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Quasilinear elliptic equations at critical growth

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Abstract

The existence of nontrivial solutions of quasilinear elliptic equations at critical growth is proved. The solutions are obtained by variational methods: as the corresponding functional is nonsmooth, the analysis of Palais-Smale sequences requires suitable generalizations of the techniques involved in the study of the corresponding semilinear problem with lack of compactness.

1 Introduction

In this paper we study the existence of positive functions $u \in H := H_0^1(\Omega)$ solving in distributional sense the quasilinear elliptic equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u)D_i u D_j u = g(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is open and bounded, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent and g is a subcritical term; this problem is a generalization of some semilinear variational problems in differential geometry and physics, see [5, 15, 21].

We endow the space H with the Dirichlet norm ($\|u\|^2 := \int_\Omega |\nabla u|^2$): weak solutions of (1.1) are critical points of the functional $J : H \to \mathbb{R}$ defined by

$$\forall u \in H \qquad J(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_i u D_j u - \int_\Omega G(x,u) - \frac{1}{2^*}\int_\Omega |u|^{2^*}$$

where $G(x,s) = \int_0^s g(x,t)dt$; as noted in [6], under reasonable assumptions on a_{ij}, g, the functional J is continuous but not even locally Lipschitz whenever the

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functions $a_{ij}(x,s)$ depend on s. However, J is weakly C_c^∞-differentiable (see [3, 6]) and the derivative of J exists in any smooth direction: for all $u \in H$ and $\varphi \in C_c^\infty$ we can evaluate

$$\begin{aligned} J'(u)[\varphi] &= \int_\Omega \sum_{i,j=1}^n \left[a_{ij}(x,u) D_i u D_j \varphi + \frac{1}{2} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \varphi \right] + \\ &\quad - \int_\Omega g(x,u)\varphi - \int_\Omega |u|^{2^*-2} u\varphi \ ; \end{aligned}$$

according to the nonsmooth critical point theory of [10, 11] it is possible to prove that critical points u of J satisfy $J'(u)[\varphi] = 0\ \forall \varphi \in C_c^\infty$ and hence solve (1.1) in distributional sense. We refer to the original papers [6, 10, 11] for the definitions of *weak slope*, *weak C_c^∞-differentiability*, *critical point*, *PS sequence* and *PS condition* in this nonsmooth context; see also the appendix of [9] for a quick overview. The tools provided by this theory have been widely used for the study of problems related to quasilinear elliptic equations of the kind of (1.1), see [3, 6, 7, 9]; we also refer to [2] for a different approach.

In the study of equation (1.1), the lack of compactness of the embedding $H \subset L^{2^*}(\Omega)$ yields some difficulties in the analysis of PS sequences: in the fundamental paper by Brezis-Nirenberg [5] (where the case $a_{ij}(x,s) \equiv \delta_{ij}$ is studied) it is shown that the corresponding functional only satisfies the PS condition at certain energy levels. Our problem possesses some analogies with that in [9]: there, the lack of compactness is due to the unboundedness of the domain (a locally compact case), here, it is due to the critical Sobolev exponent (a limit case); to prove the existence of nontrivial solutions, in [9] the quasilinear problem is assumed to "converge" to a semilinear problem as $|x| \to \infty$, here we require the same convergence when $u \to +\infty$. Then, under minimal assumptions on the lower order term g (as in [13]) we will prove that (1.1) admits a nontrivial solution. To this aim, a deep analysis of the "obstruction to compactness" of the PS sequences is needed: since $J \notin C^1$, a representation result as in [19] is not possible; for this reason, and because our assumptions do not allow to prove that the critical levels of J are positive, we cannot find a "range of compactness" as in [5]. This analysis is then performed by proving that the range of compactness of [5] becomes a "nontrivial energy range" for which the weak limit of the PS sequences cannot be 0: the solutions are obtained as weak limits of PS sequences; as the derivative J' is not weakly continuous, to prove that the weak limit is indeed a solution we apply a result of [4], following an idea of [6].

