obviously, $u^\ell=\frac{12(1-\sigma^2)}{Ed^3}u$, where $u$ is the unique solution of (14) found in Theorem 3.2. If we view the plate as a parallelepiped-shaped beam $(0,\pi)\times(-\ell,\ell)\times(-d/2,d/2)$ we are led to the problem

$$EI\psi''''=2\ell f\quad\text{in }(0,\pi)\,,\qquad \psi(0)=\psi''(0)=\psi(\pi)=\psi''(\pi)=0\,.\tag{30}$$

Here the forcing term $2\ell f$ represents a force per unit of length and $I=\frac{d^3\ell}{6}=\int_{(-\ell,\ell)\times(-d/2,d/2)}z^2dydz$ is the moment of inertia of the section of the beam with respect to its middle line parallel to the $y$-axis. Then (30) reduces to $\frac{Ed^3}{12}\psi''''=f$, the function $\psi$ is independent of $\ell$ but the corresponding total energy of the beam depends on $\ell$:

$$\mathbb{E}_T(\psi)=\frac{E\,d^3\ell}{12}\int_0^\pi\psi''(x)^2dx-2\ell\int_0^\pi f(x)\psi(x)\,dx=-\left(\frac{6\pi}{E\,d^3}\sum_{m=1}^{+\infty}\frac{\beta_m^2}{m^4}\right)\ell\,.\tag{31}$$

Then we prove

**Theorem 3.3.** *Assume* (9) *and let* $f \in L^2(\Omega)$ *be a vertical load per unit of surface depending only on* $x$*, see* (25)-(26)*. Let* $u^\ell$ *and* $\psi$ *be respectively as in* (29) *and* (30)*. Then*

$$\lim_{\ell\to 0} \sup_{(x,y)\in\Omega} \left|u^\ell(x,y) - \psi(x)\right| = 0 \qquad \text{and} \qquad \mathbb{E}_T(u^\ell) = \mathbb{E}_T(\psi) + o(\ell) \quad \text{as } \ell \to 0\,, \tag{32}$$

*where* $\mathbb{E}_T(u^\ell)$ *is given by* (10) *and* $\mathbb{E}_T(\psi)$ *is given by* (31)*.*

Theorem 3.3 states that, when $\ell \to 0$, the solution and the energy of the plate are "almost the same" as for the beam. However, one cannot neglect the $o(\ell)$ term if one wishes to display torsional oscillations.

Next, we study the oscillating modes of the rectangular plate; we consider the eigenvalue problem

$$\begin{cases} \Delta^2 w = \lambda w & \text{in } \Omega \\ w(0,y) = w_{xx}(0,y) = w(\pi,y) = w_{xx}(\pi,y) = 0 & \text{for } y \in (-\ell,\ell) \\ w_{yy}(x,\pm\ell) + \sigma w_{xx}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi) \\ w_{yyy}(x,\pm\ell) + (2-\sigma)w_{xxy}(x,\pm\ell) = 0 & \text{for } x \in (0,\pi)\,. \end{cases} \tag{33}$$

Similar to (24), problem (33) admits the following variational formulation: a nontrivial function $w \in H^2_*(\Omega)$ is an eigenfunction of (33) if

$$\int_\Omega \left[\Delta w \Delta v + (1-\sigma)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx}) - \lambda w v\right] dxdy = 0$$

for all $v \in H^2_*(\Omega)$. In such a case we say that $\lambda$ is an eigenvalue for problem (33).

In Section 7 we prove that for all $\ell > 0$ and $\sigma \in (0, \frac{1}{2})$ there exists a unique $\mu_1 \in (1-\sigma, 1)$ such that

$$\sqrt{1-\mu_1}\,\big(\mu_1 + 1 - \sigma\big)^2 \tanh(\ell\sqrt{1-\mu_1}) = \sqrt{1+\mu_1}\,\big(\mu_1 - 1 + \sigma\big)^2 \tanh(\ell\sqrt{1+\mu_1})\,. \tag{34}$$

The number $\lambda = \mu_1^2$ is the least eigenvalue.

**Theorem 3.4.** *Assume* (9)*. Then the set of eigenvalues of* (33) *may be ordered in an increasing sequence* $\{\lambda_k\}$ *of strictly positive numbers diverging to* $+\infty$ *and any eigenfunction belongs to* $C^\infty(\overline{\Omega})$*. The set of eigenfunctions of* (33) *is a complete system in* $H^2_*(\Omega)$*.*

*Moreover, the least eigenvalue of* (33) *is* $\lambda_1 = \mu_1^2$*, where* $\mu_1 \in (1-\sigma, 1)$ *is the unique solution of* (34)*; the least eigenvalue* $\mu_1^2$ *is simple and the corresponding eigenspace is generated by the positive eigenfunction*

$$\left\{ \frac{\cosh(\sqrt{1+\mu_1}\ell)}{\mu_1 - 1 + \sigma} \cosh(\sqrt{1-\mu_1}y) + \frac{\cosh(\sqrt{1-\mu_1}\ell)}{\mu_1 + 1 - \sigma} \cosh(\sqrt{1+\mu_1}y) \right\} \sin x$$

*defined for any* $(x,y) \in \overline{\Omega}$*.*

In fact, we obtain a stronger statement describing the whole spectrum and characterizing the eigenfunctions, see Theorem 7.6 in Section 7. In Proposition 7.7 we also show that if $\ell$ is small enough ($\ell \le 0.44$), then the first two eigenvalues are simple. In Figure 3 we display the qualitative behavior of the first two "longitudinal" eigenfunctions and of the first two "torsional" eigenfunctions. It appears that the maximum and minimum of these eigenfunctions are attained on the boundary and that every mode has a tendency to display a torsional behavior: as expected, the "weak" part of the plate are the two long free edges. Note also that in the limit

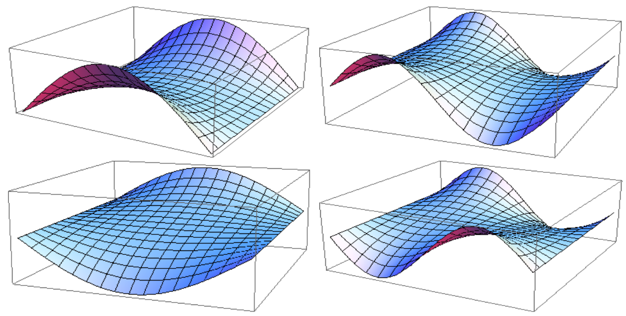

FIGURE 3. Qualitative behavior of some eigenfunctions of (33).

case $\sigma = 0$, excluded by assumption (9), the first eigenvalue is $\lambda_1 = 1$ and the first eigenfunction is $\sin x$.

We now turn to the nonlinear model. With a simple minimization argument one can prove

**Theorem 3.5.** *Assume (9), (16)-(18) and let $f \in \mathcal{H}(\Omega)$; then there exists a unique weak solution $u \in H^2_*(\Omega)$ of (20). This solution is the unique minimizer of the problem*

$$\min_{v \in H^2_*(\Omega)} \mathbb{E}_T(v)$$

*where $\mathbb{E}_T$ is the nonlinear static energy defined in (19).*

Since the proof of Theorem 3.5 is standard, we omit it.

Our last result proves well-posedness for the evolution problem (22). If $T > 0$ we say that

$$u \in C^0([0,T]; H^2_*(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; \mathcal{H}(\Omega)) \tag{35}$$

is a solution of (22) if it satisfies the initial conditions and if

$$\begin{gathered} \langle u''(t), v \rangle + \delta(u'(t), v)_{L^2} + (u(t), v)_{H^2_*} + (h(\cdot,\cdot,u(t)), v)_{L^2} = (f(t), v)_{L^2} \\ \forall v \in H^2_*(\Omega)\,,\ \forall t \in (0,T)\,. \end{gathered} \tag{36}$$

If $T = +\infty$ then the interval $[0,T]$ should be read as $[0,+\infty)$. Then we have

**Theorem 3.6.** *Assume (9), (16)-(18). Let $T > 0$ (including the case $T = +\infty$), let $f \in C^0([0,T]; L^2(\Omega))$ and let $\delta > 0$; let $u_0 \in H^2_*(\Omega)$ and $u_1 \in L^2(\Omega)$. Then*

*(i) there exists a unique solution of (22);*

*(ii) if $f \in L^2(\Omega)$ is independent of $t$, then $T = +\infty$ and the unique solution $u$ of (22) satisfies*

$$u(t) \to \overline{u} \quad \textit{in } H^2_*(\Omega) \quad \textit{and} \quad u'(t) \to 0 \quad \textit{in } L^2(\Omega) \quad \textit{as } t \to +\infty$$

*where $\overline{u}$ is the unique solution of the stationary problem (20).*

4. **Proof of Theorem 3.1.** Let $D^2w$ denote the Hessian matrix of a function $w \in H^2(\Omega)$. Thanks to the Intermediate Derivatives Theorem, see [1, Theorem 4.15], the space $H^2(\Omega)$ is a Hilbert space if endowed with the scalar product

$$(u,v)_{H^2} := \int_\Omega \left( D^2u \cdot D^2v + uv \right) dxdy \qquad \text{for all } u,v \in H^2(\Omega)\,.$$

On the closed subspace $H^2_*(\Omega)$ we may also define a different scalar product.

**Lemma 4.1.** *Assume (9). On the space $H^2_*(\Omega)$ the two norms*

$$u \mapsto \|u\|_{H^2}\,, \quad u \mapsto \|u\|_{H^2_*} := \left[\int_\Omega \left[(\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\right] dxdy\right]^{1/2}$$

*are equivalent. Therefore, $H^2_*(\Omega)$ is a Hilbert space when endowed with the scalar product*

$$(u,v)_{H^2_*} := \int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]\, dxdy\,. \tag{37}$$

*Proof.* We first get rid of the $L^2$-norm. Take any $u \in H^2_*(\Omega)$ so that $u \in C^0(\overline{\Omega})$ and for all $(x,y) \in \Omega$ we have

$$|u(x,y)| = \left|\int_0^x u_x(t,y)dt\right| \le \int_0^\pi |u_x(t,y)|dt \le \sqrt{\pi}\left[\int_0^\pi (u_x(t,y))^2 dt\right]^{\frac12}$$
$$\le \sqrt{\pi}\left[-\int_0^\pi u_{xx}(t,y)u(t,y)dt\right]^{\frac12} \le \sqrt{\pi}\left[\int_0^\pi (u_{xx}(t,y))^2 dt\right]^{\frac14}\left[\int_0^\pi (u(t,y))^2 dt\right]^{\frac14}$$

where we used an integration by parts and twice Hölder's inequality. This inequality, readily yields $\|u\|_{L^2} \le C\|D^2u\|_{L^2}$ for some $C>0$ and proves that the $H^2(\Omega)$-norm is equivalent to the norm $u \mapsto \|D^2u\|_{L^2}$ on the space $H^2_*(\Omega)$. Next, we notice that

$$(1-\sigma)\|D^2u\|^2_{L^2} \le \|u\|^2_{H^2_*} = \int_\Omega [u_{xx}^2 + u_{yy}^2 + 2(1-\sigma)u_{xy}^2 + 2\sigma u_{xx}u_{yy}]dxdy$$
$$\le (1+\sigma)\|D^2u\|^2_{L^2}$$

so that the norms $u \mapsto \|D^2u\|_{L^2}$ and $H^2_*(\Omega)$ are equivalent. These two equivalences prove the lemma. □

By combining Lemma 4.1 with the Lax-Milgram Theorem, we infer that for any $f \in \mathcal{H}(\Omega)$ there exists a unique $u \in H^2_*(\Omega)$ satisfying (24). This proves the first part of Theorem 3.1.

