Attractors for families of processes in weak topologies of Banach spaces

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Abstract

A general theory for the study of families of processes in the weak topology of some Banach space is suggested: sufficient conditions for the existence and connectedness of attractors are proved. The results apply to (nonlinear) nonautonomous evolution partial differential equations for which the behavior of the corresponding processes is better described when the phase space is endowed with its weak topology.
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1 Introduction

The study of processes on complete metric spaces started with the pioneering papers by Sell [23], Dafermos [9, 10] and Ball [4]: it looks difficult to give an exhaustive list of the subsequent results existing in literature but let us just mention the monographs [16, 18] where further references may be found. Processes are particularly useful in the study of evolution PDE's; it is well-known [2, 21, 25] that for autonomous PDE's the long-time behavior of the solution may be described by a family of operators which enjoy the semigroup properties: processes represent the generalization of semigroups to the nonautonomous case. In the last few years, the attention has turned to families of processes, see [6, 7, 8].

In order to apply the tools involved in these theories some compactness properties of the process (or semigroup) defined by the equation are requested; even for fully nonlinear PDE's, when the equation is dissipative, standard energy estimates usually suggest to study the long-time behaviour of the solution in the norm topology of some Banach space or with the metric of some nonlinear space, see e.g. [20, 25] and references therein. However, in some cases the nonlinearity in the equation only allows to prove nice compactness properties of the corresponding process in the weak topology, see e.g. [3, 5, 12, 17, 19, 24]. A general theory which also allows to treat families of processes for these equations seems not to be known: in this paper we make an attempt in this direction. To this end, we introduce a self-contained analysis which is based on the notion of *uniformly compactifying at infinity* (UCI in the sequel) family of processes; in metric spaces, this compactness assumption is exploited by Arosio [1] in the context of semigroups (see also [21]): it allows to study the compactness properties of the problem in terms of sequences. Our main results concern the case of weak topologies: assuming that the family of processes is UCI, we reach sufficient conditions for the existence and connectedness of attractors when the phase space is some reflexive separable Banach space endowed with its weak topology. These results generalize in some sense to a different context previous results by Chepyzhov-Vishik [6, 7, 8] and Gobbino-Sardella [15]. In literature, the study of evolution equations in the weak topology is usually handled in a different way, see e.g. [12, 13] [1]: one first proves the existence of an absorbing set which is bounded in the norm topology and hence, relatively compact in the weak topology; next one uses the metric induced by the weak topology on bounded sets. The aim of this paper is to insert these problems into a more general framework and to avoid the use of a metric: all definitions and proofs are stated in a more topological fashion. In Remark 1 below we show that our assumptions are actually weaker than the existence of a bounded absorbing set.

This paper is organized as follows: in Section 2 we introduce the basic tools for the study of processes in weak topologies of Banach phase spaces. In Section 3 we state our main results, sufficient conditions for the existence and connectedness of attractors for families of processes acting in the weak topology of some separable reflexive Banach space; the proofs of these results are quoted in Section 4. Finally, in Section 5 we apply our results to the study of the long-time behavior of the solutions of some nonautonomous parabolic equations when the forcing term is free to run in a suitable bounded subset of a functional space: the equations considered are the modified Navier-Stokes equations introduced by Prouse [22] and some general parabolic equations with a monotone principal part.

2 Notations and definitions

Throughout this paper we assume that X is a reflexive separable Banach space: we will essentially deal with the case where X is endowed with its weak topology and, when we wish to highlight this fact, we will denote the space by X_w; to say that a sequence $\{x_n\} \subset X$ converges weakly to x we simply denote $x_n \to x$. We denote by $\mathcal{B}$ the set of bounded subsets of X: it is well-known that if $B \in \mathcal{B}$ then B is relatively compact in X_w. In the sequel, for all $\tau \in \mathbf{R}$ we set $\mathbf{R}_\tau := [\tau, +\infty)$.

Several slightly different definitions of process are given in literature, see e.g. [6, 16, 18]; in this paper we use the following

Definition 1 *A family of operators $U(t,\tau) : X \to X$, $\tau \in \mathbf{R}$, $t \in \mathbf{R}_\tau$ is a* **process** *on X if the following two conditions hold:*
(i) $U(\tau,\tau) = I$ (identity on X) for every $\tau \in \mathbf{R}$;
(ii) $U(t,s)U(s,\tau) = U(t,\tau)$ for every $t \in \mathbf{R}_\tau$.

Consider now a family of processes $\{U_f(t,\tau),\ f \in F\}$ depending on a parameter $f \in F$, where F is a topological space; the parameter f is called the *symbol* of the process U_f. In order to define absorbing sets the topology is not needed: in the following definition, we specify that the absorbing sets we consider are uniform with respect to the symbol f, namely, that they do not depend on f.

Definition 2 *A set $B_0 \subset X$ is said to be* **uniformly absorbing** *(***u-absorbing***) for the family of processes $\{U_f(t,\tau),\ f \in F\}$ if for every $B \in \mathcal{B}$ and every $\tau \in \mathbf{R}$ there exists $T = T(\tau, B) \in \mathbf{R}_\tau$ such that*

$$\bigcup_{f \in F} U_f(t,\tau)B \subset B_0 \qquad \forall t \in \mathbf{R}_T \ .$$

[1] In fact, the attractor for the KdV equations found in [13] is in the norm topology, see [14].

