

Minimization properties of Hill's orbits and applications to some N -body problems *

by

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ABSTRACT. – We consider the periodic problem for a class of planar N -body systems in Celestial Mechanics. Our goal is to give a variational characterization of the Hill's (retrograde) orbits as minima of the action functional under some geometrical and topological constraints. The method developed here also turns out to be useful in the study of the full problem with N primaries each having at most two satellites. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – On considère le problème périodique pour une certaine classe de systèmes de N -corps en Mécanique Céleste. Notre but est de donner une caractérisation variationnelle des orbites (rétrogrades) de Hill comme minima de la fonctionnelle d'action sous certaines contraintes géométriques et topologiques. La méthode ici développée est également

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utile pour l'étude du problème complet avec N corps primaires ayant chacun au plus deux satellites. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. INTRODUCTION

This paper concerns the periodic problem for a class of planar N -body systems in Celestial Mechanics. We mainly deal with a class of 3-body problems, though our method may be applied in some particular cases of more general N -body systems. We consider the case of two major bodies and a satellite and we seek orbits when the whole system revolves with a frequency θ , while the third mass rotates around one of the other bodies with a T -periodic motion. In the restricted case, when the motion of the satellite takes place close to one of the primaries, this problem is known as Hill's problem and a simple argument based on the inverse function theorem shows the existence of periodic orbits (in the rotating frame) for small values of the quantity θT , see [12]. Our goal is to give a variational characterization of the Hill's (retrograde) orbits as minima of the action functional under some geometrical and topological constraints. The method developed here also turns out to be useful in the study of the (full) problem with N primaries each having at most two satellites.

The periodic problem for both restricted and full N -body systems has such a long story that it is impossible to give an extensive bibliography here; we refer the reader to the classical texts [12–14]. In the last two decades, a new method for finding periodic motions has been provided by the use of variational techniques, see, e.g., the book [3] and the references therein. The first variational characterization of the periodic solutions of the 2-body problem goes back to a paper by Gordon [11], where it is shown that the periodic orbits are local minima of the action under the topological constraint of non triviality of the rotation index. This constraint is used to overcome the lack of coercivity of the action integral in the space of periodic functions. However, from the functional point of view, the minimization problem (even in a local sense) is degenerate, that is it possesses a continuum of solutions. This is due to the fact that every solution to the associated differential equation is periodic, provided its energy is negative, and the period (and the associated action

value) depends only on the energy. In particular all the periodic orbits, including the degenerate ellipses, where the two bodies collide, share the same variational characterization; in other words, it is impossible to distinguish them by looking at their functional levels. On the other hand, even though they can be extended as global solutions to the differential equations, the motions of collision type are periodic only in a mathematical sense.

Starting from the subsequent paper by Gordon [10], different kinds of assumptions have been considered in order to rule out the collision solutions, in the case of 2-body and N -body problems. In the case of Keplerian interaction potentials, a fundamental remark is that the minimization problem may become non degenerate by imposing further symmetry constraints in the space of periodic functions. This fact has been first pointed out in the 2-body case in [9], and then exploited in order to obtain noncollision periodic orbits in various situations [4,6–8,16,17]. In the planar N -body problem this idea has led to associate the boundary condition $x_i(t + \tau) = R_\varphi x_i(t)$ with the equations system, where $x_i(t)$ is the position of the i -mass, and R_φ a rotation of the plane of angle φ and τ is the period of the mutual distances between the bodies see [4]. In this setting, it can be shown (see [5]) that the simple minimization argument in the space of symmetric functions leads to the *relative equilibrium motions* that are well known periodic solutions to the system [1]. In order to avoid such a triviality, Bessi and Coti Zelati [4] imposed a further topological constraint, that is one of the body couples has a non trivial rotation index in the rotating frame. However, though they ruled out the simultaneous collisions of the whole system, they were unable to avoid periodic solutions having partial collisions. The aim of this paper is to go further in the analysis in [4], and prove, by level estimates, that the minimum of the action functional under both symmetry and topological constraints is free of any collision. We first deal with the 3-body problem in the restricted and full cases: to this end, we exploit the variational structure of the problem, looking for minimizers of the action integral among the functions which satisfy the above mentioned constraints. More precisely, we require the system coordinates $X(t) = (x_1(t), x_2(t), x_3(t))$ to satisfy $x_i(t + T) = R_{\theta T} x_i(t)$, where $R_{\theta T}$ is a rotation of angle θT in the plane; in addition we require the motion of $R_{-\theta t} x_3(t)$ to have a negative rotation index with respect to the first body, that is, to be retrograde. Finally, we use similar arguments to study more general problems with more major bodies each one having at most two retrograde satellites.

2. STATEMENT OF THE RESULTS

Throughout this paper we denote by G the universal gravitation constant, by B_1, \dots, B_n ($n = 3$ or $n = 4$) the bodies of the problem we consider and by m_1, \dots, m_n their respective masses; their positions in the plane \mathbb{R}^2 are described by the n functions $x_i = x_i(t)$ ($i = 1, \dots, n$).

DEFINITION 1. – *We say that $(\bar{x}_1, \dots, \bar{x}_n)$ is a noncollision orbit on the interval $[0, T]$ if $\bar{x}_i(t) \neq \bar{x}_j(t)$ for all $1 \leq i < j \leq n$ and all $t \in [0, T]$.*

We first deal with a restricted 3-body problem $\{B_1, B_2, B_3\}$, which we briefly describe. Consider for a moment only the system $\{B_1, B_2\}$: if we assume the center of mass to be fixed in the origin and we set the period to be $2\pi/\vartheta$, then a solution of the following equations of motion

$$-m_i \ddot{x}_i = Gm_1 m_2 \frac{x_i - x_j}{|x_i - x_j|^3}, \quad x_i(0) = x_i\left(\frac{2\pi}{\vartheta}\right), \quad i, j = 1, 2, \quad i \neq j,$$

is given by

$$x_1(t) = -R_1(\cos \vartheta t, \sin \vartheta t), \quad x_2(t) = R_2(\cos \vartheta t, \sin \vartheta t), \quad (2.1)$$

where

$$\begin{aligned} R_1 &= m_2 G^{1/3} (m_1 + m_2)^{-2/3} \vartheta^{-2/3}, \\ R_2 &= m_1 G^{1/3} (m_1 + m_2)^{-2/3} \vartheta^{-2/3}. \end{aligned}$$

The restricted problem we consider consists in assigning $x_1(t), x_2(t)$ as in (2.1), while the motion of B_3 satisfies the equation

$$-m_3 \ddot{x}_3 = V'(x_3), \quad (2.2)$$

where

$$V(x_3) = -\frac{Gm_1 m_3}{|x_3 - x_1|} - \frac{Gm_2 m_3}{|x_3 - x_2|}.$$

Fix $T, \vartheta > 0$ so that $\vartheta T < 2\pi$. Let R_α be the rotation in \mathbb{R}^2 by an angle α ; we denote by R_α the corresponding matrix as well

$$R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

Note that $x_1(t) = R_{\vartheta t}x_1(0)$ and $x_2(t) = R_{\vartheta t}x_2(0)$. We look for solutions of (2.2) satisfying $x_3(T) = R_{\vartheta T}x_3(0)$; to this end we introduce the space

$$H = \{x \in H^1([0, T], \mathbb{R}^2) : x(T) = R_{\vartheta T}x(0)\}.$$

Moreover, we require B_3 to orbit around B_1 without colliding with neither B_1 nor B_2 ; more precisely we consider the noncollision set

$$\Lambda_0 = \{x \in H : x(t) \neq x_i(t) \ (i = 1, 2) \text{ for all } t \in [0, T]\}$$

and

$$\Lambda_1 = \{x \in \Lambda_0 : \text{ind}(x - x_1) \neq 0\},$$

where $\text{ind}(y)$ denotes the winding number of y in the interval $[0, T]$. Consider the following Lagrangian functional

$$L(x) = \int_0^T \frac{m_3}{2} |\dot{x}|^2 - V(x), \quad (2.3)$$

whose critical points correspond to solutions of (2.2). Then, we have

THEOREM 1. – *There exists a continuum of periodic and quasi-periodic noncollision solutions of (2.2). More precisely, there exists $\bar{\nu} \in (0, 2\pi)$ (depending only on the ratio m_2/m_1) such that if $\vartheta T \leq \bar{\nu}$ then problem (2.2) admits a solution $x_3 \in \Lambda_1$; moreover, x_3 minimizes L over Λ_1 .*

We prove Theorem 1 in Section 3. Of course, a major problem concerning the statement of Theorem 1 is to estimate $\bar{\nu}$: in order to show that our results are not perturbative we take the masses of B_1 , B_2 and B_3 to be respectively the masses of the Earth, the Sun and the Moon. In this case $m_2/m_1 \approx (3.3)10^5$; moreover, the real (direct) motion of this 3-body system has the corresponding $\vartheta T \approx 0.46$; hence, it is of some interest to show that our method avoids collisions (for the retrograde motion) when $\bar{\nu} = 0.46$. In fact, our next result states that even larger values are allowed:

THEOREM 2. – *Assume that $m_2/m_1 = (3.3)10^5$; then, if $\vartheta T \leq 0.8$ problem (2.2) admits a solution $x_3 \in \Lambda_1$; moreover, x_3 minimizes L over Λ_1 .*

This result is proved in Section 6 where we also give the pictures of some numerical experiments: such results lead us to conjecture that the upper bound for ν could be even larger.