Our next purpose is to study the regularity of the just found solution of (24).

**Lemma 4.2.** *Assume (9) and $1<p<\infty$; let $f \in L^p(\Omega)$ and let $u \in H^2_*(\Omega)$ be a (weak) solution of (14). Then $u \in W^{4,p}(\Omega)$ and there exists a constant $C(\ell,\sigma,p)$ depending only on $\ell$, $\sigma$ and $p$ such that*

$$\|u\|_{W^{4,p}} \le C(\ell,\sigma,p)\|f\|_{L^p}\,. \tag{38}$$

*Proof.* By (23) and Lemma 4.1, the assumptions make sense. The next step is to show that the boundary conditions satisfy the complementing conditions, see [2, p.633] for the definition. On the vertical edges we have Navier boundary conditions for which this property is well-known, see [11, Section 2.3]. On the horizontal edges, the polynomials $\mathbb{R}^2 \to \mathbb{R}$ in the variables $\alpha,\beta$ associated to the boundary conditions (13) are independent of $x$ and $y$ and read $B_1(\alpha,\beta) = \sigma\alpha^2 + \beta^2$ and $B_2(\alpha,\beta) = (2-\sigma)\alpha^2\beta + \beta^3$. Let $\nu = (\nu_1,\nu_2)$ denote the unit normal to $\partial\Omega$ and let $\tau = (\tau_1,\tau_2)$ be any vector tangent to $\partial\Omega$ so that $\nu_1 = \tau_2 = 0$ and $\nu_2 = \operatorname{sign} y$ while $\tau_1$ is arbitrary. Then $B_1(\tau + t\nu) = t^2 + \sigma\tau_1^2$ and $B_2(\tau+t\nu) = (\operatorname{sign} y)t[t^2 + (2-\sigma)\tau_1^2]$; therefore

$$B_1(x,y,\tau+t\nu) = 2i|\tau_1|t + (\sigma+1)\tau_1^2 \mod (t-i|\tau|)^2\,,$$
$$B_2(x,y,\tau+t\nu) = (\operatorname{sign} y)\left[-(\sigma+1)\tau_1^2 t + 2i|\tau_1|^3\right] \mod (t-i|\tau|)^2\,.$$

Since $\sigma \neq 1$, the polynomials

$$\widetilde{B}_1(t) := 2i|\tau_1|t + (\sigma+1)\tau_1^2 \qquad \text{and} \qquad \widetilde{B}_2(t) := (\operatorname{sign} y)\left[-(\sigma+1)\tau_1^2 t + 2i|\tau_1|^3\right]$$

are linearly independent: indeed,

$$-\frac{(\operatorname{sign} y)(\sigma+1)|\tau_1|}{2i}\widetilde{B}_1(t) = (\operatorname{sign} y)\left[-(\sigma+1)\tau_1^2 t + i\frac{(\sigma+1)^2|\tau_1|^3}{2}\right]$$

with $\frac{(\sigma+1)^2}{2} \neq 2$. This proves that also (13) satisfies the complementing conditions.

The lack of smoothness of $\partial\Omega$ is not a serious difficulty. By odd extension, we can view the problem in $\Omega$ as the restriction of a problem in $(-\pi, 2\pi) \times (-\ell, \ell)$. Then classical elliptic local regularity results [2, Theorem 15.1] yield (38). We also refer to [24] for more general regularity results in nonsmooth domains. □

Finally, we show that smooth weak solutions and classical solutions coincide. Note first that, for all $u \in H^2_*(\Omega)$ we have

$$u(0,y) = u(\pi,y) = u_y(0,y) = u_y(\pi,y) = u_{yy}(0,y) = u_{yy}(\pi,y) = 0 \tag{39}$$

for any $y \in (-\ell, \ell)$. Then, by adapting the Gauss-Green formula

$$\int_\Omega \Delta u \Delta v \, dx\, dy = \int_\Omega \Delta^2 u v \, dx\, dy + \int_{\partial\Omega} \Big[\Delta u\, v_\nu - v\,(\Delta u)_\nu\Big]\, ds$$

to our situation, and with some integration by parts, we obtain that if $u \in C^4(\overline{\Omega}) \cap H^2_*(\Omega)$ satisfies (24), then

$$\int_\Omega (\Delta^2 u - f) v \, dxdy + \int_{-\ell}^{\ell} [u_{xx}(\pi,y)v_x(\pi,y) - u_{xx}(0,y)v_x(0,y)]\, dy \tag{40}$$

$$+ \int_0^\pi \Big\{ [u_{yyy}(x,-\ell) + (2-\sigma)u_{xxy}(x,-\ell)]\, v(x,-\ell) - [u_{yy}(x,-\ell) + \sigma u_{xx}(x,-\ell)]\, v_y(x,-\ell)\Big\}\, dx \tag{41}$$

$$+ \int_0^\pi \Big\{ [u_{yy}(x,\ell) + \sigma u_{xx}(x,\ell)]\, v_y(x,\ell) - [u_{yyy}(x,\ell) + (2-\sigma)u_{xxy}(x,\ell)]\, v(x,\ell)\Big\}\, dx = 0 \tag{42}$$

for any $v \in H^2_*(\Omega)$. If we choose $v \in C^2_c(\Omega)$ in (40), then all the boundary terms vanish and we deduce that $\Delta^2 u = f$ in $\Omega$. Hence we may drop the double integral in (40). By arbitrariness of $v$, the coefficients of the terms $v_x(\pi,y)$, $v_x(0,y)$, $v(x,-\ell)$, $v_y(x,-\ell)$, $v_y(x,\ell)$, and $v(x,\ell)$ must vanish identically and we obtain (12)-(13); this conclusion may also be reached with particular choices of $v$ but we omit here the tedious computations.

We have so proved that if $u \in C^4(\overline{\Omega}) \cap H^2_*(\Omega)$ satisfies (24) then it is a classical solution of (14). For the converse implication let $u \in C^4(\overline{\Omega})$ be a classical solution of (14) and let $v \in H^2_*(\Omega)$. Then (40) holds true and, exploiting (39) and integrating by parts, we also recover the validity of (24) for all $v \in H^2_*(\Omega)$.

5. **Proof of Theorem 3.2.** Consider the function

$$\phi(x) := \sum_{m=1}^{+\infty} \frac{\beta_m}{m^4} \sin(mx) \tag{43}$$

and note that it solves the ODE

$$\phi''''(x) = f(x) \quad \text{in } (0,\pi)\,, \qquad \phi(0)=\phi''(0)=\phi(\pi)=\phi''(\pi)=0\,.$$

Moreover, $\phi'' \in H^2(0,\pi)$ is given by

$$\phi''(x) = -\sum_{m=1}^{+\infty} \frac{\beta_m}{m^2}\sin(mx) \tag{44}$$

and the series (44) converges in $H^2(0,\pi)$ and, hence, uniformly.

We now introduce the auxiliary function $v(x,y) := u(x,y) - \phi(x)$; if $u$ solves (14), then $v$ satisfies

$$\begin{cases} \Delta^2 v = 0 & \text{in } \Omega \\ v = v_{xx} = 0 & \text{on } \{0,\pi\}\times(-\ell,\ell) \\ v_{yy} + \sigma v_{xx} = -\sigma\phi'' & \text{on } (0,\pi)\times\{-\ell,\ell\} \\ v_{yyy} + (2-\sigma)v_{xxy} = 0 & \text{on } (0,\pi)\times\{-\ell,\ell\}\,. \end{cases} \tag{45}$$

We seek solutions of (45) by separating variables, namely we seek functions $Y_m = Y_m(y)$ such that

$$v(x,y) = \sum_{m=1}^{+\infty} Y_m(y)\sin(mx)$$

solves (45). Then

$$\Delta^2 v(x,y) = \sum_{m=1}^{+\infty} [Y_m''''(y) - 2m^2 Y_m''(y) + m^4 Y_m(y)]\sin(mx)$$

and the equation in (45) yields

$$Y_m''''(y) - 2m^2 Y_m''(y) + m^4 Y_m(y) = 0 \quad \text{for } y \in (-\ell,\ell)\,. \tag{46}$$

The solutions of (46) are linear combinations of the functions $\cosh(my)$, $\sinh(my)$, $y\cosh(my)$ and $y\sinh(my)$ but, due to the symmetry of $\Omega$ and to the uniqueness of the solution $v$ to (45), we know that $Y_m$ is even with respect to $y$. Hence, we seek functions $Y_m$ of the form

$$Y_m(y) = A\cosh(my) + Bmy\sinh(my) \tag{47}$$

where $A = A(m,\ell)$ and $B = B(m,\ell)$ are constants to be determined by imposing the boundary conditions in (45) and the coefficient $m$ is highlighted on the term $y\sinh(my)$ for later simplifications. By differentiating we obtain

$$\begin{aligned} Y_m'(y) &= m[(A+B)\sinh(my) + Bmy\cosh(my)]\,, \\ Y_m''(y) &= m^2[(A+2B)\cosh(my) + Bmy\sinh(my)]\,, \\ Y_m'''(y) &= m^3[(A+3B)\sinh(my) + Bmy\cosh(my)]\,. \end{aligned} \tag{48}$$

The two boundary conditions on $(0,\pi)\times\{-\ell,\ell\}$, see (45), become respectively

$$\sum_{m=1}^{+\infty} [Y_m''(\pm\ell) - \sigma m^2 Y_m(\pm\ell)]\sin(mx) = -\sigma\phi''(x)\,,$$

$$\sum_{m=1}^{+\infty} [Y_m'''(\pm\ell) - (2-\sigma)m^2 Y_m'(\pm\ell)]\sin(mx) = 0$$

for all $x \in (0, \pi)$, and, by (44),

$$Y_m''(\ell) - \sigma m^2 Y_m(\ell) = \sigma \frac{\beta_m}{m^2} \ , \quad Y_m'''(\ell) - (2-\sigma) m^2 Y_m'(\ell) = 0 \ ,$$

the condition for $y = -\ell$ being automatically fulfilled since $Y_m$ is even. By plugging these information into the explicit form (48) of the derivatives we find the system

$$\begin{cases} (1-\sigma)\cosh(m\ell)\, A + \Big(2\cosh(ml) + (1-\sigma) m\ell \sinh(m\ell)\Big)\, B = \sigma \frac{\beta_m}{m^4} \\ (1-\sigma)\sinh(m\ell)\, A + \Big((1-\sigma) m\ell \cosh(m\ell) - (1+\sigma)\sinh(m\ell)\Big)\, B = 0 \ , \end{cases}$$

and we finally obtain (27)-(28).

