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Multibump periodic motions of an infinite lattice of particles

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1 Introduction

We consider one dimensional lattices with infinitely many particles having nearest neighbor interaction: the interaction we consider is repulsive-attractive, that is, with a force being repulsive for small displacements and attractive for large displacements; periodic motions for purely attractive interactions can also be found, see [2, 3]. The state of the system at time t is represented by a sequence $q(t) = \{q_i(t)\}$ $(i \in \mathbb{Z})$, where $q_i(t)$ is the state of the i-th particle: let Φ_i denote the potential of the interaction between the i-th and the $(i+1)$-th particle (whose displacement is $q_i - q_{i+1}$), then the equation governing the state of $q_i(t)$ reads

$$(1) \qquad \ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}) \; ;$$

if we define the potential of the system $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ by $\Phi(q) := \sum_{i \in \mathbf{Z}} \Phi_i(q_i - q_{i+1})$, then equations (1) can be written in a vectorial form

$$(2) \qquad \ddot{q} = -\Phi'(q) \; .$$

Since the problem is clearly invariant with respect to translations of the whole system, the natural framework to study this problem is the Hilbert space

$$H := \left\{ q : S^1 \to \mathbb{R}^\infty ; \int_0^T q_0(t) dt = 0 \; , \right.$$
$$\left. \sum_{i \in \mathbf{Z}} \int_0^T \left[(\dot{q}_i(t))^2 + (q_i(t) - q_{i+1}(t))^2 \right] dt < \infty \right\}$$

endowed by the scalar product

$$(p, q) := \sum_{i \in \mathbf{Z}} \int_0^T \left[\dot{p}_i(t) \dot{q}_i(t) + (p_i(t) - p_{i+1}(t))(q_i(t) - q_{i+1}(t)) \right] dt \; ;$$

we consider the functional $J : H \to \mathbb{R}$ defined by

$$J(q) := \frac{1}{2}\int_0^T |\dot{q}(t)|^2 dt - \int_0^T \Phi(q(t))dt$$

whose critical points are periodic solutions of (2), see [8]. Assume that $\forall i \in \mathbb{Z}$

i) $\Phi_i(t) = -\alpha_i t^2 + V_i(t)$, $\alpha_i > 0$

ii) $\exists \delta > 0$ such that $V_i'(t)t \geq (2+\delta)V_i(t) \geq 0$, $\forall t \in \mathbb{R}$

iii) $\lim_{|t|\to+\infty} V_i(t) = +\infty$

iv) $V_i \in C^{1,1}_{loc}$.

We also require a spatial periodicity in order to have the functional J invariant for translations of indices (see formula (5) below):

v) $\exists m \in \mathbb{N}$ such that $\Phi_{i+m} \equiv \Phi_i$, $\forall i \in \mathbb{Z}$.

Note that conditions i), ii), iii) and iv) imply that $\forall i \in \mathbb{Z}$

- $\Phi_i(0) = 0$
- V_i is superquadratic at the origin and at infinity
- Φ_i has a strong local maximum in 0
- Φ_i admits at least two local minima
- denote by θ_{i_j} the nonzero stationary points of Φ_i; then $\max_j \Phi_i(\theta_{i_j}) < 0$.

Under the above assumptions it is proved in [2] that the functional J satisfies the geometric assumptions of the mountain pass Theorem and that, if b is the infmax level, J admits a critical level not higher than b; the motion corresponding to such a critical level are non-constant if the period is large enough.

The main result we prove in this paper is Theorem 2 which can be stated approximately as follows:

System (2) admits infinitely many periodic motions up to its symmetries; furthermore if $\exists\bar{\alpha} > 0$ such that the set of critical points at level lower than $b + \bar{\alpha}$ is $\mathbb{Z}$ times a compact set (that is, the countable disjoint union of a compact set $\mathcal{K}$ with its translations), then there exist infinitely many periodic non-constant motions of multibump type, i.e. having most of their (finite) energy concentrated in a finite number of disjoint regions of the lattice.

The variational structure of our problem possesses some analogies with that of the problem of homoclinic orbits for non-autonomous Hamiltonian systems with periodic potentials. Indeed in both cases, the functional being invariant under the action of a non-compact group $\mathbb{Z}$, the Palais-Smale compactness condition does not hold at any level; on the other hand, the same action may allow to prove the existence of solutions of multibump type by sticking suitable solutions. This structure was discovered by Séré [10] for the problem of homoclinics studied in [5]. Since then, some progress in this field has been obtained in several subsequent papers, and different techniques have been developed, see [4, 6, 11]; similar arguments are also used in [7] for the search of homoclinic type solutions of semilinear elliptic PDE's.

In all these papers the multibump solutions are obtained under the assumption that the mountain pass sublevel does not contain "too many" critical points:

it is required that either such critical points are finite, or they are countable. These assumptions do not cover the case where the functional is invariant under the action of a continuous compact group as well: since our problem is indeed invariant under an S^1-action, we extended the result to the more general case of assumption (A) in Theorem 2. To explain more precisely our result some preliminaries are in order.

2 The main result

For all $a, b \in \mathbb{R}$ such that $b \leq a$ we denote

$$J^a = J^{-1}((-\infty, a]) , \qquad J_b = J^{-1}([b, +\infty)) , \qquad J_b^a = J^{-1}([b, a]) ,$$

and

$$K = \{q \in H \setminus \{0\};\ J'(q) = 0\}$$

$$K^a = K \cap J^a , \qquad K_b = K \cap J_b , \qquad K_b^a = K^a \cap K_b .$$

In [2] it is proved that the functional J is well defined on H, $J \in C^1(H, \mathbb{R})$ and that for large period T (say $T > \bar{T}$) it admits nontrivial critical points, which are periodic solutions of (2). More precisely it is proved that $\exists \tilde{q} \in J^0 \setminus \{0\}$ and that 0 is a strong local minimum for J; therefore, if $B(q, r)$ denotes the H−ball centered in q with radius r,

$$\text{(3)} \qquad \exists \lambda > 0 \qquad \text{such that} \qquad K \bigcap B(0, \lambda) = \emptyset$$

and the functional has a mountain-pass level given by

$$\text{(4)} \qquad b := \inf_{\gamma \in \Gamma} \max_{q \in \gamma([0,1])} J(q) > 0 ,$$

where

$$\Gamma := \{\gamma \in C([0, 1], H);\ \gamma(0) = 0,\ \gamma(1) = \tilde{q}\} ,$$

hence a Palais-Smale sequence exists at level b by a standard procedure. From now on we will always assume the condition $T > \bar{T}$ to be satisfied in order to guarantee the nontriviality of the critical points.