Next we consider a full planar 3-body problem $\{B_1, B_2, B_3\}$; their motion is described by the equations

$$-m_i \ddot{x}_i = V_i(x), \quad i = 1, 2, 3, \quad (2.4)$$

where the potential is given by

$$V(x) = - \sum_{1 \leq i < j \leq 3} \frac{Gm_i m_j}{|x_j - x_i|}$$

and $V_i(x) = \partial V(x) / \partial x_i$. Consider the Hilbert space

$$H = \{x = (x_1, x_2, x_3), x_i \in H^1([0, T], \mathbb{R}^2): x_i(T) = R_{\vartheta T} x_i(0)\},$$

the noncollision set

$$\Lambda_0 = \{x \in H: x_i(t) \neq x_j(t) \ (1 \leq i < j \leq 3) \text{ for all } t \in [0, T]\}$$

and

$$\Lambda_1 = \{x \in \Lambda_0: \text{ind}(x_3 - x_1) \neq 0\}.$$

By adding the three equations in (2.4) we get $\sum_i m_i \ddot{x}_i(t) = 0$, therefore, without loss of generality we may seek solutions $x = (x_1, x_2, x_3) \in H$ of (2.4) which satisfy the constraint $\sum_i m_i x_i(t) \equiv 0$; in particular, this implies that the “interesting” degrees of freedom of the system are 4. The Lagrangian of this 3-body problem is

$$\Phi(x) = \int_0^T \sum_{i=1}^3 \frac{m_i}{2} |\dot{x}_i|^2 - V(x). \quad (2.5)$$

Also in this case we obtain infinitely many noncollision periodic or quasi-periodic solutions of (2.4): in Section 4 we prove

THEOREM 3. – *If m_3 is sufficiently small, then there exists a continuum of periodic and quasi-periodic noncollision solutions of (2.4). More precisely, there exist two constants $\bar{\nu} \in (0, 2\pi)$ and $M > 0$ depending only on m_1, m_2 such that if $\vartheta T \leq \bar{\nu}$ and $m_3 \leq M$, then problem (2.4) admits a solution $x \in \Lambda_1$; moreover, x minimizes Φ over Λ_1 .*

This result enables us to study the planar 4-body problem $\{B_1, B_2, B_3, B_4\}$ where B_3 and B_4 are two satellites of B_1 . Their motion is described by the equations

$$-m_i \ddot{x}_i = V_i(x), \quad i = 1, 2, 3, 4, \quad (2.6)$$

where the potential is given by

$$V(x) = - \sum_{1 \leq i < j \leq 4} \frac{Gm_i m_j}{|x_j - x_i|}.$$

Define the Hilbert space

$$H = \{x = (x_1, x_2, x_3, x_4), x_i \in H^1([0, T], \mathbb{R}^2): x_i(T) = R_{\vartheta T} x_i(0)\},$$

the noncollision set

$$\Lambda_0 = \{x \in H: x_i(t) \neq x_j(t) \ (1 \leq i < j \leq 4) \text{ for all } t \in [0, T]\},$$

and

$$\Lambda_1 = \{x \in \Lambda_0: \text{ind}(x_i - x_1) \neq 0, \ i = 3, 4\}.$$

The corresponding Lagrangian is

$$\Phi(x) = \int_0^T \sum_{i=1}^4 \frac{m_i}{2} |\dot{x}_i|^2 - V(x). \quad (2.7)$$

Then, in Section 5 we will prove

THEOREM 4. – *If m_3 and m_4 are sufficiently small, then there exists a continuum of periodic and quasi-periodic noncollision solutions of (2.6). More precisely, there exist two constants $\bar{\nu} \in (0, 2\pi)$ and $M > 0$ depending only on m_1, m_2 such that if $\vartheta T \leq \bar{\nu}$ and $m_3, m_4 \leq M$, then problem (2.6) admits a solution $x \in \Lambda_1$; moreover, x minimizes Φ over Λ_1 .*

Remark 1. – The proofs of our results may be naturally modified in order to obtain similar statements for N -body problems $\{B_1, \dots, B_N\}$ with k major bodies ($k < N$), each one having at most two satellites.

Remark 2. – Although we seek solutions in Λ_1 (a set having nontrivial index for some couples of bodies), in fact by construction all the solutions

we find have the index equal to -1 , therefore they represent a clockwise motion, while the two major bodies are assumed to rotate anticlockwise.

3. THE RESTRICTED PROBLEM

In this section we prove Theorem 1.

First assume that $\vartheta T < 2\pi$. We switch to a rotating coordinate frame $(0, e_1, e_2)$ so that the bodies B_1 and B_2 , whose motions in the original coordinate frame are described by (2.1), are at rest. More precisely, the position of B_1 is $-R_1 e_1$ and the position of B_2 is $R_2 e_1$. If we set $q(t) = R_{-\vartheta t} x_3(t)$ and if

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denotes the standard symplectic matrix, then we get $\dot{x}_3(t) = \dot{R}_{\vartheta t} q(t) + R_{\vartheta t} \dot{q}(t)$ and $|\dot{x}_3(t)|^2 = |\dot{q}(t) + \vartheta J q(t)|^2$; the condition $x_3(T) = R_{\vartheta T} x_3(0)$ becomes $q(T) = q(0)$. Next we rescale the period T and we translate the system in order to have B_1 at the origin by setting $y(t/T) = q(t) + R_1 e_1$, so that we get the standard Lagrangian functional of the restricted 3-body problem:

$$I(y) = \frac{1}{T} \int_0^1 \frac{m_3}{2} |\dot{y} + T \vartheta J(y - R_1 e_1)|^2 - T^2 V(y),$$

where

$$V(y) = -\frac{Gm_1 m_3}{|y|} - \frac{Gm_2 m_3}{|y - (R_1 + R_2)e_1|}.$$

Up to the addition of a constant, we may redefine the Lagrangian I as follows:

$$I(y) = \frac{1}{T} \int_0^1 \frac{m_3}{2} |\dot{y} + T \vartheta J y|^2 - m_3 \vartheta^2 T^2 R_1(y, e_1) - T^2 V(y);$$

we remark that the corresponding Euler–Lagrange equations are unchanged. We set $x(t) = (T^2 G m_1)^{-1/3} y(t)$, we introduce the adimen-

sional parameters $\rho = m_2/m_1$ and $\nu = \vartheta T$ and we define

$$\Phi(x) = m_3^{-1} T^{-1/3} (Gm_1)^{-2/3} I(x) - \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}},$$

so that we get

$$\begin{aligned} \Phi(x) &= \int_0^1 \frac{1}{2} |\dot{x} + T \vartheta Jx|^2 + \frac{1}{|x|} \\ &\quad + \frac{T^{2/3} G^{1/3} m_2 m_1^{-2/3}}{|(T^2 G m_1)^{1/3} x - (m_1 + m_2)^{1/3} G^{1/3} \vartheta^{-2/3} e_1|} \\ &\quad - \frac{\vartheta^2 T^{4/3}}{(Gm_1)^{1/3}} \frac{m_2 G^{1/3}}{(m_1 + m_2)^{2/3} \vartheta^{2/3}}(x, e_1) - \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}} \\ &= \int_0^1 \frac{1}{2} |\dot{x} + T \vartheta Jx|^2 + \frac{1}{|x|} + \frac{m_2/m_1}{|x - (m_2/m_1 + 1)^{1/3} (\vartheta T)^{-2/3} e_1|} \\ &\quad - \frac{\vartheta^{4/3} T^{4/3} m_2}{m_1^{1/3} (m_1 + m_2)^{2/3}}(x, e_1) - \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}} \\ &= \int_0^1 \frac{1}{2} |\dot{x} + \nu Jx|^2 + \frac{1}{|x|} + \frac{\rho}{|x - (\rho + 1)^{1/3} \nu^{-2/3} e_1|} \\ &\quad - \frac{\rho \nu^{4/3}}{(\rho + 1)^{2/3}}(x, e_1) - \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}}. \end{aligned}$$

For all $\xi \in \mathbb{R}^2$ define the functions

$$g(\xi) := \frac{\rho}{|\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|}$$

and

$$f(\xi) := g(\xi) - \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}} - \frac{\rho \nu^{4/3}}{(\rho + 1)^{2/3}}(\xi, e_1),$$

so that

$$\Phi(x) = \int_0^1 \frac{1}{2} |\dot{x} + \nu Jx|^2 + \frac{1}{|x|} + f(x).$$

Note that

$$\begin{aligned}\nabla g(\xi) &= -\rho \frac{\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1}{|\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^3}, \\ \nabla^2 g(\xi)[\zeta, \zeta] &= -\frac{\rho |\zeta|^2}{|\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^3} \\ &\quad + \frac{3\rho(\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1, \zeta)^2}{|\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^5},\end{aligned}$$

and, by expanding in McLaurin polynomial, there exists $s \in (0, 1)$ ($s = s(\xi)$) such that

$$\begin{aligned}g(\xi) &= \frac{\rho \nu^{2/3}}{(\rho + 1)^{1/3}} + \frac{\rho \nu^{4/3}}{(\rho + 1)^{2/3}}(\xi, e_1) - \frac{\rho |\xi|^2}{2|s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^3} \\ &\quad + \frac{3\rho(s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1, \xi)^2}{2|s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^5};\end{aligned}$$

therefore, we also have for some $s = s(\xi) \in (0, 1)$

$$\begin{aligned}f(\xi) &= -\frac{\rho |\xi|^2}{2|s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^3} \\ &\quad + \frac{3\rho(s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1, \xi)^2}{2|s\xi - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^5}.\end{aligned}\quad (3.1)$$

The Lagrangian functional L defined in (2.3) is therefore transformed into Φ while the original space H is transformed in the Hilbert space H_1^1 of 1-periodic functions in H^1 . Consider the noncollision open subset

$$\Lambda_0 = \{x \in H_1^1: x(t) \neq 0 \text{ and } x(t) \neq (\rho + 1)^{1/3} \nu^{-2/3} e_1 \text{ for all } t \in [0, 1]\}$$

and

$$\Lambda_1 = \{x \in \Lambda_0: \text{ind}(x) = -1\}.\quad (3.2)$$

Theorem 1 is proved if we show that the problem

$$\inf_{x \in \Lambda_1} \Phi(x)$$

admits a solution $x \in \Lambda_1$ and no solutions in $\partial \Lambda_1$.

LEMMA 1. – *Let $\{x^n\} \subset \Lambda_1$ be a minimizing sequence for Φ ; then there exists $\underline{x} \in \overline{\Lambda_1}$ such that $x^n \rightharpoonup \underline{x}$, up to a subsequence, and $\Phi(\underline{x}) = \inf_{x \in \Lambda_1} \Phi(x)$.*

Proof. – In the following c represents a generic positive constant which may vary within the same formula. Let $x_0 = R(\sin 2\pi t, \cos 2\pi t)$ with

$R = (2\pi)^{-2/3}$; then x_0 achieves the minimum of the functional

$$\Psi(x) = \int_0^1 \frac{|\dot{x}|^2}{2} + \frac{1}{|x|}$$

under the constraint $\text{ind}(x) = -1$; x_0 is not a minimum of Φ , therefore for large values of n we have $\Phi(x^n) \leq \Phi(x_0)$ and

$$\int_0^1 \frac{1}{2} |\dot{x}^n + vJx^n|^2 - c(x^n, e_1) \leq \Phi(x_0) + \frac{\rho v^{2/3}}{(\rho + 1)^{1/3}} = c.$$

Note that if $v \leq \bar{v}$ then

$$\inf_{\|\dot{x}\|_2=1} \int_0^1 |\dot{x} + vJx|^2 = \left(1 - \frac{v}{2\pi}\right)^2 \geq \left(1 - \frac{\bar{v}}{2\pi}\right)^2; \tag{3.3}$$

so the last inequality yields

$$c\|\dot{x}^n\|_2^2 - c\|\dot{x}^n\|_2 \leq c$$

and finally $\|\dot{x}^n\|_2 \leq c$. By the topological requirements and Poincaré inequality we infer $\|x^n\|_\infty \leq c\|\dot{x}^n\|_2 \leq c$, therefore $\{x^n\}$ is bounded in H and, up to a subsequence, it admits a weak limit $\underline{x} \in H$, which is a minimum of Φ by the weak lower semicontinuity of the functional Φ . Finally, $\underline{x} \in \bar{\Lambda}_1$ by uniform convergence of $\{x^n\}$. \square

To complete the proof of Theorem 1 we exclude the case $\underline{x} \in \partial\Lambda_1$ in the statement of the previous lemma; first of all it is possible to obtain a contradiction to a collision between B_3 and B_2 , because they are not linked by a topological constraint. Indeed, if there exists $\tau \in [0, 1]$ such that $\underline{x}(\tau) = (\rho + 1)^{1/3}v^{-2/3}e_1$, we can modify the trajectory of $\underline{x} = \underline{x}(t)$ in a neighborhood of τ in order to lower both the kinetic part and the potential part of Φ and to avoid a collision. Since the proof is standard, we omit it.

