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EXISTENCE OF GROUND STATES AND FREE-BOUNDARY PROBLEMS FOR THE PRESCRIBED MEAN-CURVATURE EQUATION*

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Abstract. Existence and nonexistence of radially symmetric ground states and compact support solutions for a quasilinear equation involving the mean-curvature operator are studied in dependence of the parameters involved. Different tools are used in the proofs, according to the cases considered. Several numerical results are also given: the experiments show a possible lack of uniqueness of the solution and a strong dependence on the space dimension.

1. Introduction

Consider the prescribed mean-curvature quasilinear elliptic equation in (subsets of) $\mathbf{R}^n$

$$-\text{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = -au^q + u^p \ , \tag{1.1}$$

where $a > 0$, $-1 < q < p < \frac{n+2}{n-2}$ and $n \geq 3$. According to [15] we call *normal case* the case where $a > 0$ and *anomalous case* the case where $a = 0$. When $a = p = 0$ equation (1.1) is known as the Delaunay equation [8] while if $q = p = 1$ equation (1.1) arises in the analysis of capillary surfaces and of pendent drops [9] depending on whether $a > 1$ or $a < 1$.

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In this paper we mainly deal with the case $p \neq 1$, and we consider either solutions u of (1.1) satisfying

$$u \in C^2(\mathbf{R}^n), \quad u \not\equiv 0 \quad u(x) \geq 0 \quad \forall x \in \mathbf{R}^n, \qquad \lim_{|x| \to \infty} u(x) = 0 \ , \tag{1.2}$$

or solutions u of (1.1) satisfying the following homogeneous Dirichlet–Neumann free-boundary conditions

$$u \in C^2(B_R), \qquad u(x) > 0 \quad \forall x \in B_R, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R \ , \tag{1.3}$$

where $R > 0$ and B_R is an open ball in $\mathbf{R}^n$ centered at the origin. It is known [20] that if $q > 0$, then any nonnegative solution u of (1.1) whose (open) support is connected is radially symmetric about one point and radially decreasing as long as it remains positive. For this reason we restrict our attention to *radial* solutions of (1.1) and we call *ground state* a nontrivial nonnegative radially symmetric solution of (1.1)–(1.2) and *compact-support solution* a positive radially symmetric solution of (1.1)–(1.3) for some $R > 0$. If $q > 0$ and u is a compact support solution, then the function

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in B_R \\ 0 & \text{if } x \notin B_R \end{cases} \tag{1.4}$$

is a ground state. When no confusion arises, we simply denote by u also its extension $\tilde{u}$.

The distinction between ground states and compact-support solutions is due to the parameter q; see [10, section 1.3]. If $q \geq 1$, then any radially symmetric solution of (1.1) is positive on $\mathbf{R}^n$ and is a ground state, while if $q < 1$ then every radial solution of (1.1) has compact support. If further $0 < q < 1$, then we have multibump phenomena (see [17]) and compact-support solutions are also ground states by means of the extension (1.4).

The differential operator in (1.1) may be seen as a perturbation of the Laplace operator thanks to a suitable rescaling (see Remark 1 below). Moreover, it may be studied in a more general context of quasilinear operators in divergence form; see [10, 14, 15, 17]. In spite of these facts and of its important applications not much is known about (1.1). In particular, we mention that even the anomalous case where $a = 0$ is not yet completely understood; see [6, 15]. The reason is that solutions of (1.1) may exhibit vertical points [1, 2, 3], and this yields several difficulties. In the normal case $a > 0$, if $p \geq \frac{n+2}{n-2}$ then (1.1) admits no ground states and no compact-support solutions; see [14]. For this reason we restrict ourselves to the case $p < \frac{n+2}{n-2}$. For general a, p and q it is known [10] (see also Theorem 1 below) that if a is too large then (1.1) admits no ground states. On the other hand, by means of a perturbation technique, Peletier–Serrin [16] prove that if $q = 1$ and $p < \frac{n+2}{n-2}$, then (1.1) admits a ground state provided a is sufficiently small.

In this paper we pursue further the study of existence and nonexistence of ground states and compact support solutions of (1.1) by allowing the three parameters a, p and q to vary more freely. In particular, we also consider the case where $q \leq 0$. Our main purpose is to give explicit bounds on the parameter a for the existence and nonexistence of solutions. Of course, the explicit constants may have unpleasant forms. As expected, we obtain existence results if a is sufficiently small and nonexistence results if a is too large. We also test our results with several numerical experiments.

First of all, we observe that the action functional associated to (1.1) contains the term $\int \sqrt{1+|\nabla u|^2}$, and therefore the natural functional space where one could argue variationally seems to be the space BV; see [13] and references therein. In this setting, the subcritical assumption is $p < \frac{1}{n-1}$. However, as just mentioned, also the critical exponent for H^1 (i.e., $p = \frac{n+2}{n-2}$) plays an important role for (1.1). This means that there are two critical exponents for (1.1). The feeling that a transition occurs when $p = \frac{1}{n-1}$ is highlighted in Theorem 2 below, for which the tools involved in the proof work precisely if and only if $0 \leq p < \frac{1}{n-1}$. We prove there that if a is sufficiently small (we have an explicit upper bound) then there exists a compact support solution of (1.1), while if a is too large (with explicit lower bound given in Theorem 1) then there exist no compact-support solutions (we deal with compact-support solutions because $q < 1$). Unfortunately, there are values of a for which we are unable to establish existence or nonexistence. In Section 8 we quote some numerical results in order to test how fine are Theorem 2 and the tools involved in its proof. It turns out that the compact-support solution may not be unique for values of a near the nonexistence threshold. This fact shows a substantial difference between (1.1) and the corresponding quasilinear equation with the m-Laplacian Δ_m $(1 < m < n)$ for which one has uniqueness of a solution for all $q < p$; see [19].

In Theorem 3 we give an existence result for sublinear problems when the exponent p varies in a wider range than the one in Theorem 2: we merely require that $0 \leq p < 1$. However, in the common range $0 \leq p < \frac{1}{n-1}$, the statement of Theorem 2 yields existence results for a wider interval of coefficients a; see the numerical computations in Section 8.

Concerning the H^1 critical exponent, in Theorem 4 below (see also Corollary 1) we show that for nearly critical values of p (i.e., $p = (\frac{n+2}{n-2})^-$) one may have existence results only if a is sufficiently small and the upper bound for a tends to 0 as $p \to \frac{n+2}{n-2}$. In other words, the lower-order term au^q must become smaller and smaller as p grows towards $\frac{n+2}{n-2}$ and it must vanish at the limit. This result shows that $p = \frac{n+2}{n-2}$ is critical in a "continuous sense." On the contrary, it is known that for the m-Laplacian (in particular for the Laplacian Δ) one has existence for all $a > 0$ and all subcritical p (see [11]) and existence for critical p only if $a = 0$. In Section 8 we also give some numerical results relative to this case. In fact, the solution exists only for very small values of a, much smaller than the upper bound given in Theorem 4.

In the proof of the existence results we reduce (1.1) to an ordinary differential equation and we make use of a shooting method following the approach introduced by [5] for semilinear problems and later extended to quasilinear problems by [10] and others. Let us emphasize that the proof of each one of Theorems 1–4 involves its own tools which are quite different one from another. For this reason, general results for $-1 < q < p < \frac{n+2}{n-2}$ appear difficult to obtain. Unfortunately, among the above statements we do not have an existence result for superlinear problems ($p > 1$). In Section 8 we give some numerical results which show that even if we merely restrict to superlinear problems, a general theoretical rule appears difficult to obtain. Existence and uniqueness of the solutions seem to depend very strongly on the space dimension n.

Finally, slightly beyond the scopes of this paper, we study the decay at infinity of the ground states of (1.1). As already mentioned, in order to have positive ground states we need to assume that $q \geq 1$. Then, one merely expects polynomial decay at infinity; see [14, Proposition 5.1]. On the other hand, since the case $q = 1$ is the "borderline" which separates compact support solutions and positive ground states, it may be of particular interest. In Theorem 5 below we show that it is indeed a limit case since the corresponding ground states have exponential decay at infinity. This result is well-known for the Laplace operator [4, Theorem 1 (iv)]. Here, with a different proof, we extend it to the mean-curvature operator.