The functional J is invariant under both a representation of $\mathbb{Z}$ which we denote $*$ following Séré [10], and a representation of S^1 which we denote by Ω. These representations are defined by:

$$(*) : \mathbb{Z} \times H \to H , \qquad (k, q) \mapsto k * q \quad \text{where} \quad (k * q(t))_i = q_{i+km}(t) - \frac{1}{T} \int_0^T q_{km}$$

and

$$\Omega : S^1 \times H \to H , \qquad [\Omega(\tau, q(t))]_i = q_i(t + \tau) .$$

Let $l \in \mathbb{N}$, $\bar{k} = (k^1, \ldots, k^l) \in \mathbb{Z}^l$ and $\bar{q} = (q^1, \ldots, q^l) \in H^l$, we define

$$\bar{k} * \bar{q} := \sum_{j=1}^{l} k^j * q^j \ ;$$

for all sequences of l-tuples of integers $\bar{k}_n = (k_n^1, \ldots, k_n^l)$, by $\bar{k}_n \to \infty$ we mean that for $i \neq j$ $|k_n^i - k_n^j| \to \infty$ as $n \to \infty$.

The previous notations are useful to define the multibump solutions:

Definition 1 *A critical point q is a multibump solution of kind $(l, \rho) \in \mathbb{N} \times \mathbb{R}^+$ if $\exists \bar{k} = (k^1, \ldots, k^l) \in \mathbb{Z}^l$ and $\bar{q} = (q^1, \ldots, q^l)$, $q^j \in K$, such that $q \in B(\bar{k} * \bar{q}, \rho)$.*

Due to the $\mathbb{Z}$ and S^1 invariance, the functional J does not satisfy the Palais-Smale condition; nonetheless the following representation theorem holds:

Theorem 1 *Assume i), ii), iii), iv) and v); let $q^{(n)} \in H$ be a Palais-Smale sequence for J at level $b > 0$. Then there exist a subsequence still denoted by $q^{(n)}$, l points $q^j \in K$ $(j = 1, \ldots, l)$, and l sequences of integers k_n^j $(n \in \mathbb{N},\ l \geq 1)$, such that if $\bar{q} = (q^1, \ldots, q^l)$ and $\bar{k}_n = (k_n^1, \ldots, k_n^l)$, then*

$$\left\| q^{(n)} - \bar{k}_n * \bar{q} \right\| \to 0 \ ,$$

$$\textstyle\sum_{j=1}^{l} J(q^j) = b \ ,$$

$$\bar{k}_n \to \infty \ .$$

Remark. The proof of this result and an upper estimate for l are given in [2].

Note that even if $l > 1$, we cannot state that there exist l different critical points, as it could be $q^i = q^j$ for $i \neq j$.

The aim of this paper is to prove a multiplicity theorem for the existence of critical points of J; we will prove the following result:

Theorem 2 *Assume i), ii), iii), iv) and v); then $K/(S^1 \times \mathbb{Z})$ contains infinitely many points. More precisely if b is defined as in (4) and*

(A) $\exists \bar{\alpha} > 0$ and a compact set $\mathcal{K} \subset H$ such that $K^{b+\bar{\alpha}} = \bigcup_{k \in \mathbf{Z}} k * \mathcal{K}$,

then b is a critical level and $\forall n \in \mathbb{N}$ and $\forall a, \rho > 0$ system (2) admits infinitely many multibump solutions of kind (n, ρ) in $J_{nb-a}^{nb+a}/(S^1 \times \mathbb{Z})$.

3 Proof of the main result

We refer to the next section for the proofs of the results stated here.

To prove Theorem 2, from now on we assume that (A) holds. For the sake of simplicity we prove the theorem in the case $n = 2$: the proof can be extended without major changes if $n > 2$.

We introduce some other notations: for all $q \in K^{b+\bar{\alpha}}$ we denote by $\mathcal{K}q$ the compact subset of $K^{b+\bar{\alpha}}$ to which q belongs; for all $k \in \mathbb{Z}$ and reals $b > a > 0$ we define

$$\mathcal{V}_a(k) := \{u \in H;\ d(u, k * \mathcal{K}) < a\}\ ,\quad \mathcal{V}_a := \mathcal{V}_a(0)\ ,\quad \mathcal{U}_a = \bigcup_{k \in \mathbf{Z}} \mathcal{V}_a(k)$$

and

$$\mathcal{T}(k, a, b) := \{u \in H;\ d(u, k * \mathcal{K}) \in (a, b)\}\ ;$$

note that for all $q \in K^{b+\bar{\alpha}}$ there exists an integer $\tilde{k}$ such that $\mathcal{K}q = \tilde{k} * \mathcal{K}$.

Finally, for all $l \in \mathbb{N}$, $\bar{k} = (k^1, \dots, k^l) \in \mathbb{Z}^l$ and reals $b > a > 0$ we define $\bar{k} * \mathcal{K} := k^1 * \mathcal{K} + \dots + k^l * \mathcal{K}$ and $\mathcal{T}(\bar{k}, a, b) := \{u \in H;\ d(u, \bar{k} * \mathcal{K}) \in (a, b)\}$.

Remark. For all $q \in H$ we denote by $\Omega q = \bigcup_\theta \{\Omega_\theta q\}$ the orbit of such point under the representation Ω: an example of particular interest of the compact set $\mathcal{K}q$ is Ωq when $q \in K$; therefore if (A) does not hold, then $K^{b+\alpha}/\mathbb{Z}$ consists of infinitely many critical orbits for all $\alpha > 0$.