LEMMA 2. – *Let \underline{x} be as in Lemma 1; then $\underline{x}(t) \neq (\rho + 1)^{1/3}v^{-2/3}e_1$ for all $t \in [0, 1]$.*

The proof that B_3 does not collide with B_1 is more subtle and requires some estimates; we exclude collisions for sufficiently small v :

LEMMA 3. – Let \underline{x} denote an orbit obtained in Lemma 1. There exists $\bar{\nu} \in (0, 2\pi)$ such that if $\nu \leq \bar{\nu}$ then $\underline{x}(t) \neq 0$ for all $t \in [0, 1]$ and

$$\inf_{\Lambda_1} \Phi < \inf_{\partial\Lambda_1} \Phi;$$

in particular, $\underline{x} \in \Lambda_1$.

Proof. – In this proof we denote by K_i ($i = 1, \dots, 4$) positive constants depending (eventually) on ρ . By contradiction, assume that $\underline{x} \in \partial\Lambda_1$, where \underline{x} is determined by the statement of Lemma 1; without loss of generality, we may assume that $\underline{x}(0) = 0$ and hence

$$\|\underline{x}\|_2 \leq \frac{1}{\pi} \|\dot{\underline{x}}\|_2, \quad \|\underline{x}\|_\infty \leq \frac{1}{2} \|\dot{\underline{x}}\|_2. \quad (3.4)$$

Consider again the function $x_0 = R(\sin 2\pi t, \cos 2\pi t)$ with $R = (2\pi)^{-2/3}$; if ν is sufficiently small, then we have $|sx_0(t) - (\rho + 1)^{1/3}\nu^{-2/3}e_1| \geq \frac{1}{2}(\rho + 1)^{1/3}\nu^{-2/3}$ for all $t \in [0, 1]$ and $s \in (0, 1)$, hence, (3.1) yields

$$f(x_0) \leq \frac{\rho R^2}{2|sx_0 - (\rho + 1)^{1/3}\nu^{-2/3}e_1|^3} \leq K_1\nu^2, \quad \forall t \in [0, 1],$$

and

$$\Phi(x_0) \leq \frac{(2\pi - \nu)^2}{2(2\pi)^{4/3}} + (2\pi)^{2/3} + K_1\nu^2 < \frac{3}{2}(2\pi)^{2/3} - K_2\nu + K_3\nu^2. \quad (3.5)$$

Since \underline{x} minimizes Φ , by (3.3) (3.4) and (3.5) we get

$$\begin{aligned} & \frac{1}{2} \left(1 - \frac{\nu}{2\pi}\right)^2 \|\dot{\underline{x}}\|_2^2 - \frac{\rho\nu^{4/3}}{\pi(\rho + 1)^{2/3}} \|\dot{\underline{x}}\|_2 - \frac{\rho\nu^{2/3}}{(\rho + 1)^{1/3}} \\ & \leq \Phi(\underline{x}) \leq \Phi(x_0) < \frac{3}{2}(2\pi)^{2/3} - K_2\nu + K_3\nu^2; \end{aligned}$$

therefore, we obtain (independently of $\nu \leq \bar{\nu} \leq \pi$) $\|\dot{\underline{x}}\|_2 \leq K_4$: in turn, if ν is sufficiently small, by (3.4) this yields

$$\|\underline{x}\|_\infty \leq \frac{1}{2} \frac{(\rho + 1)^{1/3}}{\nu^{2/3}}. \quad (3.6)$$

On the other hand, by (3.1)

$$\Phi(\underline{x}) \geq \int_0^1 \frac{1}{2} |\dot{\underline{x}} + \nu J \underline{x}|^2 + \frac{1}{|\underline{x}|} - \frac{\rho |\underline{x}|^2}{2|s\underline{x} - (\rho + 1)^{1/3} \nu^{-2/3} e_1|^3}$$

which, together with (3.6), gives

$$\Phi(\underline{x}) \geq \int_0^1 \frac{1}{2} |\dot{\underline{x}} + \nu J \underline{x}|^2 + \frac{1}{|\underline{x}|} - 4\nu^2 |\underline{x}|^2; \tag{3.7}$$

finally, note that

$$\begin{aligned} & \inf_{x \in H_0^1} \int_0^1 \frac{1}{2} |\dot{x} + \nu J x|^2 + \frac{1}{|x|} - 4\nu^2 |x|^2 \\ &= \inf_{x \in H_0^1} \int_0^1 \frac{1}{2} |\dot{x}|^2 + \frac{1}{|x|} - 4\nu^2 |x|^2. \end{aligned} \tag{3.8}$$

Therefore, since we assume that $\underline{x} \in H_0^1$ minimizes Φ , by (3.4), (3.5), (3.7) and (3.8)

$$\begin{aligned} & \frac{3}{2} (2\pi)^{2/3} - K_2 \nu + K_3 \nu^2 \\ &> \Phi(x_0) \geq \Phi(\underline{x}) \geq \inf_{x \in H_0^1} \int_0^1 \frac{1}{2} |\dot{x}|^2 + \frac{1}{|x|} - 4\nu^2 |x|^2 \\ &\geq \inf_{y \in H_0^1} \int_0^1 \frac{1}{2} \left(1 - \frac{8\nu^2}{\pi^2}\right) |\dot{y}|^2 + \frac{1}{|y|} \\ &= \left(1 - \frac{8\nu^2}{\pi^2}\right)^{1/3} \inf_{z \in H_0^1} \int_0^1 \frac{1}{2} |\dot{z}|^2 + \frac{1}{|z|} = \left(1 - \frac{8\nu^2}{\pi^2}\right)^{1/3} \frac{3}{2} (2\pi)^{2/3} \end{aligned}$$

which is impossible if $\nu > 0$ is small enough: hence, for such ν , we get a contradiction and the lemma is proved. \square

Theorem 1 is proved: if $\theta T \leq \bar{\nu}$ we obtain a solution of (2.2); such solution is periodic if $\theta T/\pi \in \mathbb{Q}$, while it is quasi-periodic if $\theta T/\pi \in \mathbb{R} \setminus \mathbb{Q}$.

4. THE FULL 3-BODY PROBLEM

In this section we prove Theorem 3.

To this end, we show that the Lagrangian functional Φ defined in (2.5) admits a global minimum in the (open) set

$$\Lambda_1 = \{x \in \Lambda_0: \text{ind}(x_3 - x_1) = -1\},$$

where Λ_0 is the noncollision open set

$$\Lambda_0 = \{x \in H: x_i(t) \neq x_j(t) \text{ for all } t \in [0, T] \text{ and } i \neq j\}.$$

We first prove

LEMMA 4. – *Let $\{x^n\} \subset \Lambda_1$ be a minimizing sequence for Φ ; then $\{x^n\}$ is bounded in H .*

Proof. – Since $\Phi(x) \geq \sum_i \int_0^T \frac{m_i}{2} |\dot{x}_i|^2$ for all $x \in \Lambda_0$, then $\|\dot{x}^n\|_2$ is bounded. Since $x_i(T) = R_{\vartheta T} x_i(0)$, then $|x_i(t+T) - x_i(t)|^2 = |(R_{\vartheta T} - R_0)x_i(t)|^2 = c|x_i(t)|^2$. But $x_i(t+T) - x_i(t) = \int_t^{t+T} \dot{x}_i(s) ds$, hence $|x_i(t)|^2 \leq c^{-1} \|\dot{x}_i^n\|_1^2 \leq Tc^{-1} \|\dot{x}_i^n\|_2^2$ for all t , so $\|x^n\|_\infty$ is bounded and finally $\|x^n\|$ is bounded. \square

We use again a rotating coordinate system by setting $Q_i(t) = R_{-\vartheta t} x_i(t)$ for all $x = (x_1, x_2, x_3) \in H$ so that $\dot{x}_i(t) = \dot{R}_{\vartheta t} Q_i(t) + R_{\vartheta t} \dot{Q}_i(t)$ and $|\dot{x}_i(t)|^2 = |\dot{Q}_i(t) + \vartheta J Q_i(t)|^2$. Next we rescale the period by setting $y(t/T) = Q(t)$. Up to a multiplication by T , the Lagrangian becomes

$$I(y) = \int_0^1 \sum_{i=1}^3 \frac{m_i}{2} |\dot{y}_i + T \vartheta J y_i|^2 - T^2 V(y);$$

clearly, the constraint $\sum_i m_i x_i(t) \equiv 0$ transforms into $\sum_i m_i y_i(t) \equiv 0$. We introduce a new Hilbert space of periodic functions (which we still denote by H), defined by

$$H = \left\{ y = (y_1, y_2, y_3) \in H^1([0, 1], \mathbb{R}^6): y(1) = y(0), \sum_{i=1}^3 m_i y_i(t) \equiv 0 \right\},$$

the corresponding noncollision open set

$$\Lambda_0 = \{y \in H: y_i(t) \neq y_j(t) \text{ for all } t \in [0, 1] \text{ and } i \neq j\}$$

and its subset

$$\Lambda_1 = \{y \in \Lambda_0: \text{ind}(y_3 - y_1) = -1\}.$$

Therefore, the original problem of minimizing Φ is reduced to the following minimization problem:

$$\inf_{y \in \Lambda_1} I(y). \tag{4.1}$$

We prove that the infimum in (4.1) is achieved:

LEMMA 5. – *There exists $\underline{y} \in \overline{\Lambda}_1$ such that $I(\underline{y}) = \inf_{y \in \Lambda_1} I(y)$.*

Proof. – Let $\{y^n\} \subset \Lambda_1$ be a minimizing sequence for I ; then, the corresponding sequence $\{x^n\}$ defined by setting $x_i^n(t) = R_{\vartheta t} y_i^n(t/T)$ is minimizing for Φ : by Lemma 4, it is bounded and so is $\{y^n\}$. Therefore, up to a subsequence, $\{y^n\}$ admits a weak limit $\underline{y} \in H$, which is a minimum of I by the lower semicontinuity of the functional I ; finally, $\underline{y} \in \overline{\Lambda}_1$ by uniform convergence of the sequence $\{y^n\}$. \square

As in Section 3, one can easily exclude collisions between B_2 and B_3 ; we have to exclude the other possible collisions.