6. **Proof of Theorem 3.3.** Let $u \in H^2_*(\Omega)$ be the solution of (14), see Theorem 3.2. By (27)-(28) we have

$$|A\cosh(my)| \le C \frac{\beta_m}{m^3} \quad \text{and} \quad Bmy\sinh(my) \le C \frac{\beta_m}{m^3}$$

for any $y \in (-\ell, \ell)$, $\ell \in (0,1)$ and $m \ge 1$ for some constant $C > 0$ depending on $\sigma$ but independent of $y$, $\ell$ and $m$. Moreover we also have

$$\lim_{\ell \to 0} A(m,\ell) = \frac{\sigma^2}{1-\sigma^2} \, \frac{\beta_m}{m^4} \, , \qquad \lim_{\ell \to 0} B(m,\ell) = \frac{\sigma}{2(1+\sigma)} \, \frac{\beta_m}{m^4} \qquad \text{for any } m \in \mathbb{N} \, . \tag{49}$$

This implies that for any $N \in \mathbb{N}$

$$\begin{aligned} &\limsup_{\ell \to 0} \sup_{(x,y)\in\Omega} \left| u(x,y) - \frac{1}{1-\sigma^2} \phi(x) \right| \\ &\le \limsup_{\ell \to 0} \sup_{y \in (-\ell,\ell)} \sum_{m=1}^{+\infty} \left| A\cosh(my) - \frac{\sigma^2}{1-\sigma^2} \frac{\beta_m}{m^4} + Bmy\sinh(my) \right| \\ &\le \lim_{\ell \to 0} \sum_{m=1}^{N} \left[ \left| A - \frac{\sigma^2}{1-\sigma^2} \frac{\beta_m}{m^4} \right| \cosh(m\ell) + \frac{\sigma^2}{1-\sigma^2} \frac{\beta_m}{m^4} (\cosh(m\ell) - 1) \right. \\ &\qquad \left. + Bm\ell \sinh(m\ell) \right] + C \sum_{m=N+1}^{+\infty} \frac{\beta_m}{m^3} \le C \sum_{m=N+1}^{+\infty} \frac{\beta_m}{m^3} \, . \end{aligned}$$

Letting $N \to +\infty$, we obtain

$$\lim_{\ell \to 0} \sup_{(x,y)\in\Omega} \left| u(x,y) - \frac{1}{1-\sigma^2} \, \phi(x) \right| = 0 \ . \tag{50}$$

Let us now recall a well-known result about Fourier series which will be repeatedly used in the sequel.

**Lemma 6.1.** *Let $\{a_m\}, \{b_m\} \in \ell^2$ and let*

$$a(x) = \sum_{m=1}^{+\infty} a_m \sin(mx) \ , \quad b(x) = \sum_{m=1}^{+\infty} b_m \sin(mx) \ .$$

*Then $a, b \in L^2(0,\pi)$ and*

$$\int_0^\pi a(x) b(x)\, dx = \frac{\pi}{2} \sum_{m=1}^{+\infty} a_m b_m \ , \quad \int_0^\pi a(x)^2\, dx = \frac{\pi}{2} \sum_{m=1}^{+\infty} a_m^2 \ .$$

By differentiating the solution $u$ we find

$$u_{xx}(x,y) = -\sum_{m=1}^{+\infty}\left[\frac{\beta_m}{m^2} + Am^2\cosh(my) + Bm^3 y\sinh(my)\right]\sin(mx)\,,$$

$$u_{yy}(x,y) = \sum_{m=1}^{+\infty} m^2[(A+2B)\cosh(my) + Bmy\sinh(my)]\sin(mx)\,,$$

$$u_{xy}(x,y) = \sum_{m=1}^{+\infty} m^2[(A+B)\sinh(my) + Bmy\cosh(my)]\cos(mx)\,,$$

and therefore

$$\Delta u(x,y) = \sum_{m=1}^{+\infty}\left[-\frac{\beta_m}{m^2} + 2Bm^2\cosh(my)\right]\sin(mx)\ .$$

Then, by Lemma 6.1, we obtain

$$\begin{aligned}\int_\Omega |\Delta u|^2 &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}\left[-\frac{\beta_m}{m^2} + 2Bm^2\cosh(my)\right]^2 dy \\ &= \pi\sum_{m=1}^{+\infty}\left[\frac{\beta_m^2}{m^4}\ell + 2B^2m^4\ell - 4\frac{B\beta_m}{m}\sinh(m\ell) + B^2m^3\sinh(2m\ell)\right]\ .\end{aligned} \tag{51}$$

Moreover, Lemma 6.1 also yields

$$\begin{aligned}\int_\Omega u_{xx}u_{yy} = -\frac{\pi}{2}\sum_{m=1}^{+\infty}\int_{-\ell}^{\ell}&[\beta_m + Am^4\cosh(my) + Bm^5 y\sinh(my)] \\ &\times[(A+2B)\cosh(my) + Bmy\sinh(my)]\,dy \\ = -\pi\sum_{m=1}^{+\infty}&\Big[\tfrac{\beta_m}{m}[(A+B)\sinh(m\ell) + Bm\ell\cosh(m\ell)] + \tfrac{B(2A+B)m^3}{4}m\ell\cosh(2m\ell) \\ &+(\tfrac{2A^2+2AB-B^2}{8} + \tfrac{B^2}{4}m^2\ell^2)\,m^3\sinh(2m\ell) + \tfrac{A(A+2B)}{2}m^4\ell - \tfrac{B^2}{6}m^6\ell^3\Big]\end{aligned} \tag{52}$$

and

$$\begin{aligned}\int_\Omega u_{xy}^2 = \frac{\pi}{2}\sum_{m=1}^{+\infty} m^4\int_{-\ell}^{\ell}&[(A+B)\sinh(my) + Bmy\cosh(my)]^2\,dy \\ = \pi\sum_{m=1}^{+\infty} m^3&\Big[\tfrac{2A^2+2AB+B^2}{8}\sinh(2m\ell) + \tfrac{B(2A+B)}{4}m\ell\cosh(2m\ell) + \tfrac{B^2}{4}m^2\ell^2\sinh(2m\ell) \\ &+\tfrac{B^2}{6}m^3\ell^3 - \tfrac{(A+B)^2}{2}m\ell\Big]\ .\end{aligned} \tag{53}$$

Finally, by (26) and a further application of Lemma 6.1,

$$\begin{aligned}\int_\Omega fu &= \frac{\pi}{2}\sum_{m=1}^{+\infty}\beta_m\int_{-\ell}^{\ell}\left[\tfrac{\beta_m}{m^4} + A\cosh(my) + Bmy\sinh(my)\right]dy \\ &= \pi\sum_{m=1}^{+\infty}\beta_m\left[\tfrac{\beta_m}{m^4}\ell + \tfrac{A-B}{m}\sinh(m\ell) + B\ell\cosh(m\ell)\right]\ .\end{aligned} \tag{54}$$

Since $u$ solves (14), the corresponding energy is given by (11) and hence collecting (51)-(54) we obtain

$$\mathbb{E}_T(u)=\pi\sum_{m=1}^{+\infty}\Bigg\{-\frac{\beta_m^2\ell}{2m^4}+\frac{\sigma+1}{2}B^2m^4\ell-\frac{\sigma\beta_m(A+B)}{m}\sinh(m\ell)+\frac{B^2m^3}{2}\sinh(2m\ell)$$
$$-\sigma\beta_m B\ell\cosh(m\ell)+\frac{1-\sigma}{2}\big[A(A+B)m^3\sinh(2m\ell)+B^2m^5\ell^2\sinh(2m\ell)$$
$$+B(2A+B)\,m^4\ell\cosh(2m\ell)\big]\Bigg\}=:\sum_{m=1}^{+\infty}a(m,\ell)\,.$$

With a direct computation one can see that by (27) and (28) we get

$$\frac{|a(m,\ell)|}{\ell}\le C\frac{\beta_m^2}{m^3}\qquad\text{for any }\ell\in(0,1)\text{ and }m\ge 1$$

for some constant $C>0$ depending on $\sigma$ but independent of $\ell$ and $m$.

This implies that for any $N\in\mathbb{N}$ we have

$$\sum_{m=1}^{N}\frac{a(m,\ell)}{\ell}-\sum_{m=N+1}^{+\infty}C\frac{\beta_m^2}{m^3}\le\sum_{m=1}^{+\infty}\frac{a(m,\ell)}{\ell}\le\sum_{m=1}^{N}\frac{a(m,\ell)}{\ell}+\sum_{m=N+1}^{+\infty}C\frac{\beta_m^2}{m^3}\,.$$

Letting $\ell\to 0$ we obtain

$$\sum_{m=1}^{N}\lim_{\ell\to 0}\left(\frac{a(m,\ell)}{\ell}\right)-\sum_{m=N+1}^{+\infty}C\frac{\beta_m^2}{m^3}\le\liminf_{\ell\to 0}\sum_{m=1}^{+\infty}\frac{a(m,\ell)}{\ell}$$
$$\le\limsup_{\ell\to 0}\sum_{m=1}^{+\infty}\frac{a(m,\ell)}{\ell}\le\sum_{m=1}^{N}\lim_{\ell\to 0}\left(\frac{a(m,\ell)}{\ell}\right)+\sum_{m=N+1}^{+\infty}C\frac{\beta_m^2}{m^3}\,.$$

Letting $N\to+\infty$, by (49), we deduce that

$$\lim_{\ell\to 0}\frac{\mathbb{E}_T(u)}{\ell}=\lim_{\ell\to 0}\sum_{m=1}^{+\infty}\frac{a(m,\ell)}{\ell}=\sum_{m=1}^{+\infty}\lim_{\ell\to 0}\frac{a(m,\ell)}{\ell}=-\frac{\pi}{2(1-\sigma^2)}\sum_{m=1}^{+\infty}\frac{\beta_m^2}{m^4}$$

and, in turn,

$$\mathbb{E}_T(u)=-\left(\frac{\pi}{2(1-\sigma^2)}\sum_{m=1}^{+\infty}\frac{\beta_m^2}{m^4}\right)\ell+o(\ell)\quad\text{as }\ell\to 0\,.\tag{55}$$

Consider now $u^\ell$ and $\psi$ as in (29) and (30); recall that $u^\ell=\frac{12(1-\sigma^2)}{Ed^3}u$ where $u$ solves (14) and that $\psi=\frac{12}{Ed^3}\phi$. Then, from (50) we deduce the first of (32). Since $u^\ell$ solves (29), the corresponding energy is given by (10) and hence, by (55) and the identity $u^\ell=\frac{12(1-\sigma^2)}{Ed^3}u$, we obtain

$$\mathbb{E}_T(u^\ell)=\frac{12(1-\sigma^2)}{E\,d^3}\int_\Omega\left[\frac{1}{2}(\Delta u)^2+(\sigma-1)\det(D^2u)-fu\right]dxdy$$
$$=\frac{12(1-\sigma^2)}{E\,d^3}\mathbb{E}_T(u)=\frac{12(1-\sigma^2)}{E\,d^3}\left[-\left(\frac{\pi}{2(1-\sigma^2)}\sum_{m=1}^{+\infty}\frac{\beta_m^2}{m^4}\right)\ell+o(\ell)\right]$$

and the second of (32) follows.