The first lemma is a consequence of Theorem 1: it gives a lower bound for $\|J'\|$ in a suitable set, and it will replace in some sense the standard Palais-Smale condition at level lower than $b + \bar{\alpha}$:

Lemma 1 *(a) $\exists r_0 > 0$ such that if $q^1, q^2 \in K^{b+\bar{\alpha}}$ and $\mathcal{K}q^1 \cap \mathcal{K}q^2 = \emptyset$, then*

$$d(\mathcal{K}q^1, \mathcal{K}q^2) := \min_{(p^1,p^2) \in \mathcal{K}q^1 \times \mathcal{K}q^2} \|p^1 - p^2\| \geq 3r_0\ .$$

(b) $\forall \rho \in (0, r_0)$, $\mu_\rho := \inf\{\|J'(q)\|;\ q \in \bigcup_{k \in \mathbf{Z}} \mathcal{T}(k, \rho, r_0)\} > 0$.
(c) For all Palais-Smale sequence $q^{(n)}$ at level $c < b + \bar{\alpha}$ satisfying $\|q^{(n+1)} - q^{(n)}\| \to 0$, there exists $k \in \mathbb{Z}$ such that $d(q^{(n)}, k * \mathcal{K}) \to 0$ (and hence $q^{(n)}$ admits a converging subsequence).

Remark. The statement (c) in the previous lemma is a weaker version of the Palais-Smale condition which is sufficient to prove a deformation lemma (see Lemma 2 below). A similar idea was used in [5] with the condition they called $(\overline{PS})$: in the particular case they considered $(\overline{PS})$ is also weaker than the classical Palais-Smale condition since they had isolated critical points, but for a more general problem $(\overline{PS})$ is not a weaker condition.

By the next results we give a "local" mountain pass characterization of the critical level b. Choose $0 < \delta < r_0/3$ and set $\mu = \inf\{\|J'(q)\|;\ q \in \bigcup_{k \in \mathbf{Z}} \mathcal{T}(k, \delta, r_0)\}$; $\mu > 0$ by Lemma 1 (b) and the following deformation lemma holds:

Lemma 2 *For all positive $\varepsilon < \min(\bar{\alpha}, \delta\mu/2)$ there exists $\eta : [0, 2\varepsilon] \times H \to H$ such that:*

(a) $\eta(s, u) = u$ for all $u \notin J_{b-\bar{\alpha}}^{b+\bar{\alpha}}$ and all $u \in \mathcal{U}_\delta$
(b) $\frac{d}{ds} J(\eta(s, u)) \leq 0$
(c) $\eta(2\varepsilon, J^{b+\varepsilon} \setminus \mathcal{U}_{3\delta}) \subset J^{b-\varepsilon}$
(d) $\eta(2\varepsilon, J^{b+\varepsilon}) \subset J^{b-\varepsilon} \bigcup \mathcal{U}_{3\delta}$.

The previous lemma is used to prove the following:

Lemma 3 *Let $r \in (0, r_0)$. There exist $\omega_r > 0$ such that for all $\omega \in (0, \omega_r)$ there exists a path $\gamma_{r,\omega} \in \Gamma$ and M $(0 < M < \infty)$ integers $\{k_i\}$ satisfying:*
(a) $\gamma_{r,\omega}([0,1]) \cap J_{b-2\omega} \subset \bigcup_{i=1}^{M} \mathscr{Z}_r(k_i)$
(b) $\gamma_{r,\omega}([0,1]) \subset J^{b+\omega}$.

We apply the previous result with $r = r_0/4$: we choose $\omega \leq \min(\omega_r, r\mu_{\frac{r}{2}}/3))$ ($\mu_{\frac{r}{2}}$ is defined in Lemma 1) and consider the path $\gamma_{r,\omega}$ given by Lemma 3; note that ω is chosen arbitrarily and this will be useful to prove that the statement of Theorem 2 holds $\forall a > 0$. We can isolate the "interesting" part of $\gamma_{r,\omega}$ as stated in

Lemma 4 *There exist $t_0, t_1 \in (0,1)$ such that, up to a translation of indices:*
(a) $\gamma_{r,\omega}([t_0, t_1]) \subset \overline{\mathscr{Z}_r}$
(b) $\gamma_{r,\omega}(t_i) \in \partial\mathscr{Z}_r$, $i = 0, 1$
(c) The set $F = (J_b \cap \mathscr{Z}_{r_0})$ disconnects $\mathscr{Z}_{r_0}$ and the endpoints $\gamma_{r,\omega}(t_0)$ and $\gamma_{r,\omega}(t_1)$ belong to different connected components of $\mathscr{Z}_{r_0}$.

We can rescale t on the path $\gamma_{r,\omega}$ given by the previous result: for $t \in [0,1]$ let $\hat{\gamma}(t) = \gamma_{r,\omega}(t_0 + t(t_1 - t_0))$; by Lemma 3 we have

$$\hat{\gamma}(\{0,1\}) \subset J^{b-2\omega} \text{ and } \hat{\gamma}([0,1]) \subset J^{b+\omega} \; ; \tag{6}$$

consider the class of paths

$$\Gamma' := \{\gamma \in C([0,1], \mathscr{Z}_r);\ \gamma(i) = \hat{\gamma}(i) \text{ for } i = 0,1\} \; ,$$

by Lemma 4(c) we have

$$\inf_{\gamma \in \Gamma'} \max_{t \in [0,1]} J(\gamma(t)) \geq b \; . \tag{7}$$

Let $\gamma_j : [0,T] \to H$ $(j = 1,2)$ (we omit the subscript k) be the paths defined by

$$[\gamma_1(t)]_i = \begin{cases} [k * \hat{\gamma}(t)]_i \text{ if } i > 0 \\ 0 \text{ if } i \leq 0 \end{cases} \tag{8}$$

and

$$[\gamma_2(t)]_i = \begin{cases} [(-k) * \hat{\gamma}(t)]_i \text{ if } i < 0 \\ 0 \text{ if } i \geq 0 \; ; \end{cases} \tag{9}$$

note that by the continuity of J we have

$$\exists \xi > 0 \quad \text{such that} \quad B(\hat{\gamma}(i), \xi) \subset J^{b-\frac{3}{2}\omega} \quad (i = 0,1), \tag{10}$$

therefore if k is sufficiently large, the endpoints of the truncated paths γ_j $(j = 1,2)$ are still "good" endpoints for the mountain pass characterization of b. We can therefore state that the following classes of paths:

$$\Gamma_1' := \{\gamma \in C([0,1], \mathscr{Z}_r(k));\ \gamma(i) = \gamma_1(i) \text{ for } i = 0,1\}$$

and

$$\Gamma_2' := \{\gamma \in C([0,1], \mathscr{Z}_r(-k));\ \gamma(i) = \gamma_2(i) \text{ for } i = 0, 1\}$$

have the same mountain pass points of the class Γ', up to translation of indices of km or $-km$.