It is well-known that the notion of attraction is related to the topology considered in the phase space X:

Definition 3 *A nonempty set $\mathcal{A} \subset X$ is said to be* w**-u-attracting** *for $\{U_f(t,\tau),\ f \in F\}$ if every open set $\mathcal{O} \subset X_w$ such that $\mathcal{O} \supset \mathcal{A}$ is u-absorbing.*

Definition 4 *A compact set $\mathcal{A} \subset X_w$ is said to be the* **minimal** w**-u-attractor** *for the family of processes $\{U_f(t,\tau),\ f \in F\}$ if the following two conditions occur:*
(i) $\mathcal{A}$ is w-u-attracting;
(ii) $\mathcal{A}$ is contained in any closed w-u-attracting set.

The above property of minimality is the natural generalization of the invariance property in the definition of semigroup's attractors.

In order to study the properties of the family of processes we need to introduce different kinds of continuity:

Definition 5 *A family of processes $\{U_f(t,\tau),\ f \in F\}$ acting on the space X_w is:*
(i) t-continuous if $\forall (x,f) \in X \times F$ the map $(t,\tau) \mapsto U_f(t,\tau)x$ is continuous;
(ii) x-continuous if $\forall (\tau,t,f) \in \mathbf{R} \times \mathbf{R}_\tau \times F$ the map $x \mapsto U_f(t,\tau)x$ is continuous;
(iii) fx-continuous if $\forall (\tau,t) \in \mathbf{R} \times \mathbf{R}_\tau$ the map $(x,f) \mapsto U_f(t,\tau)x$ is continuous.

The existence of a minimal w-attractor is related with the compactness properties of the family of processes: for this reason, we introduce the

Definition 6 *The family of processes $\{U_f(t,\tau);\ f \in F\}$ is said to be* **uniformly compactifying at infinity** *(UCI) if for all $\tau \in \mathbf{R}$, for all $\{f_n\} \subset F$, for all $\{t_n\} \subset \mathbf{R}_\tau$ such that $t_n \to +\infty$ and for all $\{x_n\} \in \mathcal{B}$ the set $\{U_{f_n}(t_n,\tau)x_n\}$ is relatively compact in X_w.*

Finally, in order to present a useful characterization of the minimal w-attractor, we introduce the notion of complete trajectory of a process:

Definition 7 *A curve $x(s)$, $s \in \mathbf{R}$ is said to be a* **complete trajectory of the process** $U(t,\tau)$ *if $U(t,\tau)x(\tau) = x(t)$ for every $\tau \in \mathbf{R}$ and $t \geq \tau$.*

3 Main results

Throughout this section we assume that

$$X \text{ is a reflexive separable Banach space} \tag{1}$$

and, for all $r > 0$ we set $B_r := \{x \in X;\ \|x\| \leq r\}$; moreover, we assume that F is a topological space.

For all $\tau \in \mathbf{R}$ and $B \in \mathcal{B}$ we define the uniform ω-limit set

$$\omega_\tau(B) := \bigcap_{s \geq \tau} \overline{\bigcup_{t \geq s} \bigcup_{f \in F} U_f(t,\tau)B}\ ;$$

as X_w satisfies the first axiom of countability, it is not difficult to verify that an equivalent characterization is the following

$$\omega_\tau(B) = \Big\{x \in X :\ \exists \{x_n\} \subset B\ ,\ \exists \{f_n\} \subset F\ ,\ \exists t_n \to +\infty \text{ s.t. } U_{f_n}(t_n,\tau)x_n \to x\Big\}\ . \tag{2}$$

Our first result consists in two necessary and sufficient conditions for the existence of the minimal uniform w-attractor: one of these conditions is the weak topology version of the standard assumption [7, 18] that the family of processes is uniformly asymptotically compact.

Theorem 1 *Assume (1) and let F be a topological space; assume that a family of processes $\{U_f(t,\tau);\ f \in F\}$ is defined on X_w. Then the following conditions are equivalent:*
(i) there exists a compact nonempty w-u-attractor A;
(ii) the following conditions hold:
(a) $\{U_f(t,\tau);\ f \in F\}$ is UCI;
(b) $\bigcup_{\tau\in\mathbf{R}}\bigcup_{n\in\mathbf{N}}\omega_\tau(B_n)$ is bounded;
(iii) there exists the minimal w-u-attractor $\mathcal{A}$ and it can be characterized by

$$\mathcal{A}=\overline{\bigcup_{\tau\in\mathbf{R}}\bigcup_{n\in\mathbf{N}}\omega_\tau(B_n)}.$$

Remark 1 In Theorem 1 no continuity assumptions on the family of processes are needed. Note also that the existence of a bounded u-absorbing set is a sufficient condition which ensures that (*iii*) holds; it is obviously not necessary as shows the following example. Take $X=\ell^2$, consider the shift operator $\Phi:\ell^2\to\ell^2$, namely

$$\forall(x^1,x^2,x^3,...)\in\ell^2 \qquad \Phi(x^1,x^2,x^3,...)=(0,x^1,x^2,...)\ ,$$

and the discrete (x-continuous) dynamical system $\{\Phi^n\}$: then, $\{0\}$ is the minimal w-attractor but $\{\Phi^n\}$ does not admit a bounded absorbing set. □