2. EXISTENCE AND NONEXISTENCE RESULTS

We start with a nonexistence result essentially due to Franchi–Lanconelli–Serrin [10]:

Theorem 1. *Assume that* $-1 < q < p < \frac{n+2}{n-2}$. *If*

$$a \geq K_{p,q} := \Big(\frac{(p+1)(q+1)}{p-q}\Big)^{(p-q)/(p+1)} ,$$

then (1.1) *admits no ground states and no compact-support solutions.*

Since we allow $q<0$ and for the sake of completeness, in next section we give an outline of the proof of Theorem 1. Note that $K_{p,q}$ *does not* depend on the space dimension n.

Our first result is an existence result for problems which are subcritical in the sense of BV:

Theorem 2. *Assume that* $-1<q<p<\frac{1}{n-1}$ *and* $p\geq 0$. *Let* $K=K_{p,q}$ *be the constant defined in Theorem* 1. *If*

$$a\leq \overline{K}(n,p,q):=K\Big(1+(n-1)\Big(\frac{p+1}{p+1-np}\Big)^{\frac{p+1}{p-q}}\Big(\frac{q+1-nq}{q+1}\Big)^{\frac{q+1}{p-q}}\Big)^{\frac{q-p}{p+1}},$$

then (1.1) *admits a compact-support solution. Such a solution is also a ground state if* $q>0$ *while if* $q\leq 0$, *then* (1.1) *admits no ground states. Moreover, there exists a constant* $\Gamma=\Gamma(p,q)$ *such that any compact-support solution* u_a *of* (1.1) *satisfies*

$$u_a(0)=\|u_a\|_\infty<\Gamma(p,q)\cdot a^{1/(p-q)}\ .$$

In particular, $u_a\to 0$ *uniformly as* $a\to 0$.

The explicit value of $\Gamma(p,q)$ is given in (4.3) below. In order to evaluate the strength of this theoretical result, we performed some numerical experiments. In Section 8 we quote some of the results we obtained and we comment on them for particular values of the parameters p and q; see Corollary 2 below. In particular, for some values of the parameter a we find *two compact-support solutions*.

We now turn to the general sublinear case, namely $-1<q<p$, $0\leq p<1$.

Let $A=A(n,p,q)>0$ be the unique solution of the equation

$$\Big(\frac{p+1}{2}\Big)^{\frac{1}{p+1}}=\frac{(n-1)^2}{n}2^{\frac{2p+1-3q}{p-q}-\frac{p}{p+1}}\frac{(p+1)^{\frac{1-q}{p-q}+\frac{p}{p+1}}}{(q+1)^{\frac{1-p}{p-q}}}A^{\frac{1-p}{p-q}}+\Big(2\frac{p+1}{q+1}A\Big)^{\frac{1}{p-q}}.$$

Moreover, we define

$$\overline{A}(n,p,q)=\min\Big\{A(n,p,q),\frac{q+1}{(p+1)^{\frac{q+1}{p+1}}}\frac{1}{2^{\frac{p-q+2}{p+1}}},\Big[\frac{(p+1)(q+1)}{2(p-q)}\Big]^{\frac{p-q}{p+1}}\Big\}.\tag{2.1}$$

Note that $\overline{A}\to 0$ if either $p\to 1$ or $q\to -1$ or $n\to\infty$. We can now state

Theorem 3. *Assume that* $-1<q<p<1$ *and* $p\geq 0$. *If*

$$a\leq \overline{A}(n,p,q),\tag{2.2}$$

then (1.1) *admits a compact-support solution which is also a ground state if* $q>0$.

If we compare Theorem 2 with Theorem 3 we remark that the former yields existence results for a wider class of coefficients a when $0 \leq p < \frac{1}{n-1}$ (see the computations in Section 8) while the latter yields existence results for a wider class of exponents p.

For the superlinear case we prove a nonexistence result:

Theorem 4. *Assume that* $1 \leq p < \frac{n+2}{n-2}$, $0 \leq q < p$ *and that*

$$p > \frac{2n(2q+1) - (n-2)(q+1)}{2n + (n-2)(q+1)} \ . \tag{2.3}$$

Then, (1.1) *admits neither ground states nor compact-support solutions if*

$$a \geq C(n,p,q)\Big(\frac{n+2}{n-2} - p\Big),$$

where

$$C(n,p,q) = \frac{(q+1)(4n^2p)^{\frac{p-q}{p+1}}}{(2^*-1-q)^{\frac{q+1}{p+1}}(p+1)^{\frac{q+1}{p+1}}[2^*(p-2q-1)+(p+1)(q+1)]^{\frac{p-q}{p+1}}}, \tag{2.4}$$

so that $C(n,p,q)$ *remains bounded as* p *approaches* $\frac{n+2}{n-2}$ *(here* $2^* = \frac{2n}{n-2}$*).*

Note that the constant in the right-hand side of (2.3) is strictly less than $\frac{n+2}{n-2}$ for all $q \in [0,p)$ so that the set of assumptions is not empty. Note also that if (2.3) holds, then $C(n,p,q)$ is well-defined and strictly positive. In several cases, this result improves Theorem 1; see Corollary 3 in Section 8. There, we also quote some numerical results which show that, in fact, the upper bound for existence results is much smaller than the one given in Theorem 4.

The most important consequence of Theorem 4 is that it shows that if p approaches the critical (H^1) exponent, then a must tend to 0 in order to hope to have existence results. We state this as

Corollary 1. *Let* $q \geq 0$ *and for all* $p > q$ *such that* $1 \leq p < \frac{n+2}{n-2}$ *let*

$$a^*(p) = \sup\{a > 0 : (1.1) \text{ admits a ground state}\} \ .$$

Then $a^*(p) > 0$ *for all* p *and*

$$\lim_{p \to \frac{n+2}{n-2}} a^*(p) = 0 \ .$$

The fact that $a^*(p) > 0$ for all $p < \frac{n+2}{n-2}$ is a consequence of the perturbation technique developed in [16, Lemma 4].

Finally, we study the behavior at infinity of the ground states of (1.1) when $q = 1$ and we prove that they decay exponentially. In other words, the next result states that in the limit case $q = 1$ the ground state "almost" behaves as a compact-support solution:

Theorem 5. *Assume that* $1 = q < p < \frac{n+2}{n-2}$, *and let* u *be a ground state of* (1.1). *Then, there exist* $\xi, \lambda, \mu > 0$ (*depending on* a *and* p) *such that if* $|x| \geq \xi$, *then*

$$u(x) \leq \mu e^{-\lambda|x|}, \qquad |Du(x)| \leq \mu e^{-\lambda|x|}, \qquad |D^2u(x)| \leq \mu e^{-\lambda|x|} \ .$$

Remark 1. All our results may be extended to the slightly more general class of equations

$$-\text{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = -au^q + bu^p \tag{2.5}$$

where $b > 0$. To see this, note that by the rescaling

$$v(x) = \Big(\frac{b^2}{a}\Big)^{1/(2p-q+1)} u\Big(\Big(\frac{b^{q-3}}{a^{p-1}}\Big)^{1/2(2p-q+1)} x\Big)$$

the previous equation becomes

$$-\text{div}\Big(\frac{Dv}{\sqrt{1+\varepsilon|Dv|^2}}\Big) = -\varepsilon v^q + v^p \ ,$$

where $\varepsilon = (a^{p+1}b^{-(q+1)})^{1/(2p-q+1)}$ (so that $\varepsilon \to 0$ as $a \to 0$ or $b \to \infty$). This may be seen as a perturbation of the corresponding equation for $\varepsilon = 0$; that is,

$$-\Delta v = v^p, \tag{2.6}$$

which is known to have no positive solutions whenever $p < \frac{n+2}{n-2}$; see [12].