Using again the continuity of J, the fact that $\gamma_j([0,1])$ (with $j = 1, 2$) is compact in H and (6), we have $\gamma_j([0,1]) \subset J^{b+\omega_k}$, with $\omega_k \to \omega$ as $k \to \infty$; we choose k so large that $\omega_k < \frac{5}{4}\omega$, i.e.

$$\max_{[0,1]} J(\gamma_j) < b + \frac{5}{4}\omega\ , \qquad j = 1, 2\ . \tag{11}$$

Define for all $k \in \mathbb{N}$:

$$\bar{B} = \{q \in H;\ d(q, (-k) * \mathscr{K} + k * \mathscr{K}) \le 4r = r_0\}\ ,$$

$$\tilde{B} = \{q \in H;\ d(q, (-k) * \mathscr{K} + k * \mathscr{K}) \le 2r\}\ ,$$

$$Q = [0, 1]^2\ ,$$

$$\tilde{\gamma} : Q \to \tilde{B},\ (t_1, t_2) \mapsto \tilde{\gamma}(t_1, t_2) = \gamma_1(t_1) + \gamma_2(t_2)\ ,$$

$$\Gamma^2 = \{\gamma \in C(Q, \bar{B});\ \gamma|_{\partial Q} = \tilde{\gamma}|_{\partial Q}\}\ ,$$

and for all $q \in H$,

$$F_i(q) = \int_0^T |\dot{q}_i|^2 + \int_0^T |q_i - q_{i+1}|^2\ ; \tag{12}$$

each functional F_i measures in some sense the amount of the norm around the particle labeled by i, indeed $\sum_i F_i(q) = \|q\|^2$ for all $q \in H$. By the properties of Φ_i, there exists a constant σ such that if $|t|$ is small enough, say $|t| < \bar{t}$, then $-\Phi_i(t) \le \sigma t^2$.

By the same arguments used to prove (11), we can choose k so large that (without restrictions we assume k even)

$$\sum_{|i| \le k/2+2} F_i(q) < r^2 \qquad \forall q \in (-k) * \mathscr{K} + k * \mathscr{K}\ ;$$

for such k we also have

$$\sum_{|i| \le k/2+1} F_i(q) < 17r^2 \qquad \forall q \in \bar{B}\ , \tag{13}$$

i.e. the particles with low index only possess a small portion of the total energy of the system.

By the embedding $H^1 \subset L^\infty$ we can also choose k so large that

$$\|q_i - q_{i+1}\|_\infty \le \bar{t} \qquad \forall q \in \bar{B},\ \forall i,\ |i| \le k/2 + 2\ .$$

Denote by k_ω an integer such that for all $k \ge k_\omega$ the above statements are fulfilled: this integer will be used in the final part of the proof.

We wish to prove the existence of a critical point in $\bar{B}$ by means of a suitable variational characterization. The following lemma will be used in order to prove some estimates on the functional level:

Lemma 5 *There exists a constant $\Lambda > 0$ which only depends on r, such that if k is the same integer as in (8) and (9), then for any map $\gamma \in \Gamma^2$, there exists a map $\gamma' \in \Gamma^2$ such that for all $\tau \in Q$:*
(a) $\gamma'(\tau)_i = \gamma(\tau)_i$ if $|i| > k/2$
(b) $[\gamma'(\tau)]_0 \equiv 0$
(c) $J(\gamma'(\tau)) \leq J(\gamma(\tau)) + \Lambda/k$.

As a consequence, the following estimate holds for all $\gamma \in \Gamma^2$:

$$J(\gamma(\tau)) \geq J(\gamma^+(\tau)) + J(\gamma^-(\tau)) - \Lambda/k \qquad \forall \tau \in Q\ , \tag{14}$$

where we set

$$[\gamma^+(\tau)]_i = \begin{cases} [\gamma(\tau)]_i \text{ if } i > 0 \\ 0 \text{ if } i \leq 0 \end{cases} \tag{15}$$

and

$$[\gamma^-(\tau)]_i = \begin{cases} [\gamma(\tau)]_i \text{ if } i < 0 \\ 0 \text{ if } i \geq 0\ . \end{cases} \tag{16}$$

Define

$$\beta = \inf_{\gamma \in \Gamma^2} \max_{q \in \gamma(Q)} J(q)\ ;$$

to give a mountain-pass structure to level β (which will be the desired critical level) we prove that the functional is low on $\tilde{\gamma}(\partial Q)$:

Lemma 6 *If k is large enough, then* $\max J(\tilde{\gamma}(\partial Q)) < 2b - \frac{\omega}{4}$.

Exploiting ideas of both [4] and [10] we give an estimate of β depending on the choice of ω:

Lemma 7 *If k is large enough, then $2b - \frac{\omega}{6} \leq \beta \leq 2b + \frac{5}{2}\omega$.*

The next result states that in $\bar{B} \setminus \tilde{B}$ the functional is "steep"; this will be useful to build the deformation of Lemma 10.

Lemma 8 *If k is large enough, then for all $q \in \bar{B} \setminus \tilde{B}$, $\|J'(q)\| > \mu_{\frac{\varepsilon}{2}}/2$.*

To build a deformation in $\bar{B}$ we also need a local Palais-Smale condition: to prove it we make use of Theorem 1 and of the fact that the "movement of a bump" implies a change of norm of at least λ.

Lemma 9 *The Palais-Smale condition holds in $\bar{B}$, i.e. every sequence $\{q^{(n)}\} \subset \bar{B}$ such that $J'(q^{(n)}) \to 0$ is precompact.*

Finally, we can define the desired deformation

Lemma 10 *Let* $U \subset \bar{B}$ *be a neighborhood of* $K_{2b-\omega/5}^{2b+5\omega/2} \cap \bar{B}$. *There exist* $\eta \in C(\tilde{B} \times [0,1], \bar{B})$ *such that*

$$\eta(J^{2b+5\omega/2} \setminus U, 1) \subset J^{2b-\omega/5}$$

$$\eta(q,t) = q \text{ if } q \notin J_{2b-\omega/5}^{2b+5\omega/2}$$

$$\eta(\cdot,t) \text{ is equivariant with respect to } \Omega.$$

Theorem 2 is now proved if we show that when (A) holds and k is large enough, then there exists a critical point in $\bar{B}$ at level β; choose k larger than k_ω and so large to satisfy the statements of Lemmas 6, 7 and 8. If such critical point does not exist, we can use Lemma 10 to deform the path $\tilde{\gamma}$. By the properties of such deformation, as $\max J(\tilde{\gamma}) < 2b + \frac{5}{2}\omega$, we get

$$\eta(\tilde{\gamma}(\cdot), 1) \in \Gamma^2$$

and

$$\max_{\eta(\tilde{\gamma}(Q),1)} < 2b - \omega/5 ,$$

which contradicts the definition of β. Theorem 2 follows now for all a because of the arbitrariness of ω (we chose it before Lemma 4) and of Lemma 7.