Assume now that on the space F it is defined a semigroup of operators $\{S_t,\ t\ge 0\}$ which satisfies the following translation relations (see [6]):

$$S_tF=F\quad\forall t\ge 0\ ,\qquad U_{S_sf}(t,\tau)=U_f(t+s,\tau+s)\quad\forall s\ge 0,\ f\in F,\ \tau\in\mathbf{R},\ t\in\mathbf{R}_\tau\ . \qquad (3)$$

With this assumption we prove that the uniform ω-limit does not depend on τ:

Proposition 1 *If (3) holds, then for all $\tau_1,\tau_2\in\mathbf{R}$ and all $B\in\mathcal{B}$ we have $\omega_{\tau_1}(B)=\omega_{\tau_2}(B)$.*

Therefore, from now on, when (3) holds we simply denote by $\omega(B)$ the uniform ω-limit set of $B\in\mathcal{B}$; in this case, under the assumptions of Theorem 1, by Proposition 1 we obtain directly the following characterization of the minimal w-u-attractor:

$$\mathcal{A}=\overline{\bigcup_{n\in\mathbf{N}}\omega(B_n)}\ .$$

However, when (3) holds and F is compact we can give a more precise characterization of $\mathcal{A}$ by adapting the approach of [6] to our context: on the space $X\times F$ define the family of operators $\{T_t\}$ ($t\in\mathbf{R}^+$) by

$$\forall(x,f)\in X\times F\qquad\forall t\ge 0\qquad T_t(x,f):=\Big(U_f(t,0)x,S_tf\Big)\ . \qquad (4)$$

We recall that a set $A\subset X\times F$ is *invariant* with respect to the family of operators $\{T_t\}$ if $T_tA=A$ for all $t\ge 0$. We prove

Theorem 2 *Assume (1) and let F be a compact space. Assume that a family of processes $\{U_f(t,\tau);\ f\in F\}$ is defined on X_w and that a semigroup of operators $\{S_t,\ t\geq 0\}$ is defined on F and satisfies (3). Assume moreover that:*
(i) $\{U_f(t,\tau);\ f\in F\}$ possesses the minimal w-u-attractor $\mathcal{A}$;
(ii) $\{U_f(t,\tau);\ f\in F\}$ is fx-continuous.
Then the family of operators $\{T_t\}$ defined in (4) is a semigroup which possesses the minimal w-attractor $\mathcal{T}\subset X\times F$ which is invariant with respect to the semigroup $\{T_t\}$. Moreover, $\mathcal{A}$ is the projection of $\mathcal{T}$ onto X and can be characterized by

$$\mathcal{A}=\Big\{x(0)\Big|\ x(t)\ \textit{is an arbitrary complete trajectory of}\ U_f(t,\tau)\ \textit{for some}\ f\in F\Big\}\ .$$

Let us underline the fact that assumption (i) of Theorem 2 may be replaced by either (i) or (ii) of Theorem 1. Finally, we are interested in sufficient conditions for the connectedness of the minimal w-u-attractor (when it exists) related to a family of processes:

Theorem 3 *Let X satisfy (1) and let F be a connected topological space; let $\{U_f(t,\tau),\ f\in F\}$ be a family of t-continuous and fx-continuous processes on X_w. If there exists the minimal w-u-attractor $\mathcal{A}$, then $\mathcal{A}$ is connected.*

4 Proofs of the main results

The proof of Theorem 1 requires two lemmas:

Lemma 1 *Make the assumptions of Theorem 1. If the family of processes is UCI and B is a (nonempty) bounded set then $\omega_\tau(B)$ is nonempty.*

Proof. Let $\tau\in\mathbf{R}$, $x\in B$, $f\in F$ and consider $t_n:=n(|\tau|+1)$ $(n\in\mathbf{N})$. As the family of processes is UCI, we infer that the sequence $\{U_f(t_n,\tau)x\}$ is relatively compact in X_w; hence, there exists $\bar{x}\in X$ such that $U_f(t_n,\tau)x\to\bar{x}$, up to a subsequence: by (2) we infer that $\bar{x}\in\omega_\tau(B)$. This proves that $\omega_\tau(B)\neq\emptyset$. □

Lemma 2 *Let $A,B\in\mathcal{B}$; if A w-u-attracts B then $\omega_\tau(B)\subseteq\bar{A}$ for all $\tau\in\mathbf{R}$.*
Assume also that $\{U_f(t,\tau);\ f\in F\}$ is UCI; if $\omega_\tau(B)\subseteq A$ for all $\tau\in\mathbf{R}$, then A w-u-attracts B.

Proof. Assume that A w-u-attracts B: we claim that $\omega_\tau(B)\subseteq\bar{A}$ for all τ. Fix $\tau\in\mathbf{R}$ and let $x\in\omega_\tau(B)$: by (2) we know that

$$\exists\{x_n\}\subset B\ ,\ \exists\{f_n\}\subset F\ ,\ \exists t_n\to+\infty\quad \text{s.t.}\quad U_{f_n}(t_n,\tau)x_n\to x\ ;$$

hence, for all open set $\mathcal{O}\ni x$ there exists $n(\mathcal{O})$ such that

$$U_{f_n}(t_n,\tau)x_n\in\mathcal{O}\qquad \forall n\geq n(\mathcal{O})\ . \tag{5}$$

By contradiction, assume that $x\not\in\bar{A}$; as X_w is a Hausdorff space and $\bar{A}$ is w-compact, there exist two open sets $\mathcal{O}_x\ni x$ and $\mathcal{O}_A\supset\bar{A}$ such that $\mathcal{O}_x\cap\mathcal{O}_A=\emptyset$. By (5), we have $U_{f_n}(t_n,\tau)x_n\in\mathcal{O}_x$ for sufficiently large n; hence, $U_{f_n}(t_n,\tau)x_n\not\in\mathcal{O}_A$ for large n: this contradicts the fact that A is w-u-attracting B.