As we do not require that $g(x,s) \geq 0$, assumptions (2.6) and (2.7) on the subcritical term are somehow weaker than those of Lemma 2.1 in [5]: the estimate of the energy level of the PS sequence considered is obtained as in [13], by a method which does not involve the computation of $\frac{d}{dt}J(tu)$ as in [5]. We also point out that assumption (2.7) in the case $n \geq 5$ seems to be the weakest possible: indeed, by Pohožaev identity [16], it follows that if Ω is strictly starshaped and $g(x,s) \leq 0$ then (1.1) with $a_{ij}(x,s) \equiv \delta_{ij}$ only has the trivial solution. Assumptions (2.6) and

(2.7) only imply the existence of a set of positive measure where $g(x,s) > 0$ for some values of s and allow $g(x,s)$ to become negative, if its negative part is not too greater than its positive part, that is if $G(x,s)$ is not too negative, see also Remark 5. For the case $n = 4$, we refer to Remark 2 in Section 5 where we discuss a possible weaker assumption. As noted in [5], the case $n = 3$ is more difficult and we need to study separately the case where $g(x,s) = \lambda s$: by arguing as in [8], we prove a result similar to that in the semilinear case.

In Section 5, by applying a generalized Pohožaev identity due to Pucci-Serrin [17], we obtain a non-existence result.

2 Existence results

Throughout this paper we require the coefficients a_{ij} $(i,j = 1,...,n)$ to satisfy

$$\begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x,\cdot) \in C^1(\mathbb{R}) \text{ for a.e. } x \in \Omega \\ a_{ij}(x,s), \dfrac{\partial a_{ij}}{\partial s}(x,s) \in L^\infty(\Omega \times \mathbb{R}) \ , \end{cases} \tag{2.1}$$

$$\exists \nu \in (0,1] \ , \qquad \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n \ , \tag{2.2}$$

$$0 \leq s \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n \ , \tag{2.3}$$

$$\begin{cases} \exists \gamma \in (0, 2^* - 2),\ \exists \bar{s} > 0 \text{ s.t. for a.e. } x \in \Omega,\ \forall |s| > \bar{s},\ \forall \xi \in \mathbb{R}^n \\ s \displaystyle\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \leq \gamma \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j \end{cases} \tag{2.4}$$

and

$$\begin{cases} \displaystyle\lim_{s \to +\infty} a_{ij}(x,s) = \delta_{ij} \ , \quad \lim_{s \to +\infty} s\frac{\partial a_{ij}}{\partial s}(x,s) = 0 \\ \forall i,j = 1,\dots,n \quad \text{and uniformly w.r.t. } x \in \Omega \ ; \end{cases} \tag{2.5}$$

assumption (2.5) implies that (1.1) converges in some sense to a semilinear problem as $u \to +\infty$.

Let λ_1 be the first eigenvalue of $-\Delta$ in Ω, let $G(x,s) = \int_0^s g(x,t)dt$ and assume:

$$\begin{cases} g : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function} \\ \forall \varepsilon > 0\ \exists f_\varepsilon \in L^{\frac{2n}{n+2}} \text{ such that} \\ |g(x,s)| \leq f_\varepsilon(x) + \varepsilon|s|^{\frac{n+2}{n-2}} \quad \text{for a.e. } x \in \Omega \ , \ \forall s \in \mathbb{R} \\ \displaystyle\limsup_{s \to 0} \frac{2G(x,s)}{s^2} < \nu\lambda_1 \qquad \text{uniformly w.r.t. } x \in \Omega \\ G(x,s) \geq 0 \quad \text{for a.e. } x \in \Omega \ , \ \forall s \in \mathbb{R} \ . \end{cases} \tag{2.6}$$

Moreover, we require that there exists a nonempty open set $\Omega_0 \subset \Omega$ such that

$$\begin{cases} \text{if } n=3 \text{ then} \\ \qquad \lim_{s\to+\infty} \dfrac{G(x,s)}{s^4} = +\infty \quad \text{uniformly w.r.t. } x\in\Omega_0 \\ \text{if } n=4 \text{ then } \exists a>0\ ,\ \exists\mu>0 \quad \text{such that} \\ \qquad \begin{cases} \text{either } G(x,s)\ge \mu s^2 \text{ for a.e. } x\in\Omega_0\ ,\ \forall s\in[0,a] \\ \text{or } G(x,s)\ge\mu(s^2-a^2) \text{ for a.e. } x\in\Omega_0\ ,\ \forall s\ge a \end{cases} \\ \text{if } n\ge5 \text{ then } \exists b>a>0\ ,\ \exists\mu>0 \quad \text{such that} \\ \qquad G(x,s)\ge\mu \quad \text{for a.e. } x\in\Omega_0\ ,\ \forall s\in[a,b]\ . \end{cases} \tag{2.7}$$

Note that by (2.6) we have $g(x,0)=0$, therefore equation (1.1) always admits the trivial solution. We prove the following result:

Theorem 2.1 *Assume (2.1)-(2.7); then, there exists at least a nonnegative nontrivial function $u \in H_0^1(\Omega)$ solving (1.1) in distributional sense.*

A case of particular interest for (1.1) is when $g(x,u)=\lambda u$, see [5, 8, 15]: we consider the equation

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_iu)+\frac12\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju=\lambda u+|u|^{2^*-2}u \quad \text{in } \Omega\ ; \tag{2.8}$$

by Theorem 2.1 we immediately infer

Corollary 2.1 *Assume (2.1)-(2.5), let $n\ge4$ and $\lambda\in(0,\nu\lambda_1)$; then, there exists at least a nonnegative nontrivial function $u\in H_0^1(\Omega)$ solving (2.8) in distributional sense.*

As for the semilinear problem [5], the case $n=3$ is more difficult. We obtain an existence result by strengthening assumption (2.2) by requiring the quasilinear elliptic operator to be, in a sense, more similar to a semilinear one. More precisely we require

$$\begin{cases} \exists\nu\in\left(1-\dfrac{S|\Omega|^{-2/3}}{\lambda_1},1\right]\ , \qquad \displaystyle\sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j\ge\nu|\xi|^2 \\ \text{for a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^n\ , \end{cases} \tag{2.9}$$

where S denotes the best Sobolev constant of the embedding $H_0^1\subset L^{2^*}$, see [20]. We prove

Theorem 2.2 *Let $n=3$, assume (2.1), (2.3)-(2.5), (2.9); there exists a constant $\lambda^*\in(0,\lambda_1-S|\Omega|^{-2/3}]$ such that if $\lambda\in(\lambda^*,\nu\lambda_1)$ then (2.8) admits at least a nonnegative nontrivial (weak) solution $u\in H_0^1(\Omega)$.*

Note that $S|\Omega|^{-2/3}<\lambda_1$ and that the lower bound for ν in (2.9) is needed to ensure that the interval $(\lambda^*,\nu\lambda_1)$ is nonempty.

3 The analysis of PS sequences

It is a standard result that with the above assumptions on a_{ij} we have

$$u \in H \Longrightarrow \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \in L^1(\Omega) \tag{3.1}$$

(see e.g. [6]) and therefore $J'(u)[u]$ is well defined for all $u \in H$ and can be written in integral form.

Define the cone of positive functions

$$\mathcal{C} := \{u \in H;\ u(x) \geq 0 \text{ for a.e. } x \in \Omega\} \tag{3.2}$$

and the functional

$$J_+(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u - \int_\Omega G(x,u^+) - \frac{1}{2^*}\int_\Omega |u^+|^{2^*} \ .$$

By the same procedure used in [9], i.e. by evaluating $J'_+(u)$ on the negative part of u and observing that $J'_+(v)[\varphi] = J'(v)[\varphi]$ for all $\varphi \in C_c^\infty$ and all $v \in \mathcal{C}$, the following lemma can be proved:

Lemma 3.1 *Assume (2.1)-(2.3), (2.6) and let $u \in H$ satisfy $J'_+(u)[\varphi] = 0$ for all $\varphi \in C_c^\infty$; then u is a weak positive solution of (1.1).*

Therefore, without loss of generality we will assume that

$$g(x,s) = 0 \qquad \forall s \leq 0\ , \quad \text{for a.e. } x \in \Omega$$

and to prove Theorem 2.1 we look for critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u - \int_\Omega G(x,u) - \frac{1}{2^*}\int_\Omega (u^+)^{2^*} \ :$$

for simplicity we have dropped the index + on J. By (2.3) and (2.5) we have

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u) D_i u D_j u \leq \int_\Omega |\nabla u|^2 \qquad \forall u \in \mathcal{C}\ . \tag{3.3}$$

For all $R, \delta > 0$ let

$$\vartheta_\delta(s) = \begin{cases} s & \text{if } |s| \leq R \\ R + \delta R - \delta s & \text{if } R < s < \frac{R+\delta R}{\delta} \\ -R - \delta R - \delta s & \text{if } -\frac{R+\delta R}{\delta} < s < -R \\ 0 & \text{if } |s| \geq \frac{R+\delta R}{\delta}\ ; \end{cases}$$

the following lemma is a restatement of Theorem 2.2.9 in [7]:

