On an isoperimetric inequality for capacity conjectured by Pólya and Szegö

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Abstract

We study a conjecture by Pólya and Szegö on the approximation of the electrostatic capacity of convex bodies in terms of their surface measure. We prove that a "local version" of this conjecture holds true and we give some results which bring further evidence to its global validity.

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1 Introduction

The electrostatic capacity of a bounded set $\Omega \subset \mathbb{R}^3$ is defined by

$$\mathrm{Cap}(\Omega) = \frac{1}{4\pi} \inf \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 \ : \ u \in \mathcal{C}_c^\infty(\mathbb{R}^3),\, u = 1 \text{ in } \Omega \right\} . \tag{1}$$

It can be equivalently obtained through the asymptotic expansion

$$u_\Omega(x) = \mathrm{Cap}(\Omega)|x|^{-1} + O(|x|^{-2}) \qquad \text{for } |x| \to +\infty ,$$

where u_Ω is the *electrostatic potential* of Ω, namely the unique function which solves the Euler-Lagrange equation for problem (1):

$$\Delta u_\Omega = 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega , \qquad u_\Omega = 1 \quad \text{on } \partial\Omega , \qquad \lim_{|x|\to\infty} u_\Omega(x) = 0 . \tag{2}$$

For general Ω, this exterior problem is quite delicate to solve, even numerically, so that it is difficult to compute the exact value of $\mathrm{Cap}(\Omega)$. Nevertheless, since capacity appears in many physical phenomena, when the exact value is missing it is of great interest to find some estimates of it (see *e.g.* the pioneering works [7, 13, 14, 15, 16]). In this spirit, one may consider the following ratio:

$$\mathcal{E}(\Omega) = \frac{\mathrm{Cap}(\Omega)}{\sqrt{\frac{S(\Omega)}{4\pi}}} , \tag{3}$$

where $S(\Omega)$ denotes the surface area of Ω, namely the 2-dimensional Hausdorff measure $\mathcal{H}^2$ of its boundary $\partial\Omega$; in case Ω is a planar set, we understand that $S(\Omega) = 2\mathcal{H}^2(\Omega)$. Thus $\mathcal{E}(\Omega)$ is well-defined if and only if $\mathcal{H}^2(\Omega) > 0$. The denominator in (3) is also known as *Russell capacity* of Ω. Indeed, as already noticed about one century ago by Russell [17] (see also [1]), for simple geometries Ω where calculations may be performed explicitly, it gives a good approximation of $\mathrm{Cap}(\Omega)$. Clearly, for a fixed Ω, the closer $\mathcal{E}(\Omega)$ is to 1, the better the Russell capacity approximates the true capacity.

The approximation is perfect for balls, as the normalization constant at the denominator ensures that the value of $\mathcal{E}$ on balls is 1 (notice that $\mathcal{E}$ is invariant under dilations, as both numerator and denominator are 1-homogeneous). If one wants to get a uniform estimate on the approximation error when Ω varies, one has to study a shape optimization problem, consisting in the minimization of $\mathcal{E}(\Omega)$ over some class of admissible sets. It is immediate to realize that, in order to keep the infimum of $\mathcal{E}$ strictly positive, the *convexity* constraint is irremissible: indeed, a nonconvex set contained into a fixed sphere may have an arbitrarily large surface area, whereas by monotonicity its capacity remains bounded from above. No further constraint is a priori needed. Thus, setting

$$\mathcal{K} := \{\Omega \subset \mathbb{R}^3 \ : \ \Omega \text{ bounded and convex}, \ \mathcal{H}^2(\Omega) > 0\}$$

one comes to the optimization problem

$$\inf_{\mathcal{K}} \mathcal{E}(\Omega) \ . \tag{4}$$

Let us point out that the analogous maximization problem is not of interest since, by considering a sequence of thinning prolate ellipsoids, one can see that $\sup_{\mathcal{K}} \mathcal{E}(\Omega) = +\infty$ [4, Section 4.1]. On the contrary, the minimization in (4) is a long-standing open problem. More than half a century ago, Pólya and Szegö [16, §I.1.18] made the following:

Conjecture. Let D be a 2-dimensional disk. Then

$$\mathcal{E}(\Omega) \geq \inf_{\mathcal{K}} \mathcal{E} = \mathcal{E}(D) = \frac{2\sqrt{2}}{\pi} \approx 0.9 \ .$$

Moreover, $\mathcal{E}(\Omega) = \inf_{\mathcal{K}} \mathcal{E}$ if and only if Ω is a disk.

For the benefit of the reader, let us recall what is known at present about this conjecture.