To prove the second statement, we also argue by contradiction. Assume that $\omega_\tau(B)\subseteq A$ for all τ; by contradiction, assume also that there exist $\tau\in\mathbf{R}$ and an open set $\mathcal{O}\supset A$ such that for all

$t \in \mathbf{R}_\tau$ there exist $\bar{t} \geq t$ and $\bar{f} \in F$ such that $U_{\bar{f}}(\bar{t},\tau)B \not\subset \mathcal{O}$: then, there exist three sequences $\{t_n\} \subset \mathbf{R}_\tau$ satisfying $t_n \to +\infty$, $\{x_n\} \subset B$ and $\{f_n\} \subset F$ such that

$$U_{f_n}(t_n,\tau)x_n \notin \mathcal{O} \ . \tag{6}$$

As the process is UCI, there exists $\bar{x} \in X$ such that

$$U_{f_n}(t_n,\tau)x_n \to \bar{x} \ , \tag{7}$$

up to a subsequence; by (2), we have $\bar{x} \in \omega_\tau(B)$ and therefore, $\bar{x} \in A \subset \mathcal{O}$, which contradicts (6)-(7). $\square$

Remark 2 By Lemmas 1 and 2, we infer that for all nonempty $B \in \mathcal{B}$ the set $\omega_\tau(B)$ is nonempty and w-compact for all $\tau \in \mathbf{R}$. $\square$

We are now ready to give the

Proof of Theorem 1. We prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$:

$\underline{(iii) \Rightarrow (i)}$: trivial.

$\underline{(i) \Rightarrow (ii)}$: we first prove that $\{U_f(t,\tau);\ f \in F\}$ is UCI. Fix $\tau \in \mathbf{R}$, $\{f_n\} \subset F$, $\{t_n\} \subset \mathbf{R}_\tau$ such that $t_n \to +\infty$ and $\{x_n\} \in \mathcal{B}$: we claim that the set $K := \{U_{f_n}(t_n,\tau)x_n;\ n \in \mathbf{N}\}$ is relatively compact in X_w; to this end, it is enough to prove that $\phi(K)$ is bounded for every ϕ in the dual space X' of X. So, let $\phi \in X'$ and fix $\varepsilon > 0$: due to the compactness of A, there exists a finite subset $\{a_1, ..., a_m\}$ of A such that the set $\mathcal{O} := \bigcup_i \{y \in X :\ |\phi(y - a_i)| < \varepsilon\}$ is an open neighbourhood of A. Since $\{x_n\} \in \mathcal{B}$ and A is a w-u-attractor, there exists $k \in \mathbf{N}$ such that $U_{f_n}(t_n,\tau)x_n \in \mathcal{O}$ for each $n \geq k$. Hence, for every $n \geq k$ there exists $i = 1, ..., m$ such that

$$|\phi(U_{f_n}(t_n,\tau)x_n - a_i)| < \varepsilon \ .$$

Now, let $c > 0$ be such that $|\phi(a)| < c$ for all $a \in A$ and $|\phi(U_{f_n}(t_n,\tau)x_n)| < c$ for all $n < k$. It follows that

$$|\phi(U_{f_n}(t_n,\tau)x_n)| < c + \varepsilon \qquad \forall n \in \mathbf{N} \ ,$$

which proves the claim. In order to prove (ii) (b), we just observe that, due to (i) and Lemma 2, one has $\omega_\tau(B_n) \subseteq A$ for all $n \in \mathbf{N}$ and $\tau \in \mathbf{R}$.

$\underline{(ii) \Rightarrow (iii)}$: let $\mathcal{A}$ be as in the statement (iii), then $\mathcal{A}$ is compact by (ii) (b). Moreover, by Lemma 1 we have $\omega_\tau(B_n) \neq \emptyset$ for all n, τ: this proves that $\mathcal{A} \neq \emptyset$. Take $B \in \mathcal{B}$, then there exists $n \in \mathbf{N}$ such that $B \subset B_n$; hence, $\omega_\tau(B) \subseteq \omega_\tau(B_n)$ for all τ. Therefore, $\omega_\tau(B) \subseteq \mathcal{A}$ for all τ, and, by Lemma 2 we infer that $\mathcal{A}$ w-u-attracts B: by arbitrariness of B, this proves that $\mathcal{A}$ is w-u-attracting. Finally, let $\mathcal{A}'$ be another closed w-u-attracting set, then by Lemma 2 we infer that $\omega_\tau(B_n) \subseteq \mathcal{A}'$ for all τ, n: hence, $\bigcup_\tau \bigcup_n \omega_\tau(B_n) \subseteq \mathcal{A}'$ and, since $\mathcal{A}'$ is closed, $\mathcal{A} \subseteq \mathcal{A}'$; this proves that $\mathcal{A}$ is minimal. $\square$