Lemma 3.2 *Assume (2.1)-(2.4), (2.6) and let* $\{u_m\}$ *be a PS sequence for* J *in* H*; then, for all* $R, \varepsilon > 0$ *there exists* $\delta > 0$ *such that the following inequality holds as* $m \to \infty$*:*

$$\int_{\{|u_m|\le R\}} \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m \le$$
$$\varepsilon \int_{\{|u_m|>R\}} \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m +$$
$$+ \int_{\Omega} [g(x,u_m) + (u_m^+)^{2^*-1}] \vartheta_\delta(u_m) + o(1) \ .$$

In particular, if $u_m \rightharpoonup 0$*, then*

$$\int_{\{|u_m|\le R\}} \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m \le$$
$$\varepsilon \int_{\{|u_m|>R\}} \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m + o(1) \ .$$

The above result is used to prove the following lemma, which implies that the PS sequences are bounded.

Lemma 3.3 *Assume (2.1)-(2.4), (2.6); then every sequence* $\{u_m\} \subset H$ *satisfying*

$$|J(u_m)| \le c \qquad \text{and} \qquad |J'(u_m)[u_m]| \le c\|u_m\|$$

is bounded in H*.*

Proof. Let $s^+ = \max\{s, 0\}$. By the assumptions on $g(x,s)$ it follows that for all $\beta \in [2, 2^*)$ there exists $s_\beta > 0$ such that

$$g(x,s)s + (s^+)^{2^*} \ge \beta \left(G(x,s) + \frac{(s^+)^{2^*}}{2^*} \right) \quad \forall s \ge s_\beta \ . \tag{3.4}$$

Consider $\{u_m\} \subset H$ satisfying the assumptions, then $\forall \beta \in [2, 2^*)$ $\exists c_\beta > 0$ such that

$$I_m^\beta := \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m - \frac{1}{\beta} \int_\Omega g(x,u_m) u_m - \frac{1}{\beta} \int_\Omega (u_m^+)^{2^*} \le c_\beta \ ;$$

by (3.1) we can compute $J'(u_m)[u_m]$ and by the assumptions we have

$$\left| \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_m) D_i u_m D_j u_m + \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m + \right.$$
$$\left. - \int_\Omega g(x,u_m) u_m - \int_\Omega (u_m^+)^{2^*} \right| \le c\|u_m\| \ .$$

Take $R > \bar{s}$ ($\bar{s}$ as in (2.4)); following Lemma 2.3.2 in [7] we choose $\gamma' \in (\gamma, 2^* - 2)$ and by (2.1) and (2.4) we obtain

$$\begin{aligned} K_m &:= \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m \\ &= \int_{\{|u_m| \le R\}} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m + \\ &\quad + \int_{\{|u_m| > R\}} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m u_m \\ &\le cR \int_{\{|u_m| \le R\}} |\nabla u_m|^2 + \gamma \int_{\{|u_m| > R\}} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m \ ; \end{aligned}$$

hence, by (2.2) and Lemma 3.2

$$\begin{aligned} K_m &\le \left(\frac{cR\varepsilon}{\nu} + \gamma\right) \int_{\{|u_m| > R\}} \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + c(R,\varepsilon) + o(1) \\ &\le \gamma' \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m + c(R,\varepsilon) + o(1) \ , \end{aligned}$$

if ε is small enough. Now, take $\beta \in (\gamma' + 2, 2^*)$ and compute $I_m^\beta - \frac{1}{\beta} J'(u_m)[u_m]$: by (3.4) we get

$$\frac{\beta - 2 - \gamma'}{2\beta} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m \le c(\|u_m\| + 1)$$

and the result follows by (2.2). □

As J' is not weakly continuous, we need to prove the following lemma to obtain a solution as weak limit of a PS sequence.

Lemma 3.4 *Assume (2.1)-(2.4), (2.6) and let $\{u_m\} \subset H$ be a PS sequence for J; then, there exists $u \in H$ such that*
(i) $u_m \rightharpoonup u$ in H, up to a subsequence
(ii) u solves (1.1) in distributional sense.

Proof. (i) follows from Lemma 3.3.

