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Existence of Minima for Nonconvex Functionals in Spaces of Functions Depending on the Distance from the Boundary

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Abstract

We prove that some nonconvex functionals admit a unique minimum in a functional space of functions which depend only on the distance from the boundary of the (plane) domain where they are defined. The domains considered are disks and regular polygons. We prove that the sequence of minima of the functional on the polygons converges to the unique minimum on the circumscribed disk as the number of sides tends to infinity. Our method also allows us to determine the explicit form of the minima.

1. Introduction

Let Ω be an open bounded domain of $\mathbf{R}^2$. We consider the cases where Ω is either a disk or a regular polygon. Let $h : \mathbf{R}^+ \to \bar{\mathbf{R}}$ be a (not necessarily convex) superlinear lower semicontinuous (l.s.c.) function and consider the functional J defined by

$$J(u) = \int_\Omega [h(|\nabla u|) + u]\, dx.$$

We study the following problem of existence of minima,

$$\min_{u \in \mathscr{K}} J(u), \tag{1}$$

where $\mathscr{K}$ is the subset of $W_0^{1,1}(\Omega)$ of functions depending only on the distance from the boundary $\partial\Omega$. We call the functions in $\mathscr{K}$ *web functions* because their level lines recall a spider's web. When Ω is a disk, web functions are nothing but radially symmetric functions.

Since no convexity on h is required, the functional J may not have a minimum on the whole space $W_0^{1,1}(\Omega)$. In such a case it is usual to introduce the relaxed functional J^* and consider its minimum, which coincides with the minimum of J

if the latter exists. In Section 6.1 we study a problem from shape optimization. In this case the minimum of J does not generally exist, so that the minimum of J^* corresponds to an optimal design which is not easily manufactured. For this reason we could decide to seek an optimal design in a simpler class of possible designs as, for instance, the class of web functions.

When $\Omega = D_R$ (an open disk of given radius $R > 0$) it is known [4] that the functional J admits a unique minimum u in $W_0^{1,1}(D_R)$ which is radially symmetric. We will prove that for all $n \geqq 3$ and for all regular polygons Ω_n of n sides inscribed in D_R, problem (1) on Ω_n admits a unique solution u_n for which we are able to give the explicit form. Then we prove that the unique solution u of (1) on D_R may be obtained as the (uniform) limit of the sequence $\{u_n\}$. This also enables us to give the explicit form of the function u.

We believe that our results may be a starting point for further research, provided answers can be found to some natural questions. First of all, since $\mathscr{K} \subset W_0^{1,1}(\Omega)$ we clearly have

$$\min_{u\in\mathscr{K}} J(u) \geqq \inf_{u\in W_0^{1,1}(\Omega)} J(u).$$

It would be interesting to understand for which kind of functionals J (i.e., for which functions h) equality holds. In other words, are there functionals for which there is a possible minimizing function in the set $\mathscr{K}$? In [2] there is an example of a convex functional on a square whose minimum does not have convex sublevels. The first step towards answering the above question would then be to determine sufficient conditions for the possible minimum to have convex sublevels.

If the radius R is smaller than some constant related to h, the results in [3] show that the minima of J on $W_0^{1,1}(\Omega_n)$ exist and depend linearly on $d(x, \partial\Omega_n)$. More general sets Ω (other than polygons or disks) are also considered. Another natural question which arises is the following. If a web function minimizes J on $W_0^{1,1}(\Omega)$, is it necessarily linear with respect to $d(x, \partial\Omega)$? When Ω is a disk the answer is negative, as our explicit form (see (4) below) clearly states (see also [4]). We believe that it is also negative when Ω is a polygon.

Can our results be extended to a wider class of functionals J or to general convex domains $\Omega \subset \mathbf{R}^2$? Probably Theorems 1 and 2 cannot be completely extended to more general problems, but maybe this is possible in a weaker form, as in Section 5.1.

The outline of this paper is as follows. In Section 2 we state our main existence, uniqueness and convergence results and we determine explicitly the solutions to the minimum problems. These results are proved in Sections 3 and 4. In Section 5 we make several remarks; in particular, we partially extend our results to a slightly more general class of functionals J. Finally, some applications are given in Section 6.

2. Main results

For simplicity, let D_R be the open disk centered at the origin of given radius $R > 0$, let $A \in \partial D_R$ and let Ω_n be the regular polygon of n ($n \geqq 3$) sides inscribed in D_R and having a vertex in A. Consider the sets of web functions

$$\mathscr{K} = \{u \in W_0^{1,1}(D_R);\ \ u(x) = u(|x|) \quad \forall\, x \in D_R\},$$
$$\mathscr{K}_n = \{u \in W_0^{1,1}(\Omega_n);\ u(x) = u(d(x, \partial\Omega_n)) \quad \forall\, x \in \Omega_n\}.$$

Note that $\mathscr{K}_n \subset C(\bar{\Omega}_n \setminus \{O\})$. Indeed, a discontinuity in some point $x \in \bar{\Omega}_n \setminus \{O\}$ would imply the discontinuity on a whole polygon and would not allow the function to belong to $W_0^{1,1}$.

We assume that the function $h \not\equiv +\infty$ satisfies the following two conditions:

$$(h1) \qquad h : \mathbf{R}^+ \to \bar{\mathbf{R}} \text{ is l.s.c.,}$$

$$(h2) \quad \begin{cases} \text{there exists a convex l.s.c. increasing function } \Phi : \mathbf{R}^+ \to \mathbf{R} \text{ such that} \\ h(t) \geqq \Phi(t) \text{ for all } t \in \mathbf{R}^+ \\ \lim\limits_{t\to\infty} \dfrac{\Phi(t)}{t} = +\infty. \end{cases}$$

We consider the problems of minimizing the functional J on $\mathscr{K}$ and $\mathscr{K}_n$:

$$(P) \qquad \min_{u\in\mathscr{K}} \int_{D_R} [h(|\nabla u|) + u]\, dx,$$
$$(P_n) \qquad \min_{u\in\mathscr{K}_n} \int_{\Omega_n} [h(|\nabla u|) + u]\, dx.$$

In order to solve these problems we introduce the function h^*, the convexification of h (the supremum of the convex functions less or equal to h), and we denote by Σ its support, namely

$$\Sigma := \{t \geqq 0;\ \ h^*(t) < +\infty\}.$$

Next, we define the functions

$$T^-(\sigma) = \min\left\{t \in \Sigma;\ \ \frac{h^*(t+\varepsilon) - h^*(t)}{\varepsilon} \geqq \frac{\sigma}{2} \quad \forall\, \varepsilon > 0\right\},$$
$$T^+(\sigma) = \max\left\{t \in \Sigma;\ \ \frac{h^*(t) - h^*(t-\varepsilon)}{\varepsilon} \leqq \frac{\sigma}{2} \quad \forall\, \varepsilon > 0\right\},$$

where we use the convention that $h^*(t+\varepsilon) - h^*(t) = -\infty$ for all $\varepsilon > 0$ and all t strictly less than any element of Σ, while $h^*(t) - h^*(t-\varepsilon) = +\infty$ for all $\varepsilon > 0$ and all t strictly greater than any element of Σ. Since h^* is convex, it has left and right derivatives at every point $t \in \Sigma$ (with the same convention as above for the points of $\partial\Sigma$, if they exist). We denote such derivatives by $(h^*)'_-(t)$ and $(h^*)'_+(t)$. Then, it is not difficult to verify that an equivalent definition of the functions $T^\pm$ is

$$T^-(\sigma) = \min\left\{t \in \Sigma;\, (h^*)'_+(t) \geqq \frac{\sigma}{2}\right\}, \quad T^+(\sigma) = \max\left\{t \in \Sigma;\, (h^*)'_-(t) \leqq \frac{\sigma}{2}\right\}. \tag{2}$$

In what follows we will make use of both the above characterizations of these functions. We also refer to Section 5.2 for some properties of $T^\pm$.