7. **Proof of Theorem 3.4.** By Lemma 4.1 the bilinear form (37) is continuous and coercive; standard spectral theory of self-adjoint operators then shows that the eigenvalues of (33) may be ordered in an increasing sequence of strictly positive numbers diverging to $+\infty$ and that the corresponding eigenfunctions form a complete system in $H^2_*(\Omega)$. The eigenfunctions are smooth in $\overline{\Omega}$: this may be obtained by making an odd extension as in Lemma 4.2 and with a bootstrap argument. This proves the first part of Theorem 3.4.

Take an eigenfunction $w$ of (33) and consider its Fourier expansion with respect to the variable $x$:

$$w(x,y)=\sum_{m=1}^{+\infty} h_m(y)\sin(mx) \qquad \text{for } (x,y)\in(0,\pi)\times(-\ell,\ell)\,. \tag{56}$$

Since $w\in C^\infty(\overline{\Omega})$, the Fourier coefficients $h_m=h_m(y)$ are smooth functions and solve the ordinary differential equation

$$h_m''''(y)-2m^2h_m''(y)+(m^4-\lambda)h_m(y)=0 \tag{57}$$

for some $\lambda>0$. The eigenfunction $w$ in (56) satisfies (12), while by imposing (13) we obtain the boundary conditions on $h_m$

$$h_m''(\pm\ell)-\sigma m^2h_m(\pm\ell)=0\,, \qquad h_m'''(\pm\ell)+(\sigma-2)m^2h_m'(\pm\ell)=0\,. \tag{58}$$

Put $\mu=\sqrt{\lambda}>0$ and consider the characteristic equation $\alpha^4-2m^2\alpha^2+m^4-\mu^2=0$ related to (57). By solving this algebraic equation we find

$$\alpha^2=m^2\pm\mu\,. \tag{59}$$

Three cases have to be distinguished.

- **The case $0<\mu<m^2$.** By (59) we infer

$$\alpha=\pm\beta \text{ or } \alpha=\pm\gamma \qquad \text{with} \qquad \sqrt{m^2-\mu}=:\gamma<\beta:=\sqrt{m^2+\mu}\,. \tag{60}$$

Hence, possible nontrivial solutions of (57)-(58) have the form

$$h_m(y)=a\cosh(\beta y)+b\sinh(\beta y)+c\cosh(\gamma y)+d\sinh(\gamma y) \qquad (a,b,c,d\in\mathbb{R})\,. \tag{61}$$

By computing the derivatives of $h_m$ and imposing the conditions (58) we find the two systems

$$\begin{cases}(\beta^2-m^2\sigma)\cosh(\beta\ell)a+(\gamma^2-m^2\sigma)\cosh(\gamma\ell)c=0\\ (\beta^3-m^2(2-\sigma)\beta)\sinh(\beta\ell)a+(\gamma^3-m^2(2-\sigma)\gamma)\sinh(\gamma\ell)c=0\,,\end{cases}$$
$$\begin{cases}(\beta^2-m^2\sigma)\sinh(\beta\ell)b+(\gamma^2-m^2\sigma)\sinh(\gamma\ell)d=0\\ (\beta^3-m^2(2-\sigma)\beta)\cosh(\beta\ell)b+(\gamma^3-m^2(2-\sigma)\gamma)\cosh(\gamma\ell)d=0\,.\end{cases} \tag{62}$$

There exists a nontrivial solution $h_m$ of (57) of the form (61) if and only if there exists a nontrivial solution of at least one of the two systems (62). The first system in (62) admits a nontrivial solution $(a,c)$ if and only if

$$\begin{aligned}(\beta^2-m^2\sigma)(\gamma^3-m^2(2-\sigma)\gamma)\cosh(\beta\ell)\sinh(\gamma\ell)\\ =(\gamma^2-m^2\sigma)(\beta^3-m^2(2-\sigma)\beta)\sinh(\beta\ell)\cosh(\gamma\ell)\,.\end{aligned}$$

By (60), this is equivalent to

$$\frac{\gamma}{(\gamma^2-m^2\sigma)^2}\,\tanh(\ell\gamma)=\frac{\beta}{(\beta^2-m^2\sigma)^2}\,\tanh(\ell\beta)\,. \tag{63}$$

Recalling that both $\beta$ and $\gamma$ depend on $\mu$, we prove

**Lemma 7.1.** *Assume* (9)*. For any* $m \geq 1$ *there exists a unique* $\mu = \mu_m \in (0, m^2)$ *such that* (63) *holds; moreover we also have* $\mu_m \in ((1-\sigma)m^2, m^2)$*.*

*Proof.* Consider the function $\eta_m(t) := \frac{t}{(t^2-m^2\sigma)^2} \cdot \tanh(\ell t)$ for any $t \in [0,+\infty) \setminus \{\sqrt{\sigma}m\}$. Then

$$\eta_m'(t) = \frac{(-3t^2 - m^2\sigma)\sinh(\ell t)\cosh(\ell t) + \ell t(t^2 - m^2\sigma)}{(t^2-m^2\sigma)^3\cosh^2(\ell t)} \qquad \forall t \in [0,+\infty) \setminus \{\sqrt{\sigma}m\}\,.$$

For any $t > \sqrt{\sigma}m$ we have

$$\eta_m'(t) < \frac{-3t^2\sinh(\ell t)\cosh(\ell t) + \ell t^3}{(t^2-m^2\sigma)^3\cosh^2(\ell t)} < -\frac{2\ell t^3}{(t^2-m^2\sigma)^3\cosh^2(\ell t)} < 0\,.$$

This shows that $\eta_m$ is decreasing in $(\sqrt{\sigma}m, +\infty)$ and, if $\beta > \gamma > \sqrt{\sigma}m$ then $\eta_m(\beta) < \eta_m(\gamma)$ so that (63) cannot hold. We have proved that if $\gamma$ and $\beta$ satisfy (63) then necessarily $\gamma \in [0, \sqrt{\sigma}m)$.

Since $\beta = \sqrt{2m^2-\gamma^2}$, identity (63) is equivalent to

$$\frac{\sqrt{2m^2-\gamma^2}\,(\gamma^2-m^2\sigma)^2}{[(2-\sigma)m^2-\gamma^2]^2}\tanh(\ell\sqrt{2m^2-\gamma^2}) = \gamma\tanh(\ell\gamma)\,. \tag{64}$$

Then we define

$$g_m(t) := \frac{\sqrt{2m^2-t^2}\,(m^2\sigma-t^2)^2}{[(2-\sigma)m^2-t^2]^2}\tanh(\ell\sqrt{2m^2-t^2}) \qquad \forall t \in [0, \sqrt{\sigma}m]\,.$$

The function $t \mapsto [m^2\sigma - t^2]/[(2-\sigma)m^2-t^2]$ is nonnegative and decreasing and hence so is its square. It then follows that $g_m$ is decreasing in $[0, \sqrt{\sigma}m]$ and $g_m(\sqrt{\sigma}m) = 0$. On the other hand, the map $t \mapsto t\tanh(\ell t)$ is increasing in $[0, \sqrt{\sigma}m]$ and vanishes at $t = 0$. This proves that there exists a unique $\gamma_m \in (0, \sqrt{\sigma}m)$ satisfying (64). The statements of the lemma now follow by putting $\mu_m = m^2 - \gamma_m^2$. □

In the next result we prove that the sequence $\{\mu_m\}$ found in Lemma 7.1 is increasing.

**Lemma 7.2.** *Assume* (9)*. For any* $m \geq 1$*, let* $\mu_m$ *be as in Lemma* 7.1*. Then* $\mu_m < \mu_{m+1}$ *for all* $m \geq 1$*.*

*Proof.* By (60), the equation (63) reduces to

$$\Phi(m,\mu) := \sqrt{\frac{m^2-\mu}{m^2+\mu}}\left(\frac{\mu+(1-\sigma)m^2}{\mu-(1-\sigma)m^2}\right)^2 \frac{\tanh(\ell\sqrt{m^2-\mu})}{\tanh(\ell\sqrt{m^2+\mu})} = 1\,. \tag{65}$$

We consider $\Phi$ as a function defined in the region of the plane $\{(m,\mu) \in \mathbb{R}^2;\, (1-\sigma)m^2 < \mu < m^2\}$. In this region, the three maps

$$(m,\mu) \mapsto \sqrt{\frac{m^2-\mu}{m^2+\mu}}\,,\ (m,\mu) \mapsto \left(\frac{\mu+(1-\sigma)m^2}{\mu-(1-\sigma)m^2}\right)^2\,,\ (m,\mu) \mapsto \frac{\tanh(\ell\sqrt{m^2-\mu})}{\tanh(\ell\sqrt{m^2+\mu})}\,,$$

are all positive, strictly increasing with respect to $m$, and strictly decreasing with respect to $\mu$. Therefore, the function $m \mapsto \mu_m$, implicitly defined by $\Phi(m,\mu_m) = 1$, is strictly increasing. □

Similarly, the second system in (62) has nontrivial solutions $(b,d)$ if and only if

$$\begin{aligned}(\beta^2 - m^2\sigma)(\gamma^3 - m^2(2-\sigma)\gamma)\sinh(\beta\ell)\cosh(\gamma\ell)\\ = (\gamma^2 - m^2\sigma)(\beta^3 - m^2(2-\sigma)\beta)\cosh(\beta\ell)\sinh(\gamma\ell)\,.\end{aligned}$$

By (60), this is equivalent to

$$\frac{\beta}{(\beta^2-m^2\sigma)^2}\,\coth(\ell\beta)=\frac{\gamma}{(\gamma^2-m^2\sigma)^2}\,\coth(\ell\gamma)\,. \tag{66}$$

Recalling that both $\beta$ and $\gamma$ depend on $\mu$, we prove

**Lemma 7.3.** *Assume (9). Then there exists a unique $\mu=\mu^m\in(0,m^2)$ satisfying (66) if and only if*

$$\ell m\sqrt{2}\,\coth(\ell m\sqrt{2})>\left(\frac{2-\sigma}{\sigma}\right)^2. \tag{67}$$

*Moreover in such a case we have $\mu^m\in((1-\sigma)m^2,m^2)$.*

*Proof.* The function $\eta_m(t):=\frac{t}{(t^2-m^2\sigma)^2}\cdot\coth(\ell t)$ is strictly decreasing for $t\in(\sqrt{\sigma}m,+\infty)$ because it is the product of two positive and strictly decreasing functions. In particular, if $\beta>\gamma>\sqrt{\sigma}m$ then $\eta_m(\beta)<\eta_m(\gamma)$ so that (66) cannot hold. This proves that if $\gamma$ and $\beta$ satisfy (66) then necessarily $\gamma\in(0,\sqrt{\sigma}m)$.