Let I_0 be the Lagrangian I corresponding to $m_3 = 0$: as I_0 does not depend on the third component y_3 of y , we may identify it with its restriction to the subspace H_0 corresponding to $y_3 = 0$, namely

$$H_0 = \left\{ y = (y_1, y_2) \in H^1([0, 1], \mathbb{R}^4): y(1) = y(0), \sum_{i=1}^2 m_i y_i(t) \equiv 0 \right\}$$

and we study the minimization (2-body) problem

$$\inf_{x \in \Omega_0} I_0(x), \tag{4.2}$$

where Ω_0 is the corresponding noncollision open set. Consider the function $f: \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$$f(\xi_1, \xi_2) = \sum_{i=1}^2 \frac{m_i}{2} \vartheta^2 |\xi_i|^2 + \frac{m_1 m_2}{|\xi_1 - \xi_2|} \quad (\xi_i \in \mathbb{R}^2);$$

one can easily check that f has a unique strict global minimum, up to a $SO(2)$ -symmetry: more precisely, there exists $\Sigma \subset \mathbb{R}^4$ (which is a $SO(2)$

orbit), such that f attains its global minima on Σ . One such minimum is

$$\vartheta^{-2/3} \left(-\frac{m_2}{(m_1 + m_2)^{2/3}}, 0, \frac{m_1}{(m_1 + m_2)^{2/3}}, 0 \right)$$

and the corresponding Hessian matrix of f has rank 3 with 3 strictly positive eigenvalues (we denote by $C > 0$ the smallest of these eigenvalues); the 0 eigenvalue corresponds to the direction tangent to Σ . Therefore, if we denote by $\xi^\Sigma = (\xi_1^\Sigma, \xi_2^\Sigma)$ the projection on Σ of any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^4$ sufficiently close to Σ we have

$$\begin{aligned} & \sum_{i=1}^2 \frac{m_i}{2} \vartheta^2 |\xi_i|^2 + \frac{m_1 m_2}{|\xi_1 - \xi_2|} \\ & \geq \sum_{i=1}^2 \frac{m_i}{2} \vartheta^2 |\xi_i^\Sigma|^2 + \frac{m_1 m_2}{|\xi_1^\Sigma - \xi_2^\Sigma|} + \frac{C}{2} \sum_{i=1}^2 |\xi_i - \xi_i^\Sigma|^2; \end{aligned} \quad (4.3)$$

in particular, any (stationary) point in Σ is also a minimum for the functional I_0 .

Fix $\vartheta > 0$: we prove that if m_3 and T are small enough and $y = (y_1, y_2, y_3)$ is the minimum obtained in Lemma 5, then (y_1, y_2) is close to Σ in the H norm topology. In particular, this shows that B_1 and B_2 do not collide.

LEMMA 6. – *There exist two constants $\bar{T}, M > 0$ depending only on θ, m_1, m_2 and a constant $c > 0$ such that for all $T \leq \bar{T}$, all $m_3 \leq M$ and all $y = (y_1, y_2, y_3) \in H$ achieving the minimum in (4.1) (as given by Lemma 5) we have*

$$\sum_{i=1}^2 (\|\dot{y}_i\|_2^2 + T^2 \|y_i - y_i^\Sigma\|_2^2) \leq c T^{4/3} m_3,$$

where $y_i^\Sigma = y_i^\Sigma(t)$ is the (pointwise) projection of $y_i(t)$ onto Σ .

Proof. – Take any $(y_E, y_S) \in \Sigma$ and consider the function $Y \in H$ defined by

$$\begin{cases} Y_1(t) \equiv y_E, \\ Y_2(t) \equiv y_S, \\ Y_3(t) = y_E + T^{2/3} (\sin 2\pi t, \cos 2\pi t), \end{cases}$$

so that

$$V_{23}(y_E, Y_3) = -\frac{m_2 m_3}{T^{2/3}}$$

and $\dot{Y}_3(t) = 2\pi T^{2/3}(\cos 2\pi t, -\sin 2\pi t)$. Then the kinetic part of Y_3 is given by

$$|\dot{Y}_3 + T\vartheta JY_3|^2 = T^{4/3}|2\pi(\cos 2\pi t, -\sin 2\pi t) + T^{1/3}\vartheta J(y_E + T^{2/3}(\sin 2\pi t, \cos 2\pi t))|^2.$$

If T is small enough, then $Y \in \Lambda_1$, and by taking into account that $\inf_{\Lambda_1} I \leq I(Y)$ we obtain

$$\inf_{\Lambda_1} I_0(y) \leq \inf_{\Lambda_1} I \leq \inf_{\Lambda_1} I_0 + c_1 T^{4/3} m_3 + c_2 T^2 m_3. \tag{4.4}$$

Let $\{m_3^n\}$ be a vanishing sequence, let I_n be the functional corresponding to m_3^n and let $y^n = y^n(m_3^n, T) \in \overline{\Lambda_1^n}$ be the minimum of I_n obtained in Lemma 5. If y_i^n ($i = 1, 2, 3$) denote the components of y^n , then by (4.4) we infer that (y_1^n, y_2^n) is a minimizing sequence for I_0 , hence it converges weakly in H and uniformly to some $(y_E, y_S) \in \Sigma$, up to a subsequence. In particular, this proves that for any given $\varepsilon > 0$ there exists $m_3^\varepsilon > 0$ such that if $m_3 \leq m_3^\varepsilon$, then

$$|y_1(t) - y_E| \leq \varepsilon \quad \text{and} \quad |y_2(t) - y_S| \leq \varepsilon \quad \text{for all } t \in [0, 1]. \tag{4.5}$$

For all $y \in H^1(S^1, \mathbb{R}^2)$ satisfying $\int_0^1 y = 0$ we have $\|y\|_\infty \leq \|\dot{y}\|_1 \leq \gamma \|\dot{y}\|_2$ (the first is Wirtinger inequality and the second is Hölder’s). Then

$$\left| \int_0^1 (J\dot{y}, y) \right| = \left| \int_0^1 \left(J\dot{y}, y - \int_0^1 y \right) \right| \leq \|\dot{y}\|_2^2. \tag{4.6}$$

Now let $\bar{y} = (y_E, y_S)$, fix $y \in H^1(S^1, \mathbb{R}^4)$ and let $\tilde{y} = y - \bar{y}$; since $\int_0^1 (J\dot{y}, \bar{y}) = 0$, then

$$\int_0^1 |\dot{y}_i + T\vartheta Jy_i|^2 = \int_0^1 (|\dot{y}_i|^2 + T^2\vartheta^2|\bar{y}_i + \tilde{y}_i|^2 - 2T\vartheta(J\dot{y}_i, \tilde{y}_i)), \quad i = 1, 2.$$

Recall that $f(\bar{y}) = f(y^\Sigma(t))$ for all $t \in [0, 1]$, using (4.3), (4.5) and (4.6) we get:

$$I_0(y) - I_0(\bar{y}) = \int_0^1 \sum_{i=1}^2 \frac{m_i}{2} (|\dot{y}_i|^2 + T^2\vartheta^2|y_i|^2 - 2T\vartheta(J\dot{y}_i, \tilde{y}_i))$$

$$\begin{aligned}
 & + \frac{T^2 m_1 m_2}{|y_1 - y_2|} - \int_0^1 \sum_{i=1}^2 \frac{m_i}{2} T^2 \vartheta^2 |y_i^\Sigma|^2 + \frac{T^2 m_1 m_2}{|y_1^\Sigma - y_2^\Sigma|} \\
 & \geq \int_0^1 \sum_{i=1}^2 \frac{m_i}{2} (|\dot{y}_i|^2 - 2T\vartheta(J\dot{y}_i, \tilde{y}_i)) + \frac{C}{2} T^2 |y_i - y_i^\Sigma|^2 \\
 & \geq \sum_{i=1}^2 \frac{m_i}{4} \|\dot{y}_i\|_2^2 + \frac{C}{2} T^2 \|y_i - y_i^\Sigma\|_2^2;
 \end{aligned}$$

the last inequality follows by Wirtinger inequality and (4.6) since, for T small enough, we have

$$\int_0^1 \sum_{i=1}^2 m_i T \vartheta (\dot{y}_i, J\tilde{y}_i) \leq \sum_{i=1}^2 \frac{m_i}{4} \|\dot{y}_i\|_2^2.$$

Finally, from (4.4) we get $\inf_{\Lambda_1} I - \inf_{\Lambda_1} I_0 \leq c_1 T^{4/3} m_3 + c_2 T^2 m_3$ and therefore

$$\sum_{i=1}^2 \frac{m_i}{4} \|\dot{y}_i\|_2^2 + \frac{C}{2} T^2 \|y_i - y_i^\Sigma\|_2^2 \leq c_1 T^{4/3} m_3 + c_2 T^2 m_3,$$

which proves the estimate. \square

Finally, to prove that the minimum obtained in Lemma 5 is a noncollision minimum, we show that B_1 and B_3 do not collide:

LEMMA 7. – *If T and m_3 are sufficiently small, then*

$$\inf_{y \in \Lambda_1} I(y) < \inf_{y \in \partial \Lambda_1} I(y);$$

in particular, there exists $\underline{y} \in \Lambda_1$ such that $I(\underline{y}) = \inf_{y \in \Lambda_1} I(y)$.

Proof. – Fix $T > 0$, let I_T be the corresponding functional and let $\underline{y}^T \in \overline{\Lambda}_1$ be the minimum of I_T over $\overline{\Lambda}_1$ obtained in Lemma 5: we claim that $\underline{y}^T \notin \partial \Lambda_1$ for T and m_3 sufficiently small. By contradiction, let $y \in \partial \Lambda_1$ be a collision minimum, that is, $I(y) = \inf_{\Lambda_1} I$, where we have set $I = I_T$. By Lemma 6 we know that (y_1, y_2) is close in the norm topology of $H^1([0, 1], \mathbb{R}^4)$ to some point in Σ ; in particular, $y_1(t) \neq y_2(t)$ for all $t \in [0, 1]$. Moreover, if we set

$$V(q) = -\frac{m_1}{|q - y_1|} - \frac{m_2}{|q - y_2|},$$

then y_3 minimizes the restricted functional

$$\Psi(q) = \int_0^1 \frac{1}{2} |\dot{q} + T \vartheta Jq|^2 - T^2 V(q)$$

on the set Ω of functions $q \in H^1([0, 1], \mathbb{R}^2)$ satisfying $q(0) = q(1)$, $q(t) \neq y_i(t)$ for all $t \in [0, 1]$ and $i = 1, 2$ and $\text{ind}(q - y_1) = -1$.