We first prove the following result.

Theorem 1. *Assume* $(h1)$ *and* $(h2)$*; then, for all* $n \geqq 3$*, problem* (P_n) *admits a unique solution* $u_n \in \mathscr{K}_n$*. Moreover,* $u_n \in W_0^{1,\infty}(\Omega_n)$*, and it is explicitly expressed by*

$$u_n(x) = -\int_0^{d(x,\partial\Omega_n)} T^-\Big(R\cos\frac{\pi}{n} - \sigma\Big)\, d\sigma. \tag{3}$$

With an abuse of notation we also denote by u_n the function in $W_0^{1,1}(D_R)$ obtained by extending u_n by 0 on $D_R \setminus \Omega_n$, and we prove that the sequence $\{u_n\}$ converges as $n \to \infty$ to the unique solution of the minimizing problem in the disk.

Theorem 2. *Assume* $(h1)$ *and* $(h2)$*; for all n let* u_n *be the solution of* (P_n) *and let u be the unique solution of* (P)*. Then, we have* $u_n \rightharpoonup^* u$ *in* $W_0^{1,\infty}(D_R)$*; moreover,* $\lim_{n\to\infty} J(u_n) = J(u)$ *and* $u_n \to u$ *uniformly. Therefore, u is explicitly expressed by*

$$u(x) = -\int_{|x|}^{R} T^-(\sigma)\, d\sigma. \tag{4}$$

3. Proof of Theorem 1

Let h^* be the convexification of h and let J^* be the corresponding functional

$$J^*(u) = \int_{\Omega_n} [h^*(|\nabla u|) + u]\, dx$$

where $n \geqq 3$ is fixed. Consider the relaxed problem

$$(P_n^*) \qquad \min_{u\in\mathscr{K}_n} \int_{\Omega_n} [h^*(|\nabla u|) + u]\, dx.$$

We first prove the following lemma.

Lemma 1. *The problem* (P_n^*) *admits a solution* $u_n \in \mathscr{K}_n$*.*

Proof. The set $\mathscr{K}_n$ is clearly a linear space. Moreover, let $\{u_m\} \subset \mathscr{K}_n$ satisfy $u_m \rightharpoonup u$ in $W_0^{1,1}(\Omega_n)$ for some $u \in W_0^{1,1}(\Omega_n)$. Then there is a subsequence of u_m converging to u a.e., which proves that $u \in \mathscr{K}_n$ and that $\mathscr{K}_n$ is weakly closed.

The functional J^* is convex and by $(h2)$ it is coercive, so that any minimizing sequence is bounded in $W_0^{1,1}$ and relatively compact in L^1. By Theorem 11 in [13] J^* is l.s.c. with respect to the L^1 norm topology and therefore it admits a minimum. □

Since problem (P_n) is autonomous we may translate and rotate Ω_n so that it lies in the half plane $x_1 > 0$ and so that one of its sides has equation $x_1 = 0$, $0 \leqq x_2 \leqq 2R\sin\frac{\pi}{n}$. Let $\ell_n = R\sin\frac{\pi}{n}$, $\lambda_n = R\cos\frac{\pi}{n}$ and $\vartheta_n = \tan\frac{\pi}{n}$, and define the triangle

$$T_n = \{(x_1, x_2) \in \Omega_n;\ x_1 \in (0, \lambda_n),\ x_2 \in (\vartheta_n x_1, 2\ell_n - \vartheta_n x_1)\}. \tag{5}$$

Then, by symmetry properties of functions in $\mathscr{K}_n$, we have

$$\forall\, u \in \mathscr{K}_n, \quad J(u) = n \int_{T_n} [h(|\nabla u|) + u]\, dx, \quad J^*(u) = n \int_{T_n} [h^*(|\nabla u|) + u]\, dx. \tag{6}$$

Moreover, any solution $\bar{u}$ of (P_n^*) may be represented by

$$\forall\, (x_1, x_2) \in T_n, \quad \bar{u}(x_1, x_2) = -\int_0^{x_1} |\nabla \bar{u}(s, x_2)|\, ds. \tag{7}$$

Indeed, if $\bar{u}$ minimizes J^*, we clearly have $\bar{u}(x) \leqq 0$ for all $x \in \Omega_n$, and since $\bar{u} \in \mathscr{K}_n$ its level lines in T_n have equations $x_1 = c$ and $\nabla \bar{u}$ is a.e. orthogonal to such lines, see Lemma A.2 p. 50 in [7].

Existence of a solution. We claim that any solution $\bar{u}$ of (P_n^*) is also a solution of (P_n).

Note that there exists at most a countable set of intervals $[t_1^m, t_2^m]$ $(m \in \mathbf{N})$ on which h^* is affine; denote by a_m the slope of h^* in such intervals, that is,

$$(h^*)'(t) = a_m \quad \forall\, t \in (t_1^m, t_2^m).$$

To prove the claim it suffices to show that

$$|\nabla \bar{u}(x)| \notin \bigcup_{m \in \mathbf{N}} (t_1^m, t_2^m) \quad \text{for a.e. } x \in \Omega_n.$$

Since (P_n^*) is a minimizing problem we have

$$\forall\, m \in \mathbf{N}, \quad a_m \leqq 0 \implies |\nabla \bar{u}(x)| \notin (t_1^m, t_2^m) \quad \text{for a.e. } x \in \Omega_n. \tag{8}$$

Indeed, for contradiction, assume that there exists $T \subset T_n$ of positive measure such that $|\nabla \bar{u}(x)| \in (t_1^m, t_2^m)$ for all $x \in T$. Characterize the function $v \in \mathscr{K}_n$ by $\nabla v(x) = \frac{t_2^m}{|\nabla \bar{u}(x)|} \nabla \bar{u}(x)$ for all $x \in T$ and by $\nabla v(x) = \nabla \bar{u}(x)$ for a.e. $x \in T_n \setminus T$. Then, $h^*(|\nabla v(x)|) < h^*(|\nabla \bar{u}(x)|)$ in T and, by (7), $v(x) \leqq \bar{u}(x)$ in T_n which yield $J^*(v) < J^*(\bar{u})$, a contradiction.

By using (8) we can prove that $|\nabla \bar{u}(x)| \notin (t_1^m, t_2^m)$ for all m such that $a_m > 0$. We fix any such m and consider two different cases according to the value of R.