By (60) identity (66) is equivalent to

$$\frac{\sqrt{2m^2-\gamma^2}\,(\gamma^2-m^2\sigma)^2}{[(2-\sigma)m^2-\gamma^2]^2}\,\coth(\ell\sqrt{2m^2-\gamma^2})=\gamma\,\coth(\ell\gamma)\,. \tag{68}$$

Then we define

$$g_m(t)=\frac{\sqrt{2m^2-t^2}\,(m^2\sigma-t^2)^2}{[(2-\sigma)m^2-t^2]^2}\,\coth(\ell\sqrt{2m^2-t^2})\qquad\forall t\in[0,\sqrt{\sigma}m]\,. \tag{69}$$

We have

$$g_m'(t)=\frac{\ell t(m^2\sigma-t^2)^2}{[(2-\sigma)m^2-t^2]^2\,\sinh^2(\ell\sqrt{2m^2-t^2})} \tag{70}$$

$$-\frac{8(1-\sigma)m^2(2m^2-t^2)+(m^2\sigma-t^2)[(2-\sigma)m^2-t^2]}{\sqrt{2m^2-t^2}\,[(2-\sigma)m^2-t^2]^3}\,(m^2\sigma-t^2)t\coth(\ell\sqrt{2m^2-t^2})$$

$$<\frac{t(m^2\sigma-t^2)^2\,[(2-\sigma)m^2-t^2][\ell\sqrt{2m^2-t^2}-\sinh(\ell\sqrt{2m^2-t^2})\cosh(\ell\sqrt{2m^2-t^2})]}{\sqrt{2m^2-t^2}[(2-\sigma)m^2-t^2]^3\,\sinh^2(\ell\sqrt{2m^2-t^2})}$$

which is negative for any $t\in(0,\sqrt{\sigma}m)$. Therefore $g_m$ is decreasing in $(0,\sqrt{\sigma}m)$ with $g_m(0)=\sqrt{2}m(\frac{\sigma}{2-\sigma})^2\,\coth(\ell m\sqrt{2})$ and $g_m(\sqrt{\sigma}m)=0$. On the other hand, the map $t\mapsto t\,\coth(\ell t)$ is increasing in $(0,\sqrt{\sigma}m)$ and tends to $1/\ell$ as $t\to 0^+$. This proves that there exists a unique $\gamma^m\in(0,\sqrt{\sigma}m)$ satisfying (68) if and only if (67) holds. The proof of the lemma now follows by putting $\mu^m=m^2-(\gamma^m)^2$. □

Note also that (67) holds if and only if $m$ is large enough, that is,

$$\exists m_\sigma\ge 1\ \text{ such that (67) holds if and only if }\ m\ge m_\sigma\,. \tag{71}$$

In particular, if $\ell\sqrt{2}\coth(\ell\sqrt{2})>(\frac{\sigma}{2-\sigma})^2$ then $m_\sigma=1$. We now prove that the sequence $\{\mu^m\}$, found in Lemma 7.3, is increasing.

**Lemma 7.4.** *Assume (9). For any $m\ge 1$, let $\mu^m$ as in the statement of Lemma 7.3. Then $\mu^m<\mu^{m+1}$ for any $m\ge m_\sigma$, see (71).*

*Proof.* Let $m \geq m_\sigma$; by Lemma 7.3 we know that $\mu^m < m^2$ and $\mu^{m+1} > (1-\sigma)(m+1)^2$. Therefore, we may restrict our attention to the case where $(1-\sigma)(m+1)^2 < m^2$ and $\mu^m, \mu^{m+1} \in ((1-\sigma)(m+1)^2, m^2)$ since otherwise the statement follows immediately. For

$$(m,\mu) \in A := \{(m,\mu) \in \mathbb{R}^2;\, m \geq m_\sigma,\, (1-\sigma)(m+1)^2 < \mu < m^2\}\,,$$

consider the functions

$$\Gamma(m,\mu) := \frac{\sqrt{\mu+m^2}\,[\mu-(1-\sigma)m^2]^2}{[\mu+(1-\sigma)m^2]^2}\,\coth(\ell\sqrt{\mu+m^2})\,,$$
$$K(m,\mu) := \sqrt{m^2-\mu}\,\coth(\ell\sqrt{m^2-\mu})\,.$$

On the interval $\mu < s < \frac{\mu}{1-\sigma}$, both the positive maps

$$s \mapsto \frac{\sqrt{\mu+s}\,[\mu-(1-\sigma)s]^2}{[\mu+(1-\sigma)s]^2} \quad \text{and} \quad s \mapsto \coth(\ell\sqrt{\mu+s})$$

have strictly negative derivatives. Moreover, if $g_m$ is as in (69), then $\Gamma(m,\mu) = g_m(\sqrt{m^2-\mu})$ and (70) proves that $\mu \mapsto \Gamma(m,\mu)$ has strictly positive derivative. Summarizing,

$$\frac{\partial\Gamma}{\partial m}(m,\mu) < 0 \quad \text{and} \quad \frac{\partial\Gamma}{\partial\mu}(m,\mu) > 0 \quad \forall (m,\mu) \in A\,. \tag{72}$$

It is also straightforward to verify that

$$\frac{\partial K}{\partial m}(m,\mu) > 0 \quad \text{and} \quad \frac{\partial K}{\partial\mu}(m,\mu) < 0 \quad \forall (m,\mu) \in A\,. \tag{73}$$

Finally, put

$$\Psi(m,\mu) := \frac{K(m,\mu)}{\Gamma(m,\mu)} \quad \forall (m,\mu) \in A\,. \tag{74}$$

The function $m \mapsto \mu^m$ is implicitly defined by $\Psi(m,\mu^m) = 1$, see (66) and (60). By (72)-(73) we infer

$$\frac{\partial\Psi}{\partial m}(m,\mu) > 0 \quad \text{and} \quad \frac{\partial\Psi}{\partial\mu}(m,\mu) < 0 \quad \forall (m,\mu) \in A\,.$$

This proves that the map $m \mapsto \mu^m$ is strictly increasing. □

We finally compare $\mu_m$ with $\mu^m$.

**Lemma 7.5.** *Assume (9). Let $\mu_m$ and $\mu^m$ be, respectively, as in Lemmas 7.1 and 7.3. Then for any $m \geq m_\sigma$ we have $\mu_m < \mu^m$.*

*Proof.* Let $\Phi$ and $\Psi$ be as in (65) and (74); then $\Phi(m,\mu) < \Psi(m,\mu)$ for all $(m,\mu) \in A$. Since $\mu^m$ is implicitly defined by $\Psi(m,\mu^m) = 1$, we have $\Phi(m,\mu^m) < 1$. Moreover, in Lemma 7.2 we saw that $\mu \mapsto \Phi(m,\mu)$ is strictly decreasing. Hence, $\Phi(m,\mu^m) < 1 = \Phi(m,\mu_m)$ implies $\mu_m < \mu^m$. □

• **The case $\mu = m^2$.** By (59) we infer that possible nontrivial solutions of (57)-(58) have the form

$$h_m(y) = a\cosh(\sqrt{2}my) + b\sinh(\sqrt{2}my) + c + dy \qquad (a,b,c,d \in \mathbb{R})\,.$$

By differentiating $h_m$ and by imposing the boundary conditions (58) we get

$$\begin{cases} (2-\sigma)\cosh(\sqrt{2}m\ell)a - \sigma c = 0 \\ \sigma \sinh(\sqrt{2}m\ell)a = 0\,, \end{cases}$$
$$\begin{cases} (2-\sigma)\sinh(\sqrt{2}m\ell)b - \sigma\ell d = 0 \\ \sqrt{2}m\sigma\cosh(\sqrt{2}m\ell)b + (\sigma-2)d = 0\,. \end{cases} \tag{75}$$

The first system in (75) has the unique solution $a = c = 0$ under the assumption (9). The second system in (75) admits a nontrivial solution $(b,d)$ if and only if

$$\tanh(\sqrt{2}m\ell) = \left(\frac{\sigma}{2-\sigma}\right)^2 \sqrt{2}m\ell\,. \tag{76}$$

By (9) the equation $\tanh(s) = \left(\frac{\sigma}{2-\sigma}\right)^2 s$ admits a unique solution $\overline{s} > 0$. But if $m_* := \overline{s}/\ell\sqrt{2}$ is not an integer, then (76) admits no solution. If $m_* \in \mathbb{N}$, then the second system in (75) admits a nontrivial solution $(b,d) \neq (0,0)$ whenever $m = m_*$. If $m \in \mathbb{N}$ does not satisfy (76), then the second system in (75) only admits the trivial solution $b = d = 0$.

• **The case** $\mu > m^2$**.** By (59) we infer that

$$\alpha = \pm\beta \text{ or } \alpha = \pm i\gamma \quad \text{with} \quad \sqrt{\mu - m^2} = \gamma < \beta = \sqrt{\mu + m^2}\,. \tag{77}$$

Therefore, possible nontrivial solutions of (57) have the form

$$h_m(y) = a\cosh(\beta y) + b\sinh(\beta y) + c\cos(\gamma y) + d\sin(\gamma y) \qquad (a,b,c,d \in \mathbb{R})\,.$$

Differentiating $h_m$ and imposing the boundary conditions (58) yields the two systems:

$$\begin{cases} (\beta^2 - m^2\sigma)\cosh(\beta\ell)a - (\gamma^2 + m^2\sigma)\cos(\gamma\ell)c = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\sinh(\beta\ell)a + (\gamma^3 + m^2(2-\sigma)\gamma)\sin(\gamma\ell)c = 0\,, \end{cases} \tag{78}$$

$$\begin{cases} (\beta^2 - m^2\sigma)\sinh(\beta\ell)b - (\gamma^2 + m^2\sigma)\sin(\gamma\ell)d = 0 \\ (\beta^3 - m^2(2-\sigma)\beta)\cosh(\beta\ell)b - (\gamma^3 + m^2(2-\sigma)\gamma)\cos(\gamma\ell)d = 0\,. \end{cases} \tag{79}$$

Due to the presence of trigonometric sine and cosine, for any integer $m$ there exists a sequence $\zeta_k^m \uparrow +\infty$ such that $\zeta_k^m > m^2$ for all $k \in \mathbb{N}$ and such that if $\mu = \zeta_k^m$ for some $k$ then one of the above systems admits a nontrivial solution.

Not only the above arguments prove all the statements of Theorem 3.4, but they also prove the following result.