We set $e(t) = T^{-2/3} y_1(t)$, $q(x) = T^{2/3}(x + e)$ and $\overline{\Phi}(x) = T^{-4/3} \Psi(q(x))$; then, we infer that $y_0(t) = T^{-2/3}(y_3(t) - y_1(t))$ minimizes the functional

$$\begin{aligned} \overline{\Phi}(x) = \int_0^1 & \left[\frac{1}{2} |\dot{x}|^2 + \frac{m_1}{|x|} + (\dot{x}, \dot{e}) + T \vartheta(\dot{x}, Jx) + 2T \vartheta(\dot{e}, Jx) \right. \\ & \left. + \frac{T^2 \vartheta^2}{2} |x|^2 + T^2 \vartheta^2(e, x) + T^{2/3} \frac{m_2}{|T^{2/3}(x + e) - y_2|} \right] + c(e), \end{aligned}$$

where

$$c(e) = \int_0^1 \frac{|\dot{e}|^2}{2} + T \vartheta(\dot{e}, Je) + \frac{T^2 \vartheta^2}{2} |e|^2.$$

Next we define

$$\Phi(x) = \overline{\Phi}(x) - c(e) - \int_0^1 \frac{m_2 T^{2/3}}{|y_2|},$$

so that y_0 also minimizes the functional

$$\begin{aligned} \Phi(x) = \int_0^1 & \left[\frac{1}{2} |\dot{x}|^2 + \frac{m_1}{|x|} + (\dot{x}, \dot{e}) + T \vartheta(\dot{x}, Jx) + 2T \vartheta(\dot{e}, Jx) \right. \\ & \left. + \frac{T^2 \vartheta^2}{2} |x|^2 + T^2 \vartheta^2(e, x) + T^{4/3} m_2 G(x) \right], \end{aligned}$$

where $G(x)$ is the smooth function given by

$$G(x) = T^{-2/3} \left(\frac{1}{|T^{2/3}(x + e) - y_2|} - \frac{1}{|y_2|} \right);$$

moreover, since $y_2(t) \neq 0$ for all t , by the asymptotic expansion

$$\frac{1}{|\varepsilon u - v|} = \frac{1}{|v|} \left(1 + \varepsilon \frac{(u, v)}{|v|} + o(\varepsilon) \right)$$

(which holds for all $u, v \in \mathbb{R}^2 \setminus \{0\}$ as $\varepsilon \rightarrow 0$) we have as $T \rightarrow 0$

$$G(x) = \left(x + e, \frac{y_2}{|y_2|^3} \right) + o(1).$$

Let $\mathcal{E} = \{t \in [0, 1]: y_0(t) = 0\}$. It is well known [2] that \mathcal{E} has measure zero and that $y_0(t)$ satisfies the Euler equation corresponding to Φ for all $t \notin \mathcal{E}$, namely

$$\begin{aligned} \ddot{y}_0 &= -2T \vartheta J \dot{y}_0 - T^2 \vartheta^2 y_0 + T^2 \vartheta^2 e - 2T \vartheta J \dot{e} - \ddot{e} - m_1 \frac{y_0}{|y_0|^3} \\ &+ (T^{4/3} m_2) \nabla G(y_0) \end{aligned}$$

and if we compute the scalar product of this equation by Jy_0 we obtain

$$\begin{aligned} \frac{d}{dt} (\dot{y}_0, Jy_0) &= (\ddot{y}_0, Jy_0) = (-2T \vartheta J \dot{y}_0 + T^2 \vartheta^2 e - 2T \vartheta J \dot{e} - \ddot{e}, Jy_0) \\ &+ T^{4/3} m_2 (\nabla G(y_0), Jy_0); \end{aligned}$$

since $y_0 \in \partial A_1$ we may assume that 0 is a collision time, i.e. $(\dot{y}_0, Jy_0)(0) = 0$: hence, by integrating the previous equation on the interval $[0, t]$ with $0 < t \leq 1$, we get

$$\begin{aligned} (\dot{y}_0, Jy_0)(t) &= -2T \vartheta \int_0^t (\dot{y}_0, y_0) + T^2 \vartheta^2 \int_0^t (e, Jy_0) - 2T \vartheta \int_0^t (\dot{e}, y_0) \\ &+ \int_0^t (\dot{e}, J \dot{y}_0) + T^{4/3} m_2 \int_0^t (\nabla G(y_0), Jy_0). \end{aligned} \tag{4.7}$$

Now we estimate the integrals in (4.7): since, $y_0 = y_0(T)$ is bounded in H^1 as $T \rightarrow 0$, we have

$$\left| 2T \vartheta \int_0^t (\dot{y}_0, y_0) \right| \leq 2T \vartheta \|\dot{y}_0\|_2 \|y_0\|_2 \leq 2\lambda T \tag{4.8}$$

for a.e. $t \in [0, 1]$. Choose a constant $\mu > 0$ and let

$$m_3 \leq \mu T^{8/3}; \tag{4.9}$$

since $e(t) = T^{-2/3}y_1(t)$, by Lemma 6 we have $\|\dot{e}\|_2 \leq \mu T^{4/3}$ and therefore

$$T^2 \vartheta^2 \int_0^t (e, Jy_0) - 2T \vartheta \int_0^t (\dot{e}, y_0) + \int_0^t (\dot{e}, J\dot{y}_0) = o(T) \tag{4.10}$$

for a.e. $t \in [0, 1]$. Hence, by (4.7), (4.8) and (4.10) we obtain for small T

$$\|(\dot{y}_0, Jy_0)\|_\infty \leq \lambda T.$$

Consider the function

$$X(t) = R(\sin 2\pi t, \cos 2\pi t), \quad \text{where } R = \sqrt[3]{m_1/(4\pi^2)},$$

which minimizes the functional

$$\hat{I}(x) = \int_0^1 \frac{|\dot{x}|^2}{2} + \frac{m_1}{|x|}$$

under the constraint $\text{ind}(x) = -1$; we have $(\dot{X}, JX) \equiv -2\pi R^2$ and therefore

$$\begin{aligned} &\Phi(X) - \Phi(y_0) \\ &= \hat{I}(X) - \hat{I}(y_0) + T \vartheta \int_0^1 [(\dot{X}, JX) - (\dot{y}_0, Jy_0)] + o(T) < 0 \end{aligned}$$

for small T . This contradicts the assumption that y_0 minimizes Φ and proves the lemma. \square

To complete the proof of Theorem 3, note that a noncollision critical point $x = (x_1, x_2, x_3)$ of Φ satisfies (2.4); then, it also satisfies

$$(\dot{x}_i(T), p(T)) = (\dot{x}_i(0), p(0)) = (R_{\vartheta T} \dot{x}_i(0), p(T)), \quad i = 1, 2, 3,$$

for all $p \in H^1([0, T], \mathbb{R}^2)$; hence, $\dot{x}_i(T) = R_{\vartheta T} \dot{x}_i(0)$. This proves that the motion \underline{x} is periodic if $\theta T/\pi \in \mathbb{Q}$ while it is quasi-periodic if $\theta T/\pi \in \mathbb{R} \setminus \mathbb{Q}$.

5. THE 4-BODY PROBLEM

In this section we prove Theorem 4.

We may assume that $m_4 \leq m_3$. As in the previous section, by rotating and rescaling, the Lagrangian becomes

$$I(y) = \int_0^1 \sum_{i=1}^4 \frac{m_i}{2} |\dot{y}_i + T \vartheta J y_i|^2 - T^2 V(y),$$

where

$$V(y) = - \sum_{1 \leq i < j \leq 4} \frac{G m_i m_j}{|y_j - y_i|};$$

we consider the Hilbert space H defined by

$$H = \left\{ y = (y_1, y_2, y_3, y_4) \in H^1([0, 1], \mathbb{R}^8) : y(1) = y(0), \right. \\ \left. \sum_{i=1}^4 m_i y_i(t) \equiv 0 \right\},$$

the corresponding noncollision open set Λ_0 and its subset $\Lambda_1 = \{y \in \Lambda_0 : \text{ind}(y_i - y_1) = -1, i = 3, 4\}$. By arguing as in Lemmas 4 and 5 one can prove that the Lagrangian achieves a minimum on $\overline{\Lambda_1}$:

LEMMA 8. – *There exists $\underline{y} \in \overline{\Lambda_1}$ such that $I(\underline{y}) = \inf_{y \in \Lambda_1} I(y)$.*

We can exclude the collisions between B_2 and B_3 and the collisions between B_2 and B_4 .

Let I_0 be the functional corresponding to the case $m_3 = m_4 = 0$, let H_0 and Σ be as in the previous section and consider again the problem (4.2); then we obtain

LEMMA 9. – *There exist two constants $\overline{T}, M > 0$ depending only on θ, m_1, m_2 and a constant $c > 0$ such that for all $T \leq \overline{T}$, all $m_4 \leq m_3 \leq M$ and all $y = (y_1, y_2, y_3, y_4) \in H$ achieving the minimum in (4.1) (as given by Lemma 8) the following holds*

$$\sum_{i=1}^2 (\|\dot{y}_i\|_2^2 + T^2 \|y_i - y_i^\Sigma\|_2^2) \leq c T^{4/3} m_3,$$

where $y_i^\Sigma = y_i^\Sigma(t)$ is the (pointwise) projection of $y_i(t)$ onto Σ .

Proof. – Take any $(y_E, y_S) \in \Sigma$ and consider the function $Y \in H$ defined by

$$\begin{cases} Y_1(t) \equiv y_E, \\ Y_2(t) \equiv y_S, \\ Y_3(t) = y_E + T^{2/3}(\sin 2\pi t, \cos 2\pi t), \\ Y_4(t) = y_E - T^{2/3}(\sin 2\pi t, \cos 2\pi t), \end{cases}$$

so that (4.4) holds (recall that $m_4 \leq m_3$). The proof may now be completed as in Lemma 6. \square

In particular, Lemma 9 excludes collisions between B_1 and B_2 .