We also refer to [7] for perturbation existence results concerning (2.5) in general bounded domains. These results are obtained for b large enough, that is, when (2.5) is sufficiently close to (2.6). □

3. An overview of the corresponding ODE

In this section we reduce (1.1) to an ordinary differential equation. If $v = v(|x|)$ is a smooth nonnegative radially symmetric solution of (1.1) then $v = v(r)$ is a (nonnegative) solution of the problem

$$\begin{cases} \Big(\dfrac{v'}{\sqrt{1+|v'|^2}}\Big)' + \dfrac{n-1}{r}\dfrac{v'}{\sqrt{1+|v'|^2}} - av^q + v^p = 0 \\ v(0) = v_0, \quad v'(0) = 0 \end{cases} \tag{3.1}$$

for some $v_0 > 0$. The "boundary conditions" (1.2) and (1.3) become respectively

$$v(r) \to 0 \text{ as } r \to \infty \tag{3.2}$$

and

$$\exists \rho > 0 \quad \text{such that} \quad v(\rho) = v'(\rho) = 0 \ . \tag{3.3}$$

We call *ground state* any nonnegative nontrivial solution of (3.1) defined for all $r > 0$ and satisfying (3.2). We call *compact-support solution* any solution of (3.1) which satisfies (3.3) and is positive on $[0, \rho)$.

In the sequel, we denote

$$f(s) = -as^q + s^p \quad \text{and} \quad F(s) = -\frac{a}{q+1}s^{q+1} + \frac{1}{p+1}s^{p+1} \ .$$

We also introduce the constants

$$\delta = a^{1/(p-q)}, \qquad \overline{F} = \frac{p-q}{(p+1)(q+1)}a^{(p+1)/(p-q)}, \qquad \beta = \Big(\frac{p+1}{q+1}a\Big)^{1/(p-q)} \ .$$

Note that $f(\delta) = 0$, $F(\delta) = -\overline{F} = \min_s F(s)$ and $F(\beta) = 0$ (clearly, $\delta < \beta$).

3.1. **Proof of Theorem 1.** We state here in our setting some results from [10]. Since the results in [10] are given for general quasilinear operators, we obtain somehow simpler versions. Moreover, since we also deal with possibly singular f (if $q < 0$) we also briefly sketch the proofs. In the final part of this section we extend the proof of Theorem 1 given in [10] to the case $q \leq 0$.

We first recall a local existence and uniqueness result:

Lemma 1. *For all $v_0 > 0$, (3.1) has a unique (classical) solution u in a neighborhood of the origin.*

Proof. Local existence and uniqueness of solutions of the Cauchy problem (3.1) are well-known; see for example [14, 15] and Propositions A1 and A4 in [10]. The fact that f may be singular at 0 is not important since $v_0 > 0$. $\square$

Next, we recall a crucial identity; see (1.1.4) in [10]:

Lemma 2. *Let $v_0 > 0$ and assume that $v = v(r)$ is a (positive) solution of (3.1) which can be continued on some (open) interval $(0, R)$, $R \in (0, \infty]$. Then for all $r, r_0 \in [0, R)$, we have*

$$\begin{aligned} &-\frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)) \\ &= -\frac{1}{\sqrt{1+|v'(r_0)|^2}} + F(v(r_0)) - (n-1)\int_{r_0}^{r} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}}\frac{dt}{t} \ . \end{aligned} \tag{3.4}$$

Proof. We first claim that the map

$$r \mapsto \frac{|v'(r)|^2}{r\sqrt{1+|v'(r)|^2}}$$

is locally integrable on $[0,R)$. Indeed, by (3.1) we have

$$\frac{|v'(r)|}{\sqrt{1+|v'(r)|^2}} \le \frac{1}{r^{n-1}} \int_0^r t^{n-1}|f(v(t))|dt \ .$$

Therefore, since $|f(v(r))|$ and $|v'(r)|$ are locally bounded on $[0,R)$, we have

$$\frac{|v'(r)|^2}{r\sqrt{1+|v'(r)|^2}} \le c(r)\frac{1}{r^{n-1}}r^n\frac{1}{r} = c(r) < \infty \qquad \forall r < R,$$

and the claim is proved. We now introduce the energy function

$$E(r) = -\frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)), \qquad r \ge 0. \tag{3.5}$$

By a straightforward calculation which makes use of (3.1), we find that E is continuously differentiable on J and

$$E'(r) = -\frac{n-1}{r}\frac{|v'(r)|^2}{\sqrt{1+|v'(r)|^2}} \ . \tag{3.6}$$

Let $r, r_0 \in [0,R)$; then (3.4) follows by replacing (3.5) and (3.6) in the identity

$$E(r) = E(r_0) + \int_{r_0}^r E'(t)dt \ . \qquad \square$$

As a straightforward consequence of Lemma 2 we obtain a necessary condition for a ground state of (3.1) to exist:

Lemma 3. *Let $v_0 > 0$ and assume that $v = v(r)$ is a ground state of* (3.1)–(3.2). *Then $\lim_{r\to\infty} v'(r) = 0$, and for all $r \ge 0$ we have*

$$-\frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)) = -1 + (n-1)\int_r^\infty \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}}\frac{dt}{t} \ . \tag{3.7}$$

In particular,

$$F(v_0) = (n-1)\int_0^\infty \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}}\frac{dt}{t} > 0 \ . \tag{3.8}$$

Proof. Let $r \to \infty$ in (3.4). Then, since $F(v(r)) \to 0$ and since the map

$$r \mapsto \int_{r_0}^{r} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}} \frac{dt}{t}$$

is nondecreasing for $r > r_0$, we infer that $\frac{1}{\sqrt{1+|v'(r)|^2}}$ admits a limit. Therefore, $v'(r)$ admits a limit (necessarily 0 by (3.2)) as $r \to \infty$.

Identity (3.7) follows by letting $r_0 \to \infty$ in (3.4), and (3.8) follows by taking $r = 0$ in (3.7). □

Similarly, we have

Lemma 4. *Let $v_0 > 0$ and assume that $v = v(r)$ is a compact-support solution of* (3.1)–(3.3)*. Then, for all $r \in [0, \rho)$, we have*

$$-\frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)) = -1 + (n-1) \int_{r}^{\rho} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}} \frac{dt}{t} \ . \tag{3.9}$$

In particular,

$$F(v_0) = (n-1) \int_{0}^{\rho} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}} \frac{dt}{t} > 0 \ . \tag{3.10}$$

We are now in position to complete the proof of Theorem 1; see Proposition 1.2.2 in [10]:

Proposition 1. *If $\overline{F} \geq 1$, then* (3.1) *has no ground states and no compact support solutions. Therefore, the condition*

$$a < \Big(\frac{(p+1)(q+1)}{p-q}\Big)^{(p-q)/(p+1)} \tag{3.11}$$

is a necessary condition for the existence of ground states and/or compact support solutions for (3.1)*.*

Proof. We prove the result for ground states, the proof being similar for compact-support solutions.

For contradiction, let v be a ground state of (3.1). Then, (3.8) implies that $v_0 > \beta$ (recall $F(\beta) = 0$). Since v is a ground state, $v(r)$ takes all the values in $(0, \beta]$. In particular, there exists R_δ such that $v(R_\delta) = \delta$ (recall $f(\delta) = 0$). By taking $r_0 = 0$ and $r = R_\delta$ in (3.4), we obtain

$$F(v_0) + \overline{F} = 1 - \frac{1}{\sqrt{1+|v'(R_\delta)|^2}} + (n-1) \int_{0}^{R_\delta} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}} \frac{dt}{t} \ .$$

By (3.8) we also obtain

$$\overline{F} = 1 - \frac{1}{\sqrt{1+|v'(R_\delta)|^2}} - (n-1)\int_{R_\delta}^{\infty} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}}\frac{dt}{t},$$

which contradicts $\overline{F} \geq 1$ and proves the first statement.

A simple computation shows that $\overline{F} < 1$ if and only if (3.11) holds. Hence, (3.11) is a necessary condition for the existence of ground states for (3.1). □

3.2. **A lower bound for the shooting level.** By (3.8) we know that a solution of (3.1) may exist only if $v_0 > \beta$. With this restriction, consider the local solution v of (3.1) given by Lemma 1. As proved in Lemma 1.1.1 in [10], we have

$$\frac{d}{dr}\Big(\frac{v'(r)}{\sqrt{1+|v'(r)|^2}}\Big)_{(r=0)} = -\frac{1}{n}f(v_0)\ . \tag{3.12}$$

As $v_0 > \beta$, we have $f(v_0) > 0$. This shows that $v'(r) < 0$ for small $r > 0$. Continuation of the solution given by Lemma 1 is standard: we denote by $J = (0, R)$, $R = R(v_0) \leq \infty$, the maximal open interval of continuation of v under the restriction

$$v(r) > 0, \quad -\infty < v'(r) < 0 \quad \text{in } J\ . \tag{3.13}$$

Clearly, for all $v_0 > \beta$ the corresponding interval J is uniquely determined. In the sequel we understand that every solution v of (3.1) is continued exactly to the corresponding maximal domain J.