**Theorem 7.6.** *Assume* (9) *and consider the eigenvalue problem* (33)*. Then:*

*(i) for any* $m \geq 1$ *there exists a sequence of eigenvalues* $\lambda_{k,m} \uparrow +\infty$ *such that* $\lambda_{k,m} > m^4$ *for all* $k \geq 1$*; the corresponding eigenfunctions are of the kind*

$$\left[a\cosh\left(y\sqrt{\lambda_{k,m}^{1/2} + m^2}\right) + b\sinh\left(y\sqrt{\lambda_{k,m}^{1/2} + m^2}\right)\right.$$
$$\left. + c\cos\left(y\sqrt{\lambda_{k,m}^{1/2} - m^2}\right) + d\sin\left(y\sqrt{\lambda_{k,m}^{1/2} - m^2}\right)\right]\sin(mx)$$

*for suitable constants* $a, b, c, d$*, depending on* $m$ *and* $k$*;*

*(ii) if the unique positive solution $m$ of (76) is an integer $m_* \in \mathbb{N}$, then $\lambda = m_*^4$ is an eigenvalue with corresponding eigenfunction*

$$\Big[\sigma\ell\sinh(\sqrt{2}m_*y) + (2-\sigma)\sinh(\sqrt{2}m_*\ell)\,y\Big]\ \sin(m_*x)\,;$$

*(iii) for any $m \geq 1$, there exists an eigenvalue $\lambda_m \in ((1-\sigma)^2m^4, m^4)$ with corresponding eigenfunction*

$$\Bigg[\big(\sqrt{\lambda_m}-(1-\sigma)m^2\big)\,\frac{\cosh\Big(y\sqrt{m^2+\sqrt{\lambda_m}}\Big)}{\cosh\Big(\ell\sqrt{m^2+\sqrt{\lambda_m}}\Big)} + \big(\sqrt{\lambda_m}+(1-\sigma)m^2\big)\,\frac{\cosh\Big(y\sqrt{m^2-\sqrt{\lambda_m}}\Big)}{\cosh\Big(\ell\sqrt{m^2-\sqrt{\lambda_m}}\Big)}\Bigg]\ \sin(mx)\,;$$

*(iv) for any $m \geq 1$, satisfying (67), there exists an eigenvalue $\lambda^m \in (\lambda_m, m^4)$ with corresponding eigenfunction*

$$\Bigg[\big(\sqrt{\lambda_m}-(1-\sigma)m^2\big)\,\frac{\sinh\Big(y\sqrt{m^2+\sqrt{\lambda^m}}\Big)}{\sinh\Big(\ell\sqrt{m^2+\sqrt{\lambda^m}}\Big)} + \big(\sqrt{\lambda_m}+(1-\sigma)m^2\big)\,\frac{\sinh\Big(y\sqrt{m^2-\sqrt{\lambda^m}}\Big)}{\sinh\Big(\ell\sqrt{m^2-\sqrt{\lambda^m}}\Big)}\Bigg]\ \sin(mx)\,;$$

*(v) There are no eigenvalues other than the ones characterized in (i) − (iv).*

Note that the eigenfunctions in *(iii)* are even with respect to $y$ whereas the eigenfunctions in *(iv)* are odd. In the next result we give a precise description of the first two eigenvalues when $\ell$ is small enough.

**Proposition 7.7.** *Assume (9) and consider the eigenvalue problem (33). If $\ell \leq \frac{1}{5}$ then the first two eigenvalues are simple and they coincide with the numbers $\lambda_1, \lambda_2$ defined by Lemma 7.1. Therefore,*

$$(1-\sigma)^2 < \lambda_1 < 1 < 16(1-\sigma)^2 < \lambda_2 < 16\,. \tag{80}$$

*Proof.* By (9) we know that (67) may hold only if $\ell m\sqrt{2}\coth(\ell m\sqrt{2}) > 9$. In turn, since $\ell \leq \frac{1}{5}$, this necessarily yields $m > 31$. From Theorem 7.6 we readily obtain (80). In order to prove the statement it is therefore enough to show that all the other eigenvalues found in Theorem 7.6 are larger than or equal to 16 for $\ell \leq \frac{1}{5}$.

We start by showing that for $\ell \leq \frac{1}{5}$ the numbers $\mu$ corresponding to the case $\mu > m^2$ are larger than or equal to 4. We take $m = 1$ since if $m \geq 2$ we immediately obtain $\mu > 4$ and we are done.

When $m = 1$ system (78) admits a nontrivial solution if and only if

$$(\beta^2-\sigma)(\gamma^3+(2-\sigma)\gamma)\cosh(\beta\ell)\sin(\gamma\ell)+(\gamma^2+\sigma)(\beta^3-(2-\sigma)\beta)\sinh(\beta\ell)\cos(\gamma\ell) = 0\,.$$

This may happen only if the two terms $\sin(\gamma\ell)$ and $\cos(\gamma\ell)$ have opposite sign: this yields

$$\frac{\gamma}{5} \geq \gamma\ell > \frac{\pi}{2} \implies \gamma > \frac{5\pi}{2} \implies \mu > 1 + \left(\frac{5\pi}{2}\right)^2 > 4\,.$$

Consider now the system (79) with $m=1$. Put

$$h(\mu,\sigma,\ell) := (\mu+1-\sigma)^2\sqrt{\mu-1}\sinh(\ell\sqrt{\mu+1})\cos(\ell\sqrt{\mu-1}) \\ -(\mu-1+\sigma)^2\sqrt{\mu+1}\sin(\ell\sqrt{\mu-1})\cosh(\ell\sqrt{\mu+1})$$

so that, by (77), (79) admits a nontrivial solution if and only if

$$h(\mu,\sigma,\ell)=0. \tag{81}$$

We prove that $\mu\geq 4$ by showing that

$$h(\mu,\sigma,\ell)>0 \qquad \forall\mu\in(1,4)\,,\ \sigma\in\left(0,\frac{1}{2}\right)\,,\ \ell\in\left(0,\frac{1}{5}\right]\,. \tag{82}$$

It is readily verified that $h_\sigma(\mu,\sigma,\ell)<0$ so that (82) is satisfied provided that

$$h\left(\mu,\frac{1}{2},\ell\right)>0 \qquad \forall\mu\in(1,4)\,,\ \ell\in\left(0,\frac{1}{5}\right]\,. \tag{83}$$

By differentiating we obtain

$$h_\ell\left(\mu,\frac{1}{2},\ell\right)=\frac{\mu}{2}\Big[4\sqrt{\mu^2-1}\cosh(\ell\sqrt{\mu+1})\cos(\ell\sqrt{\mu-1}) \\ -(4\mu^2-3)\sinh(\ell\sqrt{\mu+1})\sin(\ell\sqrt{\mu-1})\Big]$$

and by the inequality $s\cosh s>\sinh s$, valid for any $s>0$, we get

$$h_\ell\left(\mu,\frac{1}{2},\ell\right)>\frac{\mu}{2}\ell\sqrt{\mu-1}\cos(\ell\sqrt{\mu-1})\sinh(\ell\sqrt{\mu+1})\times \\ \times\left[\frac{4}{\ell^2}-(4\mu^2-3)\frac{\tan(\ell\sqrt{\mu-1})}{\ell\sqrt{\mu-1}}\right]\,.$$

Since the map $x\mapsto\frac{\tan x}{x}$ is increasing in $(0,\pi/2)$ and $\ell\sqrt{\mu-1}<\sqrt{3}/5$, we have that

$$h_\ell\left(\mu,\frac{1}{2},\ell\right)>\frac{\mu}{2}\ell\sqrt{\mu-1}\cos(\ell\sqrt{\mu-1})\sinh(\ell\sqrt{\mu+1})\left(100-61\cdot\frac{\tan(\sqrt{3}/5)}{\sqrt{3}/5}\right)>0$$

for $1<\mu<4$ and $\ell\leq\frac{1}{5}$ so that (83) follows and completes the proof in the case $\mu>m^2$.

By (9) the equation $\tanh(s)=\left(\frac{\sigma}{2-\sigma}\right)^2 s$ admits a unique positive solution $\overline{s}>8$. Hence, if $\mu=m^2$ is the square root of an eigenvalue, then by (76) we have

$$m=\frac{\overline{s}}{\ell\sqrt{2}}>20\sqrt{2} \implies \mu>800\,.$$

We have so shown that, in any case, $\mu>4$; hence, $\lambda>16$. □

If $\ell\leq 0.44$, (67) implies $m>14$. Moreover, numerical computations show that (82), and hence Proposition 7.7, are true for all $\ell\leq 0.44$.

## 8. **Proof of Theorem 3.6.**

In order to prove existence of solutions of (22), we perform a Galerkin-type procedure directly on the nonlinear problem (22). Uniqueness of solutions of (22) is obtained from suitable estimates coming from an energy identity. We start by proving global existence for solutions of (22).

**Lemma 8.1.** *Assume (9), let $T\in(0,+\infty)$, $\delta>0$, $f\in C^0([0,T];L^2(\Omega))$ and let $h$ satisfy (16)-(18); let $u_0\in H^2_*(\Omega)$ and $u_1\in L^2(\Omega)$. Then (22) admits a solution.*

*Proof.* We divide the proof in several steps.

**Step 1.** We construct a sequence of solutions of approximated problems in finite dimensional spaces. By Theorem 3.4 we may consider an orthogonal complete system $\{w_k\}_{k\ge1}\subset H^2_*(\Omega)$ of eigenfunctions of (33) such that $\|w_k\|_{L^2}=1$. Let $\{\lambda_k\}_{k\ge1}$ be the corresponding eigenvalues and, for any $k\ge1$, put $W_k:=\mathrm{span}\{w_1,\dots,w_k\}$. For any $k\ge1$ let

$$u_0^k:=\sum_{i=1}^k (u_0,w_i)_{L^2}w_i=\sum_{i=1}^k \lambda_i^{-1}(u_0,w_i)_{H^2_*}\,w_i \quad\text{and}\quad u_1^k=\sum_{i=1}^k (u_1,w_i)_{L^2}\,w_i$$

so that $u_0^k\to u_0$ in $H^2_*(\Omega)$ and $u_1^k\to u_1$ in $L^2(\Omega)$ as $k\to+\infty$. For any $k\ge1$ we seek a solution $u_k\in C^2([0,T];W_k)$ of the variational problem

$$\begin{cases}(u''(t),v)_{L^2}+\delta(u'(t),v)_{L^2}+(u(t),v)_{H^2_*}+(h(\cdot,\cdot,u(t)),v)_{L^2}=(f(t),v)_{L^2}\\ u(0)=u_0^k\,,\quad u'(0)=u_1^k\,.\end{cases} \tag{84}$$

for any $v\in W_k$ and $t\in(0,T)$. If we write $u_k$ in the form $u_k(t)=\sum_{i=1}^k g_i^k(t)w_i$ and we put $g^k(t):=(g_1^k(t),\dots,g_k^k(t))^T$ then the vector valued function $g^k$ solves

$$\begin{cases}(g^k(t))''+\delta(g^k(t))'+\Lambda_k g^k(t)+\Phi_k(g^k(t))=F_k(t) \qquad \forall t\in(0,T)\\ g^k(0)=((u_0,w_1)_{L^2},\dots,(u_0,w_k)_{L^2})^T\,,\quad (g^k)'(0)=((u_1,w_1)_{L^2},\dots,(u_1,w_k)_{L^2})^T\end{cases} \tag{85}$$

where $\Lambda_k:=\mathrm{diag}(\lambda_1,\dots,\lambda_k)$, $\Phi_k:\mathbb{R}^k\to\mathbb{R}^k$ is the map defined by

$$\Phi_k(y_1,\dots,y_k):=\left(\Big(h\Big(\cdot,\cdot,\sum_{j=1}^k y_jw_j\Big),w_1\Big)_{L^2},\dots,\Big(h\Big(\cdot,\cdot,\sum_{j=1}^k y_jw_j\Big),w_k\Big)_{L^2}\right)^T$$

and $F_k(t):=((f(t),w_1)_{L^2},\dots,(f(t),w_k)_{L^2})^T\in C^0([0,T];\mathbb{R}^k)$.