The case $R \cos \frac{\pi}{n} \leqq 2a_m$. In this case we will prove that $|\nabla \bar{u}(x)| \leqq t_1^m$ for a.e. $x \in \Omega_n$. For contradiction, assume that there exist $\varepsilon > 0$ and a subset $\omega \subset \Omega_n$ of positive measure such that $|\nabla \bar{u}(x)| \geqq t_1^m + \varepsilon$ for all $x \in \omega$; then, by the symmetry of $\bar{u}$ there exists a set $I \subset [0, \lambda_n]$ of positive one-dimensional measure such that $|\nabla \bar{u}(x_1, x_2)| \geqq t_1^m + \varepsilon$ for a.e. $(x_1, x_2) \in T_n$ such that $x_1 \in I$. Consider the function $v \in \mathscr{K}_n$ defined for all $(x_1, x_2) \in T_n$ by

$$|\nabla v(x_1, x_2)| = \begin{cases} |\nabla \bar{u}(x_1, x_2)| & \text{if } x_1 \notin I \\ |\nabla \bar{u}(x_1, x_2)| - \varepsilon & \text{if } x_1 \in I \end{cases} \quad v(x_1, x_2) = -\int_0^{x_1} |\nabla v(s, x_2)|\, ds. \tag{9}$$

Then by (7) and by Fubini Theorem we get

$$
\begin{aligned}
&\int_0^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}[v-\bar{u}]\,dx_2\,dx_1\\
&\qquad=\int_0^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}\int_0^{x_1}[|\nabla\bar{u}(s,x_2)|-|\nabla v(s,x_2)|]\,ds\,dx_2\,dx_1\\
&\qquad=\int_0^{\lambda_n}\int_s^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}[|\nabla\bar{u}(s,x_2)|-|\nabla v(s,x_2)|]\,dx_2\,dx_1\,ds\\
&\qquad=2\varepsilon\int_I\int_s^{\lambda_n}(\ell_n-\vartheta_n x_1)\,dx_1\,ds\\
&\qquad=\varepsilon\int_I(\vartheta_n x_1^2-2\ell_n x_1-\vartheta_n\lambda_n^2+2\ell_n\lambda_n)\,dx_1.
\end{aligned}
\tag{10}
$$

On the other hand, since $|\nabla\bar{u}(x_1,x_2)|-\varepsilon\geqq t_1^m$ whenever $x_1\in I$, by definition of v we also have

$$
\begin{aligned}
&\int_0^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}[h^*(|\nabla v|)-h^*(|\nabla\bar{u}|)]\,dx_2\,dx_1\\
&\qquad=\varepsilon\int_I\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}\frac{h^*(|\nabla\bar{u}|-\varepsilon)-h^*(|\nabla\bar{u}|)}{\varepsilon}\,dx_2\,dx_1\\
&\qquad\leqq-\varepsilon a_m\int_I\int_{\vartheta_n x_1}^{2\ell_n-\vartheta_n x_1}dx_2\,dx_1=-2\varepsilon a_m\int_I(\ell_n-\vartheta_n x_1)\,dx_1;
\end{aligned}
$$

this, together with (6) and (10) implies that

$$
\frac{J^*(v)-J^*(\bar{u})}{n}\leqq\varepsilon\vartheta_n\int_I[x_1^2+2(a_m-\lambda_n)x_1+\lambda_n^2-2a_m\lambda_n]\,dx_1. \tag{11}
$$

Consider the function $f(s)=s^2+2(a_m-\lambda_n)s+\lambda_n^2-2a_m\lambda_n$; it is not difficult to verify that $f(s)<0$ for all $s\in(0,\lambda_n)$ and all $\lambda_n\leqq 2a_m$: therefore, (11) proves that $J^*(v)-J^*(\bar{u})<0$ and contradicts the assumption that $\bar{u}$ minimizes J^*.

The case $R\cos\frac{\pi}{n}>2a_m$. From the previous case we know that $|\nabla\bar{u}(x)|\leqq t_1^m$ for a.e. $(x_1,x_2)\in T_n$ such that $x_1>\lambda_n-2a_m$; we claim that $|\nabla\bar{u}(x)|\geqq t_2^m$ for a.e. $(x_1,x_2)\in T_n$ such that $x_1\leqq\lambda_n-2a_m$. We argue as above, but here we achieve the contradiction by showing that $|\nabla\bar{u}(x)|$ may be increased. Assume that there exist $\varepsilon>0$ and a set $I\subset[0,\lambda_n-2a_m]$ of positive one-dimensional measure such that $|\nabla\bar{u}(x_1,x_2)|\leqq t_2^m-\varepsilon$ for a.e. $(x_1,x_2)\in T_n$ such that $x_1\in I$. Consider the function $v\in\mathscr{K}_n$ defined for all $(x_1,x_2)\in T_n$ by

$$
|\nabla v(x_1,x_2)|=\begin{cases}|\nabla\bar{u}(x_1,x_2)| & \text{if } x_1\notin I\\ |\nabla\bar{u}(x_1,x_2)|+\varepsilon & \text{if } x_1\in I\end{cases}\qquad v(x_1,x_2)=-\int_0^{x_1}|\nabla v(s,x_2)|\,ds.
$$

Then, by reasoning corresponding to (11) we get

$$
\frac{J^*(v)-J^*(\bar{u})}{n}\leqq\varepsilon\vartheta_n\int_I[-x_1^2+2(\lambda_n-a_m)x_1-\lambda_n^2+2a_m\lambda_n]\,dx_1.
$$

Consider the function $g(s) = -s^2 + 2(\lambda_n - a_m)s - \lambda_n^2 + 2a_m\lambda_n$; then, $g(s) < 0$ for all $s \in [0, \lambda_n - 2a_m)$ and this proves that $J^*(v) < J^*(\bar{u})$ which is a contradiction. The existence of a solution u_n to problem (P_n) is proved for all $R > 0$. □

Uniqueness of the solution. Let $u, v \in \mathscr{K}_n$ be two solutions of (P_n^*). Since u and v have the same level lines we infer

$$|\nabla v(x)|\, \nabla u(x) = |\nabla u(x)|\, \nabla v(x) \qquad \text{a.e. in } \Omega_n.$$

Uniqueness then follows by reasoning as in the proof of Theorem 10 in [6].

Proof of (3)**.** Let $u_n \in \mathscr{K}_n$ be the unique solution of (P_n). Exactly as in the proof of existence above, we may obtain the following results:

$$\begin{aligned} &|\nabla u_n(x)| \leqq T^-(\sigma) \qquad \text{for a.e. } x \in \Omega_n \text{ such that } d(x, \partial\Omega_n) \geqq R\cos\frac{\pi}{n} - \sigma. \\ &|\nabla u_n(x)| \geqq T^+(\sigma) \qquad \text{for a.e. } x \in \Omega_n \text{ such that } d(x, \partial\Omega_n) \leqq R\cos\frac{\pi}{n} - \sigma. \end{aligned} \tag{12}$$

Note also that (2) and the convexity of h^* entail

$$\forall\sigma \geqq 0 \qquad T^-(\sigma) \leqq T^+(\sigma). \tag{13}$$

Now we prove that

$$T^-(\sigma) < T^+(\sigma) \quad \Longrightarrow \quad h^* \text{ is affine in the interval } [T^-(\sigma), T^+(\sigma)]. \tag{14}$$

Indeed, for $\varepsilon > 0$ small enough we have $T^-(\sigma) + \varepsilon < T^+(\sigma)$. For such ε the map $t \mapsto h^*(t) - h^*(t - \varepsilon)$ is non-decreasing, therefore by taking into account the definition of $T^\pm$ we get

$$\frac{\sigma}{2} \leqq \frac{h^*(T^-(\sigma) + \varepsilon) - h^*(T^-(\sigma))}{\varepsilon} \leqq \frac{h^*(T^+(\sigma)) - h^*(T^+(\sigma) - \varepsilon)}{\varepsilon} \leqq \frac{\sigma}{2}$$

which proves that $h^* \in C^1[T^-(\sigma), T^+(\sigma)]$ and that $(h^*)'(t) = \frac{\sigma}{2}$ for all $t \in (T^-(\sigma), T^+(\sigma))$, that is, (14).