From the Ni–Serrin identity [14], we obtain a Pohožaev-type inequality:

Lemma 5. *Let $v_0 > 0$ and assume that $v = v(r)$ is a ground state of* (3.1)–(3.2)*. Then*

$$\int_0^{\infty} s^{n-1}\left(2^* F(v(s)) - v(s)f(v(s))\right) ds > 0\ . \tag{3.14}$$

Let $v_0 > 0$ and assume that $v = v(r)$ is a compact-support solution of (3.1)–(3.3)*. Then,*

$$\int_0^{\rho} s^{n-1}\left(2^* F(v(s)) - v(s)f(v(s))\right) ds > 0\ . \tag{3.15}$$

Proof. Consider first the case of a ground state of (3.1). The identity corresponding to (3.1) in [14] is

$$\frac{d}{dr}\Big[r^n\Big(1 - \frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)) + \frac{\alpha v(r)v'(r)}{r\sqrt{1+|v'(r)|^2}}\Big)\Big]$$

$$= r^{n-1}\Big(n - \frac{n}{\sqrt{1+|v'(r)|^2}} + (\alpha+1-n)\frac{|v'(r)|^2}{\sqrt{1+|v'(r)|^2}}$$
$$+ nF(v(r)) - \alpha v(r) f(v(r))\Big) \qquad \forall \alpha \in \mathbf{R}.$$

Choose $\alpha = \frac{n-2}{2}$ and integrate over $[0,r]$:

$$r^n\Big(1 - \frac{1}{\sqrt{1+|v'(r)|^2}} + F(v(r)) + \frac{n-2}{2}\frac{v(r)v'(r)}{r\sqrt{1+|v'(r)|^2}}\Big)$$
$$= \int_0^r s^{n-1}\Big(n - \frac{n}{\sqrt{1+|v'(s)|^2}} - \frac{n}{2}\frac{|v'(s)|^2}{\sqrt{1+|v'(s)|^2}}$$
$$+ nF(v(s)) - \frac{n-2}{2}v(s)f(v(s))\Big)ds.$$

By the decay estimates in [14, Proposition 5.1], as $r \to \infty$ one sees that the left-hand side vanishes, and therefore

$$\int_0^\infty s^{n-1}\Big(n - \frac{n}{\sqrt{1+|v'(s)|^2}} - \frac{n}{2}\frac{|v'(s)|^2}{\sqrt{1+|v'(s)|^2}}\Big)ds$$
$$= \int_0^\infty s^{n-1}\Big(\frac{n-2}{2}v(s)f(v(s)) - nF(v(s))\Big)ds\ .$$

The first integral may be rewritten as

$$n\int_0^\infty s^{n-1}\phi(|v'(s)|^2)ds \quad \text{where} \quad \phi(t) = 1 - \frac{1+\frac{t}{2}}{\sqrt{1+t}}\ .$$

Since $\phi(t) < 0$ for all $t > 0$, we finally have

$$\int_0^\infty s^{n-1}\Big(\frac{n-2}{2}v(s)f(v(s)) - nF(v(s))\Big)ds < 0\ ,$$

that is, (3.14).

The proof of (3.15) in the case of compact-support solutions is similar: instead of the decay estimates one should use the boundary conditions $v(\rho) = v'(\rho) = 0$. □

We apply the previous result to obtain a lower bound for the shooting level of any possible ground state or compact-support solution:

Lemma 6. *Let v be either a ground state of* (3.1)–(3.2) *or a compact-support solution of* (3.1)–(3.3)*. Then,*

$$v_0 > \Big(\frac{2^* - (q+1)}{2^* - (p+1)}\frac{p+1}{q+1}a\Big)^{\frac{1}{p-q}}\ .$$

Proof. As they are similar, we give the proof only for ground states. Assume for contradiction that the converse inequality holds for a ground state v. Since v is decreasing, we have

$$v(r) \leq \Big(\frac{2^* - (q+1)}{2^* - (p+1)}\frac{p+1}{q+1}a\Big)^{\frac{1}{p-q}} \qquad \forall r \geq 0,$$

and therefore

$$2^* F(v(r)) - v(r)f(v(r) \leq 0 \qquad \forall r \geq 0$$

in contradiction with (3.14). □

3.3. **Nonexistence of vertical points and the shooting method.** In this section we study the possibility of vertical points for the solution of (3.1) in dependence of the shooting level v_0. Roughly speaking, we show that if v_0 is too large then v has a vertical point, while if v_0 is sufficiently small then v has no vertical points.

We first adapt a result by Serrin [18] to the situation of Theorem 4:

Lemma 7. *Let* $1 \leq p < \frac{n+2}{n-2}$, $0 \leq q < p$. *Assume that* $v_0 \geq \beta$, *that* $v_0^{q+1}(v_0^{p-q} - 2a) \geq 4n^2 p$ *and let* v *be the* (*local*) *solution of* (3.1). *Then,* v *ends at a vertical point.*

Proof. By arguing as in the proof of [18, Theorem 2] we obtain that the graph of v (i.e., the couples $(r, v(r))$) does not exit the bounded region

$$\left\{(r,y) \in \mathbf{R}^2 : \, r > 0, \, v_0 - \frac{1}{\sqrt{f'(v_0)}} \leq y \leq v_0 - \frac{1}{\sqrt{f'(v_0)}} + \sqrt{\frac{1}{f'(v_0)} - r^2}\right\}$$

whenever

$$\frac{1}{v_0^2} \leq f'(v_0) \leq \frac{1}{4n^2} f^2(v_0) \; . \tag{3.16}$$

The only difference with the just-mentioned proof is that in our case f may not be convex (this happens when $q > 1$). But since $1 \leq p$ and $0 \leq q < p$ we still have

$$f(v_0) = \max_{s \in [0,v_0]} f(s), \qquad f'(v_0) = \max_{s \in [0,v_0]} f'(s) > 0,$$

which are the conditions needed in that proof.

Thanks to the hint given in the Corollary following Theorem 2 in [18] and recalling that $v_0 \geq \beta$, after some calculations one finds that the condition $v_0^{q+1}(v_0^{p-q} - 2a) \geq 4n^2 p$ implies (3.16). □

By Proposition 1, in order to have ground states or compact support solutions of (3.1) it is necessary to assume (3.11) or, equivalently,

$$\overline{F} < 1 \ . \tag{3.17}$$

Define the auxiliary function

$$M(s) := F(s) + \overline{F} - 1 \ . \tag{3.18}$$

By (3.17), it consists in a downwards translation of the function F, and therefore

$$\exists! \gamma > \beta \quad \text{such that} \quad M(\gamma) = 0 \ . \tag{3.19}$$

We prove that if the shooting level v_0 is below γ then the corresponding local solution v has no vertical points:

Lemma 8. *Assume that* (3.11) *holds and let*

$$v_0 \in (\beta, \gamma] \ .$$

Let v be the local solution of (3.1) *given by Lemma* 1. *Then, v can be continued until either $v(r)$ or $v'(r)$ reaches zero.*

Proof. We use a strengthened version of the technique employed in Lemma 1 in [16]. By (3.4) with $r_0 = 0$ and $r \in J$, and by taking the limit as $r \to R$, we have

$$F(v_0) - \lim_{r\to R} F(v(r)) = 1 - \lim_{r\to R} \frac{1}{\sqrt{1+|v'(r)|^2}} + (n-1) \lim_{r\to R} \int_0^r \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}} \frac{dt}{t}.$$

But $v_0 \leq \gamma$; therefore

$$F(v_0) - \lim_{r\to R} F(v(r)) \leq F(\gamma) + \overline{F} = 1 \ .$$

This, together with the previous inequality, yields

$$1 > 1 - \lim_{r\to R} \frac{1}{\sqrt{1+|v'(r)|^2}},$$

which proves that $\lim_{r\to R} |v'(r)| < \infty$. □

The upper bound γ in the previous statement is probably not optimal; see [1, 2] and the numerical results quoted in Section 8. Nevertheless, (3.8) and Lemma 8 suggest taking the shooting level v_0 such that

$$v_0 \in (\beta, \gamma] \ . \tag{3.20}$$

In such a case, as in [11, section 2], we can give a more detailed behavior of solutions to (3.1):

Proposition 2. *Assume* (3.11) *and* (3.20)*, and let* $J = (0, R)$ *be the maximal open interval of continuation* (*in the sense of* (3.13)) *for the solution* v *of* (3.1)*. Then*

(*i*) *for any* R*,* $L = \lim_{r\to R} v(r)$ *exists and* $L \in [0, \delta]$;

(*ii*) *if* $R = \infty$*, then* $\lim_{r\to\infty} v'(r) = 0$;

(*iii*) *if* $R < \infty$ *and* $v(r) \to 0$ *as* $r \to R$ *then* $v'(R) = \lim_{r\to R} v'(r)$ *exists and* $v'(R) \le 0$.