4 Proofs of the lemmas

Proof of Lemma 1. (a) Set

$$c := \lim_{n\to\infty} \min_{(p^1,p^2)\in\mathcal{K}^2} \|p^1 - n * p^2\| = 2 \min_{p\in\mathcal{K}} \|p\| \geq 2\lambda > 0 .$$

Then the inequality $d(\mathcal{K}, n * \mathcal{K}) < c/3$ holds only for a finite number of integers and (a) is proved by the compactness of $\mathcal{K}$; it is not restrictive to assume $r_0 < \lambda/2$, λ given by (3).

(b) Suppose the contrary: then by the invariance of the functional there exists a sequence $\{q^{(n)}\} \subset H$ such that $q^{(n)} \in \mathcal{T}(0, \rho, r_0)$ and $J'(q^{(n)}) \to 0$. As $\mathcal{T}$ is bounded, $\{q^{(n)}\}$ is a PS sequence and by Theorem 1 there exist l critical points $q^1, \ldots, q^l$ and a sequence $\{\bar{k}_n\} \subset \mathbb{Z}^l$, $\bar{k}_n \to \infty$, such that, up to a subsequence

$$\|q^{(n)} - \bar{k}_n * \bar{q}\| \to 0 ,$$

that is, for large enough n

$$\bar{k}_n * \bar{q} \in \mathcal{T}(0, \rho/2, 2r_0) ,$$

but this is not possible, indeed as $\bar{k}_n \to \infty$ we have $k_n^i \to k^i \in \mathbb{Z} \cup \{+\infty\} \cup \{-\infty\}$ for $n \to \infty$ and $k^i \in \mathbb{Z}$ for at most one value of i. The assertion follows bearing in mind that $\|q^i\| \geq \lambda > 2r_0$ and that each bump of the PS sequence carries at least an amount of norm λ.

(c) By Theorem 1 there exists a subsequence $q^{(n_k)}$ such that $\|q^{(n_k)} - \bar{h}_k * \bar{u}\| \to 0$; if $\|q^{(n+1)} - q^{(n)}\| \to 0$ and (c) is false, then there exists $\rho \in (0, r_0)$ such that for all k there exists m_k and $q^{(m_k)} \in \mathcal{T}(\bar{h}_k, \rho, r_0)$. This is impossible because for all $\rho \in (0, r_0)$, $l \in \mathbb{N}$ and for all sequences $\{\bar{k}_n\} \subset \mathbb{Z}^l$ such that $\bar{k}_n \to \infty$ for $n \to \infty$ we have

$$\liminf_{n\to\infty}\{\inf\{\|J'(q)\|;\ q \in \mathcal{T}(\bar{k}_n, \rho, r_0)\}\} > 0 \,. \tag{17}$$

Indeed, it is easy to prove (17) by contradiction: assume that there exists a sequence $\{q^{(n)}\}$ with $q^{(n)} \in \mathcal{T}(\bar{k}_n, \rho, r_0)$, $J'(q^{(n)}) \to 0$ and $\bar{k}_n \to \infty$. By Theorem 1 (up to a subsequence) there exist l' critical points $(u^1, \dots, u^{l'}) =: \bar{u}$ and a sequence $\bar{h}_n$ such that $\bar{h}_n \to \infty$ and $\|q^{(n)} - \bar{h}_n * \bar{u}\| \to 0$, hence for n large enough $\bar{h}_n * \bar{u} \in \mathcal{T}(\bar{k}_n, \rho/2, 2r_0)$. This is not possible because of part (a) and $2r_0 < \lambda$. □

Proof of Lemma 2. Let $h : \mathbb{R} \to [0, 1]$ be any Lipschitz continuous function satisfying $h(s) = 1$ if $s \in [b - \varepsilon, b + \varepsilon]$ and $h(s) = 0$ if $s \notin [b - \bar{\alpha}, b + \bar{\alpha}]$; let $\psi : H \to [0, 1]$ be another Lipschitz continuous function such that $\psi(u) = 1$ if $u \notin \mathcal{U}_{2\delta}$ and $\psi(u) = 0$ if $u \in \mathcal{U}_\delta$. Let η be the flow defined by

$$\begin{cases} \frac{d\eta}{ds} = -h(J(\eta))\psi(\eta)\frac{J'(\eta)}{\|J'(\eta)\|^2} \\ \eta(0, u) = u \end{cases} \tag{18}$$

Obviously (18) admits a unique solution η in a suitable right neighborhood of 0; assume for the moment that for all $u \in H$ such a neighborhood is the whole half line $[0, +\infty)$: properties (a) and (b) are straightforward.

Take any $u \in J^{b+\varepsilon} \setminus \mathcal{U}_{3\delta}$, then $\eta(s, u) \notin \mathcal{U}_{2\delta}$ for all $s \in [0, 2\varepsilon]$ (i.e. $h(J(\eta)) \equiv 1$), indeed

$$\|\eta(2\varepsilon, u) - \eta(0, u)\| = \left\| \int_0^{2\varepsilon} h(J(\eta))\psi(\eta)\frac{J'(\eta)}{\|J'(\eta)\|^2} ds \right\| \le \frac{2\varepsilon}{\mu} < \delta \,.$$

By contradiction, if $\eta(2\varepsilon, u) \notin J^{b-\varepsilon}$, then for all $s \in [0, 2\varepsilon]$ we have $J(\eta(s, u)) > b - \varepsilon$ and therefore

$$J(\eta(2\varepsilon, u)) - J(u) = \int_0^{2\varepsilon} \frac{d}{ds} J(\eta(s, u)) ds = \int_0^{2\varepsilon} J'(\eta(s, u))[\eta'(s, u)] ds = -2\varepsilon \,,$$

from which we get the contradiction $J(\eta(2\varepsilon, u)) \le b - \varepsilon$ and (c) is proved.