Since h^* is affine on at most a countable set of intervals, from (13) and (14) we infer that

$$T^-(\sigma) = T^+(\sigma) \qquad \text{for a.e. } \sigma \geqq 0. \tag{15}$$

By their definitions, the maps $T^\pm$ are non-decreasing and therefore they admit at most a countable set of discontinuities of the first kind (i.e., with left and right limits both finite but different). Such discontinuities correspond with $\sigma = 2a_m$ (double the slope of an affine part of h^*). We also refer to Section 5.2 for a more precise characterization of $T^\pm$. Now let $\bar{\sigma}$ be a point of continuity for T^+ and T^-, so that

$$\lim_{\sigma \to \bar{\sigma}} T^+(\sigma) = T^+(\bar{\sigma}) = T^-(\bar{\sigma}) = \lim_{\sigma \to \bar{\sigma}} T^-(\sigma).$$

Then, by (12), for all $\delta > 0$ we have

$$|\nabla u_n(x)| \leqq T^-(\sigma+\delta) \quad \text{for a.e. } x \in \Omega_n \text{ such that } d(x, \partial\Omega_n) \geqq R\cos\tfrac{\pi}{n} - \sigma - \delta,$$
$$|\nabla u_n(x)| \geqq T^+(\sigma-\delta) \quad \text{for a.e. } x \in \Omega_n \text{ such that } d(x, \partial\Omega_n) \leqq R\cos\tfrac{\pi}{n} - \sigma + \delta.$$

By letting $\delta \to 0$ we infer that $|\nabla u_n(x)| = T^-(\sigma)$ for a.e. $\sigma \geqq 0$ and a.e. $x \in \Omega_n$ such that $d(x, \partial\Omega_n) = R\cos\frac{\pi}{n} - \sigma$. Then (3) follows by (7). □

4. Proof of Theorem 2

For the proof of Theorem 2 we need the following density result which we think may be of independent interest.

Proposition 1. *Let Ω be a disk, and for all $n \geqq 3$ let Ω_n be a regular polygon of n sides inscribed in Ω. Let $\mathscr{K}_n$ denote the set of web functions relative to Ω_n, extended by* 0 *in $\Omega \setminus \Omega_n$. Let $p \in [1, \infty)$. Then, for any radially symmetric function $w \in W_0^{1,p}(\Omega)$ there exists a sequence $\{w_n\} \subset W_0^{1,p}(\Omega)$ such that $w_n \in \mathscr{K}_n$ and $w_n \to w$ in the $W_0^{1,p}(\Omega)$ norm topology. Moreover, the sequence $\{w_n\}$ may be chosen so that $|\nabla w_n(x)| \leqq |\nabla w(x)|$ for a.e. $x \in \Omega$.*

Proof. For simplicity, let Ω be the unit disk centered at the origin, and assume that all the Ω_n are symmetric with respect to the x_1-axis and such that one of their sides has equation $x_1 = \cos\frac{\pi}{n}$. To get the result, it suffices to prove that for all $\varepsilon > 0$ there exists $n \in \mathbf{N}$ ($n \geqq 3$) and a function $w_n \in \mathscr{K}_n \cap W_0^{1,p}(\Omega)$ such that $\|w_n - w\|_{1,p} < \varepsilon$.

Thus, fix $\varepsilon > 0$ and take a radially symmetric function $w_\varepsilon \in C_0^1(\Omega)$ such that $|\nabla w_\varepsilon(x)| \leqq |\nabla w(x)|$ for a.e. $x \in \Omega$ and

$$\|w_\varepsilon - w\|_{1,p} < \frac{\varepsilon}{3}. \tag{16}$$

This is always possible by a density argument. Since ∇w_ε is uniformly continuous in $\bar\Omega$ there exists $\delta > 0$ such that

$$\forall\, x^1, x^2 \in \bar\Omega \quad |x^1 - x^2| < \delta \quad \Longrightarrow \quad |\nabla w_\varepsilon(x^1) - \nabla w_\varepsilon(x^2)| < \frac{\varepsilon}{6|\Omega|^{1/p}}.$$

Next choose n large enough so that

$$\sin\frac{\pi}{n} < \delta \quad \text{and} \quad \left(\int_{\Omega\setminus\Omega_n} |\nabla w_\varepsilon(x)|^p dx\right)^{1/p} < \frac{\varepsilon}{3}. \tag{17}$$

For such n consider the triangle

$$T_n = \left\{(x_1, x_2) \in \Omega_n; \quad 0 \leqq x_1 \leqq \cos\frac{\pi}{n},\ 0 \leqq x_2 \leqq x_1 \tan\frac{\pi}{n}\right\}.$$

We wish to define a suitable function $w_n \in \mathscr{K}_n \cap W_0^{1,p}(\Omega)$ in T_n so that we obtain its definition on the whole Ω_n by symmetrization. We first set $w_n(\cos\frac{\pi}{n}, x_2) = 0$ for all $x_2 \in [0, \sin\frac{\pi}{n}]$. Next, for all $x_1 \in (0, \cos\frac{\pi}{n}]$ let $\bar{x}_2$ be such that $|\nabla w_\varepsilon(x_1, \bar{x}_2)| \leqq |\nabla w_\varepsilon(x_1, x_2)|$ for all $x_2 \in [0, x_1 \tan\frac{\pi}{n}]$ (such $\bar{x}_2$ exists because the map $x \mapsto |\nabla w_\varepsilon(x)|$ is continuous and the segment is compact). Then we set

$$\nabla w_n(x_1, x_2) = \nabla w_\varepsilon(x_1, 0)\frac{|\nabla w_\varepsilon(x_1, \bar{x}_2)|}{|\nabla w_\varepsilon(x_1, 0)|}$$
$$\forall x_1 \in (0, \cos\frac{\pi}{n}], \quad \forall x_2 \in [0, x_1 \tan\frac{\pi}{n}).$$

If $\nabla w_\varepsilon(x_1, 0) = \underline{0}$ (i.e. $\bar{x}_2 = 0$), we simply set $\nabla w_n(x_1, 0) = \underline{0}$. By (17), for all $(x_1, x_2) \in T_n$ we have

$$|(x_1, x_2) - (x_1, 0)| = x_2 \leqq x_1 \tan\frac{\pi}{n} \leqq \sin\frac{\pi}{n} < \delta \quad \text{and} \quad |(x_1, \bar{x}_2) - (x_1, 0)| < \delta.$$