Proof. Since v is decreasing and positive on J the limit $L = \lim_{r\uparrow R} v(r)$ exists and is nonnegative. This proves the first part of (*i*).

Let $R = \infty$. Consider the energy function $E = E(r)$ defined in (3.5). Since by (3.6) it is decreasing and by (3.17) (i.e., (3.11)) it is bounded from below by -2, it is convergent as $r \to \infty$. Then, by definition of E and the fact that $F(v(r)) \to F(L)$, we infer that $v'(r)$ also approaches a limit, necessarily zero, as $r \to \infty$. This proves (*ii*).

Statement (*iii*) is obtained by arguing as above. It is clear that v' approaches a limit, but $v' < 0$ in J and the conclusion follows.

Finally, suppose for contradiction that $L > \delta$. Then, $v > \delta$ in J, so by (3.1) we get

$$\Big(r^{n-1}\frac{v'(r)}{\sqrt{1+|v'(r)|^2}}\Big)' = -r^{n-1}f(v(r)) < 0 \qquad \text{on } J\ ; \tag{3.21}$$

that is, $\omega(r) := r^{n-1}\frac{v'(r)}{\sqrt{1+|v'(r)|^2}}$ is decreasing on J. Now, if R is finite, then by Lemma 8 we have $v'(r) \to 0$ as $r \to R$. In turn, $\omega(r) \to 0$ as $r \to R$, while also $\omega(0) = 0$. But this is absurd since ω is decreasing on J. If $R = \infty$, then by (3.1) and (*ii*) we get

$$\lim_{r\to\infty}\Big(\frac{v'(r)}{\sqrt{1+|v'(r)|^2}}\Big)' = -f(L)\ ,$$

which is strictly negative because $L > \delta$. This is impossible since

$$\frac{v'(r)}{\sqrt{1+|v'(r)|^2}} \to 0 \text{ as } r \to \infty. \qquad \square$$

We can now outline the core of the shooting method. With the same assumptions and the same notation as in Proposition 2 we introduce the sets

$$I^- = \{\alpha \in [\beta,\gamma] :\ R < \infty,\ L = 0,\ v'(R) < 0\}, \quad I^+ = \{\alpha \in [\beta,\gamma] :\ L > 0\}\ .$$

Clearly I^- and I^+ are disjoint. By arguing as in [11] (where the case of possibly singular f is considered) one can prove that

Fact 1. $\beta \in I^+$,
Fact 2. I^+ is open in $[\beta, \gamma]$,
Fact 3. I^- is open in $[\beta, \gamma]$.
These facts allow us to obtain a sufficient condition for the existence of a ground state or a compact-support solution of (3.1):

Proposition 3. *Assume that $I^- \neq \emptyset$. Then* (3.1) *admits either a ground state or a compact-support solution v such that $v(0) < \alpha$ for all $\alpha \in I^-$.*

Proof. By Lemma 8 and Proposition 2, it follows that $v_0 \in [\beta, \gamma]$ is a shooting level for a ground state or a compact-support solution of (3.1) if and only if $v_0 \notin I^+ \bigcup I^-$. The existence of such a v_0 is ensured by Facts 1–3 and the assumption that I^- is nonempty.

Again by Facts 1–3, we infer that $\inf I^- > \beta$ and $\inf I^- \notin I^+ \bigcup I^-$. Then, by what we have just proved, there exists a ground state v of (3.1) such that $v_0 = \inf I^-$. □

4. Proof of Theorem 2

We start with a technical result, which we also use in the proof of Theorem 3:

Lemma 9. *Assume $p \geq 0$, let $v_0 \in [\beta, \gamma]$, let v be the corresponding solution of* (3.1) *whose maximal interval is J and let Ψ be the continuous function defined on $\overline{J}$ by*

$$\Psi(r) = \frac{|v'(r)|}{r\sqrt{1 + |v'(r)|^2}} \qquad \forall r \in J.$$

Then $\Psi(0) = \frac{f(v_0)}{n}$ and Ψ is strictly decreasing on J.

Proof. By continuity we have $\Psi(0) = \lim_{r \to 0} \Psi(r)$, and the first statement follows from (3.12). By taking into account that v solves (3.1) we infer that

$$\Psi'(r) = \frac{r f(v(r)) - n \frac{|v'(r)|}{\sqrt{1+|v'(r)|^2}}}{r^2} = \frac{G(r)}{r^2} \ .$$

Therefore, the proof is complete if we show that $G(r) < 0$ for all $r \in J$. Let R_δ be the unique solution r of the equation $v(r) = \delta$ (recall $f(\delta) = 0$). By Proposition 2 we know that it may be that $R_\delta = R$ (eventually $+\infty$). If this case occurs the set $r \geq R_\delta$ is empty; otherwise we have $G(r) < 0$ for all $r \geq R_\delta$ since $f(v(r))$ is negative. Hence, we restrict our attention to the interval $(0, R_\delta)$. As $p \geq 0$, in this interval we have $f'(v(r)) > 0$. Therefore, by using (3.1), we obtain

$$\frac{d}{dr}\left(r^{n-1}G(r)\right) = r^n f'(v(r))v'(r) < 0 \qquad \forall r \leq R_\delta,\ r \neq 0\ .$$

This shows that the map $r \mapsto r^{n-1}G(r)$ is strictly decreasing on $[0, R_\delta]$. Since it takes the value 0 for $r = 0$ we obtain the result. □

We now turn to the proof of Theorem 2. Since $q + 1 < p + 1 < \frac{n}{n-1} < 2$, in view of [17] we seek compact support solutions of (3.1)–(3.3).

We first obtain an upper bound for the shooting level:

Lemma 10. *Assume that* $0 \leq p < \frac{1}{n-1}$ *and assume there exists a compact-support solution* v *of* (3.1)–(3.3)*. Then*

$$v_0 < \Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}a\Big)^{1/(p-q)}\ . \tag{4.1}$$

Proof. By (3.10) and Lemma 9, we have

$$F(v_0) = (n-1)\int_0^\rho \Psi(t)|v'(t)|dt < \frac{n-1}{n}f(v_0)\int_0^\rho |v'(t)|dt = \frac{n-1}{n}v_0 f(v_0)\ .$$

By replacing f and F we infer

$$\Big(\frac{1}{p+1} - \frac{n-1}{n}\Big)v_0^{p-q} < a\Big(\frac{1}{q+1} - \frac{n-1}{n}\Big)\ ,$$

which proves the result. □

We are now in position to prove

Proposition 4. *Assume that* $0 \leq p < \frac{1}{n-1}$*. Then* (3.1)–(3.3) *admits a compact-support solution provided*

$$a \leq \Big(\frac{(p+1)(q+1)}{p-q}\Big)^{\frac{p-q}{p+1}}\Big[1+(n-1)\Big(\frac{p+1}{p+1-np}\Big)^{\frac{p+1}{p-q}}\Big(\frac{q+1-nq}{q+1}\Big)^{\frac{q+1}{p-q}}\Big]^{\frac{q-p}{p+1}}\ . \tag{4.2}$$

Proof. Let M and γ be as in (3.18) and (3.19) respectively. Then,

$$M\Big(\Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}a\Big)^{1/(p-q)}\Big) \leq 0$$

if (4.2) holds. Hence, for all $a > 0$ satisfying (4.2) we have

$$\Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}a\Big)^{1/(p-q)} \leq \gamma\ .$$

By Lemma 8 the corresponding solution of (3.1) has no vertical points if we take as initial value

$$v_0 \in \Big[\Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}a\Big)^{1/(p-q)}, \gamma\Big].$$

By Lemma 10 we know that such a solution is not a compact-support solution. Moreover, if we assume that $v_0 \in I^+$, we may apply (3.4) with $r = 0$ and $r_0 = R$ (recall $v'(R) = 0$ and $F(v(R)) < 0$), and by arguing as in the proof of Lemma 10 we obtain again (4.1). This shows that for all $a > 0$ satisfying (4.2) we have

$$\Big[\Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}a\Big)^{1/(p-q)}, \gamma\Big] \subset I^-$$

so that $I^- \neq \emptyset$. The statement follows from Proposition 3. □

Proof of Theorem 2. By Proposition 4 we know that (4.2) is a sufficient condition for the existence of a compact-support solution of (3.1)–(3.3). By the results in [17] we know that if $q > 0$ then the compact-support solution is also a ground state, while if $q \leq 0$ then (3.1) admits no ground states.