- $\inf_{\mathcal{K}} \mathcal{E} \geq 2/\pi$ (see [16, (4), p.165]);
- the infimum $\inf_{\mathcal{K}} \mathcal{E}$ is attained (see [4]);
- the disk is the only minimizer of $\mathcal{E}$ if the class of admissible sets is restricted to the class of *planar* sets (see [15, p.14] and [16, §VII.7.3, p.157]).

The purpose of the present paper is to bring some new contributions to this challenging problem. We show that the inequality

$$\mathcal{E}(\Omega) > \mathcal{E}(D) \tag{5}$$

holds for Ω in some subclasses of $\mathcal{K}$. We distinguish our results into *local* and *global* ones.

In our results of local type we aim at proving inequality (5) for sets $\Omega \in \mathcal{K}$ which are close to D in the Hausdorff metric. We start with perturbations defined through a generic parametric family of concave functions (see Proposition 1), to deal then with perturbations obtained either by "flattening" a given graph (see Proposition 3) or via Minkowski addition (see Proposition 4). It is fascinating to compare the effect produced by different perturbations separately on surface area and capacity. For some perturbations surface area has a null derivative and capacity has a strictly positive derivative, whereas for some others surface area has a finite derivative and capacity has an infinite derivative. In any case, it turns out that the quotient $\mathcal{E}$ strictly increases as soon as the family of convex sets "detaches" from the disk. So, we arrive at the local minimality result for one-parameter perturbations stated in Theorem 5, and at the version for sequences converging to D in Hausdorff metric stated in Theorem 6.

In our results of global type, we prove inequality (5) for different classes of convex bodies, which are possibly far away from D in Hausdorff distance. First we show that (5) holds when Ω is any convex combination (through Minkowski addition) of the disk and a ball, see Theorem 7. The proof of this result is based on the Brunn-Minkowski inequality for capacity.
Then we turn our attention to the class of ellipsoids. We give numerical evidence that inequality (5) holds when Ω is a generic triaxial ellipsoid (see the plot in Figure 3), thus extending the computations made in [4] for the particular cases of prolate and oblate ellipsoids. Moreover, in Theorem 8 we prove analytically that no open triaxial ellipsoid except balls satisfies the stationarity condition for problem (4): it amounts to a Neumann condition for the electrostatic potential in terms of the mean curvature of the boundary, and thus leads to an intriguing overdetermined problem for harmonic functions on exterior domains. It was conjectured in [4] that such overdetermined exterior problem admits a solution only on the complement of a ball; in Theorem 8 we show that it does not have solution on the complement of any other ellipsoid.

The paper is organized as follows. Local results are stated in Section 2 and proved in Section 4. Global results are stated in Section 3 and proved in Section 5. Finally the Appendix is devoted to the proof of a general first variation formula for capacity, which is needed in Section 4, and may deserve an autonomous interest.

2 Local results

In this section we establish local minimality properties of D. We consider different kinds of one-parameter families of convex sets D_t, with $t \in [0, 1]$. In all the situations under study, the Hausdorff distance between D_t and $D_0 = D$ will be infinitesimal as $t \to 0^+$: in fact D_t will be contained into a cylinder with height of order t. In order to determine the behaviour of the quotient energy $\mathcal{E}(D_t)$ for small $t > 0$, we need to analyze separately the behaviour of the two functions

$$C(t) := \mathrm{Cap}(D_t) \qquad \text{and} \qquad S(t) := \mathcal{H}^2(\partial D_t) \ .$$

More specifically, we are led to investigate the properties of their incremental ratios in a right neighbourhood of $t = 0$. Although they are merely *right* derivatives, we set

$$C'(0) := \lim_{t\to 0^+} \frac{C(t) - C(0)}{t} \qquad \text{and} \qquad S'(0) := \lim_{t\to 0^+} \frac{S(t) - S(0)}{t} \ ,$$

whenever these limits exist. For convenience, we think of D as centered at the origin and contained into the plane $x_3 = 0$, namely

$$D = \{(x_1, x_2, 0) \in \mathbb{R}^3 \ : \ 0 \le r < 1\} \ ; \tag{6}$$

here and below, for any $(x_1, x_2, x_3) \in \mathbb{R}^3$, we set $r = \sqrt{x_1^2 + x_2^2}$.
We consider first the case when the D_t's are the subgraphs of a one-parameter family of concave radially symmetric functions defined on D. We call them a *parametric family* which we denote by $\phi(r; t)$, and we assume that

$$\phi \in \mathcal{C}\left([0, 1]^2\right) \text{ with } t \mapsto \phi(\cdot; t) \text{ nondecreasing and } \phi(\cdot; 0) \equiv 0; \tag{7}$$

$$r \mapsto \phi(r; t) \text{ nonincreasing, concave, with } \phi(0; t) = t \text{ and } \phi(1; t) = 0 \ . \tag{8}$$

The next proposition is the key result from which the local minimality of D will stem. It is obtained by using as main tools the first variation formula for capacity proved in Appendix (see Theorem 15), the explicit expression for the potential of D (see Lemma 13), and careful estimates on (a regularization of) the parametric family $\phi(r; t)$.