To prove (ii), let $\beta_m := g(x,u_m) + (u_m^+)^{2^*-1}$: then, up to a subsequence, we have $\beta_m \rightharpoonup \beta$ in $L^{2n/(n+2)}(\Omega)$ for some $\beta \in L^{2n/(n+2)}(\Omega)$; as $\{u_m\}$ is a PS sequence we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j \varphi + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_m) D_i u_m D_j u_m \varphi = \langle \beta_m + \gamma_m, \varphi \rangle \qquad \forall \varphi \in C_c^\infty$$

with $\gamma_m \to 0$ in $H^{-1}(\Omega)$. The above equations can also be written as

$$-\mathrm{div}[a_{ij}(x,u_m)D_iu_m] = -\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_m + \beta_m + \gamma_m \quad \text{in } \mathcal{D}'(\Omega)$$

with $\{\beta_m\}$ bounded in $H^{-1}(\Omega)\cap L^1(\Omega)$; therefore, the hypotheses of Theorem 2.1 in [4] are satisfied and $\nabla u_m(x) \to \nabla u(x)$ for a.e. $x \in \Omega$, up to a subsequence. Finally, by arguing just as in Lemma 2.3 in [6] we obtain

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi \ = \langle \beta, \varphi\rangle \quad \forall \varphi \in C_c^\infty \ ;$$

indeed, the only difference here is that β_m does not necessarily converge strongly in $H^{-1}(\Omega)$ to β but one can still deduce (2.3.5) from (2.3.4) in [6] by using the strong convergence of $\varphi\exp\{-Mu_m^+\}$ in $L^{2^*}(\Omega)$. □

Let S denote the best constant of the embedding $H^1_0 \subset L^{2^*}$; we determine a nontrivial energy range:

Lemma 3.5 *Assume (2.1)-(2.4), (2.6), let $\mathcal{C}$ be as in (3.2), let $\{u_m\} \subset \mathcal{C}$ be a PS sequence for J at level $c \in \left(0, \frac{S^{n/2}}{n}\right)$ and assume that $u_m \rightharpoonup u$; then $u \not\equiv 0$.*

Proof. Assume $u \equiv 0$: then $u_m \to 0$ in L^p $(1 \le p < 2^*)$ and $\int_\Omega g(x,u_m)u_m \to 0$; therefore, from $J'(u_m)[u_m] = o(1)$ we get

$$\begin{aligned} o(1) = &\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_m)D_iu_mD_ju_m + \\ &+\frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m - \int_\Omega (u_m)^{2^*} \ . \end{aligned} \tag{3.5}$$

By (2.1), (2.2), (2.5) and Lemma 3.2 we have for all $R, \varepsilon > 0$ (in particular, for $\varepsilon = R^{-2}$)

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m = \\ &\int_{\{u_m\le R\}} \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m + \\ &+\int_{\{u_m> R\}} \sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u_m)D_iu_mD_ju_mu_m \le \\ &cR\int_{\{u_m\le R\}} \sum_{i,j=1}^{n} a_{ij}(x,u_m)D_iu_mD_ju_m + \beta(R)\int_\Omega |\nabla u_m|^2 \le \\ &(c'R\varepsilon + \beta(R))\int_\Omega |\nabla u_m|^2 + o(1) \end{aligned}$$

where $\beta(R)$ is a function independent on ε and vanishing as $R \to +\infty$, therefore the second integral in (3.5) vanishes by the arbitrariness of R. We now study the first integral and observe that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_m) D_i u_m D_j u_m = \int_\Omega |\nabla u_m|^2 + o(1) \tag{3.6}$$

as $m \to \infty$; indeed again by (2.2), (2.5) and Lemma 3.2, we have for all $R, \varepsilon > 0$

$$\left| \int_\Omega \sum_{i,j=1}^n (\delta_{ij} - a_{ij}(x,u_m)) D_i u_m D_j u_m \right| \leq$$
$$\left| \int_{\{u_m \leq R\}} \sum_{i,j=1}^n (\delta_{ij} - a_{ij}(x,u_m)) D_i u_m D_j u_m \right| +$$
$$+ \left| \int_{\{u_m > R\}} \sum_{i,j=1}^n (\delta_{ij} - a_{ij}(x,u_m)) D_i u_m D_j u_m \right| \leq$$
$$c\varepsilon + \left| \int_{\{u_m > R\}} \sum_{i,j=1}^n (\delta_{ij} - a_{ij}(x,u_m)) D_i u_m D_j u_m \right| + o(1) \leq$$
$$c\varepsilon + \beta(R) + o(1) \ .$$