Finally, from Lemma 10 we obtain that

$$\|v_a\|_\infty = v_a(0) < \Big(\frac{p+1}{q+1}\frac{q+1-nq}{p+1-np}\Big)^{1/(p-q)} \cdot a^{1/(p-q)} = \Gamma(p,q) \cdot a^{1/(p-q)} \tag{4.3}$$

for any compact-support solution v_a of (3.1). The proof of Theorem 2 is complete.

5. Proof of Theorem 3

We maintain here the same notation as in Propositions 2 and 3. We prove a result in the spirit of [10, Lemma 2.1.1]:

Lemma 11. *Let v be a (local) solution of (3.1) with $v_0 \in (\beta, \gamma]$. Let $y \in (\beta, v_0)$ and let $R_y \in [0, R)$ be the unique solution of $v(R_y) = y$. Then $v_0 \in I^-$ provided*

$$R_y^2 > (n-1)^2 \frac{y^2}{F^2(y)}(\overline{F} + F(y))(2 - \overline{F} - F(y)) \ . \tag{5.1}$$

Proof. Note first that R_y is well-defined by Proposition 2.

For contradiction, assume that (5.1) holds and that $v_0 \notin I^-$. Then, by (3.13) and Proposition 2, we know that $\lim_{r\to R} v'(r) = 0$ and $\lim_{r\to R} v(r) = L \in [0, \delta]$ (so that $F(L) \leq 0$). Therefore, there exists $R_0 \in [R_y, R)$ such that

$$M = \sup_{r\in[R_y,R)} |v'(r)| = |v'(R_0)| \ .$$

By (3.4) with $r_0 = R_0$ and $r = R$ (eventually taking the limit) and recalling that $F(L) \leq 0$, by Lemma 9, we obtain

$$\begin{aligned} 1 - \frac{1}{\sqrt{1+M^2}} &= F(L) - F(v(R_0)) + (n-1)\int_{R_0}^{R} \frac{|v'(t)|^2}{\sqrt{1+|v'(t)|^2}}\frac{dt}{t} \\ &\leq \overline{F} + (n-1)\frac{M}{\sqrt{1+M^2}}\frac{1}{R_0}\int_{R_0}^{R} |v'(t)|dt \\ &\leq \overline{F} + \frac{n-1}{R_y}\frac{M}{\sqrt{1+M^2}}y \ . \end{aligned} \tag{5.2}$$

Write now (3.4) with $r_0 = R_y$ and $r = R$. Then, by monotonicity of the map $s \mapsto \frac{s}{\sqrt{1+s^2}}$ and the same tools used for (5.2), we obtain

$$F(y) \leq \frac{n-1}{R_y}\frac{M}{\sqrt{1+M^2}}\, y \ . \tag{5.3}$$

Next, note that by elementary calculus one has the inequality

$$\frac{s}{\sqrt{1+s^2}} \leq \frac{1}{\sqrt{\omega^2+2\omega}}\Big(\omega + 1 - \frac{1}{\sqrt{1+s^2}}\Big) \qquad \forall \omega > 0 \ \forall s \geq 0.$$

In particular,

$$\frac{M}{\sqrt{1+M^2}} \leq \frac{1}{\sqrt{\omega^2+2\omega}}\Big(\omega + 1 - \frac{1}{\sqrt{1+M^2}}\Big) \qquad \forall \omega > 0 \ .$$

Since $y < \gamma$, we have $1 - \overline{F} - F(y) > 0$; see (3.18)–(3.19). Therefore, we may choose $\omega = \frac{\overline{F}+F(y)}{1-\overline{F}-F(y)}$ in the previous inequality so that, by (5.2) and after some calculations, we obtain

$$\begin{aligned} &\frac{M}{\sqrt{1+M^2}}\Big(1 - \frac{1-\overline{F}-F(y)}{\sqrt{(\overline{F}+F(y))(2-\overline{F}-F(y))}}\frac{(n-1)y}{R_y}\Big) \\ &\leq \frac{F(y) + 2\overline{F} - \overline{F}^2 - \overline{F}F(y)}{\sqrt{(\overline{F}+F(y))(2-\overline{F}-F(y))}}. \end{aligned} \tag{5.4}$$

As noticed above, we know that $1 - \overline{F} - F(y) > 0$; but (5.1) shows that the left-hand side of (5.4) is positive. Hence, by (5.3) and (5.4) we obtain

$$\frac{F(y)}{(n-1)y}\, R_y \leq \sqrt{(\overline{F}+F(y))(2-\overline{F}-F(y))},$$

which contradicts (5.1) and completes the proof. □

Next, we give a lower bound for R_y:

Lemma 12. *Let v, v_0, y and R_y be as in Lemma* 11. *Then*

$$R_y^2 > 2n\frac{1-F(v_0)}{f(v_0)}(v_0 - y) \ .$$

Proof. By Proposition 2, we may fix $r > 0$ such that $v(r) > \beta$. By integrating (3.1) on $[0, r]$, we obtain

$$\frac{v'(r)}{\sqrt{1+|v'(r)|^2}} = -r^{1-n}\int_0^r s^{n-1}f(v(s))\,ds > -r^{1-n}f(v_0)\int_0^r s^{n-1}\,ds$$

giving

$$v'(r) > -r\frac{f(v_0)}{n}\sqrt{1+|v'(r)|^2} \ . \tag{5.5}$$

By identity (3.4) with $r_0 = 0$, we get

$$F(v_0) - F(v(r)) \geq 1 - \frac{1}{\sqrt{1+|v'(r)|^2}},$$

and since $F(v(r)) > 0$ whenever $v(r) > \beta$, we also get $F(v_0) \geq 1 - \frac{1}{\sqrt{1+|v'(r)|^2}}$. As $F(v_0) \leq F(\gamma) = 1 - \overline{F} < 1$ (see (3.18)–(3.19)), this yields

$$\sqrt{1+|v'(r)|^2} \leq \frac{1}{1-F(v_0)} \ .$$

Hence, by (5.5) we estimate $v'(r)$ by

$$v'(r) > -r\frac{f(v_0)}{n}\frac{1}{1-F(v_0)}$$

so that, by integrating on $[0, r]$, we also obtain

$$v(r) - v_0 > -r^2\frac{f(v_0)}{2n}\frac{1}{1-F(v_0)} \ .$$

The result follows by taking $r = R_y$ in the previous inequality. □

Finally, thanks to Proposition 3, the proof of Theorem 3 will be complete as soon as we prove

Lemma 13. *If a satisfies* (2.2), *there results $I^- \neq \emptyset$.*

Proof. Fix a satisfying (2.2) and let

$$v_0 = \Big(\frac{p+1}{2}\Big)^{1/(p+1)} \qquad \text{and} \qquad y = \Big(2\frac{p+1}{q+1}a\Big)^{1/(p-q)} \tag{5.6}$$

so that $v_0 \in (\beta, \gamma)$ and $y \in (\beta, v_0)$. Indeed, $\beta < v_0 < \gamma$ follows from (2.1) and (2.2) since

$$a \leq \Big[\frac{(p+1)(q+1)}{2(p-q)}\Big]^{(p-q)/(p+1)},$$

and this implies $F(v_0) < F(\gamma)$. Also $\beta < y < v_0$ follows from (2.1) and (2.2) since

$$a \leq \frac{q+1}{(p+1)^{(q+1)/(p+1)}} \frac{1}{2^{(p-q+2)/(p+1)}} < \frac{q+1}{(p+1)^{(q+1)/(p+1)}} \frac{1}{2^{(2p-q+1)/(p+1)}},$$

and this yields $y < v_0$; moreover, $F(y) > 0$, which shows that $y > \beta$.