Therefore

$$\begin{aligned}
|\nabla w_n(x_1, x_2) - \nabla w_\varepsilon(x_1, x_2)| &= \left|\nabla w_\varepsilon(x_1, 0)\frac{|\nabla w_\varepsilon(x_1, \bar{x}_2)|}{|\nabla w_\varepsilon(x_1, 0)|} - \nabla w_\varepsilon(x_1, x_2)\right| \\
&\leqq |\nabla w_\varepsilon(x_1, 0) - \nabla w_\varepsilon(x_1, x_2)| \\
&\quad + \Big||\nabla w_\varepsilon(x_1, 0)| - |\nabla w_\varepsilon(x_1, \bar{x}_2)|\Big| \\
&< \frac{\varepsilon}{6|\Omega|^{1/p}} + |\nabla w_\varepsilon(x_1, 0) - \nabla w_\varepsilon(x_1, \bar{x}_2)| \\
&< \frac{\varepsilon}{3|\Omega|^{1/p}}.
\end{aligned}$$

By symmetry of w_ε and w_n, the previous inequality holds a.e. in Ω_n. Hence,

$$\begin{aligned}
&\|w_n - w\|_{1,p} \leqq \|w_n - w_\varepsilon\|_{1,p} + \|w_\varepsilon - w\|_{1,p}, \\
&\quad \text{by (16)} \; < \left(\int_{\Omega\setminus\Omega_n} |\nabla w_\varepsilon(x)|^p dx + \int_{\Omega_n} |\nabla w_n(x) - \nabla w_\varepsilon(x)|^p dx\right)^{1/p} + \frac{\varepsilon}{3}, \\
&\quad \text{by (17)} \; < \left(\frac{\varepsilon^p}{3^p}\Big(1 + \frac{|\Omega_n|}{|\Omega|}\Big)\right)^{1/p} + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

Moreover, $|\nabla w_n(x)| \leqq |\nabla w_\varepsilon(x)| \leqq |\nabla w(x)|$ for a.e. $x \in \Omega$ and the proposition is proved. □

Proof of Theorem 2. From Theorem 1 we deduce that $\|u_n\|_{1,\infty} \leqq T^-(R\cos\frac{\pi}{n}) \leqq T^-(R)$. Hence, there exists a subsequence and $v \in W_0^{1,\infty}(D_R)$ such that $u_n \rightharpoonup^* v$ in $W_0^{1,\infty}$. Since the reasoning below is available on all subsequences, we have $u_n \rightharpoonup^* v$ in $W_0^{1,\infty}$ on the whole sequence. Then, as J^* is l.s.c. with respect to the weak* $W^{1,\infty}$-topology, we have

$$J^*(v) \leqq \liminf_{n\to\infty} J^*(u_n). \tag{18}$$

Let $u \in W_0^{1,1}(D_R)$ be the unique solution of (P): by Theorem 3 in [4] we know that u is radially symmetric. Now we wish to prove that $v \equiv u$, and to this end we consider two distinct cases.

The case $0 \in \Sigma$. Let $\{w_n\} \subset W_0^{1,1}(D_R)$ be the sequence relative to u given by Proposition 1 (with $p = 1$). We claim that

$$\lim_{n\to\infty} J^*(w_n) = J^*(u). \tag{19}$$

By adding a constant to h^* we may assume that $h^*(0) = 0$, and by $(h1)$ we know that $h^* \in C(\Sigma)$. Since $\nabla w_n \to \nabla u$ in $L^1(D_R)$, we know that $|\nabla w_n(x)| \to |\nabla u(x)|$ a.e. in D_R (up to a subsequence) and therefore $h^*(|\nabla w_n(x)|) \to h^*(|\nabla u(x)|)$ for a.e. $x \in D_R$. Moreover, since $|\nabla w_n(x)| \leqq |\nabla u(x)|$ a.e. in D_R, by $(h2)$ we also have $h^*(|\nabla w_n(x)|) \leqq h^*(|\nabla u(x)|) + C$ for a.e. $x \in D_R$ with $C = -\min_{t\geqq 0} h^*(t) \geqq 0$. Then, we apply the Lebesgue Theorem and obtain $\int_{\Omega_n} h^*(|\nabla w_n|) \to \int_{D_R} h^*(|\nabla u|)$, so that (19) follows.

Since u_n is a solution of (P_n^*), we have $J^*(u_n) \leqq J^*(w_n)$ for all n. This, together with (18) and (19) yields $J^*(v) \leqq J^*(u)$, which proves that v is a solution of (P). By uniqueness of the solution of such a problem, we get $v \equiv u$.

The case $0 \notin \Sigma$. In this case we cannot guarantee that $h^*(|\nabla w_n|) \in L^1(\Omega)$. Thus, let $\alpha := \max\{t \in \Sigma;\ h^*(t) \leqq h^*(s)\ \forall s \geqq 0\}$. Since $0 \notin \Sigma$ we clearly have $\alpha > 0$. Consider the function

$$\tilde{h}(t) = \begin{cases} h^*(\alpha) & \text{if } t \in [0, \alpha] \\ h^*(t) & \text{if } t \in [\alpha, +\infty) \end{cases}$$

and the corresponding functional defined by

$$\tilde{J}(w) = \int_{D_R} [\tilde{h}(|\nabla w|) + w]\, dx \qquad \forall\, w \in W_0^{1,1}(D_R).$$

Since u minimizes J^*, by arguing as for (8), we have $|\nabla u(x)| \geqq \alpha$ for a.e. $x \in D_R$ and therefore $J^*(u) = \tilde{J}(u)$. Hence, by using the Lebesgue Theorem as for (19) we can prove that

$$\lim_{n\to\infty} \tilde{J}(w_n) = \tilde{J}(u) = J^*(u). \tag{20}$$

If for all n we have $|\nabla w_n(x)| \geqq \alpha$ for a.e. $x \in \Omega_n$, then we can finish as in the previous case. Otherwise, for all n denote by T_n the triangle in (5) and let $T^n \subset T_n$ be the set where $|\nabla w_n(x)| < \alpha$; define $\bar{w}_n \in \mathscr{K}_n$ so that $\nabla \bar{w}_n$ coincides with ∇w_n in $T_n \setminus T^n$ and so that $\nabla \bar{w}_n(x) = (-\alpha, 0)$ for all $x \in T^n$. Then we have $\tilde{h}(|\nabla \bar{w}_n(x)|) = \tilde{h}(|\nabla w_n(x)|)$ for a.e. $x \in \Omega_n$ and by a representation like (7) we get $\bar{w}_n(x) \leqq w_n(x)$ in Ω_n. Therefore, we obtain $\tilde{J}(w_n) \geqq \tilde{J}(\bar{w}_n) = J^*(\bar{w}_n)$, and since u_n is a solution of (P_n^*) we have $J^*(u_n) \leqq J^*(\bar{w}_n)$ for all n. This, together with (18) and (20), yields $J^*(v) \leqq J^*(u)$ which proves again that $v \equiv u$.

To conclude, note that the uniform convergence $u_n \to u$ follows from the boundedness of $\{u_n\}$ in $W_0^{1,\infty}$ and from the Ascoli-Arzela Theorem, while by pointwise convergence ($u_n(x) \to u(x)$) and by a change of variables ($\sigma \mapsto R - \sigma$) we get (4). $\square$

5. Remarks and further results

5.1. An extension of Theorems 1 and 2

In this section we extend part of the statements of Theorems 1 and 2 to a slightly more general class of functionals J, that is,

$$J(u) = \int_\Omega [h(|\nabla u|) + g(u)]\, dx,$$

where g satisfies the following assumptions

$$(g) \qquad \begin{cases} g \text{ is convex,} \\ g \text{ is non-decreasing,} \\ g(t+s) - g(t) \leqq s \quad \forall t \leqq 0 \quad \forall s \geqq 0. \end{cases}$$

Let A be the set of strictly positive slopes of the affine parts of h^*. The set A may be (countably) infinite, finite or empty, this last case occurring when h^* is strictly convex whenever it is increasing. We consider first the case $A \neq \emptyset$ and assume that

$$a := \inf A > 0. \tag{21}$$

Let D_R, Ω_n, $\mathscr{K}$ and $\mathscr{K}_n$ have the same meanings as before.