Hence, by (3.5) we get

$$\|u_m\|^2 - \|u_m\|_{2^*}^{2^*} = o(1) \tag{3.7}$$

and we can use the Sobolev inequality $\|u\|^2 \geq S\|u\|_{2^*}^2$ to obtain

$$o(1) \geq \|u_m\|^2 (1 - S^{-2^*/2} \|u_m\|^{2^*-2}) \ :$$

if $\|u_m\| \to 0$ we contradict $c > 0$; therefore, $\|u_m\|^2 \geq S^{n/2} + o(1)$ and by (3.6), (3.7) we get

$$J(u_m) = \frac{1}{n}\|u_m\|^2 + \frac{n-2}{2n}(\|u_m\|^2 - \|u_m\|_{2^*}^{2^*}) + o(1) \geq \frac{1}{n} S^{n/2} + o(1)$$

which contradicts $c < \frac{1}{n} S^{n/2}$. □

4 Proof of the results

Proof of Theorem 2.1. Without loss of generality we may assume that the origin $0 \in \Omega_0$, Ω_0 being as in (2.7); to achieve the proof, we need to build a PS sequence in the nontrivial range of the functional.

We first prove the existence of a PS sequence in $\mathcal{C}$, where $\mathcal{C}$ is as in (3.2). For all $e \in \mathcal{C} \setminus \{0\}$ there exists $t_e > 0$ such that $J(t_e e) < 0$: this is a consequence of the fact that the critical term is superquadratic at $+\infty$; define the class

$$\Gamma := \{\gamma \in C([0,1];\ H),\ \gamma(0) = 0,\ \gamma(1) = e\}$$

and the minimax value

$$\alpha := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) \ .$$

We obtain a PS sequence for J at level α by applying the mountain pass Lemma [1] in the nonsmooth version [11]: indeed, in a standard way one can verify that the functional J has such geometrical structure; hence, $\alpha > 0$. Moreover, as $J(u) \geq J(|u|)$ for all $u \in H$, we can assume that the PS sequence is in $\mathcal{C}$.

To prove that $\alpha < \frac{S^{n/2}}{n}$ we determine $v \in \mathcal{C}$ such that $\sup_{t\geq 0} J(tv) < \frac{S^{n/2}}{n}$. We follow the idea of [5] and consider the family of functions

$$u_\varepsilon^*(x) := \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}}$$

which solve the equation $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^n$ and satisfy $\|u_\varepsilon^*\|^2 = \|u_\varepsilon^*\|_{2^*}^{2^*} = S^{n/2}$. Let η be a positive smooth cut-off function with compact support in $B_\rho \subset \Omega_0$ and let $u_\varepsilon = \eta u_\varepsilon^*$. In order to prove that if ε is small enough, then

$$\sup_{t\geq 0} J(tu_\varepsilon) < \frac{1}{n} S^{n/2} \ , \tag{4.1}$$

we argue by contradiction and assume that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\begin{aligned} J(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{t_\varepsilon^2}{2}\int_\Omega \sum_{i,j=1}^n (a_{ij}(x,t_\varepsilon u_\varepsilon) - \delta_{ij}) D_i u_\varepsilon D_j u_\varepsilon + \\ &\quad - \int_\Omega G(x,t_\varepsilon u_\varepsilon) - \frac{t_\varepsilon^{2^*}}{2^*}\|u_\varepsilon\|_{2^*}^{2^*} \\ &\geq \frac{1}{n} S^{n/2} \ . \end{aligned} \tag{4.2}$$

Note that the sequence $\{t_\varepsilon\}$ is upper and lower bounded by two positive constants; indeed, if $t_\varepsilon \to +\infty$ then $J(t_\varepsilon u_\varepsilon) \to -\infty$, while if $t_\varepsilon \to 0$ then $J(t_\varepsilon u_\varepsilon) \to 0$ (recall that $\{u_\varepsilon\}$ is uniformly bounded in H): in both cases we contradict (4.2).

Next, we estimate the nonvanishing terms in $J(t_\varepsilon u_\varepsilon)$. Recall the following estimates (see [5]) as $\varepsilon \to 0$:

$$\|u_\varepsilon\|^2 = S^{n/2} + O(\varepsilon^{n-2}) \qquad \|u_\varepsilon\|_{2^*}^{2^*} = S^{n/2} + O(\varepsilon^n) \ ;$$

then, by reasoning as in [13], one obtains (as $\varepsilon \to 0$)

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon\|^2 - \frac{1}{2^*}\|t_\varepsilon u_\varepsilon\|_{2^*}^{2^*} \leq \frac{1}{n} S^{n/2} + O(\varepsilon^{n-2}) \ . \tag{4.3}$$