Proposition 1. (Parametric families) *For $t \in [0,1]$, let D_t be given by*

$$D_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; 0 \le r < 1,\, 0 < x_3 < \phi(r;t)\}\ ,$$

where $\phi(r;t)$ is a parametric family satisfying assumptions (7) *and* (8)*. Then:*

(i) We have

$$\liminf_{t \to 0^+} \frac{C(t) - C(0)}{t} \ge \frac{2}{\pi^2}(1 - \log 2).$$

(ii) We have

$$\limsup_{t \to 0^+} \frac{S(t) - S(0)}{t} \le 2\pi\ .$$

(iii) If, for a sequence $\{t_k\}$ decreasing to zero, it holds

$$\lim_{k \to +\infty} \frac{S(t_k) - S(0)}{t_k} > 0,$$

then there exists a subsequence $\{t^l\} := \{t_{k_l}\}$ such that

$$\lim_{l \to +\infty} \frac{C(t^l) - C(0)}{t^l} = +\infty.$$

In particular, if $S'(0)$ exists and $S'(0) > 0$, then $C'(0)$ exists and $C'(0) = +\infty$.

Remark 2. One may wonder if for some parametric family $\phi(r;t)$, satisfying assumptions (7) and (8), $S'(0)$ might not exist. The answer is affirmative, and an explicit example can be constructed as follows. Let

$$\alpha(t) := t\Big(\frac{1}{2} + \frac{1}{4}\sin(\log t)\Big)\ ,$$

and define

$$\phi(r;t) := \begin{cases} t - r\,\frac{2t - \sqrt{3}\alpha}{2 - \alpha} & \text{if } r \in [0, 1 - \frac{\alpha}{2}] \\ \sqrt{\alpha^2 - (r - 1 + \alpha)^2} & \text{if } r \in [1 - \frac{\alpha}{2}, 1]\ . \end{cases}$$

We leave to the reader to check that this family satisfies assumptions (7) and (8), and that the surface area increment admits the asymptotic development

$$\frac{S(t) - S(0)}{t} = 2\pi\Big(\frac{\pi}{3} - \frac{1}{2}\Big)\frac{\alpha(t)}{t} + o(1) \qquad \text{as } t \to 0^+\ ,$$

so that by the definition of $\alpha(t)$ the derivative $S'(0)$ does not exist.

Statement (i) of Proposition 1 implies at once the local minimality of $\mathcal{E}(D_t)$ at $t = 0$, when combined with the information that $S'(0) = 0$. This occurs for instance in the case of flattening graphs considered in Proposition 3 below, when the parametric family takes the special form $\phi(r;t) = t\varphi(r)$. On the other hand, by a comparison argument, statement (iii) of Proposition 1 allows to deal with (possibly non axially-symmetric) perturbations D_t obtained from D through a Minkowski addition. Again the outcoming information on the derivatives $C'(0)$ and $S'(0)$, stated in Proposition 4 below, implies immediately the local minimality of $\mathcal{E}(D_t)$ at $t = 0$.

Proposition 3. (Flattening graphs)
For $t \in [0,1]$, let

$$D_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \le r < 1, \, 0 < x_3 < t\varphi(r)\} \ , \tag{9}$$

being $\varphi \in \mathcal{C}([0,1])$ a concave, nonincreasing function, with $\varphi(0) = 1$ and $\varphi(1) = 0$. Then statement (i) of Proposition 1 holds and

$$S'(0) = 0 \ .$$

Proposition 4. (Minkowski sums)
Let $\Omega \subset \mathbb{R}^3$ be an open bounded convex set. For $t \in [0,1]$, let

$$D_t = (1-t)D + t\Omega := \Big\{(1-t)x + ty \ : \ x \in D \ , \ y \in \Omega\Big\} \ . \tag{10}$$