From (18) we deduce that $\Phi_k\in\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^k;\mathbb{R}^k)$ and hence (85) admits a unique local solution. We have shown that the function $u_k(t)=\sum_{j=1}^k g_j^k(t)w_j$ belongs to $C^2([0,\tau_k);H^2_*(\Omega))$ is a local solution in some maximal interval of continuation $[0,\tau_k)$, $\tau_k\in(0,T]$, of the problem

$$\begin{cases}u_k''(t)+\delta u_k'(t)+Lu_k(t)+P_k(h(\cdot,\cdot,u_k(t)))=P_k(f(t)) \qquad \text{for any } t\in[0,\tau_k)\\ u_k(0)=u_0^k\,,\quad u_k'(0)=u_1^k\end{cases} \tag{86}$$

where $L:H^2_*(\Omega)\to\mathcal{H}(\Omega)$ is implicitly defined by $\langle Lu,v\rangle:=(u,v)_{H^2_*}$ for any $u,v\in H^2_*(\Omega)$ and $P_k:H^2_*(\Omega)\to W_k$ is the orthogonal projection onto $W_k$.

**Step 2.** In this step we prove a uniform bound on the sequence $\{u_k\}$.

Testing (86) with $u_k'(t)$ and integrating over $(0,t)$ we obtain

$$\|u_k(t)\|^2_{H^2_*}+\|u_k'(t)\|^2_{L^2}+2\int_\Omega H(x,y,u_k(x,y,t))\,dxdy=\|u_0^k\|^2_{H^2_*}+\|u_1^k\|^2_{L^2} \tag{87}$$

$$+2\int_\Omega H(x,y,u_0^k(x,y))\,dxdy-2\delta\int_0^t\|u_k'(s)\|^2_{L^2}ds+2\int_0^t(f(s),u_k'(s))_{L^2}ds$$

for any $t\in[0,\tau_k)$.

The embedding $H^2_*(\Omega)\subset C^0(\overline{\Omega})$ yields a constant $C>0$ such that

$$\|v\|_{C^0(\overline{\Omega})}\le C\|v\|_{H^2_*(\Omega)} \qquad \text{for any } v\in H^2_*(\Omega)\,. \tag{88}$$

Since $\|u_0^k\|_{H^2_*} \le \|u_0\|_{H^2_*}$, by (17) and (88) we deduce that $\int_\Omega H(x,y,u_0^k(x,y))\,dxdy$ is bounded with respect to $k$. Hence, by Hölder and Young inequalities and the fact that $\delta > 0$, we obtain

$$\|u_k(t)\|^2_{H^2_*} + \|u_k'(t)\|^2_{L^2} \le \overline{C} \qquad \text{for any } t \in [0,\tau_k) \text{ and } k \ge 1 \tag{89}$$

for some constant $\overline{C}$ independent of $k \ge 1$ and $t \in (0,\tau_k)$. This uniform estimate shows that the solution $u_k$ is globally defined in $[0,T]$ and that the sequence $\{u_k\}$ is bounded in $C^0([0,T];H^2_*(\Omega)) \cap C^1([0,T];L^2(\Omega))$.

<u>**Step 3.**</u> We prove that $\{u_k\}$ admits a strongly convergent subsequence in $C^0([0,T];H^2_*(\Omega)) \cap C^1([0,T];L^2(\Omega))$.

In what follows, any subsequence of $\{u_k\}$ will be denoted in the same way. By (89) we deduce that $\{u_k\}$ is bounded and equicontinuous in $C^0([0,T];L^2(\Omega))$ and moreover for any $t \in [0,T]$, $\{u_k(t)\}$ is precompact in $L^2(\Omega)$ thanks to the compact embedding $H^2_*(\Omega) \subset L^2(\Omega)$.

By applying the Ascoli-Arzelà Theorem to the sequence $\{u_k\}$ we deduce that, up to subsequences, there exists $u \in C^0([0,T];L^2(\Omega))$ such that $u_k \to u$ strongly in $C^0([0,T];L^2(\Omega))$.

For any $n > m \ge 1$ define $u_{n,m} := u_n - u_m$, $u_0^{n,m} = u_0^n - u_0^m$, $u_1^{n,m} := u_1^n - u_1^m$ so that

$$\begin{cases} u''_{n,m} + \delta u'_{n,m} + Lu_{n,m} + P_n\Big(h(\cdot,\cdot,u_n)\Big) - P_m\Big(h(\cdot,\cdot,u_m)\Big) = (P_n - P_m)(f) \\ u_{n,m}(0) = u_0^{n,m}, \quad u'_{n,m}(0) = u_1^{n,m}. \end{cases} \tag{90}$$

By compactness it follows that, up to subsequences, $P_n\Big(h(\cdot,\cdot,u_n)\Big) \to h(\cdot,\cdot,u)$ in $C^0([0,T];L^2(\Omega))$ as $n \to +\infty$. Moreover $P_n f \to f$ in $C^0([0,T];L^2(\Omega))$ as $n \to +\infty$. Hence, as $n,m \to \infty$,

$$\Psi_{n,m} := -P_n\Big(h(\cdot,\cdot,u_n)\Big) + P_m\Big(h(\cdot,\cdot,u_m)\Big) + (P_n - P_m)f \to 0 \ \text{ in } C^0([0,T];L^2(\Omega)).$$

Testing (90) with $u'_{n,m}$ and integrating over $(0,t)$, up to enlarging $\overline{n}$, we obtain

$$\begin{aligned} &\|u'_{n,m}(t)\|^2_{L^2} + \|u_{n,m}(t)\|^2_{H^2_*} \\ &\quad = \|u_1^{n,m}\|^2_{L^2} + \|u_0^{n,m}\|^2_{H^2_*} - 2\delta \int_0^t \|u'_{n,m}(s)\|^2_{L^2} ds + 2\int_0^t (\Psi_{n,m}(s), u'_{n,m}(s))_{L^2} ds \\ &\quad \le \|u_1^{n,m}\|^2_{L^2} + \|u_0^{n,m}\|^2_{H^2_*} + \tfrac{T}{2\delta}\|\Psi_{n,m}\|^2_{C^0([0,T];L^2)} \to 0 \ \text{ as } n,m \to \infty \end{aligned}$$

for any $t \in [0,T]$. This shows that $\{u_k\}$ is a Cauchy sequence in the space $C^0([0,T];H^2_*(\Omega)) \cap C^1([0,T];L^2(\Omega))$. But we have seen before that $u_k \to u$ in $C^0([0,T];L^2(\Omega))$ so that $u$ belongs to the same space and, up to subsequences,

$$u_k \to u \qquad \text{in } C^0([0,T];H^2_*(\Omega)) \cap C^1([0,T];L^2(\Omega)) \quad \text{as } k \to +\infty,$$

thus completing the proof of the claim.

<u>**Step 4.**</u> We take the limit in (86) and we prove the existence of a solution of (22).

Let $\varphi \in C_c^\infty(0,T)$, let $v \in H^2_*(\Omega)$ and let $v_k = P_k v$. Then by (86) we have that for any $k \geq 1$

$$-\int_0^T (u_k'(t), v_k)_{L^2}\, \varphi'(t)\, dt$$
$$= \int_0^T \left[-\delta(u_k'(t), v_k)_{L^2} - (u_k(t), v_k)_{H^2_*} - (h(\cdot,\cdot,u_k(t)), v_k)_{L^2} + (f(t), v_k)_{L^2}\right]\varphi(t)\, dt\,.$$

Letting $k \to +\infty$ we obtain that $u'' \in C^0([0,T]; \mathcal{H}(\Omega))$ and $u$ solves the equation $u'' = -Lu - \delta u' - h(\cdot,\cdot,u) + f$. Moreover $u_0^k = u_k(0) \to u(0)$ in $H^2_*(\Omega)$ and $u_1^k = u_k'(0) \to u'(0)$ in $L^2(\Omega)$ so that $u(0) = u_0$ and $u'(0) = u_1$. We proved that $u$ is a solution of (22). □

In the next lemma we provide an energy identity for the nonlinear problem (22) and, by exploiting it, we show uniqueness of the solution.

**Lemma 8.2.** *Assume (9), let $T \in (0,+\infty)$, $\delta > 0$, $f \in C^0([0,T]; L^2(\Omega))$ and let $h$ satisfy (16)-(18); let $u_0 \in H^2_*(\Omega)$ and $u_1 \in L^2(\Omega)$. Then (22) admits a unique solution $u$ which, moreover, satisfies the following identity*

$$\|u'(t)\|^2_{L^2} + \|u(t)\|^2_{H^2_*} + 2\delta \int_0^t \|u'(s)\|^2_{L^2} ds + 2\int_\Omega H(x,y,u(x,y,t))\, dxdy \tag{91}$$
$$= \|u_1\|^2_{L^2} + \|u_0\|^2_{H^2_*} + 2\int_\Omega H(x,y,u_0(x,y))\, dxdy + 2\int_0^t (f(s), u'(s))_{L^2} ds$$

*for any $t \in [0,T]$.*

*Proof.* We first consider the case where $h \equiv 0$: the existence of a solution of (22) is a consequence of Lemma 8.1. Let us prove uniqueness. Take two solutions $u_1, u_2$ of (22) and their difference $v = u_1 - u_2$. Consider the function $\widetilde{v}(t) = \int_0^t v(s)\, ds$ so that $\widetilde{v} \in W^{1,\infty}(0,T; H^2_*(\Omega)) \cap W^{2,\infty}(0,T; L^2(\Omega))$ and

$$\begin{cases} \widetilde{v}''(t) + \delta \widetilde{v}'(t) + L\widetilde{v} = 0 \\ \widetilde{v}(0) = \widetilde{v}'(0) = 0\,. \end{cases}$$

We use $\widetilde{v}' \in L^\infty(0,T; H^2_*(\Omega))$ as a test function to obtain after integration over $(0,t)$

$$\|\widetilde{v}'(t)\|^2_{L^2} + \|\widetilde{v}(t)\|^2_{H^2_*} = -2\delta \int_0^t \|\widetilde{v}'(s)\|^2_{L^2} ds \leq 0\,.$$

from which it immediately follows that $\widetilde{v} = 0$ and hence $v = 0$. This proves uniqueness of the solution of (22) under the assumption $h \equiv 0$: let $u$ be the unique solution of (22). By the proof of Lemma 8.1 we infer that the sequence $\{u_k\}$ introduced in the Galerkin procedure, converges itself, without extracting a subsequence, strongly to $u$ in $C^0([0,T]; H^2_*(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. Applying (87) in our situation we obtain

$$\|u_k(t)\|^2_{H^2_*} + \|u_k'(t)\|^2_{L^2} + 2\delta \int_0^t \|u_k'(s)\|^2_{L^2} ds = \|u_0^k\|^2_{H^2_*} + \|u_1^k\|^2_{L^2} + 2\int_0^t (f(s), u_k'(s))_{L^2} ds$$

for any $t \in [0,T]$. Letting $k \to +\infty$ and exploiting the strong convergence of $\{u_k\}$, we infer that

$$\|u'(t)\|^2_{L^2} + \|u(t)\|^2_{H^2_*} + 2\delta \int_0^t \|u'(s)\|^2_{L^2} ds = \|u_1\|^2_{L^2} + \|u_0\|^2_{H^2_*} + 2\int_0^t (f(s), u'(s))_{L^2} ds \tag{92}$$

for all $t \in [0,T]$.