We prove that there exists a function $\tau = \tau(\varepsilon)$ such that $\lim_{\varepsilon\to 0} \tau(\varepsilon) = +\infty$ and such that for ε small enough we have

$$\int_\Omega G(x,t_\varepsilon u_\varepsilon) \geq \tau(\varepsilon)\cdot \varepsilon^{n-2} \ . \tag{4.4}$$

If $n = 3$, this can be done exactly as in [13]; if $n = 4$, one can follow the proof of Corollary 2.2 in [5] (or again the arguments of [13]). So, let us prove the result in the case $n \geq 5$: by a direct calculation we get

$$t_\varepsilon u_\varepsilon^*(x) = \gamma \quad \Longleftrightarrow \quad |x| = \Phi(\gamma) := \sqrt{\left(\frac{t_\varepsilon}{\gamma}\right)^{2/(n-2)} \sqrt{n(n-2)} \cdot \varepsilon - \varepsilon^2} \ ;$$

note that for all $\gamma > 0$ there exist $c_2 > c_1 > 0$ such that, for ε small enough we have $c_2\sqrt{\varepsilon} > \Phi(\gamma) > c_1\sqrt{\varepsilon}$. Therefore, by (2.6), (2.7) we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \mu \int_{\Phi(b)}^{\Phi(a)} r^{n-1}\ dr\ \geq c \int_{c_1\sqrt{\varepsilon}}^{c_2\sqrt{\varepsilon}} r^{n-1}\ dr\ \geq c\varepsilon^{n/2} \tag{4.5}$$

and the estimate (4.4) is proved for all $n \geq 3$.

Finally, note that $t_\varepsilon u_\varepsilon \in \mathcal{C}$ and that by (3.3) we have

$$\frac{t_\varepsilon^2}{2} \int_\Omega \sum_{i,j=1}^n (a_{ij}(x, t_\varepsilon u_\varepsilon) - \delta_{ij}) D_i u_\varepsilon D_j u_\varepsilon \leq 0 \ ;$$

therefore, if (4.2) held, by (4.3) and (4.4) we would obtain

$$J(t_\varepsilon u_\varepsilon) \leq \frac{1}{n} S^{n/2} + (c - \tau(\varepsilon)) \cdot \varepsilon^{n-2} \ .$$

We achieve a contradiction for ε small enough: hence (4.1) holds. So, we obtained a PS sequence (in $\mathcal{C}$) for J at level $\alpha \in \left(0, \frac{S^{n/2}}{n}\right)$: its weak limit is positive and nontrivial by Lemmas 3.1 and 3.5 and it solves (1.1) by Lemma 3.4. □

Proof of Theorem 2.2. As in the previous proof, we determine a PS sequence in $\mathcal{C}$ in the range of compactness. Following an idea of [8], instead of u_ε as in (4.1), to estimate the maximum of the functional J we take the direction of e_1, the first (positive) eigenfunction of $-\Delta$ in Ω. Let $u = te_1$ for some $t > 0$; by (3.3) and Hölder inequality we obtain

$$J(u) \leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{6}\|u\|_6^6 \leq \frac{\lambda_1 - \lambda}{2}\|u\|_2^2 - \frac{1}{6|\Omega|^2}\|u\|_2^6 \leq \frac{(\lambda_1 - \lambda)^{3/2}}{3}|\Omega| \ ,$$

the last inequality being consequence of

$$\forall a, b > 0 \qquad \max_{x \geq 0}(ax - bx^3) = \frac{2a}{3}\left(\frac{a}{3b}\right)^{1/2} \ .$$

Then, if $\lambda \in (\lambda_1 - S|\Omega|^{-2/3}, \nu\lambda_1)$, we have

$$\max_{t \geq 0} J(te_1) \leq \frac{(\lambda_1 - \lambda)^{3/2}}{3}|\Omega| < \frac{1}{3}S^{3/2}$$

and the existence of a solution follows as for Theorem 2.1: this also implies that $\lambda^* \leq \lambda_1 - S|\Omega|^{-2/3}$. □

5 Further remarks and a non-existence result

Remark 1 From (3.3) we infer, in particular, that the functional J on the cone $\mathcal{C}$ is not greater than the functional relative to the semilinear equation (i.e. for $a_{ij}(x,s) \equiv \delta_{ij}$). However, as we have seen, this fact does not modify the compactness level $\frac{1}{n}S^{n/2}$: indeed, (2.5) implies that the difference between the two functionals vanishes on the "bad" sequence $\{u_\varepsilon\}$ which is responsible of the non convergence of PS sequences, see [8, 19]. □