Proof of Proposition 1. Let $B \in \mathcal{B}$, assume that $\tau_2 > \tau_1$ and let $\bar{\tau} = \tau_2 - \tau_1$; by (3), for all $t \in \mathbf{R}_{\tau_2}$ we have

$$\bigcup_{f \in F} U_f(t,\tau_2)B = \bigcup_{f \in F} U_{S_{\bar{\tau}}f}(t - \bar{\tau},\tau_1)B = \bigcup_{f \in F} U_f(t - \bar{\tau},\tau_1)B \ .$$

Therefore, for all $s \in \mathbf{R}_{\tau_2}$ we infer

$$\bigcup_{t \geq s} \bigcup_{f \in F} U_f(t,\tau_2)B = \bigcup_{t \geq s} \bigcup_{f \in F} U_f(t - \bar{\tau},\tau_1)B = \bigcup_{t \geq s - \bar{\tau}} \bigcup_{f \in F} U_f(t,\tau_1)B \ ;$$

finally, by taking the closure and intersecting for $s \geq \tau_2$ we get

$$\bigcap_{s\geq\tau_2} \overline{\bigcup_{t\geq s} \bigcup_{f\in F} U_f(t,\tau_2)B} = \bigcap_{s-\bar{\tau}\geq\tau_1} \overline{\bigcup_{t\geq s-\bar{\tau}} \bigcup_{f\in F} U_f(t,\tau_1)B} \ ,$$

that is, the result. □

In order to prove Theorem 2, consider the space $X_w \times F$ endowed with the product topology and note that by Proposition 3.1 in [6] the family of operators $\{T_t\}$ is a semigroup. We say that a subset $Y \subseteq X \times F$ is *bounded in* $X_w \times F$ if its projection $\Pi_X(Y)$ on the space X is bounded. Extending Definition 3 we say that a set $A \subseteq X \times F$ w-attracts a bounded set B if for each open set $\mathcal{O} \supset A$ of $X_w \times F$ there exists $\bar{t}$ such that $T_t(B) \subseteq \mathcal{O}$ for all $t \geq \bar{t}$. The minimal w-attractor $\mathcal{T}$ (if it exists) is a compact invariant set which attracts every bounded subset of $X_w \times F$. For every bounded set B, we define $\omega(B) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} T_t(B)}$.

Lemma 3 *Make the assumptions of Theorem 2; then, for each nonempty bounded set $B \subset X_w \times F$ $\omega(B)$ is a (nonempty) invariant set. Moreover, for all compact set A and all bounded set B one has that A attracts B if and only if $\omega(B) \subseteq A$.*

Proof. The fact that $\omega(B)$ is invariant follows from the definition because $\{T_t\}$ is a semigroup.

By Theorem 1, the compactness of F and assumption (i) of Theorem 2, one readily shows that the dynamical system $\{T_t\}$ satisfies the following property: for each bounded sequence $\{(x_n, f_n)\} \subset X_w \times F$ and every $t_n \to +\infty$ the set $\{T_{t_n}(x_n, f_n);\ n \in \mathbf{N}\}$ is relatively compact in $X_w \times F$. Then the proof follows similarly to the ones of Lemmas 1 and 2. □

The first part of Theorem 2 is proved by means of

Lemma 4 *Under the hypotheses of Theorem 2, there exists the minimal w-attractor $\mathcal{T}$ of the dynamical system (4).*

Proof. Denote by Σ the set of compact w-attractors of $X \times F$: we have $\Sigma \neq \emptyset$ because $\mathcal{A} \times F \in \Sigma$; hence, we can define $\Sigma_0 := \bigcap_{A\in\Sigma} A$. From Lemma 3 it follows that $\omega(B) \subseteq \Sigma_0$ for each bounded set $B \subset X \times F$. Therefore, Σ_0 is a (nonempty) compact w-attractor; moreover, since Σ_0 is bounded, one has $\omega(\Sigma_0) \subseteq \Sigma_0$. On the other hand, by Lemma 3, for all bounded set $B \subset X \times F$ and all $A \in \Sigma$ we have $\omega(B) \subseteq A$; hence, $\omega(B) \subseteq \Sigma_0$. Then, as $\omega(B)$ is invariant by Lemma 3, we get

$$\omega(B) = \omega^2(B) \subseteq \omega(\Sigma_0) \ :$$

again by Lemma 3 this implies that $\omega(\Sigma_0) \in \Sigma$; this proves that $\Sigma_0 \subseteq \omega(\Sigma_0)$, that is, $\Sigma_0 = \omega(\Sigma_0)$. It follows that $\mathcal{T} := \Sigma_0$ is invariant and, by its definition, it is also the minimal w-attractor of $\{T_t\}$. □

Let us now complete the

Proof of Theorem 2. Let $\mathcal{T}$ denote the minimal w-attractor of (4) found in Lemma 4 and set $U := \Pi_X(\mathcal{T})$. Obviously, U is a nonempty w-compact subset of X; we prove that U is a w-u-attracting set. Since F is compact, for each $\tau \in \mathbf{R}$ and every bounded subset B of X, one has that $\mathcal{T}$ attracts $B \times F$: then, by (4), for all open set $\mathcal{O} \subset X_w$ satisfying $\mathcal{O} \supset U$ there exists $\bar{t} \geq 0$ such that $U_f(t,0)B \subseteq \mathcal{O}$ for all $t \geq \bar{t}$ and all $f \in F$; by (3) this implies that for all $\tau \in \mathbf{R}$ there exists $T = T(\tau, B) \in \mathbf{R}_\tau$ such that

$$\bigcup_{f\in F} U_f(t,\tau)B \subseteq \mathcal{O} \ ,$$

that is, U is a compact w-u-attracting set. By definition of $\mathcal{A}$, this proves that $\mathcal{A} \subseteq U$.