Let v and R_y have the same meaning as in Lemma 11. We claim that $v_0 \in I^-$. To show this, by Lemmas 11 and 12, it suffices to prove that

$$2n \frac{1 - F(v_0)}{f(v_0)}(v_0 - y) \geq (n-1)^2 \frac{y^2}{F^2(y)}(\overline{F} + F(y))(2 - \overline{F} - F(y)),$$

which is a consequence of the following inequality:

$$n \frac{1 - F(v_0)}{f(v_0)}(v_0 - y) \geq (n-1)^2 \frac{y^2}{F^2(y)}(\overline{F} + F(y)) \ . \tag{5.7}$$

In turn, with our choice of v_0, we have

$$\frac{1 - F(v_0)}{f(v_0)} > \frac{1}{2}\Big(\frac{2}{p+1}\Big)^{p/(p+1)} \qquad \forall a > 0 \ .$$

Moreover, with our choice of y and after some calculations, one sees that

$$\begin{aligned} \frac{y^2}{F^2(y)}(\overline{F} + F(y)) &= \frac{(p+1)^{\frac{1-q}{p-q}}}{2^{\frac{2q}{p-q}}(q+1)^{\frac{1-p}{p-q}}}\Big[2^{\frac{q+1}{p-q}} + (p-q)\frac{(q+1)^{\frac{q+1}{p-q}}}{(p+1)^{\frac{p+1}{p-q}}}\Big] a^{\frac{1-p}{p-q}} \\ &< 2^{\frac{p+1-2q}{p-q}} \frac{(p+1)^{\frac{1-q}{p-q}}}{(q+1)^{\frac{1-p}{p-q}}} a^{\frac{1-p}{p-q}}. \end{aligned}$$

Hence, (5.7) is a consequence of the inequality

$$\begin{aligned} &\frac{(n-1)^2}{n} 2^{\frac{2p+1-3q}{p-q} - \frac{p}{p+1}} \frac{(p+1)^{\frac{1-q}{p-q} + \frac{p}{p+1}}}{(q+1)^{\frac{1-p}{p-q}}} a^{\frac{1-p}{p-q}} + \Big(2\frac{p+1}{q+1}a\Big)^{\frac{1}{p-q}} \\ &\leq \Big(\frac{p+1}{2}\Big)^{\frac{1}{p+1}} \ . \end{aligned}$$

This last inequality is indeed true as it is a direct consequence of the definition of $A(n,p,q)$ and of the crucial assumption (2.2). Therefore, (5.7) also holds and the statement is proved. □

6. Proof of Theorem 4

Let v be either a ground state of (3.1)–(3.2) (if $q > 0$) or a compact-support solution of (3.1)–(3.3) (if $0 \leq q < 1$). By Lemmas 6 and 7 we infer that

$$4n^2 p > a v_0^{q+1} \Big(\frac{2^* - (q+1)}{2^* - (p+1)} \frac{p+1}{q+1} - 2 \Big),$$

and using again Lemma 6 we get

$$4n^2 p > a \Big(\frac{2^* - (q+1)}{2^* - (p+1)} \frac{p+1}{q+1} a \Big)^{\frac{q+1}{p-q}} \Big(\frac{2^* - (q+1)}{2^* - (p+1)} \frac{p+1}{q+1} - 2 \Big) \ .$$

Finally, since (2.3) holds, we have $2^*(p - 2q - 1) + (p+1)(q+1) > 0$, and we obtain

$$a < C(n,p,q) \Big(\frac{n+2}{n-2} - p \Big)$$

where $C(n,p,q)$ is as in (2.4).

7. Proof of Theorem 5

Consider the energy function E introduced in (3.5). By (3.6) and Proposition 2 (ii) we obtain

$$E(r) > \lim_{t \to \infty} E(t) = -1 \qquad \forall r \geq 0,$$

and hence

$$E(r) + 1 = \frac{\sqrt{1 + |v'(r)|^2} - 1}{\sqrt{1 + |v'(r)|^2}} + F(v(r)) > 0 \qquad \forall r \geq 0 \ . \tag{7.1}$$

Since $v(r) \to 0$ as $r \to \infty$, for sufficiently large r we have

$$-F(v(r)) = \frac{a}{2} v^2(r) - \frac{1}{p+1} v^{p+1}(r) \geq \frac{a}{3} v^2(r),$$

which together with (7.1) yields

$$\frac{\sqrt{1 + |v'(r)|^2} - 1}{\sqrt{1 + |v'(r)|^2}} > -F(v(r)) \geq \frac{a}{3} v^2(r) \ . \tag{7.2}$$

On the other hand, using again Proposition 2 (ii), for sufficiently large r we get

$$\frac{\sqrt{1 + |v'(r)|^2} - 1}{\sqrt{1 + |v'(r)|^2}} < \sqrt{1 + |v'(r)|^2} - 1 < \frac{2}{3} |v'(r)|^2,$$

which placed into (7.2) gives (for some $R > 0$)

$$\frac{|v'(r)|}{v(r)} \geq \sqrt{\frac{a}{2}} =: \lambda \qquad \forall r \geq R \ .$$

Finally, integrate the previous inequality on the interval $[R, r]$ ($r > R$) to obtain

$$v(r) \leq v(R) e^{\lambda R} e^{-\lambda r} \qquad \forall r \geq R \ , \tag{7.3}$$

which proves the first estimate.

By Lemma 5.1 in [14], we know that there exists $C \leq 0$ such that

$$\lim_{r \to \infty} r^{n-1} \frac{v'(r)}{\sqrt{1 + |v'(r)|^2}} = C \ .$$

If $C < 0$, by noting that $v(r) = -\int_r^\infty v'(t)dt$, we contradict (7.3); hence

$$\lim_{r \to \infty} r^{n-1} v'(r) = 0 \ . \tag{7.4}$$

By (7.3) we may integrate (3.21) over $[r, \infty)$ ($r \geq R$) and by (7.4) we obtain

$$r^{n-1} \frac{|v'(r)|}{\sqrt{1 + |v'(r)|^2}} = \int_r^\infty t^{n-1} f(v(t)) dt \ .$$

Then, if r is sufficiently large (say $r \geq R'$) by Proposition 2 (ii) we get

$$|v'(r)| \leq \frac{2}{r^{n-1}} \int_r^\infty t^{n-1} |f(v(t))| \, dt,$$

and therefore by (7.3)

$$|v'(r)| \leq \frac{K}{r^{n-1}} \int_r^\infty t^{n-1} e^{-\lambda t} dt$$

for some constant $K > 0$. With $n - 1$ integration by parts we finally obtain (for some $K' > 0$)

$$|v'(r)| \leq K' e^{-\lambda r} \qquad \forall r \geq R' \ . \tag{7.5}$$

In order to prove the estimate for v'' we write (3.1) as

$$v''(r) = \frac{n-1}{r} |v'(r)| (1 + |v'(r)|^2) - f(v(r)) (1 + |v'(r)|^2)^{3/2} \ .$$

Then, by (7.3) and (7.5) we infer that

$$|v''(r)| \leq K'' e^{-\lambda r} \qquad \forall r \geq R''$$

for some $K'', R'' > 0$.

To complete the proof of Theorem 5, take $\xi = \max\{R, R', R''\}$ and $\mu = \max\{v(R) e^{\lambda R}, K', K''\}$.

8. Numerical results and open problems

• In order to determine which, between Theorems 2 and 3, gives an existence result for a varying in a wider range, we start this section with some numerical computations of the constants $\overline{K}$ and $\overline{A}$ in these statements. It turns out that in the cases considered (we do not quote all the results we obtained) the constant $\overline{K}$ is considerably larger. This leads us to conjecture that the statement of Theorem 2 gives a stronger result for all $p \in [0, \frac{1}{n-1})$.