Then

$$C'(0) = +\infty \qquad \textit{and} \qquad S'(0) \in (-4\pi, +\infty) \ .$$

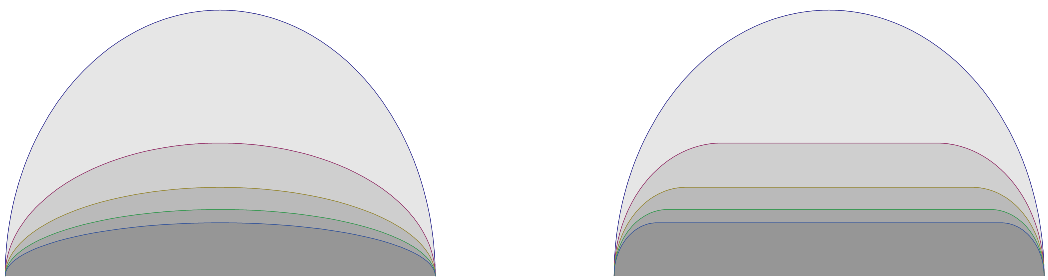

Figure 1: D_t in (9) with $\varphi(r) = \sqrt{1-r^2}$ and the upper half of D_t in (10) with $\Omega = B$

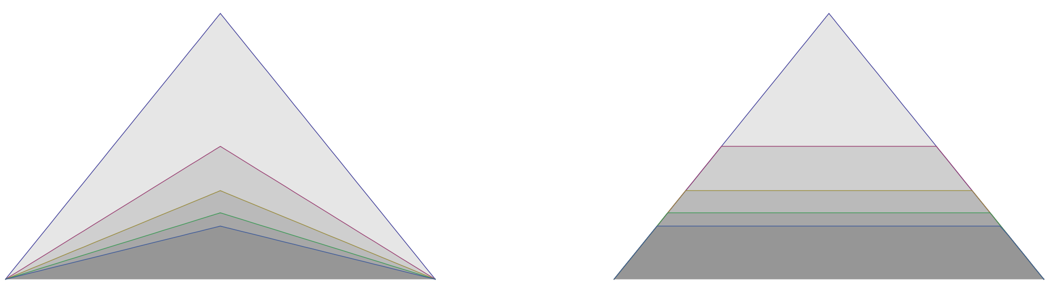

Figure 2: D_t in (9) with $\varphi(r) = 1 - r$ and the upper half of D_t in (10) with Ω a right cone

Propositions 3 and 4 reveal how perturbations which look very similar may produce dramatically different increments of $C(t)$ and $S(t)$. For instance, in Figure 1 we have drawn a comparison between D_t given by (9) with $\varphi(r) = \sqrt{1-r^2}$ and (the upper half of) D_t given by (10) with $\Omega = B$. Capacity behaves in opposite ways: in the former case $C'(0) \in (0, +\infty)$ (notice indeed that, since the involved sets are hemi-ellipsoids, their capacity may be computed explicitly as in [4, Section 4], yielding $C'(0) = 2/\pi^2$), while in the latter case $C'(0) = +\infty$ (by Proposition 4). Also surface measure behaves differently: in the former case $S'(0) = 0$ (by Proposition 3), while in the latter case $S'(0) \in (-4\pi, +\infty)$ (by Proposition 4).
A similar comparison may be traced between D_t given by (9) with $\varphi(r) = 1 - r$ and (the upper half of) D_t given by (10) with Ω a right circular cone, see Figure 2.

Though with all the differences highlighted in the above discussion, for each of the families D_t considered so far, the increments of capacity and surface measure arrange so that the value of $\mathcal{E}(D_t)$ increases as soon as t becomes strictly positive. Indeed, as a consequence of Propositions 1, 3, and 4, we obtain

Theorem 5. (local minimality for families) *For any of the families D_t as in Propositions 1, 3, and 4, there holds*

$$\mathcal{E}(D) < \mathcal{E}(D_t) \qquad \text{for } 0 < t \ll 1 \ .$$

Finally, as a byproduct of the previous results, we also obtain a local minimality statement for sequences of arbitrary axially-symmetric convex sets converging to the disk in the Hausdorff metric.

Theorem 6. (local minimality for sequences) *Let $\{D_n\}$ be a decreasing sequence of axially-symmetric convex sets converging to D in the Hausdorff metric. Then, there exists $N \in \mathbb{N}$ such that*

$$\mathcal{E}(D) < \mathcal{E}(D_n) \qquad \text{for } n \geq N.$$

3 Global results

In this section we establish two global results: although they are of different nature, both show the minimality of D for $\mathcal{E}$ within classes of convex sets not necessarily close to it in Hausdorff distance. Let us first go back to the Minkowski sums considered in Proposition 4. When D_t is defined by (10), for any Ω belonging to the class

$$\mathcal{K}_o := \{\Omega \in \mathcal{K} \, : \, \Omega \text{ open}\} \ , \tag{11}$$