Remark 2 To prove Theorem 2.1 in the case $n = 4$, one may also weaken the second alternative of (2.7) with

$$\exists b > a > 0 \ , \ \exists \mu > 0 \quad \text{such that } G(x,s) \geq \mu \quad \text{for a.e. } x \in \Omega_0 \ , \ \forall s \in [a,b] \ ;$$

with this assumption, proceeding as in the case $n \geq 5$ in the proof of Theorem 2.1, we achieve again (4.5), which reads $\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \mu\varepsilon^2$ and therefore, for μ is large enough and $\varepsilon \to 0$ we have

$$J(t_\varepsilon u_\varepsilon) \leq \frac{1}{4}S^2 + (c - c\mu)\varepsilon^2 + o(\varepsilon^2) < \frac{1}{4}S^2 \ .$$

The requirement that μ is large enough is not only technical: even if $b = +\infty$, the equation may not have solutions if μ is small, see the curious example of Theorem 2.3 in [5]. □

Remark 3 The inequality $\lambda^* \leq \lambda_1 - S|\Omega|^{-2/3}$ in Theorem 2.2 can be strict: as an example, consider the cube $\Gamma := (0,\pi)^3$. The first eigenvector is $e_1(x,y,z) = \sin x \sin y \sin z$ and the corresponding eigenvalue is $\lambda_1 = 3$: a direct computation (by means of Hölder and Poincaré inequalities) yields

$$\max_{t \geq 0} J(te_1) \leq \pi^3(3-\lambda)^{3/2}\frac{2^{3/2}}{3 \cdot 5^{3/2}} \ ;$$

therefore $\lambda^* \leq 3 - \dfrac{15}{2^{7/3} \cdot \pi^{2/3}} < 3 - S|\Gamma|^{-2/3}$. □

Remark 4 An equation of the kind of (1.1) has also been studied in [22] by minimization methods and by a generalization of the arguments of [15]: the existence result obtained there does not require the positivity assumption (2.3), but it only holds on strictly star-shaped domains and requires $a_{ij}(x,s)$ to be even with respect to s; moreover, the result of [22] is up to the multiplication of suitable Lagrange multipliers. We also refer to [18] for a similar problem on $\mathbb{R}^n$. □

Remark 5 In high dimensions ($n \geq 4$), the positivity assumption of (2.6) may be relaxed so that also the primitive G of the lower order perturbation g is allowed to change sign. Assume that there exists $\delta > 0$ and $\alpha < \frac{n}{n-2}$ such that

$$G(x,s) \geq -\delta|s|^\alpha \qquad \text{for a.e. } x \in \Omega \quad \forall s \in \mathbb{R} \ ;$$

then if δ is small enough one can still obtain (4.5) and hence a nontrivial solution of (1.1), see [13]. However, in this case, the inequality $J_+(|u|) \leq J_+(u)$ does not hold for all u and one may lose the positivity of the solution of (1.1). □

By using a generalized Pohožaev identity due to Pucci-Serrin [17] we obtain a non-existence result:

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be bounded and star-shaped with respect to the origin, assume that the coefficients a_{ij} do not depend on x and that (2.1), (2.2) hold; assume moreover that*

$$s \sum_{i,j=1}^{n} a'_{ij}(s)\xi_i\xi_j < 0 \qquad \forall s \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \; . \tag{5.1}$$

Then the equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} a'_{ij}(u)D_i u D_j u = |u|^{2^*-2}u \tag{5.2}$$

has no nontrivial solutions $u \in H \cap L^\infty(\Omega)$.

Proof. By the regularity results of [14], if $u \in H \cap L^\infty(\Omega)$ is a solution of (5.2), then $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Define $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{F}(s,\xi) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(s)\xi_i\xi_j - \frac{|s|^{2^*}}{2^*} \; ;$$

take $a = \frac{n-2}{2}$ and replace in the variational identity (5) of [17]: as Ω is star-shaped w.r.t. 0 we obtain

$$\int_\Omega \sum_{i,j=1}^{n} a'_{ij}(u)D_i u D_j u u \geq 0$$

which, together with (5.1) implies $u \equiv 0$. □

The above statement would be more interesting if the nonexistence result held in the whole H. However, without suitable assumptions on the coefficients a_{ij}, a solution of (5.2) may not be bounded, see [12]: a possible assumption to ensure that $u \in L^\infty$ is (2.3) (see [7]) but here we have precisely assumed the "contrary" in (5.1); on the other hand, the technique used in [22] to prove the boundedness of u seems to apply only to minimization problems.

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