In order to prove the converse relation, note that the steps used to obtain (3.13) in the proof of Corollary 3.1 in [6] do not depend on the topology involved and hence, we have:

$$U = \left\{ x(0) \middle| \, x(t) \text{ is an arbitrary bounded complete trajectory of } U_f(t,\tau) \text{ for some } f \in \Pi_Y(\mathcal{T}) \right\} .$$

So, fix $\bar{x} \in U$; then, there exists $f_0 \in F$ and a complete bounded trajectory $x(t)$ of $U_{f_0}(t,\tau)$ such that $x(0) = \bar{x}$. As F is invariant, for all $n \in \mathbf{N}$ there exists $f^n \in F$ such that $S_n f^n = f_0$; then, by (3), for all $n \in \mathbf{N}$ we have

$$\bar{x} = x(0) = U_{f_0}(0,-n)x(-n) = U_{S_n f^n}(0,-n)x(-n) = U_{f^n}(n,0)x(-n) .$$

Since $B_0 := \{x(-n) : n \in \mathbf{N}\}$ is bounded, by (2) we have $\bar{x} \in \omega(B_0)$; then, by Lemma 2, we get $\bar{x} \in \mathcal{A}$, which proves that $U \subseteq \mathcal{A}$. □

Proof of Theorem 3. By contradiction, assume that $\mathcal{A}$ is not connected and let $\mathcal{A} = A_1 \cup A_2$, where A_1 and A_2 are nonempty, disjoint, compact subsets of X_w; since X_w is a Hausdorff space, there exist two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{O}_1 \supset A_1$, $\mathcal{O}_2 \supset A_2$ and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$.

Take $\tau \in \mathbf{R}$, then, as $\mathcal{A}$ is w-u-attracting, for all $n \in \mathbf{N}$ there exists $t_n \in \mathbf{R}_\tau$ such that

$$\bigcup_{f \in F} U_f(t,\tau)B_n \subseteq \mathcal{O}_1 \cup \mathcal{O}_2 \qquad \forall t \geq t_n \; ; \tag{8}$$

since $F \times B_n$ is connected and by fx-continuity, the set $\cup_{f\in F} U_f(t,\tau)B_n$ is also connected for all t. Therefore, (8) implies that for all $t \geq t_n$ there exists $i = i(t) \in \{1,2\}$ such that

$$\bigcup_{f \in F} U_f(t,\tau)B_n \subseteq \mathcal{O}_i \;\; :$$

by t-continuity we infer that i does not depend on t; hence, there exists $i \in \{1,2\}$ such that

$$\bigcup_{f \in F} U_f(t,\tau)B_n \subseteq \mathcal{O}_i \qquad \forall t \geq t_n \; .$$

By repeating the above arguments for B_{n+1} we infer that there exists $j \in \{1,2\}$ such that

$$\bigcup_{f \in F} U_f(t,\tau)B_{n+1} \subseteq \mathcal{O}_j \qquad \forall t \geq t_{n+1} \;\; :$$

since $B_n \subset B_{n+1}$ we infer that $i = j$. Therefore, there exists $i \in \{1,2\}$ such that for all $B \in \mathcal{B}$ we have

$$\bigcup_{f \in F} U_f(t,\tau)B \subseteq \mathcal{O}_i \qquad \text{for all } t \text{ large enough;}$$

this implies that $\mathcal{A} \subset \mathcal{O}_i$ and contradicts the assumption that $\mathcal{A}$ is not connected. □

5 Some applications

In this section we apply the results of Section 3 to prove the existence and connectedness of attractors of processes associated to some differential equations; to this end, we introduce some functional spaces. Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with smooth boundary; we denote by L^p

the space of p^{th} power absolutely integrable functions, by $W^{m,p}$ the Sobolev spaces of functions in L^p with their first m generalized derivatives in L^p, by $H^m = W^{m,2}$ the Hilbertian Sobolev spaces, by H_0^m the H^m-closure of the space of smooth functions with compact support in Ω and by γ_n the normal trace operator. To simplify notations we delete the domain of definition Ω and we denote $\partial_t = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1, ...n)$.