Now we return to the original problem: in order to prove Theorem 4 it suffices to show that the functional

$$\Phi(x) = \int_0^T \sum_{i=1}^4 \frac{m_i}{2} |\dot{x}_i|^2 + \sum_{1 \leq i < j \leq 4} \frac{Gm_i m_j}{|x_j - x_i|}$$

satisfies

$$\inf_{\Lambda_1} \Phi < \inf_{\partial \Lambda_1} \Phi;$$

indeed, by Lemma 8, this would imply that Φ achieves a minimum over Λ_1 . Here Φ is defined on the space

$$H = \{x = (x_1, x_2, x_3, x_4), x_i \in H^1([0, T], \mathbb{R}^2) : x_i(T) = R_{\theta T} x_i(0)\}$$

and Λ_1 is given by

$$\Lambda_1 = \{x \in \Lambda_0, \text{ind}(x_i - x_1) = -1, i = 3, 4\},$$

where $\Lambda_0 = \{x \in H, x_i(t) \neq x_j(t) \ \forall t \in [0, T] \ \forall i \neq j\}$.

From now on we denote by $\underline{x} = (\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4)$ the minimum of Φ over $\overline{\Lambda_1}$ corresponding to the orbit \underline{y} obtained in Lemma 8; we exclude the collisions of the two satellites with B_1 .

LEMMA 10. – *If T, m_3 (and m_4) are sufficiently small, then $\underline{x}_3(t) \neq \underline{x}_1(t)$ and $\underline{x}_4(t) \neq \underline{x}_1(t)$ for all $t \in [0, T]$.*

Proof. – Consider the functional

$$F(x) = \int_0^T \frac{|\dot{x}|^2}{2} + \frac{Gm_1}{|x - \underline{x}_1|} + \frac{Gm_2}{|x - \underline{x}_2|}$$

defined on the space $K_1 = \{x \in H^1([0, T], \mathbb{R}^2)\}$ and the corresponding noncollision set

$$\Omega_1 = \{x \in K_1, x(t) \neq \underline{x}_1(t) \ \forall t \in [0, T], \text{ind}(x - \underline{x}_1) = -1\}.$$

Consider also the functionals

$$\overline{\Phi}(x_3, x_4) = \int_0^T \left(\sum_{i=3}^4 \frac{m_i}{2} |\dot{x}_i|^2 - V(x) \right),$$

where

$$V(x) = -\frac{Gm_1m_3}{|x_3 - \underline{x}_1|} - \frac{Gm_1m_4}{|x_4 - \underline{x}_1|} - \frac{Gm_2m_3}{|x_3 - \underline{x}_2|} - \frac{Gm_2m_4}{|x_4 - \underline{x}_2|} - \frac{Gm_3m_4}{|x_3 - x_4|}$$

and

$$\Psi(x_3, x_4) = \sum_{i=3}^4 m_i F(x_i) = \overline{\Phi}(x_3, x_4) - \int_0^T \frac{Gm_3m_4}{|x_4 - x_3|}$$

defined on the space $K_2 = \{x = (x_3, x_4), x_i \in H^1([0, T], \mathbb{R}^2)\}$; finally, consider the noncollision set

$$\Omega_2 = \{x = (x_3, x_4) \in K_2, x_3, x_4 \in \Omega_1, x_3(t) \neq x_4(t) \ \forall t \in [0, T]\}.$$

Since \underline{x} minimizes Φ over $\overline{\Lambda}_1$, the couple $(\underline{x}_3, \underline{x}_4)$ minimizes the functional $\overline{\Phi}$ over $\overline{\Omega}_2$.

Take $T > 0$ sufficiently small and $m_3 \leq \mu T^{8/3}$ as in (4.9); then, by Lemma 7, there exists $C_T > 0$ (independent of m_3 and m_4) such that

$$\inf_{x \in \partial\Omega_1} F(x) - \inf_{x \in \Omega_1} F(x) \geq C_T,$$

the two infima being in fact two minima: let X be one minimum of F over Ω_1 . By definition of Ψ we also have

$$\inf_{\partial\Omega_2} \Psi \geq \inf_{\Omega_2} \Psi + C_T m_4 \tag{5.1}$$

and the minimum of Ψ over Ω_2 is achieved by $(X(t), X(t + s))$ for any $s \in [0, T]$. Now take $X_3(t) = X(t)$ and take $X_4(t) = X(t + T/2)$ so that X_4 is also a noncollision minimum of F . Since X minimizes F , then a

simple argument shows that $X_3(t) \neq X_4(t)$ for all t and we can define $C_0 = \int_0^T \frac{G}{|x_3 - x_4|}$. Assume by contradiction that $(\underline{x}_3, \underline{x}_4) \in \partial\Omega_2$; then, by (5.1), we have

$$\begin{aligned} \overline{\Phi}(\underline{x}_3, \underline{x}_4) &= \Psi(\underline{x}_3, \underline{x}_4) + \int_0^T \frac{Gm_3m_4}{|\underline{x}_4 - \underline{x}_3|} \geq \inf_{\partial\Omega_2} \Psi + \int_0^T \frac{Gm_3m_4}{|\underline{x}_4 - \underline{x}_3|} \\ &\geq \overline{\Phi}(X_3, X_4) - \int_0^T \frac{Gm_3m_4}{|X_4 - X_3|} + \int_0^T \frac{Gm_3m_4}{|\underline{x}_4 - \underline{x}_3|} + C_T m_4 \\ &\geq \overline{\Phi}(X_3, X_4) - C_0 m_3 m_4 + C_T m_4 \end{aligned}$$

and therefore $\overline{\Phi}(\underline{x}_3, \underline{x}_4) > \overline{\Phi}(X_3, X_4)$ for sufficiently small m_3 (and m_4): this contradicts the assumption that $(\underline{x}_3, \underline{x}_4)$ minimizes $\overline{\Phi}$ over $\overline{\Omega}_2$ and proves the lemma. \square

Finally, we exclude the case where the two satellites collide with each other:

LEMMA 11. – $\underline{x}_3(t) \neq \underline{x}_4(t)$ for all $t \in [0, T]$.

Proof. – We make the following change of variables: let

$$\begin{cases} X = \frac{m_3 x_3 + m_4 x_4}{m_3 + m_4}, \\ r = x_4 - x_3, \end{cases}$$

and we denote by (X, r) the couple corresponding to $(\underline{x}_3, \underline{x}_4)$. We focus our attention on x_3 and x_4 and we consider the restricted Lagrangian

$$\Psi(X, r) = \int_0^T \frac{m_3 + m_4}{2} |\dot{X}|^2 + \frac{m_3 m_4}{2(m_3 + m_4)} |\dot{r}|^2 + \frac{Gm_3 m_4}{|r|} - V_R(X, r),$$

where

$$V_R(X, r) = -\frac{Gm_1 m_3}{|x_3 - \underline{x}_1|} - \frac{Gm_1 m_4}{|x_4 - \underline{x}_1|} - \frac{Gm_2 m_3}{|x_3 - \underline{x}_2|} - \frac{Gm_2 m_4}{|x_4 - \underline{x}_2|}.$$

Obviously,

$$\Phi(\underline{x}_1, \underline{x}_2, x_3, x_4) = \Psi(X, r) + \int_0^T \frac{m_1}{2} |\dot{\underline{x}}_1|^2 + \frac{m_2}{2} |\dot{\underline{x}}_2|^2 + \frac{Gm_1 m_2}{|\underline{x}_2 - \underline{x}_1|}.$$

By contradiction, assume that in the motion described by $\underline{x} = \underline{x}(t)$ the satellites B_3 and B_4 collide with each other, but not with B_1 , at $t = 0$, that is $\underline{r}(0) = 0$ and $\underline{X}(0) \neq \underline{x}_1(0)$. Let $\varepsilon > 0$ satisfy

$$\varepsilon < \frac{1}{2} \min\{|\underline{X}(0) - \underline{x}_1(0)|, |\underline{x}_3(0) - \underline{x}_2(0)|, |\underline{x}_4(0) - \underline{x}_2(0)|\}$$

and let $t_\varepsilon < 0 < t^\varepsilon$ be such that $|\underline{r}(t_\varepsilon)| = |\underline{r}(t^\varepsilon)| = \varepsilon$ and $|\underline{r}(t)| < \varepsilon$ if $t \in (t_\varepsilon, t^\varepsilon)$; if ε is sufficiently small (say $\varepsilon < \bar{\varepsilon}$), then $\underline{r}(t) \neq 0$ for all $t \in (t_\varepsilon, t^\varepsilon) \setminus \{0\}$. We will get a contradiction by showing that there exists $\varepsilon \in (0, \bar{\varepsilon})$ and $\hat{r} : [t_\varepsilon, t^\varepsilon] \rightarrow \mathbb{R}^2 \setminus \{0\}$ such that the function

$$R(t) = \begin{cases} \underline{r}(t) & \text{if } t \notin [t_\varepsilon, t^\varepsilon], \\ \hat{r}(t) & \text{if } t \in [t_\varepsilon, t^\varepsilon], \end{cases}$$

satisfies $R \in H^1([0, T], \mathbb{R}^2)$ and $\Psi(\underline{X}, \underline{r}) > \Psi(\underline{X}, R)$.