Theorem 3. *Assume* $(h1)$, $(h2)$, (g) (21), *and that* $R \leqq 2a$. *Then, for all* $n \in \mathbf{N}$ $(n \geqq 3)$ *there exists a unique solution* u_n *of the problem*

$$(Q_n) \qquad \min_{u \in \mathscr{K}_n} \int_{\Omega_n} [h(|\nabla u|) + g(u)]\, dx.$$

Furthermore, $\|u_n\|_{1,\infty} \leqq T^-(2a \cos \frac{\pi}{n})$.

Moreover, there exists a radially symmetric solution u *of the problem*

$$(Q) \qquad \min_{u \in W_0^{1,1}} \int_{D_R} [h(|\nabla u|) + g(u)]\, dx$$

which satisfies (up to a subsequence):

(i) $u_n \rightharpoonup^* u$ *in* $W_0^{1,\infty}(D_R)$ *and* $u_n \to u$ *uniformly,*
(ii) $\lim_{n\to\infty} J(u_n) = J(u)$,
(iii) $\|u\|_{1,\infty} \leqq T^-(2a)$.

Proof. This follows closely the proof of Theorems 1 and 2, so we just give a sketch.

Let h^* and J^* have the same meaning as in Section 3, and consider the corresponding relaxed problem (Q_n^*). As in Lemma 1, (Q_n^*) admits a solution $u_n \in \mathscr{K}_n$. Again, we translate and rotate Ω_n and consider the triangle T_n in (5) so that

$$\forall u \in \mathscr{K}_n, \quad J(u) = n \int_{T_n} [h(|\nabla u|) + g(u)]\, dx,$$
$$J^*(u) = n \int_{T_n} [h^*(|\nabla u|) + g(u)]\, dx.$$

Let $\bar{u}$ be any solution of (Q_n^*). Since $(g)_2$ holds, we still have (7); in particular $\bar{u} \leqq 0$. By $(g)_2$ we also know that $|\nabla \bar{u}|$ never belongs to the (possible) interval where h^* is non-increasing, namely $|\nabla \bar{u}(x)| \geqq T^+(0)$ for a.e. $x \in \Omega_n$. We show that $\bar{u}$ is also a solution of (Q_n). To this end, it suffices to prove that $|\nabla \bar{u}(x)| \leqq T^-(2a \cos \frac{\pi}{n})$ for a.e. $x \in \Omega_n$.

For contradiction, assume that there exist $\varepsilon > 0$ and a set $I \subset [0, \lambda_n]$ of positive one-dimensional measure such that $|\nabla \bar{u}(x)| \geqq T^-(2a \cos \frac{\pi}{n}) + \varepsilon$ for a.e. $(x_1, x_2) \in T_n$ such that $x_1 \in I$. Define the function $v \in \mathscr{K}_n$ as in (9) and note that by $(g)_3$ and (10) we get

$$\begin{aligned}\int_0^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n - \vartheta_n x_1} [g(v) - g(\bar{u})]\, dx_2\, dx_1 &\leqq \int_0^{\lambda_n}\int_{\vartheta_n x_1}^{2\ell_n - \vartheta_n x_1} [v - \bar{u}]\, dx_2\, dx_1 \\ &= \varepsilon \int_I (\vartheta_n x_1^2 - 2\ell_n x_1 - \vartheta_n \lambda_n^2 + 2\ell_n \lambda_n)\, dx_1.\end{aligned}$$

Therefore, we obtain again (11) and we arrive at a contradiction proving that $\bar{u}$ also solves (Q_n) and that $\|u_n\|_{1,\infty} \leqq T^-(2a \cos \frac{\pi}{n})$.

Uniqueness follows again by reasoning as in the proof of Theorem 10 in [6].

Now we may proceed as in Section 4 with two slight differences. We know that v (the weak limit of u_n) is radially symmetric by Remark 8 below, and the existence of a radial solution u of (P) follows here by Theorem 2 in [4] but we do not know if it is unique. We take one such solution and by reasoning as in Section 4 we obtain (i) and (ii); finally, (iii) follows by the l.s.c. of the $W^{1,\infty}$ norm with respect to weak* convergence. □

Similarly, when $A = \emptyset$ we obtain the following result.

Theorem 4. *Assume* $(h1)$, $(h2)$ *and* (g). *Then, for all* $R > 0$ *and all* $n \in \mathbf{N}$ $(n \geqq 3)$ *there exists a unique solution* u_n *of the problem* (Q_n). *Furthermore,* $\|u_n\|_{1,\infty} \leqq T^-(R \cos \frac{\pi}{n})$.

Moreover, there exists a solution u *of the problem (Q) which satisfies* (i), (ii) *of Theorem 3 and* $\|u\|_{1,\infty} \leqq T^-(R)$.

Remark 1. Assumptions $(g)_1$ and $(g)_2$ are needed in order to ensure the existence of at least one radial solution of the minimum problem in $W_0^{1,1}(D_R)$, see [4]. Moreover, they allow us to prove that the (possible) minimum in $\mathscr{K}_n$ is negative. Assumption $(g)_3$ is a one-sided Lipschitz condition which is used to obtain the crucial inequality (11): obviously, if the Lipschitz constant of g is $L > 1$, it suffices to divide both h and g by L in order to apply the above results. Finally, not that other conditions of the same kind may be considered.

5.2. Properties of the functions T^+ and T^-

Proposition 2. *The function* T^- *is left continuous and the function* T^+ *is right continuous.*

Proof. We first prove that for all $\sigma > 0$ we have $\lim_{\delta\to 0^+} T^-(\sigma-\delta) = T^-(\sigma)$.

By definition of T^-, for all $\delta > 0$ we have

$$\frac{h^*(T^-(\sigma-\delta)+\varepsilon) - h^*(T^-(\sigma-\delta))}{\varepsilon} \geqq \frac{\sigma-\delta}{2} \qquad \forall\, \varepsilon > 0. \tag{22}$$

Let $T = \liminf_{\delta\to 0^+} T^-(\sigma-\delta)$ and let $\delta_n \to 0^+$ be a sequence such that $T = \lim_{n\to\infty} T^-(\sigma-\delta_n)$. Then, by taking $\delta = \delta_n$ in (22), by letting $n \to \infty$, and by continuity of h^* we get

$$\frac{h^*(T+\varepsilon) - h^*(T)}{\varepsilon} \geqq \frac{\sigma}{2} \qquad \forall\, \varepsilon > 0,$$

which proves that $T \geqq T^-(\sigma)$ since $T^-(\sigma)$ is the minimum satisfying the above property. On the other hand, as the map T^- is non-decreasing, we have

$$\limsup_{\delta\to 0^+} T^-(\sigma-\delta) \leqq T^-(\sigma),$$

which proves the left continuity of T^-.