To prove (d) take $u \in J^{b+\varepsilon}$ and note that if $\forall s \in [0, 2\varepsilon]$ we have $\eta(s, u) \notin \mathcal{U}_{2\delta}$ then by the previous reasoning $\eta(2\varepsilon, u) \in J^{b-\varepsilon}$. So, assume that there exists $\bar{s} \in [0, 2\varepsilon]$ such that $\eta(\bar{s}, u) \in \mathcal{U}_{2\delta}$, then $\eta(s, u) \in \mathcal{U}_{3\delta}$ for all $s \in [0, 2\varepsilon]$ because in $\mathcal{U}_{3\delta} \setminus \mathcal{U}_\delta$ we have

$$\|\eta'\| \le \mu^{-1} < \frac{\delta}{2\varepsilon} \qquad \text{and} \qquad \|\eta(s, u) - \eta(\bar{s}, u)\| \le \max \|\eta'\| \cdot |s - \bar{s}| < \delta \,.$$

To complete the proof of the lemma we still have to prove the existence of the solution η of (18) for all initial data $u \in H$ and for all "times" $s \in [0, +\infty)$: by contradiction, assume there exists $u \in J_{b-\bar{\alpha}}^{b+\bar{\alpha}}$ such that the corresponding flow η is defined only on $[0, S)$, $S < +\infty$: then $\limsup_{s \to S^-} \|\eta'(s, u)\| = +\infty$. Consider the following Cauchy problem:

$$\text{(19)} \qquad \begin{cases} \frac{d\varphi}{dz} = -h(J(\varphi))\psi(\varphi)\frac{J'(\varphi)}{\|J'(\varphi)\|} \\ \varphi(0, u) = u \end{cases} :$$

since $\|\varphi'\| \leq 1$ its solution $\varphi(z)$ is defined for all $z \in \mathbb{R}^+$. Note also that

$$\eta(s, u) = \varphi\left(\int_0^s \frac{1}{\|J'(\varphi(t, u))\|} dt, u\right)$$

$$\text{(20)} \qquad \text{and} \qquad \varphi(z, u) = \eta\left(\int_0^z \|J'(\varphi(t, u))\| dt, u\right) ,$$

indeed it is easy to see that the above functions satisfy (18) and (19) respectively; by (20) we see that the curves defined by η and φ for a given u are equal up to a reparametrization and therefore we have $\lim_{z \to \infty} \varphi(z, u) = \lim_{s \to S^-} \eta(s, u)$ which gives

$$\int_0^{+\infty} \|J'(\varphi(t, u))\| dt = S < +\infty ;$$

from the previous equality we easily infer that there exists a sequence $z_n \to +\infty$ such that $|z_n - z_{n-1}| \to 0$ and $\|J'(\varphi(z_n, u))\| \to 0$, hence, by the Lipschitz continuity of φ we get $\|\varphi(z_n, u) - \varphi(z_{n-1}, u)\| \to 0$. By the properties of h we know that $\forall s \in [0, S)$ we have $\eta(s, u) \in J_{b-\bar{\alpha}}^{b+\bar{\alpha}}$, hence, by (b) $J(\eta(z_n, u)) \to c \in [b - \bar{\alpha}, b + \bar{\alpha}]$: the sequence $\eta(z_n, u)$ has all the properties of Lemma 1 (c), therefore for n large enough (say when $z_n > S'$), $\eta(z_n, u) \in \mathcal{U}_\delta$, hence, $\psi \equiv 0$ which contradicts $S < +\infty$. □

Proof of Lemma 3. Choose $\bar{\gamma} \in \Gamma$ satisfying $\max J(\bar{\gamma}) < b + \omega$. As $\text{Im}\bar{\gamma}$ is compact there exist M integers $k^1, \dots, k^M$ ($M > 0$ otherwise the deformation η of Lemma 2 would lead to a contradiction of the definition of b) such that

$$\text{Im}\bar{\gamma} \bigcap \left[\bigcup_{k \in \mathbf{Z}} \mathcal{Z}_r(k)\right] = \text{Im}\bar{\gamma} \bigcap \left[\bigcup_{i=1}^{M} \mathcal{Z}_r(k^i)\right] .$$

If $\varepsilon > 0$ satisfies the hypothesis of Lemma 2 and $\omega_r = \varepsilon/2$, then there exists a deformation η whose properties imply that $\gamma(t) := \eta(2\varepsilon, \bar{\gamma}(t))$ satisfies (a) and (b). □

Proof of Lemma 4. By a compactness argument we infer that there exists an integer l and $2l$ real numbers $0 < t_1^I < t_1^O \leq t_2^I < t_2^O \leq \ldots \leq t_l^I < t_l^O < 1$ such that if we denote by $\mathcal{G}$ the union of the connected components of $\gamma_{r,\omega}([0, 1]) \cap J_{b-2\omega}$ which intersect J_b, we have

$$\mathcal{G} = \bigcup_{j=1}^{l} \gamma_{r,\omega}([t_j^I, t_j^O]) ,$$

that is, the path $\gamma_{r,\omega}$ enters and exits l times the set $\bigcup_i \mathcal{Z}_r(k^i)$ and t_j^I, t_j^O are the values assumed by the parameter on $\partial\mathcal{Z}_r(k^i)$.

By the variational characterization of b and by the properties of the path $\gamma_{r,\omega}$ we also infer that there exists $i \in (1, \dots, l)$ such that $J_b \cap \mathcal{Z}_{r_0}(k^i)$ disconnects $\mathcal{Z}_{r_0}(k^i)$ (denote by B_0 and B_1 the disconnected parts) and there exists $j \in (1, \dots, l)$ such that $\gamma_{r,\omega}(t_j^I) \in B_0$ and $\gamma_{r,\omega}(t_j^O) \in B_1$. It is not restrictive to assume $k^i = 0$ and we satisfy easily points (a), (b) and (c) by defining $t_0 = t_j^I$ and $t_1 = t_j^O$. □