(n,p,q)	$(3,\frac{2}{5},0)$	$(3,\frac{1}{3},0)$	$(3,\frac{1}{3},-\frac{1}{4})$	$(4,\frac{1}{2},0)$	$(4,\frac{1}{5},-\frac{9}{10})$	$(7,\frac{1}{8},-\frac{1}{3})$	$(10,\frac{1}{20},-\frac{1}{2})$
$\overline{A}(n,p,q)$	0.033	0.063	0.013	0.006	0.0001	0.022	0.01
$\overline{K}(n,p,q)$	0.168	0.297	0.158	0.167	0.011	0.054	0.052

• We now give some numerical results concerning the statement of Theorem 2. Recalling that $\Gamma(p,q)$ is defined in (4.3), in the particular case where $n=3$, $q=0$ and $p=1/3$, Theorems 1 and 2 may be restated as follows:

Corollary 2. *If $a \geq \sqrt{2}$, then the equation*

$$-\text{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = -a + \sqrt[3]{u} \tag{8.1}$$

admits no compact-support solutions in $\mathbf{R}^3$ while if

$$a \leq \frac{\sqrt{2}}{\sqrt[4]{513}} \approx 0.297$$

then (8.1) *admits at least a compact-support solution in $\mathbf{R}^3$. Moreover, any compact-support solution u_a of* (8.1) *in $\mathbf{R}^3$ satisfies*

$$u_a(0) = \|u_a\|_\infty < 64a^3 \ .$$

This result leaves the indeterminacy on the exact value

$$a^* \in \Big(\frac{\sqrt{2}}{\sqrt[4]{513}}, \sqrt{2}\Big) \approx (0.297, 1.414)$$

where existence breaks down. It is therefore interesting to further investigate this point.

With the help of Mathematica we performed several numerical computations by studying the behavior of the solutions of the corresponding ODE (3.1), namely

$$\frac{v''}{(1+|v'|^2)^{3/2}} + \frac{2}{r}\frac{v'}{\sqrt{1+|v'|^2}} - a + \sqrt[3]{v} = 0 \ .$$

Since this equation has a singularity at $r=0$, according to (3.12) we studied the following Cauchy problem,

$$v(0.01)=\alpha \qquad v'(0.01)=-\frac{0.01}{3}\left(\sqrt[3]{\alpha}-a\right),$$

for varying values of the shooting level α and of the parameter a. For each experiment, the program drew the corresponding solution and told us whether $\alpha\in I^+$, $\alpha\in I^-$ or $\alpha\in I^\infty$, where, taking the same assumptions and the same notation as in Proposition 2,

$$I^-=\Big\{\alpha\geq\big(\frac{4a}{3}\big)^3 : R<\infty,\ L=0,\ v'(R)<0\Big\},$$
$$I^+=\Big\{\alpha\geq\big(\frac{4a}{3}\big)^3 :\ L>0\Big\},$$
$$I^\infty=\Big\{\alpha\geq\big(\frac{4a}{3}\big)^3 :\ \lim_{r\to R^-}v'(r)=-\infty\Big\}\ .$$

Note that $\beta=(\frac{4a}{3})^3$ and that γ solves the equation

$$\frac{3}{4}\gamma^{4/3}-a\gamma+\frac{a^4}{4}-1=0\ .$$

We collect some of our results in the following table:

a	0.1	0.2	0.6	0.8	0.9	0.92	0.93	0.94
$\exists$	yes	yes	yes	yes	yes	yes	yes	no
β	0.00237	0.01896	0.512	1.214	1.728	1.846	1.907	1.969
$64a^3$	0.064	0.512	13.824	32.768	46.656	49.836	51.479	53.157
$\inf I^-$	0.0064	0.0524	1.4425	3.5732	5.4781	6.0681	6.4619	–
$\sup I^-$	4.1865	4.5444	6.3895	7.6144	8.2238	7.9931	7.7831	–
$\inf I^\infty$	4.1865	4.5444	6.3895	7.6144	8.3172	8.4674	8.5426	8.6186
γ	1.3658	1.5124	2.3968	3.0809	3.501	3.5915	3.6378	3.6847

According to these numerical results, we have existence of a compact-support solution for $a\leq 0.938$ and nonexistence for $a\geq 0.939$. Moreover, if $a\leq 0.892$ the solution is unique while if $0.893\leq a\leq 0.938$ there exist two solutions. Nonuniqueness is obtained when $\inf I^\infty>\sup I^-$. The solutions have shooting level $\alpha_1=\inf I^-$ (see the proof of Proposition 3) and, if $\sup I^-<\inf I^\infty$, also $\alpha_2=\sup I^-$.

We conjecture that at the *turning point* a^* (which satisfies $0.938<a^*<0.939$) the solution exists and is unique. If this conjecture were true, we would have existence even if $I^-\neq\emptyset$, showing that Proposition 3 merely gives a sufficient condition for existence results.

Finally, note that there is a wide gap between γ and $\inf I^\infty$, showing that Lemma 8 may be improved.

• Let us turn to the statement of Theorem 4 in the case where $n = 5$, $q = 1$ and $p = 2$:

Corollary 3. *The equation*

$$-\mathrm{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = -au + u^2$$

admits no ground states in $\mathbf{R}^5$ *if*

$$a \geq \frac{5^{2/3}}{2^{1/3}3^{2/3}} \approx 1.1157\ .$$

Note that in this case the constant K defined in Theorem 1 is given by $K_{1,2} = \sqrt[3]{6} \approx 1.817$ and is larger than the one in Corollary 3. Therefore, in this case, Theorem 4 improves Theorem 1.

Using again Mathematica we studied the following Cauchy problem,

$$\begin{cases} \dfrac{v''}{(1+|v'|^2)^{3/2}} + \dfrac{4}{r}\dfrac{v'}{\sqrt{1+|v'|^2}} - av + v^2 = 0 \\ v(0.01) = \alpha \qquad v'(0.01) = -\frac{0.01}{5}(\alpha^2 - a\alpha), \end{cases}$$

for varying values of a and α. We obtained existence of a solution for all $a \leq 0.046$ and nonexistence if $a \geq 0.047$. In this case the solution always turned out to be unique. It is worth mentioning that also solutions with $\alpha \in I^-$ exhibited vertical points *after* crossing the r-axis, that is, at negative levels.

Note that the nonexistence constant 1.1157 in Corollary 3 is fairly large with respect to the real one, which is about 0.046. In this case, a ground state exists only for very small values of a!

Next, we kept the same function $f(s) = -as + s^2$ and we studied the equation in other dimensions. When $n = 4$ we find two solutions for some values of a (e.g. for $a = 0.15$), while if $n = 3$ we even find three solutions (e.g. for $a = 0.28$). This shows a strong dependence of uniqueness results on the dimension n.

• To conclude this paper, we quote some open problems which seem interesting to us, especially after seeing the above numerical results.

Uniqueness. When is the ground state or compact-support solution of (1.1) unique? As far as we are aware the only uniqueness result in literature is Theorem B in [10], which does not apply to (1.1) since it states uniqueness of ground states when existence fails. It would be interesting to determine some uniqueness criteria. Thanks to the above numerical results, it seems that uniqueness depends not only on p, q and a but also on n. A reasonable conjecture is that the solution may not be unique only for large values of a and that it is unique for small values of a. Moreover, uniqueness seems more likely to hold in high space dimension n.

Properties of $a^*(p)$. For simplicity, take $q = 1$ and let $a^* = a^*(p)$ be the map introduced in Corollary 1 and defined on $(1, \frac{n+2}{n-2})$.

A first natural question is the following. Is it true that for all $a \in (0, a^*)$ problem (1.1) admits a ground state? We believe that the answer is positive.

Moreover, when $a = a^*$ does the ground state exist? A continuous dependence result seems difficult to obtain. According to the above numerical results, we conjecture that the solution exists for $a = a^*$ provided one has nonuniqueness in a left neighborhood of a^* and that the solution does not exist when $a = a^*$ if one has uniqueness in a left neighborhood of a^*.

Further, is the map $a^* : (1, \frac{n+2}{n-2}) \mapsto (0, +\infty)$ decreasing? This seems reasonable, according to Corollary 1 and the above numerical results. Note also that in our setting the space dimension n may be seen as any real parameter strictly greater than 2. Therefore, we can reverse Corollary 1 and obtain a map $a_* = a_*(n)$ such that $a_* : (2, \frac{2(p+1)}{p-1}) \mapsto (0, +\infty)$ and such that $\lim_{n \to \frac{2(p+1)}{p-1}} a_*(n) = 0$. Is this map a_* decreasing?

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