Proposition 4 ensures that the energy $\mathcal{E}(D_t)$ increases with an infinite slope at $t = 0$; hence, as stated in Theorem 5, $\mathcal{E}(D) < \mathcal{E}(D_t)$ for $0 < t \ll 1$. Clearly, if we were able to show that the same inequality continues to hold *globally*, for all t in the interval $(0, 1]$, the proof of Pólya and Szegö conjecture would be achieved by choosing $t = 1$ and taking into account the arbitrariness of Ω in $\mathcal{K}_o$. Our next result states that the required inequality $\mathcal{E}(D) < \mathcal{E}(D_t)$ holds in fact for any $t \in (0, 1]$, in the special case when the set Ω that we add through a Minkowski sum is a ball. In other words, D may be deformed continuously into a ball so that at any stage the energy remains larger than $\mathcal{E}(D)$.

Theorem 7. (from the disk to a ball) *Let B be the unit ball. For any $r > 0$ and any $t \in (0, 1]$ we have*

$$\mathcal{E}(D) < \mathcal{E}\big((1-t)D + trB\big) \ .$$

We point out that, in the proof of Theorem 7 (which relies on Borell's Brunn-Minkowski inequality for capacity [3]), the knowledge of the exact values of $S(B)$ and $\mathrm{Cap}(B)$ plays a fundamental role. Another setting in which we may take advantage of some explicit formulae for capacity and surface area is the minimization of $\mathcal{E}$ in the class of ellipsoids, which is actually the second goal of this section. We give a numerical proof that D minimizes $\mathcal{E}$ in the class $\mathcal{T}$ of triaxial ellipsoids. For $a \geq b \geq c > 0$, let

$$E_{a,b,c} = \Big\{(x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} < 1\Big\}$$

denote the generic 3-dimensional ellipsoid. Similarly, for $a \geq b > 0$ let

$$E_{a,b,0} = \Big\{(x, y, 0) \in \mathbb{R}^3; \ \frac{x^2}{a} + \frac{y^2}{b} < 1\Big\}$$

denote the generic planar ellipse.
Since the shape functional $\mathcal{E}$ is invariant under scaling, we may restrict our attention to minimize the function $(b,c) \mapsto \mathcal{E}(E_{1,b,c})$ over the planar triangle $T = \{(b,c) \in \mathbb{R}^2 \,:\, 1 \geq b \geq c \geq 0\}$.
We point out that the three sides of T correspond to planar ellipses ($c = 0$), to oblate ellipsoids ($b = 1$) and to prolate ellipsoids ($c = 1$); in these cases computations of capacity and surface area are simpler, and it is shown in [4] that the point $(1,0)$, corresponding to the disk, is the unique minimizer for $\mathcal{E}$ over ∂T. Note also that the origin $(0,0)$ corresponds to the unit segment and that $\mathcal{E}$ diverges as $(b,c) \to (0,0)$.
Here, we extend the analysis on the interior of T. To this purpose we exploit the fact that both the surface measure and the capacity of a triaxial ellipsoid are known in terms of elliptic integrals, respectively as [12, (4,15) p.38]

$$\mathrm{Cap}(E_{1,b,c}) = 2 \left(\int_0^\infty \frac{ds}{\sqrt{(1+s)(b+s)(c+s)}} \right)^{-1} , \tag{12}$$

and

$$S(E_{1,b,c}) = 4 \int_0^{\pi/2} \int_0^1 \sqrt{bc + \frac{(b\cos^2\theta + c\sin^2\theta)r}{1-r}}\, dr\, d\theta \,, \tag{13}$$

see also [20] for several different expressions of capacity and surface area of ellipsoids.
Through the numerical computation of (12) and (13), we have obtained the plots in Figure 3 , where we display the behaviour respectively over the triangle T and near the point $(1,0)$ of the *inverse* function

$$(b,c) \mapsto \frac{1}{\mathcal{E}(E_{1,b,c})} \,. \tag{14}$$

This gives numerical evidence that the maximum over T of the function in (14) is attained at the point $(1,0)$, and hence that D minimizes $\mathcal{E}$ in the class $\mathcal{T}$ of triaxial ellipsoids.
Incidentally, Figure 4 shows the behaviour of the function $1/\mathcal{E}(E_{1,b,c})$ for c taking three different fixed values and b ranging in $(0.1, 0.2)$, and for $b = 0.15$ and c ranging over $(0, 0.15)$. This enlightens a rather surprising lack of monotonicity.