Proof of Lemma 5. For all $m \in \{-k/2 - 1, \dots, -2\}$ and $n \in \{2, \dots, k/2 + 1\}$ we define a map $P_{mn} : \bar{B} \to [0, 34r^2]^2$ by $P_{mn}(q) = (F_m(q) + F_{m+1}(q), F_n(q) + F_{n+1}(q))$. Choose any $\gamma \in \Gamma^2$ and let

$$A_{mn} = \gamma^{-1}(P_{mn}^{-1}[[0, 68r^2/k)^2]) \ : \tag{21}$$

by (13) the collection $\{A_{mn}\}$ is a finite open cover of Q; let $\Delta_{mn}(\tau) \in H$ be defined for all $\tau \in Q$ by

$$[\Delta_{mn}(\tau)]_i = \begin{cases} 0 & \text{if } i \notin \{m, \dots, n\} \text{ or } \tau \notin A_{mn} \\ \gamma(\tau)_i - \frac{1}{T}\int_0^T \gamma(\tau)_i(t)dt & \text{otherwise.} \end{cases}$$

We first prove that there exists $\Lambda > 0$ such that for all couples (m, n) and for all $\tau \in A_{mn}$ we have $J(\gamma(\tau)) \geq J(\gamma(\tau) - \Delta_{mn}(\tau)) - \Lambda/k$; by the definition of Δ_{mn} it suffices to prove that

$$\begin{aligned} \textstyle\sum_{i=m}^{n} \int_0^T \dot{q}_i^2 - \sum_{i=m-1}^{n} \int_0^T \Phi_i(q_i - q_{i+1}) \\ \geq -\int_0^T \Phi_{m-1}(q_{m-1} - \bar{q}_m) - \int_0^T \Phi_n(\bar{q}_n - q_{n+1}) \\ - \textstyle\sum_{i=m}^{n-1} \int_0^T \Phi_i(\bar{q}_i - \bar{q}_{i+1}) - \Lambda/k \ , \end{aligned}$$

where for all i we set $\bar{q}_i = \frac{1}{T}\int_0^T q_i$.

By the convexity of $-\Phi_i$ for small values of its argument (recall that $\|q_i - q_{i+1}\|_\infty \leq \bar{t}$) and by Jensens' inequality we have

$$-\sum_{i=m}^{n-1} \int_0^T \Phi_i(q_i - q_{i+1}) \geq -T\sum_{i=m}^{n-1} \Phi_i(\bar{q}_i - \bar{q}_{i+1}) = -\sum_{i=m}^{n-1} \int_0^T \Phi_i(\bar{q}_i - \bar{q}_{i+1})$$

and finally by (21)

$$\left| \int_0^T \Phi_{m-1}(q_{m-1} - \bar{q}_m) + \int_0^T \Phi_n(\bar{q}_n - q_{n+1}) \right.$$
$$\left. - \int_0^T \Phi_{m-1}(q_{m-1} - q_m) - \int_0^T \Phi_n(q_n - q_{n+1}) \right|$$
$$\le \sigma \left[\int_0^T (q_{m-1} - \bar{q}_m)^2 + \int_0^T (\bar{q}_n - q_{n+1})^2 + \int_0^T (q_{m-1} - q_m)^2 + \int_0^T (q_n - q_{n+1})^2 \right]$$
$$\le 2\sigma \left[\int_0^T (q_m - \bar{q}_m)^2 + \int_0^T (q_n - \bar{q}_n)^2 + 2\int_0^T (q_{m-1} - q_m)^2 + 2\int_0^T (q_n - q_{n+1})^2 \right]$$
$$\le c(F_m(q) + F_{m+1}(q) + F_n(q) + F_{n+1}(q)) \le \Lambda/k \ ,$$

where we used the embedding $H^1 \subset L^2$.

To complete the proof choose any partition of the unity $\{p_{mn}\}$ subjected to $\{A_{mn}\}$ and let $\Delta(\tau) = \sum_{mn} p_{mn}(\tau)\Delta_{mn}(\tau)$. Finally define $\gamma'(\tau) = \gamma(\tau) - \Delta(\tau)$: properties (a) and (b) are straightforward (taking into account that $\int_0^T q_0 dt = 0$ for all $q \in H$), while property (c) is deduced from the fact that in a neighborhood of zero the functional is convex, and therefore

$$J(\gamma(\tau) - \textstyle\sum_{mn} p_{mn}(\tau)\Delta_{mn}(\tau)) = J(\textstyle\sum_{mn} p_{mn}(\tau)(\gamma(\tau) - \Delta_{mn}(\tau)))$$
$$\le \textstyle\sum_{mn} p_{mn}(\tau) J(\gamma(\tau) - \Delta_{mn}(\tau)) \le \textstyle\sum_{mn} p_{mn} J(\gamma(\tau)) + \frac{\Lambda}{k} = J(\gamma(\tau)) + \frac{\Lambda}{k} \ .$$

□

Proof of Lemma 6. Let $\tau = (\tau_1, \tau_2) \in \partial Q$. By (8) and (9) γ_1 and γ_2 have disjoint support; by (10), (11) and the definition of $\tilde{\gamma}$ we have

$$J(\tilde{\gamma}(\tau)) = J(\gamma_1(\tau_1) + \gamma_2(\tau_2)) = J(\gamma_1(\tau_1)) + J(\gamma_2(\tau_2)) < \left(b - \frac{3}{2}\omega\right) + \left(b + \frac{5}{4}\omega\right) \ .$$

□

Proof of Lemma 7. To prove the lower estimate, choose $k > 6\Lambda/\omega$; we have to prove that $\max_{\tau\in Q} J(\gamma(\tau)) \ge 2b - \omega/6$ for all $\gamma \in \Gamma^2$: take any such γ and consider the functions γ^+ and γ^- defined in (15) and (16). Let $\varphi_1 \in C([0,1]; Q)$ be such that $\varphi_1(j) \in \{j\} \times [0,1]$, $j = 0, 1$, then $\gamma^+(\varphi_1(\cdot)) \in \Gamma'$; hence, by (7)

$$\max_{t\in[0,1]} J[\gamma^+(\varphi_1(t))] \ge b \ . \tag{22}$$

We also define $Q_b = \{\tau \in Q,\ J(\gamma^+(\tau)) \ge b\}$; by (22) we know that Q_b separates $\{1\} \times [0,1]$ and $\{0\} \times [0,1]$. Let Q^+ denote the connected component of $Q \setminus Q_b$ containing $\{0\} \times [0,1]$ and let

$$\sigma_1(\tau) = \begin{cases} d(\tau, Q_b) \text{ if } \tau \in Q^+ \\ -d(\tau, Q_b) \text{ if } \tau \in Q \setminus Q^+ \ ; \end{cases}$$

obviously $\sigma_1 \in C(Q, \mathbb{R})$, $\sigma_1 > 0$ on $\{0\} \times [0,1]$ and $\sigma_1 < 0$ on $\{1\} \times [0,1]$. In a similar way we define φ_2 instead of φ_1, and σ_2 in a dual way.