For T^+ one can proceed similarly. □

Remark 2. If $h^* \in C^1(\mathbf{R}^+)$ and h^* is strictly convex, then by (2) we have

$$\forall\, \sigma \geqq 0 \qquad T^-(\sigma) = T^+(\sigma) = [(h^*)']^{-1}(\tfrac{\sigma}{2}).$$

Remark 3. The above proposition and remark give a precise picture of the functions $T^{\pm}$:

– If $(h^*)'_-(t) < (h^*)'_+(t)$ then $T^{\pm}(\sigma) = t$ for all $\sigma \in [2(h^*)'_-(t), 2(h^*)'_+(t)]$.
– $T^{\pm}$ are discontinuous and different only at the points $2a_m$ where a_m is the slope of some affine part of h^*;

$$\lim_{\delta\to 0^+} T^{\pm}(2a_m-\delta) = T^-(2a_m) < T^+(2a_m) = \lim_{\delta\to 0^+} T^{\pm}(2a_m+\delta).$$

Remark 4. By (15) we infer that (3) and (4) may be replaced respectively by

$$u_n(x) = -\int_0^{d(x,\partial\Omega_n)} T^+\Big(R\cos\tfrac{\pi}{n} - \sigma\Big)\, d\sigma \quad \text{and} \quad u(x) = -\int_{|x|}^{R} T^+(\sigma)\, d\sigma.$$

5.3. Miscellaneous remarks

Remark 5. The solutions of (P_n) and (P) are Lipschitz continuous since they belong respectively to $C(\bar{\Omega}_n \setminus \{O\}) \cap W_0^{1,\infty}$ and $C(\bar{D}_R \setminus \{O\}) \cap W_0^{1,\infty}$; and they are piecewise C^1 by their explicit forms (3) and (4), respectively. No more regularity is to be expected of (P_n), as ∇u_n is certainly discontinuous on the n radii of D_R corresponding to the vertices of Ω_n if $u_n \not\equiv 0$. On the other hand, further regularity for (P) is related to the smoothness of h^* by means of (4).

Remark 6. If we assume that $h^* \in C^1(\mathbf{R}^+)$, then the solution u_n of (P_n) satisfies the following generalized Euler equation

$$\int_{\Omega_n} \left(\operatorname{div}\Big[(h^*)'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|}\Big] - 1\right)\varphi = 0 \qquad \forall\, \varphi \in \mathscr{K}_n.$$

To see this, it suffices to consider the function $F(t) = J^*(u+t\varphi)$ and to require that $F'(0) = 0$. Note also that $\frac{\nabla u_n}{|\nabla u_n|}$ is a constant vector in each one of the n isosceles triangles which compose Ω_n.

Remark 7. The extension of our results to higher dimensional problems seems to be purely technical: we could consider regular polyhedra inscribed in a ball of $\mathbf{R}^N$ ($N \geqq 3$) and define web functions in a completely similar way. We believe that Theorems 1 and 2 continue to hold.

Remark 8. The statement of Proposition 1 may be inverted. Indeed, by using the a.e. pointwise convergence, we find that if $\{w_n\} \subset W_0^{1,p}(\Omega)$ satisfies $w_n \in \mathscr{K}_n$ and $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$ for some $p \in [1, +\infty)$, then w is radially symmetric in Ω.

Remark 9. In our original proof of Theorem 2 there was no distinction between the two cases $0 \in \Sigma$ and $0 \notin \Sigma$ and the functional $\tilde{J}$ was not introduced. Indeed, Proposition 1 may be proved in a slightly stronger version by obtaining as well the bound $|\nabla w_n(x)| \geqq \inf_\Omega |\nabla w(x)| = I$. In this case the function $w_\varepsilon \in C_0^1(\Omega)$, for which the previous inequality may not be fulfilled, is constructed. Then the function w_ε is modified by a piecewise affine function with slope equal to I in the region where $|\nabla w_\varepsilon(x)| < I$ and this new approximating error is estimated. This method requires too many calculations, which is why we prefer the above proof of Theorem 2.

6. Some applications

6.1. A problem from optimal design

Let $h_1(t) = \alpha t^2$, $h_2(t) = \beta t^2 + \gamma$ (with $\alpha > \beta > 0$, $\gamma > 0$) and

$$h(t) = \min\{h_1(t), h_2(t)\}, \tag{23}$$

and consider the functional

$$J(u) = \int_\Omega [h(|\nabla u|) + u]\, dx.$$

The problem of minimizing J on the space H_0^1 arises from elasticity [1,5,6]. We wish to place two different linearly elastic materials (of shear moduli $\frac{1}{2\alpha}$ and $\frac{1}{2\beta}$) in the plane domain Ω so as to maximize the torsional rigidity of the resulting rod when the proportions of these materials are prescribed. Such a problem may not have a solution, but one can construct new composite materials by mixing them together on a microscopic scale. Mathematically, this corresponds to the introduction of the relaxed problem which does have a minimum. Hence, there exists an optimal design if one is allowed to incorporate composites. However, the resulting design may not be so easy to manufacture and therefore one may have to try to find an optimal design in a simpler class of possible designs as, for instance, the class of web functions. When Ω is a square, numerical results [5,6] lead to the conjecture that in general J does not admit a minimum because a composite region seems to appear. By seeking the minimum in the class of web functions we avoid the possibility that the composite region is different from a frame (the part between two squares) and thus raise the natural question: Is the minimum of J attained in $\mathscr{K}_4$? As we have seen, the answer to this question is positive. Let us also mention that by Remarque 41 in [9], if Ω is simply connected and if J has a minimum in H_0^1 with a C^1 interface separating the two regions containing the two different materials, then Ω must be a disk and the optimal design consists of an annulus of strong material filled with a disk of soft material. We also refer to [12] where the limiting case of the soft material being replaced by empty regions is considered.

We now show how our results apply to this problem. Straightforward calculations yield

$$h^*(t) = \begin{cases} h_1(t) & \text{if } t \leqq t_1, \\ at+b & \text{if } t_1 \leqq t \leqq t_2, \\ h_2(t) & \text{if } t_2 \leqq t, \end{cases}$$

where

$$t_1 = \sqrt{\frac{\beta\gamma}{\alpha(\alpha-\beta)}}, \quad t_2 = \sqrt{\frac{\alpha\gamma}{\beta(\alpha-\beta)}}, \quad a = 2\sqrt{\frac{\alpha\beta\gamma}{\alpha-\beta}}, \quad b = \frac{\beta\gamma}{\beta-\alpha}.$$