In order to corroborate our numerical results, we proceed by giving an analytical proof that no triaxial ellipsoid in $\mathcal{K}_o$ is a stationary domain for $\mathcal{E}$. We recall from [4, Theorem 7] that, when $\Omega \in \mathcal{K}_o$ is $\mathcal{C}^2$ and strictly convex, the stationarity condition

$$\frac{d}{dt}\mathcal{E}(\Omega + t\omega)|_{t=0} = 0 \qquad \forall \omega \in \mathcal{K}_o \,,$$

is satisfied if and only if the following pointwise identity holds:

$$|\nabla u_\Omega(x)|^2 = \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} H_{\partial\Omega}(x) \qquad \forall x \in \partial\Omega \,; \tag{15}$$

here u_Ω is the electrostatic potential of Ω, while $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$, namely $H_{\partial\Omega} = (k_1 + k_2)/2$, being k_i the principal curvatures of $\partial\Omega$. In [4] the following conjecture was formulated:

Conjecture. Assume that $\Omega \in \mathcal{K}_o$ has a $\mathcal{C}^2$ boundary and is strictly convex. If there exists a solution to the overdetermined problem given by (2)-(15), then Ω is a ball.

Since the value of $\mathcal{E}$ on balls is 1, this conjecture would imply that no smooth strictly convex domain in $\mathcal{K}_o$ can be the minimum for $\mathcal{E}$ over $\mathcal{K}$.

Here, we establish that the conjecture is true in the class of ellipsoids. Through a careful comparison between their electrostatic potential and their mean curvature we prove:

Theorem 8. (no stationary ellipsoid apart balls) *The only domains in $\mathcal{T} \cap \mathcal{K}_o$ whose potential satisfies the stationarity condition* (15) *are balls.*

As a consequence of Theorem 8, since the value of $\mathcal{E}$ on balls is 1, and since only disks minimize $\mathcal{E}$ among planar convex sets, we readily obtain

Corollary 9. (no minimizing ellipsoid) *For every $\Omega \in \mathcal{T}$ different from a 2-dimensional disk,*

$$\mathcal{E}(\Omega) > \inf_{\mathcal{K}} \mathcal{E} \ .$$

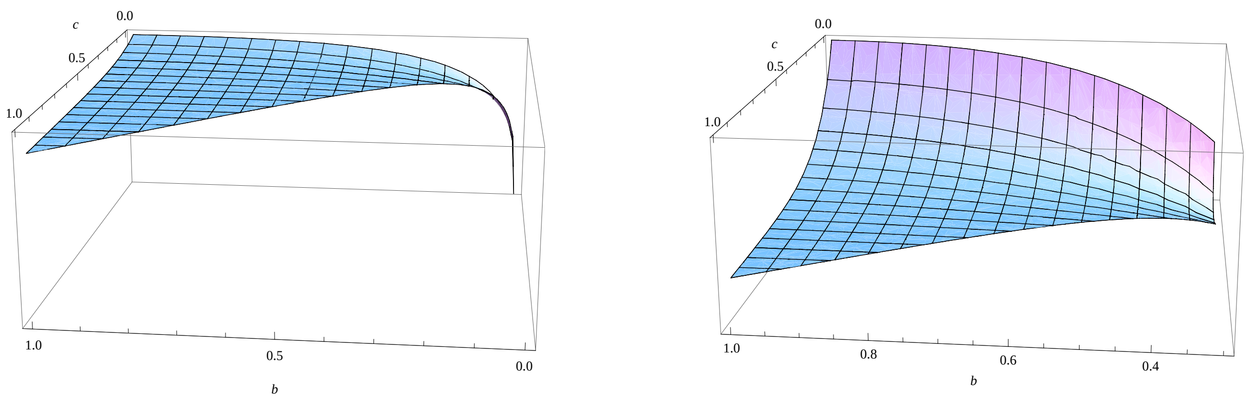

Figure 3: Plots of the map (14) for $(b, c) \in T$ and for (b, c) near $(1, 0)$.

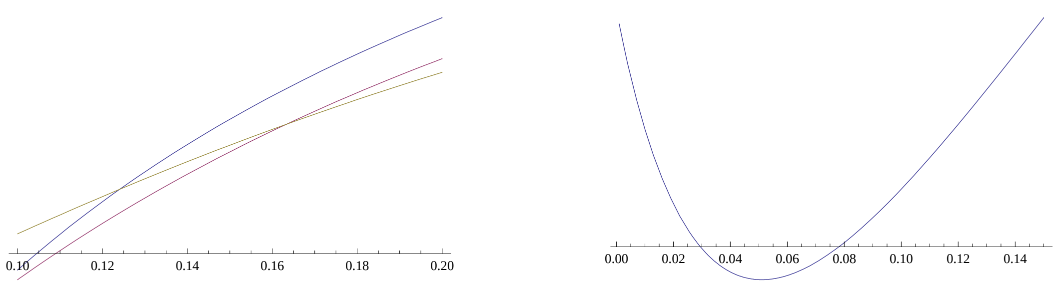

Figure 4: Plots of the maps $b \mapsto 1/\mathcal{E}(1, b, c)$ over $(0.1, 0.2)$ for $c = 0.01, 0.04, 0.08$, and plot of the map $c \mapsto 1/\mathcal{E}(1, 0.15, c)$ over $(0, 0.15)$