We may assume that $\underline{r}(t_\varepsilon) = (\varepsilon, 0)$ and $\underline{r}(t^\varepsilon) = \varepsilon e^{i\alpha}$ for some $\alpha \in (-\pi, \pi]$; let $p = \varepsilon^{3/2} e^{i\alpha/2}$. Let

$$\ell = \frac{|\underline{r}(t_\varepsilon) - p|}{\varepsilon} = \frac{|\underline{r}(t^\varepsilon) - p|}{\varepsilon};$$

since $|\alpha|/2 \leq \pi/2$, we have

$$\ell^2 = \frac{|\underline{r}(t_\varepsilon)|^2 + |p|^2 - 2(\underline{r}(t_\varepsilon), p)}{\varepsilon^2} = \frac{\varepsilon^2 + \varepsilon^3 - 2(\underline{r}(t_\varepsilon), p)}{\varepsilon^2} \leq 1 + \varepsilon. \tag{5.2}$$

Let S_1 and S_2 be respectively the segments connecting $\underline{r}(t_\varepsilon)$ and $\underline{r}(t^\varepsilon)$ with p and let $\hat{r}(t)$ be such that

$$\begin{aligned} \hat{r}(t) \in S_1 & \quad \text{and} \quad |\hat{r}(t) - p| = \ell |\underline{r}(t)| \quad \forall t \in [t_\varepsilon, 0], \\ \hat{r}(t) \in S_2 & \quad \text{and} \quad |\hat{r}(t) - p| = \ell |\underline{r}(t)| \quad \forall t \in [0, t^\varepsilon]. \end{aligned}$$

Since the motion of \hat{r} is straight, for all $t \in [t_\varepsilon, 0]$ we get

$$\begin{aligned} |\hat{r}(t) - p| \cdot \left| \frac{d}{dt} \hat{r}(t) \right| &= \left| \left(\hat{r}(t) - p, \frac{d}{dt} (\hat{r}(t) - p) \right) \right| \\ &= \left| \frac{1}{2} \frac{d}{dt} |\hat{r}(t) - p|^2 \right| = \frac{\ell^2}{2} \left| \frac{d}{dt} |\underline{r}(t)|^2 \right| \\ &\leq \ell^2 |\underline{r}(t)| \cdot \left| \frac{d}{dt} \underline{r}(t) \right|, \end{aligned}$$

and by (5.2)

$$\left| \frac{d}{dt} \hat{r}(t) \right|^2 \leq \ell^2 \left| \frac{d}{dt} \underline{r}(t) \right|^2 \leq (1 + \varepsilon) \left| \frac{d}{dt} \underline{r}(t) \right|^2; \tag{5.3}$$

similarly, we obtain the same inequality when $t \in [0, t^\varepsilon]$. Next, note that

$$V_R(\underline{X}(t), \hat{r}(t)) - V_R(\underline{X}(t), \underline{r}(t)) \geq -c\varepsilon \quad \text{for all } t \in [t_\varepsilon, t^\varepsilon];$$

this, together with (5.3) yields

$$\begin{aligned} & \int_{t_\varepsilon}^{t^\varepsilon} \frac{m_3 m_4}{2(m_3 + m_4)} \left(\left| \frac{d\underline{r}}{dt} \right|^2 - \left| \frac{d\hat{r}}{dt} \right|^2 \right) \\ & + V_R(\underline{X}(t), \hat{r}(t)) - V_R(\underline{X}(t), \underline{r}(t)) \geq -c\varepsilon. \end{aligned} \tag{5.4}$$

To estimate $\frac{1}{|\underline{z}|} - \frac{1}{|\hat{r}|}$, we argue as in [17]: by conservation of the total energy E we obtain for all $t \in [t_\varepsilon, t^\varepsilon]$

$$\begin{aligned} c|\dot{\underline{z}}(t)|^2 & \leq \frac{m_1}{2} |\dot{\underline{x}}_1(t)|^2 + \frac{m_2}{2} |\dot{\underline{x}}_2(t)|^2 + \frac{m_3}{2} |\dot{\underline{x}}_3(t)|^2 + \frac{m_4}{2} |\dot{\underline{x}}_4(t)|^2 \\ & = E + \frac{Gm_1 m_2}{|\underline{x}_2(t) - \underline{x}_1(t)|} + \frac{Gm_1 m_3}{|\underline{x}_3(t) - \underline{x}_1(t)|} + \frac{Gm_1 m_4}{|\underline{x}_4(t) - \underline{x}_1(t)|} \\ & \quad + \frac{Gm_2 m_3}{|\underline{x}_3(t) - \underline{x}_2(t)|} + \frac{Gm_2 m_4}{|\underline{x}_4(t) - \underline{x}_2(t)|} + \frac{Gm_3 m_4}{|\underline{r}(t)|} \leq \frac{c}{|\underline{r}(t)|}, \end{aligned}$$

the latter inequality being consequence of the fact that there exists $K > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ we have

$$\inf_{t \in [t_\varepsilon, t^\varepsilon]} \min\{|\underline{x}_j(t) - \underline{x}_i(t)|, 1 \leq i < j \leq 4, (i, j) \neq (3, 4)\} \geq K.$$

Therefore, if we let $\rho(t) = |\underline{r}(t)|$, we have

$$\rho \dot{\rho} = \frac{1}{2} \frac{d}{dt} \rho^2 = (\underline{r}(t), \dot{\underline{r}}(t)) \leq |\underline{r}(t)| \cdot |\dot{\underline{r}}(t)| \leq \rho \frac{c}{\sqrt{\rho}};$$

this proves that

$$\rho(t) \leq ct^{2/3} \quad \forall t \in [t_\varepsilon, t^\varepsilon]. \tag{5.5}$$

Let $t_\varepsilon < s_1 < 0 < s_2 < t^\varepsilon$ satisfy $|\underline{r}(s_i)| = \varepsilon^{3/2}/2$ and $|\underline{r}(t)| < \varepsilon^{3/2}/2$ for all $t \in (s_1, s_2)$; by (5.5) we obtain $|s_i| \geq c\varepsilon^{9/4}$. Moreover, since

$|\hat{r}(t)| > |\underline{r}(t)|$ for all $t \in (t_\varepsilon, t^\varepsilon)$ we infer that

$$\int_{t_\varepsilon}^{s_1} \frac{1}{|\underline{r}|} > \int_{t_\varepsilon}^{s_1} \frac{1}{|\hat{r}|} \quad \text{and} \quad \int_{s_2}^{t^\varepsilon} \frac{1}{|\underline{r}|} > \int_{s_2}^{t^\varepsilon} \frac{1}{|\hat{r}|};$$

on the other hand, $|\hat{r}(t)| \geq \varepsilon^{3/2} \geq 2|\underline{r}(t)|$ for all $t \in (s_1, s_2)$ so that, by (5.5), we obtain

$$\begin{aligned} Gm_3m_4 \int_{t_\varepsilon}^{t^\varepsilon} \frac{1}{|\underline{r}|} - \frac{1}{|\hat{r}|} &> Gm_3m_4 \int_{s_1}^{s_2} \frac{1}{|\underline{r}|} - \frac{1}{|\hat{r}|} \geq c \int_{-c\varepsilon^{9/4}}^{c\varepsilon^{9/4}} \frac{1}{2|\underline{r}|} \\ &\geq c \int_{-c\varepsilon^{9/4}}^{c\varepsilon^{9/4}} \frac{1}{t^{2/3}} \geq c\varepsilon^{3/4}. \end{aligned}$$

This, together with (5.4), implies that $\Psi(\underline{X}, \underline{r}) - \Psi(\underline{X}, R) \geq c\varepsilon^{3/4} - c\varepsilon > 0$ if ε is small. \square

The proof of Theorem 4 may now be completed as in the previous section.

6. NUMERICAL RESULTS

In this section we prove Theorem 2 and we give some numerical results illustrating the orbits determined in Theorem 1. Since all the solutions we obtained are minima, it is possible to numerically compute them by a rather simple procedure. The technique is standard: one chooses a finite dimensional vector space H_f which approximates the Hilbert space H , introduces a functional $\Phi_f : H_f \rightarrow \mathbb{R}$ approximating Φ and looks for minima of Φ_f by choosing an arbitrary starting point $x_0 \in H_f$ and defining a sequence $\{x_n\}$ by setting $x_{n+1} = x_n - h_n \nabla \Phi_f(x_n)$, where $\nabla \Phi_f : H_f \rightarrow H_f$ is the gradient of Φ and represents the maximum slope direction of the functional Φ_f , while h_n is computed at each step in order to minimize the function $h \mapsto \Phi(x_{n+1}(h))$. If the approximated functional maintains the properties of Φ , then the sequence $\{x_n\}$ converges to a minimum point of the functional Φ_f .

We only treat the restricted problem (although there are no obstructions to the treatment of the complete problem). As an approximate space H_f we chose the set of closed m -gonals and we let m vary between 100 and 300, depending on the values of the parameters. A function $x \in H_f$ is

uniquely characterized by the coordinates of the vertices, therefore it can be represented by a point in \mathbb{R}^{2m} . Given $x \in H_f$ by $x_i \in \mathbb{R}^2$ we denote the coordinates of the i th vertex. The (full) functional we consider is

$$\Phi(x) = \int_0^1 \frac{1}{2} |\dot{x} + vJx|^2 + \frac{1}{|x|} + \frac{\rho}{|x - (\rho + 1)^{1/3}v^{-2/3}e_1|} - \frac{\rho v^{4/3}}{(\rho + 1)^{2/3}}(x, e_1),$$

and its representation in H_f is given by

$$\Phi_f(x) = \sum_{i=1}^m \frac{1}{2} \left| \frac{x_{i+1} - x_i}{h} + vJx_i \right|^2 + \frac{1}{|x_i|} + \frac{\rho}{|x_i - (\rho + 1)^{1/3}v^{-2/3}e_1|} - \frac{\rho v^{4/3}}{(\rho + 1)^{2/3}}(x_i, e_1),$$

where $h = 1/m$. We also have

$$\frac{\partial \Phi_f(x)}{\partial x_i} = -\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} - \frac{x_i}{|x_i|^3} - 2vJ \frac{x_{i+1} - x_i}{h} + v^2 x_i - \rho \frac{x_i - (\rho + 1)^{1/3}v^{-2/3}e_1}{|x_i - (\rho + 1)^{1/3}v^{-2/3}e_1|^3} - \frac{\rho v^{4/3}}{(\rho + 1)^{2/3}}e_1.$$

Clearly, also in the numerical approximation we have to cope with the presence of the singularity in the potential; furthermore is it not very clear how to implement the topological constraint. In order to overcome these problems we introduced a naive method, i.e. we checked that at every step no vertices of x_n were too close to the singularities. More precisely we checked that the minimum of the distances of the vertices from the singularity was larger than the maximum of the length of the sides of the m -gonal. In fact this condition was satisfied at all times during all our computations.