We also find

$$T^-(\sigma) = \begin{cases} \frac{\sigma}{4\alpha} & \text{if } \sigma \leqq 2a, \\ \frac{\sigma}{4\beta} & \text{if } \sigma > 2a, \end{cases}$$

so that, by (3), the unique solution $u_n \in \mathscr{K}_n$ of (P_n) is given by $u_n(x) = \frac{1}{8\alpha}[d_n^2(x) - 2R\cos\frac{\pi}{n} d_n(x)]$ (where $d_n(x) = d(x, \partial\Omega_n)$) if $R\cos\frac{\pi}{n} \leqq 2a$, and by

$$u_n(x) = \begin{cases} \frac{1}{8\beta}(4a^2 - R^2\cos^2\frac{\pi}{n}) + \frac{1}{8\alpha}(d_n^2(x) - 2R\cos\frac{\pi}{n} d_n(x) + R^2\cos^2\frac{\pi}{n} - 4a^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } d_n(x) \geqq R\cos\frac{\pi}{n} - 2a \\ \frac{1}{8\beta}[d_n^2(x) - 2R\cos\frac{\pi}{n} d_n(x)] \qquad \text{if } d_n(x) < R\cos\frac{\pi}{n} - 2a \end{cases}$$

if $R\cos\frac{\pi}{n} > 2a$; while, by (4), the unique solution $u \in \mathscr{K}$ of (P) is given by $u(x) = \frac{1}{8\alpha}(|x|^2 - R^2)$ if $R \leqq 2a$, and by

$$u(x) = \begin{cases} \frac{1}{8\alpha}(|x|^2 - 4a^2) + \frac{1}{8\beta}(4a^2 - R^2) & \text{if } |x| \leqq 2a \\ \frac{1}{8\beta}(|x|^2 - R^2) & \text{if } |x| > 2a \end{cases}$$

if $R > 2a$. In the latter case $u \in H_0^1$ satisfies the Euler equation

$$\Delta u = \begin{cases} (2\alpha)^{-1} & \text{if } |x| \leqq 2a \\ (2\beta)^{-1} & \text{if } |x| > 2a \end{cases}$$

and has been determined in Remarque 40 in [9].

6.2. An approximating problem

Consider the case where

$$h(t) = \begin{cases} 0 & \text{if } t = 1, \\ 1 & \text{if } t = 2, \\ \infty & \text{elsewhere.} \end{cases}$$

This case was studied in [2] in an attempt to simplify the function h in (23) by retaining its essential feature of lacking convexity. It is not difficult to verify that

$$h^*(t) = \begin{cases} t - 1 & \text{if } t \in [1, 2], \\ \infty & \text{elsewhere,} \end{cases}$$

so that

$$T^-(\sigma) = \begin{cases} 1 & \text{if } \sigma \in [0, 2], \\ 2 & \text{if } \sigma \in (2, \infty). \end{cases}$$

Then, by (3), the unique solution of (P_n) is given by $u_n(x) = -d_n(x)$ if $R \cos \frac{\pi}{n} \leqq 2$, and it is given by

$$u_n(x) = \begin{cases} -2d_n(x) & \text{if } d_n(x) \leqq R \cos \frac{\pi}{n} - 2 \\ 2 - R \cos \frac{\pi}{n} - d_n(x) & \text{if } d_n(x) > R \cos \frac{\pi}{n} - 2 \end{cases}$$

if $R \cos \frac{\pi}{n} > 2$.

Note that if $R \cos \frac{\pi}{n} \leqq 1$, our solution is the "true" solution, namely, the minimum of J on $W_0^{1,1}(\Omega_n)$ (see [3]). Indeed, with the notation introduced in that paper, we have $W_{\Omega_n} = R \cos \frac{\pi}{n}$ and $\Lambda = 1$. On the other hand, the functional J is known to have no minimum in $W_0^{1,1}(\Omega_4)$ (the square) if $R \cos \frac{\pi}{4} \in (1, 1 + \varepsilon)$ for sufficiently small ε (see [2]). Therefore, when $R \cos \frac{\pi}{n} > 1$ we may conjecture that u_n just furnishes an approximate solution for the minimum of J^* on $W_0^{1,1}(\Omega_n)$. It would be interesting to estimate the "error"

$$E_n(R) = J(u_n) - \min_{W_0^{1,1}(\Omega_n)} J^*.$$

It is conceivable that

$$E_n(R) \to 0 \quad \text{as} \quad R \to \frac{1}{\cos \frac{\pi}{n}}.$$

Consider now the case of the disk D_R. By (4) we infer that $u(x) = |x| - R$ if $R \leqq 2$, and that

$$u(x) = \begin{cases} |x| + 2 - 2R & \text{if } |x| \leqq 2 \\ 2(|x| - R) & \text{if } |x| > 2 \end{cases}$$

if $R > 2$. This is the solution already described in the introduction of [2].

The natural extension of the above example is when the function h is defined by

$$h(t) = \begin{cases} \frac{t(t-1)}{2} & \text{if } t \in \mathbf{N}, \\ \infty & \text{elsewhere.} \end{cases}$$

In this case $T^+(\sigma) = [\frac{\sigma}{2}] + 1$ ($[x]$ denotes the integer part of x). Here we make use of T^+ since it has an elegant form (see Remark 4). Then, the explicit form of u_n (and u) is easily derived: the polygon (or disk) is the union of a central polygon (or disk) and of a finite number of frames (or annuli) of width 2; in the central polygon (or disk) the slope of u_n (or u) is 1 and the slope increases by 1 every time one skips into the following frame (or annulus).

6.3. A problem from glaciology

We consider the degenerate elliptic problem

$$\begin{cases} \Delta_p u = 1 & \text{in } D_R, \\ u = 0 & \text{on } \partial D_R, \end{cases} \tag{24}$$

where $p > 1$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. This equation (with nonlinear boundary conditions) has been applied to the description of some phenomena in glaciology (see [10,11]). In this case D_R is the cross-section of the glacier, u is the surface velocity, and $\Delta_p u$ represents the traction. Problem (24) has been widely studied and the explicit form of its solution is well-known (see, e.g., [8] and references therein) and may be obtained with ordinary differential equation methods. Here we determine it by means of our approach. Equation (24) is the Euler equation of the potential energy functional

$$J(u) = \int_{D_R} \Big(\frac{|\nabla u|^p}{p} + u \Big)\, dx,$$

and critical points of J are solutions of (24). In this case we have $h(t) = h^*(t) = \frac{t^p}{p}$ so that by Remark 2 we get

$$T^-(\sigma) = \Big(\frac{\sigma}{2}\Big)^{\frac{1}{p-1}}.$$

Inserting this in (4), we find that the unique (radial) minimum of J is given by

$$u(x) = \frac{p-1}{p 2^{1/(p-1)}} (|x|^{p/(p-1)} - R^{p/(p-1)}).$$

6.4. An estimate for the solution of a quasilinear problem

Let D_R be a disk of radius $R > 0$ and consider the problem

$$\begin{cases} \Delta_p u = \frac{1}{1+u^2} & \text{in } D_R, \\ u = 0 & \text{on } \partial D_R. \end{cases} \tag{25}$$

Define the function

$$\bar{g}(t) = \begin{cases} \arctan t & \text{if } t \leqq 0, \\ t & \text{if } t \geqq 0, \end{cases}.$$

which satisfies assumption (g). By Theorem 3 in [4] we know that the functional

$$J(u) = \int_{D_R} \left(\frac{|\nabla u|^p}{p} + \bar{g}(u) \right) dx$$

admits a unique minimum $\bar{u} \in W_0^{1,1}(D_R)$ which is radially symmetric. By Theorem 4 the function $\bar{u}$ is negative and satisfies

$$\|\bar{u}\|_{1,\infty} \leqq \left(\frac{R}{2} \right)^{\frac{1}{p-1}}. \tag{26}$$

Since $\bar{u}$ belongs to the cone of negative functions, it solves (25): hence, (25) admits a negative radially symmetric solution satisfying (26).

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