4 Proofs of local results

We begin with some preliminary lemmas needed to prove Proposition 1. First, as a straightforward application of Theorem 15 in the Appendix, we obtain

Lemma 10. *Under the assumptions of Proposition 1, at any $t \in (0, 1)$ the derivative of the function $C(t)$ is given by*

$$C'(t) = \frac{1}{2} \int_0^1 |\nabla u^t(r)|^2 \phi_t(r; t) \, r \, dr \ ,$$

where $\nabla u^t(r) := \nabla u_{D_t}(r, \phi(r; t))$, being u_{D_t} the electrostatic potential of D_t.

Proof. We notice that, for any fixed $t \in (0,1)$, $D_t \in \mathcal{K}_0$, and D_{t+h} may be obtained as a deformation of D_t by mapping the point $(r, \phi(r;t)) \in \partial D_t$ into the point $(r, \phi(r;t+h)) \in \partial D_{t+h}$. Since $\phi(r;t+h) = \phi(r;t) + h\phi_t(r;t) + o(h)$, the velocity of the deformation is given by $(0,0,\phi_t(r;t))$. On the other hand, the third component of the unit outer normal to ∂D_t is given by $1/\sqrt{1+\phi_r(r;t)^2}$. Then, Theorem 15 (where Ω and t play the role of Ω_t and h!) gives

$$C'(t) = \frac{1}{4\pi} \int_{\partial D_t} \frac{\phi_t(r;t)}{\sqrt{1+\phi_r(r;t)^2}} |\nabla u^t(r)|^2 \, d\mathcal{H}^2 = \frac{1}{2} \int_0^1 |\nabla u^t(r)|^2 \phi_t(r;t) \, r \, dr \ .$$

□

In view of Lemma 10, in order to prove Proposition 1 it is crucial to control the behaviour of the potential gradients ∇u^t as $t \to 0^+$. Since the family of convex sets D_t is not smooth enough to allow such control, we set up a regularization procedure based on the use of Yosida approximation. In the first part of the proof of Proposition 1 the "regularization parameter" ϵ will remain fixed, whereas it will become infinitesimal only in the proof of statement (iii).

Lemma 11. *For ϕ satisfying* (7)-(8) *and for all $\epsilon > 0$, let ϕ^ϵ be defined on $[0,1]^2$ as the solution of*

$$\phi^\epsilon(r;t) - \epsilon \phi^\epsilon_{rr}(r;t) = \phi(r;t) \ , \quad \phi^\epsilon(0;t) = t \ , \quad \phi^\epsilon(1;t) = 0.$$

Then ϕ^ϵ also satisfies (7)-(8)*. Moreover, if*

$$D_{\epsilon,t} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \ : \ 0 \le r < 1 \, , \ 0 < x_3 < \phi^\epsilon(r;t)\} \ ,$$

then the electrostatic potentials $u^{\epsilon,t}$ of $D_{\epsilon,t}$ satisfy

$$\lim_{t \to 0^+} |\nabla u^{\epsilon,t}(r, \phi^\epsilon(r;t))|^2 = |\nabla u_D(r,0)|^2 \qquad \forall r \in (0,1) \ . \tag{16}$$

Proof. Writing (in weak form)

$$(\phi^\epsilon - \phi) - \epsilon(\phi^\epsilon - \phi)_{rr} = \epsilon \phi_{rr} \le 0 \ , \quad (\phi^\epsilon - \phi)(0;t) = (\phi^\epsilon - \phi)(1;t) = 0 \ ,$$

we see by the maximum principle that $\phi^\epsilon \le \phi$ and, therefore, $\phi^\epsilon_{rr} \le 0$. By concavity and since $\phi^\epsilon \le \phi$, we also deduce that $\phi^\epsilon_r \le 0$.
Similarly, if $0 \le s \le t$, we write for $w(r) := \phi^\epsilon(r;t) - \phi^\epsilon(r;s)$:

$$w(r) - \epsilon w''(r) = \phi(r;t) - \phi(r;s) \ge 0 \ , \qquad w(0) = w(1) = 0 \ ,$$

to deduce that $w \ge 0$ and hence that

$$t \mapsto \phi^\epsilon(r;t) \quad \text{is nondecreasing for any } r. \tag{17}$$