To describe the long-time behavior of the solutions of a differential equation we need to define some other spaces. If $p \in [1, +\infty)$ and X is a Banach space we denote by $L^p(\tau, t; X)$ the space of functions defined on $[\tau, t]$ with values in X for which the p^{th} power of the X-norm is integrable on $[\tau, t]$; by $L^\infty(\tau, t; X)$ we denote the space of functions whose X-norm is essentially bounded on $[\tau, t]$. For all $p \in [1, +\infty)$ we define the Banach space of L^p_{loc}-*translation bounded* (t.b.) functions on $\mathbf{R}$ taking values in a Banach space X:

$$L^p_{tb}(\mathbf{R}; X) := \left\{ f \in L^p_{loc}(\mathbf{R}; X) : \sup_{s \in \mathbf{R}} \int_s^{s+1} \|f(t)\|_X^p dt < \infty \right\}$$

endowed with the norm

$$\|f\|_{L^p_{tb}(X)} = \sup_{s \in \mathbf{R}} \left(\int_s^{s+1} \|f(t)\|_X^p dt \right)^{1/p} .$$

5.1 A modified Navier-Stokes equation

We consider the modification of the Navier-Stokes equations for incompressible fluids suggested by Prouse [22]: here u and p denote respectively the velocity vector and the pressure of the fluid. The modified Navier-Stokes equations subject to an external force f read as follows:

$$\begin{cases} \partial_t u - \Delta\varphi(u) + (u \cdot \nabla)u + \nabla p - \nabla(\nabla \cdot \varphi(u)) = f & \text{in } \Omega \times [\tau, T] \\ \nabla \cdot u = 0 \quad \text{in } \Omega \times [\tau, T] \\ u = 0 \quad \text{on } \partial\Omega \times [\tau, T] \\ u(x, \tau) = u_0(x) \quad \text{if } x \in \Omega \end{cases} \tag{9}$$

where $\tau \in \mathbf{R}$ and $T > \tau$; we refer to [22] for physical motivations of (9). We consider the Hilbert spaces

$$H := \{u \in L^2;\ \nabla \cdot u = 0 \ \text{ and } \gamma_n u = 0\} \qquad V := \{u \in H_0^1;\ \nabla \cdot u = 0\}$$

and the dual space V^* of V. We assume that the function φ in (9) satisfies

$$\begin{cases} \varphi(u) = \sigma(|u|)u \\ \sigma \in C^1(\mathbf{R}^+), \quad \sigma(\xi) \geq \mu > 0, \quad \sigma'(\xi) \geq 0 \qquad \forall \xi \in \mathbf{R}^+ \\ \text{if } n \geq 3 \quad \exists s \geq n+1,\ \exists \alpha, \beta, \xi_0 > 0 \quad \text{such that} \quad \beta\xi^{s-1} \geq \sigma(\xi) \geq \alpha\xi^{s-1} \quad \forall \xi \geq \xi_0 \\ \text{if } n = 2 \quad \exists s \geq 1,\ \exists \alpha, \beta, \xi_0 > 0 \quad \text{such that} \quad \beta\xi^{s-1} \geq \sigma(\xi) \geq \alpha\xi^{s-1} \quad \forall \xi \geq \xi_0. \end{cases} \tag{10}$$

Let $f \in L^1(\tau, T; H) + L^2(\tau, T; V^*)$ and $u_0 \in H$; we say that u solves (9) if

$$\begin{cases} u \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^{s+1}(\tau, T; L^{s+1}) \\ \langle \partial_t u - \Delta\varphi(u) + (u \cdot \nabla)u - f, h \rangle = 0 \\ \forall h \in L^2(\tau, T; V) \cap L^\infty(\tau, T; H) \cap L^{s+1}(\tau, T; W^{2,s+1}) \\ u(x, \tau) = u_0(x) \ . \end{cases}$$

Extending previous results of [11, 22], in [12] a family of processes associated to (9) has been studied independently of the dimension $n \geq 2$. By taking into account the results of [12] and of the previous sections we obtain

Proposition 2 *Let φ be as in (10), assume that $f \in L^1(\tau, T; H) + L^2(\tau, T; V^*)$ and $u_0 \in H$; then there exists a unique solution u of (9) and u is continuous in $[\tau, T]$ in the H_w-topology.*
Moreover, if F is any bounded subset of $L^2_{tb}(V^*)$, then the family of processes $\{U_f(t,\tau),\ f \in F\}$ associated to (9) possesses a minimal connected H_w-u-attractor $\mathcal{A}$.

Proof. Existence, uniqueness and continuity of the solution of (9) are proved in Theorem 2.2 in [12].

Let F be a bounded subset of $L^2_{tb}(V^*)$, then F is w-compact in $L^2_{loc}(\mathbf{R}; V^*)$, see [8]: Theorem 4.6 in [12], Theorems 1 and 2 yield the existence (and characterization) of a minimal H_w-u-attractor $\mathcal{A}$ for the family of processes $\{U_f(t,\tau),\ f \in F\}$; by Proposition 4.8 in [12], all the assumptions of Theorem 3 are satisfied: hence, the w-u-attractor $\mathcal{A}$ is connected. □

Remark 3 The existence of a w-u-attractor for (9) is proved in [12] under more restrictive assumptions; more precisely, f is required to be translation compact (t.c.) in $L^1_{loc}(\mathbf{R}; H) + L^2_{loc}(\mathbf{R}; V^*)$, that is, it is assumed that the *hull* of f, defined by

$$\mathcal{H}(f) := \overline{\Big\{f(\cdot + s) :\ s \in \mathbf{R}\Big\}}^{L^1_{loc}(\mathbf{R};H)+L^2_{loc}(\mathbf{R};V^*)}\ ,$$

is compact in $L^1_{loc}(\mathbf{R}; H) + L^2_{loc}(\mathbf{R}; V^*)$: we denote by $L^1_{tc}(H) + L^2_{tc}(V^*)$ the set of such functions. Let $f \in L^1_{tc}(H) + L^2_{tc}(V^*)$, take $X = H$ and $F = \mathcal{H}(f)$, then all the assumptions of Theorem 3 are satisfied: hence, the attractor $\mathcal{A}$ defined in Theorem 2.3 in [12] is connected.
Finally, for all $f \in L^2_{tb}(V^*)$, one may apply Theorems 1-3 by letting F be the *weak hull* of f (see [8]), namely the weak $L^2_{loc}(\mathbf{R}; V^*)$-closure of the set $\{f(\cdot + s) :\ s \in \mathbf{R}\}$. □

5.2 General parabolic equations with a monotone principal part

In this section we consider a class of degenerate parabolic equations which have been studied in the autonomous case by Babin-Vishik [2] and other authors.