By a theorem of Miranda [9], we know that $\exists\bar{\tau} = (\tau_1, \tau_2) \in Q$ such that $\sigma_1(\bar{\tau}) = \sigma_2(\bar{\tau}) = 0$. By (22) and (14) we have

$$\max_{\tau\in Q} J(\gamma(\tau)) \geq J(\gamma(\bar{\tau})) \geq J[\gamma^+(\varphi_1(\tau_1))] + J[\gamma^-(\varphi_2(\tau_2))] - \frac{\omega}{6} > 2b - \frac{\omega}{6} .$$

The upper estimate can be obtained using (11):

$$\beta \leq \max_{\tau\in Q} J(\bar{\gamma}(\tau)) = \max_{\tau\in Q} J(\gamma_1(\tau_1) + \gamma_2(\tau_2)) = 2\max_{t,j} J(\gamma_j(t)) < 2b + \frac{5}{2}\omega .$$

□

Proof of Lemma 8. Given any point $q \in \bar{B} \setminus \tilde{B}$, we split it into the sum $q^+ + q^-$ such that at least one between q^+ and q^- is in $\mathcal{T}(k, r/2, r_0)$ for a certain k; then by Lemma 1 we get $\|J'(q^\pm)\| \geq \mu_{\frac{r}{2}}$.

More precisely, let F_i be as in (12); choose k so large that

$$\begin{aligned} &\max_{q\in(-k)*\mathcal{K}} \textstyle\sum_{i>-k/2} F_i(q) < \frac{r^2}{16} , \\ &\max_{q\in k*\mathcal{K}} \textstyle\sum_{i<k/2} F_i(q) < \frac{r^2}{16} \end{aligned} \tag{23}$$

and $k > \frac{102r^2}{\mu_{\frac{r}{2}}}$.

Let $q \in \bar{B} \setminus \tilde{B}$ and set $G_i(q) = F_{i-1}(q) + F_i(q) + F_{i+1}(q)$: by (13) there exists an integer j, $|j| \leq k/2$, such that

$$G_j(q) \leq 51r^2/k < \frac{\mu_{\frac{r}{2}}}{2} . \tag{24}$$

Define q^+ by

$$q_i^+ = \begin{cases} q_i & \text{if } i > j \\ \bar{q}_j & \text{if } i \leq j \end{cases}$$

if $j \leq 0$ and by

$$q_i^+ = \begin{cases} q_i - \bar{q}_j\ . & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

if $j > 0$. Then define $q^- = q - q^+$; this cumbersome definition is necessary because we need to have $q^+, q^- \in H$ and therefore we need to have $\int_0^T q_0^+ = \int_0^T q_0^- = 0$. For all $p \in (-k) * \mathcal{K} + k * \mathcal{K}$ define p^+ and p^- in the same way as q^+ and q^-; for such q and p we have

$$2r \leq \|q - p\| = \|q^+ + q^- - p^+ - p^-\| \leq \|q^+ - p^+\| + \|q^- - p^-\| ,$$

therefore either $\|q^+ - p^+\| \geq r$ or $\|q^- - p^-\| \geq r$. Assume the first inequality holds; by our choice of p there exist $p^1 \in (-k) * \mathcal{K}$ and $p^2 \in k * \mathcal{K}$ such that $p = p^1 + p^2$: by (23) $\|p^- - p^1\| < r/2$ and $\|p^+ - p^2\| < r/2$. Then

$q^+ \in \mathcal{T}(k, r/2, r_0)$ and by Lemma 1 $\|J'(q^+)\| \geq \mu_{\frac{r}{2}} > 0$, which implies that for all $\varepsilon > 0$ $\exists p^\varepsilon \in H$, $\|p^\varepsilon\| = 1$ such that

$$J'(q^+)[p^\varepsilon] > \mu_{\frac{r}{2}} - \varepsilon \, . \tag{25}$$

Assume that $j \leq 0$ (if $j > 0$ the same result can be obtained with slight modifications). By the definition of J' it is not restrictive to assume $p_i^\varepsilon = \bar{p}_j^\varepsilon$ for all $i < j$: therefore

$$\begin{aligned} |J'(q^+)[p^\varepsilon] - J'(q)[p^\varepsilon]| \leq & \left| \int_0^T \Phi_j'(\bar{q}_j - q_{j+1})(p_j^\varepsilon - p_{j+1}^\varepsilon) \right| \\ & + \left| \int_0^T \Phi_{j-1}'(q_{j-1} - q_j)(\bar{p}_{j-1}^\varepsilon - p_j^\varepsilon) \right| \\ & + \left| \int_0^T \Phi_j'(q_j - q_{j+1})(p_j^\varepsilon - p_{j+1}^\varepsilon) \right| \, . \end{aligned} \tag{26}$$

By Hölder inequality, the embedding $H^1 \subset L^2$, (24) and (26) we get $|J'(q^+)[p^\varepsilon] - J'(q)[p^\varepsilon]| \leq \frac{\mu_{\frac{r}{2}}}{2}$; by the arbitrariness of ε and (25) we infer the result. □

Proof of Lemma 9. Let $q^{(n)}$ satisfy the hypothesis. By Theorem 1 there exists k_n^i such that

$$\left\| q^{(n)} - \sum_{i=1}^{l} k_n^i * q^i \right\| \to 0 \, ,$$

where q^i are critical points for J. The assertion follows because $r_0 < \lambda/2$ (see Proof of Lemma 1), and because $\bar{B}$ is a neighborhood of radius r_0 of a compact set. □

Proof of Lemma 10. The proof is standard (see e.g. [8] Lemma 6.5), except for an important detail: the Palais-Smale condition (which is needed in order to give a lower estimate of $\|J'\|$) does not hold in the whole space but only in $\bar{B}$, then we have to ensure that the deformation does not bring any point in $\tilde{B}$ out of $\bar{B}$; this information is provided by Lemma 8, which yields a lower estimate for $\|J'\|$ in $\bar{B} \setminus \tilde{B}$ according to which any point traveling from $\tilde{B}$ to $\partial\bar{B}$ along the flow associated to $-J'$ must decrease its level (as assigned by J) of at least 3ω. □

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