The following pictures represent the results we obtained for various values of the parameters v and ρ . As a starting point x_0 we chose the orbit corresponding to the solution for the case $v = 0$, that is the circular orbit of radius $R = (2\pi)^{-2/3}$. Although analytically we could exclude collisions in the case $\rho = (3.3)10^5$ only for values of v smaller than 0.8, numerically it clearly appears that the minima of the functional is very close to a circular orbit even for values of v up to 3. For larger values the orbit becomes more similar to an ellipse, but still it does not come

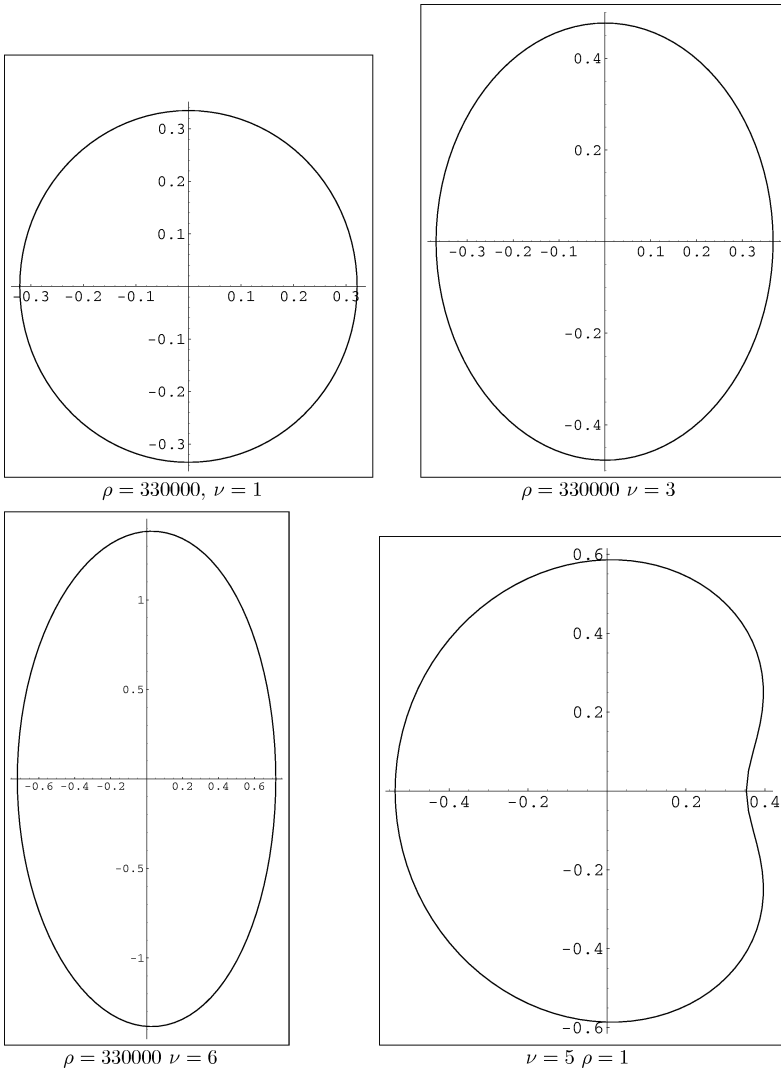


Fig. 1.

close to a collision orbit. In order to show some different behavior, we also show a picture (Fig. 1) in the case $\rho = 1$ and $\nu = 5$.

Proof of Theorem 2. – Let $\rho = (3.3)10^5$, $\nu \leq 0.8$ and let \underline{x} denote a corresponding orbit found in Lemma 1. We argue as in the proof of Lemma 3 making finer estimates. By contradiction, assume that $\underline{x} \in \partial \Lambda_1$

and consider again the function

$$x_0 = (2\pi)^{-2/3}(\sin 2\pi t, \cos 2\pi t).$$

By arguing corresponding to (3.5) we find that (if $\nu \leq 0.8$)

$$\Phi(x_0) < \frac{3}{2}(2\pi)^{2/3} - (0.541)\nu + (0.175)\nu^2 < \frac{3}{2}(2\pi)^{2/3}. \quad (6.1)$$

Since \underline{x} minimizes Φ , by (3.3) and (6.1) we get

$$\frac{1}{2} \left(1 - \frac{\nu}{2\pi}\right)^2 \|\dot{\underline{x}}\|_2^2 - \frac{\rho\nu^{4/3}}{\pi(\rho+1)^{2/3}} \|\dot{\underline{x}}\|_2 - \frac{\rho\nu^{2/3}}{(\rho+1)^{1/3}} \leq \frac{3}{2}(2\pi)^{2/3}; \quad (6.2)$$

by (3.4) this proves that

$$\|\dot{\underline{x}}\|_2 < 130, \quad \|\underline{x}\|_\infty < 65 < (0.811) \frac{(\rho+1)^{1/3}}{\nu^{2/3}}. \quad (6.3)$$

Now we claim that for all $t \in [0, 1]$ we have

$$\begin{aligned} h(\underline{x}(t)) &:= \frac{\rho}{|\underline{x}(t) - (\rho+1)^{1/3}\nu^{-2/3}e_1|} - \frac{\rho\nu^{4/3}}{(\rho+1)^{2/3}}(\underline{x}(t), e_1) \\ &> -(0.035) \frac{\rho\nu^{2/3}}{(\rho+1)^{1/3}}. \end{aligned} \quad (6.4)$$

If $(\underline{x}(t), e_1) \leq 0$, (6.4) follows readily. If $(\underline{x}(t), e_1) > 0$, then by (6.3) we get

$$|\underline{x}(t) - (\rho+1)^{1/3}\nu^{-2/3}e_1|^2 < \|\underline{x}\|_\infty^2 + \frac{(\rho+1)^{2/3}}{\nu^{4/3}} < (1.658) \frac{(\rho+1)^{2/3}}{\nu^{4/3}};$$

then (6.4) follows by estimating $(\underline{x}(t), e_1)$ with $\|\underline{x}\|_\infty$ and by (6.3).

Since $\nu \leq 0.8$, by (6.4) we may replace (6.2) with

$$(0.3807)\|\dot{\underline{x}}\|_2^2 - (1.035) \frac{\rho\nu^{2/3}}{(\rho+1)^{1/3}} \leq \frac{3}{2}(2\pi)^{2/3}, \quad (6.5)$$

which, by (3.4), proves that

$$\|\dot{\underline{x}}\|_2 < 106, \quad \|\underline{x}\|_\infty < 53 < (0.661) \frac{(\rho+1)^{1/3}}{\nu^{2/3}}; \quad (6.6)$$

this allows to prove that

$$h(\underline{x}(t)) > (0.173) \frac{\rho v^{2/3}}{(\rho + 1)^{1/3}} \quad \forall t \in [0, 1]. \quad (6.7)$$

Indeed, if $(\underline{x}(t), e_1) \leq 0$, we have

$$h(\underline{x}(t)) \geq \frac{\rho}{\|\underline{x}\|_\infty + (\rho + 1)^{1/3} v^{-2/3}} > (0.6) \frac{\rho v^{2/3}}{(\rho + 1)^{1/3}}.$$

If $(\underline{x}(t), e_1) > 0$, then by arguing as for (6.4) and by taking into account (6.6) we get (6.7).

By (6.7) the inequality (6.5) becomes

$$(0.3807) \|\dot{\underline{x}}\|_2^2 - (0.827) \frac{\rho v^{2/3}}{(\rho + 1)^{1/3}} \leq \frac{3}{2} (2\pi)^{2/3}, \quad (6.8)$$

which, by (3.4), proves that

$$\|\dot{\underline{x}}\|_2 < 94.8, \quad \|\underline{x}\|_\infty < 47.4 < (0.592) \frac{(\rho + 1)^{1/3}}{v^{2/3}}.$$

Finally, repeating once more the whole procedure, we get $h(\underline{x}(t)) > (0.267) \rho v^{2/3} (\rho + 1)^{-1/3}$ which yields

$$\|\dot{\underline{x}}\|_2 < 89.1, \quad \|\underline{x}\|_\infty < 44.55 < (0.556) \frac{(\rho + 1)^{1/3}}{v^{2/3}}.$$

Hence, $|s_{\underline{x}}(t) - (\rho + 1)^{1/3} v^{-2/3} e_1| > (0.444) (\rho + 1)^{1/3} v^{-2/3}$ and

$$\int_0^1 \frac{\rho |\underline{x}|^2}{2 |s_{\underline{x}} - (\rho + 1)^{1/3} v^{-2/3} e_1|^3} < \frac{v^2 \|\dot{\underline{x}}\|_2^2}{2 (0.444)^3 \pi^2} \leq (0.3705) \|\dot{\underline{x}}\|_2^2.$$

Therefore, by (3.1) and (6.1) we obtain

$$\begin{aligned} \frac{3}{2} (2\pi)^{2/3} > \Phi(\underline{x}) &\geq (0.3807) \|\dot{\underline{x}}\|_2^2 - \int_0^1 \frac{\rho |\underline{x}|^2}{2 |s_{\underline{x}} - (\rho + 1)^{1/3} v^{-2/3} e_1|^3} \\ &> (0.0102) \|\dot{\underline{x}}\|_2^2; \end{aligned}$$

this proves that

$$\|\dot{\underline{x}}\|_2 < 22.4, \quad \|\underline{x}\|_\infty < 11.2 < (0.14) \frac{(\rho + 1)^{1/3}}{\nu^{2/3}}.$$

Repeating this procedure we get

$$\int_0^1 \frac{\rho|\underline{x}|^2}{2|s\underline{x} - (\rho + 1)^{1/3}\nu^{-2/3}e_1|^3} < \frac{\nu^2\|\dot{\underline{x}}\|_2^2}{2(0.86)^3\pi^2} \leq (0.051)\|\dot{\underline{x}}\|_2^2.$$

Therefore, by (3.1), (6.1) and Jensen’s inequality we obtain

$$\frac{3}{2}(2\pi)^{2/3} > (0.3297)\|\dot{\underline{x}}\|_2^2 + \frac{\pi}{\|\dot{\underline{x}}\|_2},$$

which proves that

$$\|\dot{\underline{x}}\|_2 < 3.6, \quad \|\underline{x}\|_\infty < 1.8 < (0.023) \frac{(\rho + 1)^{1/3}}{\nu^{2/3}}$$

and

$$\int_0^1 \frac{\rho|\underline{x}|^2}{2|s\underline{x} - (\rho + 1)^{1/3}\nu^{-2/3}e_1|^3} < \frac{\nu^2\|\underline{x}\|_2^2}{2(0.977)^3} \leq (0.537)\nu^2\|\underline{x}\|_2^2.$$

Therefore, by arguing as in the proof of Lemma 3 we get

$$\begin{aligned} & \frac{3}{2}(2\pi)^{2/3} - (0.541)\nu + (0.175)\nu^2 \\ & > \Phi(x_0) \geq \Phi(\underline{x}) \geq \int_0^1 \frac{1}{2}|\dot{\underline{x}} + \nu J\underline{x}|^2 + \frac{1}{|\underline{x}|} - (0.537)\nu^2|\underline{x}|^2 \\ & \geq \left(1 - \frac{(1.074)\nu^2}{\pi^2}\right)^{1/3} \inf_{z \in H_0^1} \int_0^1 \frac{1}{2}|\dot{z}|^2 + \frac{1}{|z|} \\ & = \left(1 - \frac{(1.074)\nu^2}{\pi^2}\right)^{1/3} \frac{3}{2}(2\pi)^{2/3}. \end{aligned}$$

Since $\nu \leq 0.8$ we have $(1 - \frac{(1.074)\nu^2}{\pi^2})^{1/3} \geq 1 - (0.038)\nu^2$ and therefore the last sequence of inequalities yields $(0.37)\nu > 0.541$ which contradicts $\nu \leq 0.8$ and proves the statement. \square

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