Consider n functions a_i $(i = 1, ..., n)$ satisfying $a_i \in C^2 \cap W^{1,\infty}(\mathbf{R}^n, \mathbf{R})$ and

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \zeta_j}(\zeta)\xi_i\xi_j \geq 0 \qquad \forall \zeta, \xi \in \mathbf{R}^n$$

$$\exists \mu_0, \mu_1 > 0 \quad \text{s.t.} \quad \mu_1(1 + |\zeta|^2) \geq \sum_{i=1}^{n} a_i(\zeta)\zeta_i \geq \mu_0|\zeta|^2 \qquad \forall \zeta \in \mathbf{R}^n\ .$$

Consider also n^2 functions b_{ij} $(i, j = 1, ..., n)$ satisfying $b_{ij} \in L^\infty(\Omega)$ and

$$b_{ij} \equiv b_{ji} \qquad \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \geq 0 \qquad \forall \xi \in \mathbf{R}^n,\ \forall x \in \Omega\ .$$

Finally, consider a function $b \in L^\infty(\Omega)$ and n functions b_i $(i = 1, ..., n)$ satisfying $b_i \in C^1(\bar{\Omega})$. Under these assumptions we study the problem

$$\begin{cases} \partial_t u - \sum_{i,j=1}^{n} \partial_i [b_{ij}(x)\partial_j u] + \sum_{i=1}^{n} b_i(x)\partial_i u - \sum_{i=1}^{n} \partial_i [a_i(\nabla u)] + u^3 + b(x)u = f \quad \text{in } \Omega \times [\tau, T] \\ u = 0 \quad \text{on } \partial\Omega \times [\tau, T] \\ u(x, \tau) = u_0(x) \quad \text{if } x \in \Omega \ . \end{cases} \tag{11}$$

The following result holds:

Proposition 3 *Assume that $u_0 \in L^2(\Omega)$ and $f \in L^2(\mathbf{R}; L^2)$; then, under the above assumptions, (11) admits a unique solution*

$$u \in L^\infty(\tau, T; L^2) \cap L^2(\tau, T; H_0^1) \cap L^4(\tau, T; L^4) \cap C(\tau, T; L_w^2) \ .$$

Moreover, if $F = \{g \in L^2(\mathbf{R}; L^2);\ \|g\|_{L^2(\mathbf{R};L^2)} \le R\}$ for some $R > 0$, then the family of processes $\{U_f(\tau, t);\ f \in F\}$ admits a minimal connected L_w^2-u-attractor $\mathcal{A}$.

Proof. In the autonomous case, the proof of existence is performed by the standard Galerkin method, see e.g. Theorem 1.3.1 in [2]: when f depends on t it suffices to remark that the estimates of step 2 pp. 40-41 in [2] still hold. Uniqueness follows by arguing by contradiction.

In order to apply our abstract results to (11), let us first draw an energy estimate. By reasoning as in the proof of Theorem 1.3.1 in [2], i.e. by multiplying (11) by $u(t)$ in L^2 and by taking into account Proposition 1.3.1 in [2] we arrive at

$$\frac{d}{dt}\|u(t)\|_2^2 + \|u(t)\|_2^2 \le \|f(t)\|_2^2 + K$$

where K is a positive constant depending only on the functions a_i, b_{ij}, b and b_i; therefore, we obtain

$$\frac{d}{dt}\Big(\|u(t)\|_2^2 e^t\Big) \le \Big(\|f(t)\|_2^2 + K\Big)e^t \ ,$$

and finally, by integrating over $[\tau, t]$, we get

$$\|u(t)\|_2^2 \le \|u_0\|_2^2 e^{\tau - t} + \|f\|_{L^2(\mathbf{R};L^2)}^2 + K(1 - e^{\tau - t}) \ . \tag{12}$$

Take $X = L^2$, $F = \{g \in L^2(\mathbf{R}; L^2);\ \|g\|_{L^2(\mathbf{R};L^2)} \le R\}$ for some $R > 0$ so that F is w-compact in $L^2(\mathbf{R}; L^2)$; then, (12) implies that the set $A := \{v \in L^2;\ \|v\|_2^2 \le K + 1 + R^2\}$ is a bounded u-absorbing set. Theorems 1 and 2 then yield the existence (and characterization) of a minimal attractor $\mathcal{A}$. Moreover, the family of processes is ftx-continuous: hence, Theorem 3 implies that $\mathcal{A}$ is connected. □

Remark 4 The previous result may also be stated for a wider class of forcing terms f and when in (11) the term u^3 is replaced by a more general function $g(u)$. □

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