Consider now the case where $h \not\equiv 0$ and let us first prove (91). Since $u \in C^0([0,T];H^2_*(\Omega))$ and $H^2_*(\Omega) \subset C^0(\overline{\Omega})$ then $u \in C^0(\overline{\Omega} \times [0,T])$ and hence by (16)-(18) we deduce that $h(\cdot,\cdot,u) \in C^0([0,T];L^2(\Omega))$. Then by (92) we obtain

$$\|u'(t)\|^2_{L^2} + \|u(t)\|^2_{H^2_*} + 2\delta \int_0^t \|u'(s)\|^2_{L^2} ds \tag{93}$$
$$= \|u_1\|^2_{L^2} + \|u_0\|^2_{H^2_*} + 2\int_0^t (h(\cdot,\cdot,u(s)),u'(s))_{L^2} ds + 2\int_0^t (f(s),u'(s))_{L^2} ds$$

It remains to show that

$$\int_0^t (h(\cdot,\cdot,u(s)),u'(s))_{L^2} ds = \int_\Omega H(x,y,u(x,y,t))\,dxdy - \int_\Omega H(x,y,u_0(x,y))\,dxdy\,. \tag{94}$$

For proving this it is sufficient to construct the sequence $u_k(t) := \sum_{j=1}^k (u(t),w_j)_{L^2} w_j$ where $\{w_j\}$ is the orthogonal complete system introduced in the proof of Lemma 8.1. For any $k \ge 1$ we have that $u_k \in C^2([0,T];H^2_*(\Omega))$ and hence (94) trivially holds with $u_k$ in place of $u$. Letting $k \to +\infty$ and exploiting the fact that $u_k \to u$ in $C^0([0,T];H^2_*(\Omega)) \cap C^1([0,T];L^2(\Omega))$ the identity (94) also holds for $u$.

Finally we prove uniqueness of solutions of (22). Let $u,v$ two solutions of (22) and define $w := u - v$. Then $w$ solves the problem

$$\begin{cases} w''(t) + \delta w'(t) + Lw(t) = h(\cdot,\cdot,v(t)) - h(\cdot,\cdot,u(t)) & \text{in } [0,T] \\ w(0) = 0\,, \quad w'(0) = 0\,. \end{cases}$$

Let $I \subset \mathbb{R}$ be an interval satisfying $\|u\|_{C^0(\overline{\Omega}\times[0,T])}, \|v\|_{C^0(\overline{\Omega}\times[0,T])} \in I$. Applying (93) to $w$ and using (18) we obtain

$$\|w'(t)\|^2_{L^2} + \|w(t)\|^2_{H^2_*} \le \sqrt{2\ell\pi}\, L_I C \left( \int_0^t \|w(s)\|^2_{H^2_*} ds + \int_0^t \|w'(s)\|^2_{L^2} ds \right)$$

where $C$ is the constant defined in (88). Standard Gronwall estimates then implies $w \equiv 0$ thus completing the proof of the lemma. □

In the last part of this section we consider problem (22) with $f \in L^2(\Omega)$ independent of $t$. We want to study the behavior of the solution $u(\cdot,t)$ of (22) as $t \to +\infty$: its global existence and uniqueness is an easy consequence of Lemmas 8.1 and 8.2. Consider the energy function

$$\mathcal{E}_u(t) := \frac{1}{2}\|u'(t)\|^2_{L^2} + \frac{1}{2}\|u(t)\|^2_{H^2_*} - (f,u(t))_{L^2} + \int_\Omega H(x,y,u(x,y,t))\,dxdy\,.$$

By (91) we have that

$$\mathcal{E}_u(t) = \frac{1}{2}\|u_1\|^2_{L^2} + \frac{1}{2}\|u_0\|^2_{H^2_*} + \int_\Omega H(x,y,u_0(x,y))\,dxdy - (f,u_0)_{L^2} - \delta\int_0^t \|u'(s)\|^2_{L^2} ds$$

so that $\mathcal{E}_u$ is nonincreasing in $[0,+\infty)$ and in particular it is bounded from above.

On the other hand by Hölder and Young inequalities, continuous embedding $H^2_*(\Omega) \subset L^2(\Omega)$, (17) it follows the existence of two constants $C_1, C_2 > 0$ such that

$$C_1(\|u'(t)\|^2_{L^2} + \|u(t)\|^2_{H^2_*}) \le \mathcal{E}_u(t) + C_2\|f\|^2_{L^2} \qquad \text{for any } t > 0\,.$$

Then

$$\sup_{t\ge 0}\left(\|u(t)\|_{H^2_*}+\|u'(t)\|_{L^2}+\|u''(t)\|_{\mathcal{H}}\right)<+\infty\,. \tag{95}$$

This bound allows us to study the long-time behavior of the global solution.

**Lemma 8.3.** *Assume (9), let $f\in L^2(\Omega)$, $h$ satisfy (16)-(18) and $\delta>0$; let $u_0\in H^2_*(\Omega)$ and $u_1\in L^2(\Omega)$. Then the unique global solution $u$ of (22) satisfies:*

$$u(t)\to\overline{u}\quad \text{in } H^2_*(\Omega)\quad \text{and}\quad u'(t)\to 0\quad \text{in } L^2(\Omega)\quad \text{as } t\to+\infty$$

*where $\overline{u}$ is the unique solution of the stationary problem (20).*

*Proof.* By (93) and boundedness of $\mathcal{E}_u$ we have that $\int_0^{+\infty}\|u'(s)\|^2_{L^2}ds<+\infty$ and hence there exists a sequence $t_n\uparrow+\infty$ such that

$$\lim_{n\to+\infty}\int_{t_n}^{t_n+1}\|u'(s)\|^2_{L^2}ds=0\quad\text{and}\quad \lim_{n\to+\infty}\left(\|u'(t_n)\|_{L^2}+\|u'(t_n+1)\|_{L^2}\right)=0\,. \tag{96}$$

For any $v\in H^2_*(\Omega)$ we then have

$$\lim_{n\to+\infty}\int_{t_n}^{t_n+1}\langle u''(s),v\rangle\,ds=\lim_{n\to+\infty}\left[(u'(t_n+1),v)_{L^2}-(u'(t_n),v)_{L^2}\right]=0\,. \tag{97}$$

Note that (96) and (97) yield

$$\lim_{n\to+\infty}\int_{t_n}^{t_n+1}\Big[\langle u''(s),v\rangle+\delta(u'(s),v)_{L^2}\Big]\,ds=0\qquad\forall v\in H^2_*(\Omega)$$

which, in turn, implies that

$$\forall v\in H^2_*(\Omega)\quad \exists t_n^v\in(t_n,t_n+1)\quad\text{such that}\quad \lim_{n\to+\infty}\Big[\langle u''(t_n^v),v\rangle+\delta(u'(t_n^v),v)_{L^2}\Big]=0\,.$$

Fix $v\in H^2_*(\Omega)$ and note that, by (95), the sequence $\{u(t_n^v)\}$ is bounded in $H^2_*(\Omega)$ so that

$$u(t_n^v)\rightharpoonup\overline{u}_v\in H^2_*(\Omega)$$

up to a subsequence. In turn, by compact embedding, $u(t_n^v)\to\overline{u}_v$ in $L^2(\Omega)$. Take a function $w\in H^2_*(\Omega)$ such that $w\neq v$ and consider the corresponding sequence $\{t_n^w\}$. Then $u(t_n^w)\rightharpoonup\overline{u}_w$ in $H^2_*(\Omega)$ and $u(t_n^w)\to\overline{u}_w$ in $L^2(\Omega)$. By (93), Hölder inequality and Fubini-Tonelli Theorem, we obtain

$$\begin{aligned}\|u(t_n^v)-u(t_n^w)\|^2_{L^2}&=\int_\Omega\left|\int_{t_n^v}^{t_n^w}u'(s)ds\right|^2\\&\le|t_n^w-t_n^v|\int_{t_n^v}^{t_n^w}\|u'(s)\|^2_{L^2}ds\le\frac{1}{\delta}\,|\mathcal{E}_u(t_n^v)-\mathcal{E}_u(t_n^w)|\to 0\end{aligned}$$

as $n\to+\infty$, showing that the limit $\overline{u}_v$ is independent of $v$, let us simply denote it by $\overline{u}$. Summarizing, we have proved that

$$\begin{aligned}&\forall v\in H^2_*(\Omega)\qquad (\overline{u},v)_{H^2_*}+(h(\cdot,\cdot,\overline{u}),v)_{L^2}-(f,v)_{L^2}\\&=\lim_{n\to+\infty}\Big[\langle u''(t_n^v),v\rangle+\delta(u'(t_n^v),v)_{L^2}+(u(t_n^v),v)_{H^2_*}+(h(\cdot,\cdot,u(t_n^v)),v)_{L^2}-(f,v)_{L^2}\Big]=0\,.\end{aligned}$$

This shows that $\overline{u}$ is the unique solution to (20).

By subtracting the weak form of (20) from (36) we obtain

$$\langle u''(t),v\rangle+\delta(u'(t),v)_{L^2}+(u(t)-\overline{u},v)_{H^2_*}+(h(\cdot,\cdot,u(t))-h(\cdot,\cdot,\overline{u}),v)_{L^2}=0$$

for all $v \in H^2_*(\Omega)$. In fact, (95) enables us to take $v = \overline{v}(t) := u(t) - \overline{u}$ as test function; then, integrating by parts over $(t_n, t_n + 1)$ and using (17) and (96), we infer that

$$\lim_{n\to+\infty} \int_{t_n}^{t_n+1} \|u(s) - \overline{u}\|^2_{H^2_*}\, ds = 0\ .$$

By combining this fact with (96) we infer that there exists $\bar{t}_n \in [t_n, t_n+1]$ such that

$$u(\bar{t}_n) \to \overline{u} \quad \text{in } H^2_*(\Omega)\,, \qquad u'(\bar{t}_n) \to 0 \quad \text{in } L^2(\Omega)$$

and therefore

$$\lim_{n\to+\infty} \mathcal{E}_u(\bar{t}_n) = \mathbb{E}_T(\overline{u}) = I := \min_{v\in H^2_*(\Omega)} \mathbb{E}_T(v)\,.$$

But $t \mapsto \mathcal{E}_u(t)$ is decreasing so that, in fact, we also have $\lim_{t\to+\infty} \mathcal{E}_u(t) = I$.

Moreover $\mathcal{E}_u(t) = \frac{1}{2}\|u'(t)\|^2_{L^2} + \mathbb{E}_T(u(t)) \geq \frac{1}{2}\|u'(t)\|^2_{L^2} + I$ and passing to the limit as $t \to +\infty$ we infer

$$I = \lim_{t\to+\infty} \mathcal{E}_u(t) \geq I + \limsup_{t\to+\infty} \frac{1}{2}\|u'(t)\|^2_{L^2}\,.$$

This proves that $u'(t) \to 0$ in $L^2(\Omega)$ as $t \to +\infty$. In turn, this implies that

$$\lim_{t\to+\infty} \mathbb{E}_T(u(t)) = \lim_{t\to+\infty} \mathcal{E}_u(t) - \lim_{t\to+\infty} \frac{1}{2}\|u'(t)\|^2_{L^2} = I = \mathbb{E}_T(\overline{u})\,.$$

Direct methods of calculus of variations then allow to conclude that $u(t) \to \overline{u}$ in $H^2_*(\Omega)$, on the whole flow. □

The proof of Theorem 3.6 follows from Lemmas 8.1, 8.2, 8.3.

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