Thus the convex sets $D_{\epsilon,t}$ satisfy $D_{\epsilon,t} \subset D_t$ and also decrease to D as $t \to 0^+$, but in such a way that, for any fixed $\epsilon > 0$, the second derivatives $\phi^\epsilon_{rr}(\cdot;t)$ remain bounded in $\mathcal{C}^{0,\alpha}([\eta, 1-\eta])$ for all $\eta \in (0,1)$; this regularity follows from the fact that ϕ is concave and therefore Lipschitzian on $[\eta, 1-\eta]$.
As a consequence, by the Schauder regularity theory (see *e.g.* [8], Lemma 6.18), their electrostatic potentials $u^{\epsilon,t}$ are bounded in $\mathcal{C}^{2,\alpha}(B^\eta_t)$, independently of t, where

$$B^\eta_t = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \ : \ \eta \le r \le 1-\eta \, , \ \phi^\epsilon(r;t) \le x_3 \le 1\}.$$

It follows that, up to subsequences, $\nabla u^{\epsilon,t}$ converges uniformly on B^η_t. Since we know that the limit of any subsequence is equal to ∇u_D, by uniqueness, the whole function $t \mapsto \nabla u^{\epsilon,t}$ converges as $t \to 0^+$. In particular, for any fixed $\epsilon > 0$, $\nabla u^{\epsilon,t}(r, \phi^\epsilon(r;t))$ converges pointwise to $\nabla u_D(r,0)$, and we obtain (16). □

Remark 12. Since we do not need to use it, we have not made explicit the form of ϕ^ϵ which may be found by standard calculus. For instance,

$$\phi^1(r;t) = c_1 e^r + c_2 e^{-r} - \int_0^r \sinh(r-s)\phi(s;t)\,ds$$

for constants c_1 and c_2 to be determined with the boundary conditions.

To make the convergence in (16) more precise, we give the explicit expression for the potential gradient of the disk. Note that the potential u_D is not in $\mathcal{C}^1(\mathbb{R}^3)$ because the gradient fails to be continuous when crossing the disk. However, we are merely interested in its length $|\nabla u_D|$ which is continuous across the (open) disk.

Lemma 13. *Let D be given by* (6)*. Then its electrostatic potential u_D satisfies*

$$|\nabla u_D|^2(r,0) = \frac{4}{\pi^2}\frac{1}{1-r^2} \qquad \forall r \in [0,1)\ .$$

Proof. By [10] we know that $u_D = \frac{F(\lambda)}{F(0)}$, and hence $(u_D)_{x_3} = \frac{F'(\lambda)}{F(0)}\lambda_{x_3}$, where

$$F(\lambda) = \int_\lambda^{+\infty} \frac{ds}{(s+1)\sqrt{s}}\ ,$$

and the function $\lambda = \lambda(r, x_3)$ is implicitly defined on the complement of D by the equation

$$\frac{r^2}{1+\lambda} + \frac{x_3^2}{\lambda} = 1\ . \tag{18}$$

By differentiating (18) with respect to x_3, and noticing that $F(0) = \pi$, we obtain

$$(u_D)_{x_3}^2(r,x_3) = \frac{4}{\pi^2}\frac{x_3^2}{\lambda}\frac{(1+\lambda)^2}{[(1+\lambda)^2-r^2]^2} = \frac{4}{\pi^2}\Big(1-\frac{r^2}{1+\lambda}\Big)\frac{(1+\lambda)^2}{[(1+\lambda)^2-r^2]^2}\ .$$

The statement follows by letting x_3 (and hence λ) tend to zero, and by taking into account that $|\nabla u_D(r,0)| = |(u_D)_{x_3}(r,0)|$. □

We are now in a position to give the

Proof of Proposition 1.

Proof of (i). Consider the domain $D_{\epsilon,t}$ introduced in Lemma 11 and denote by $C_\epsilon(t)$ the capacity of $D_{\epsilon,t}$. We also set for brevity $\nabla u^{\epsilon,t}(r) := \nabla u^{\epsilon,t}(r,\phi^\epsilon(r;t))$ and $\nabla u_D(r) := \nabla u_D(r,0)$. Since $D_{\epsilon,t} \subset D_t$, we have

$$\frac{C(t)-C(0)}{t} \geq \frac{C_\epsilon(t)-C_\epsilon(0)}{t} = \frac{1}{t}\int_0^t C'_\epsilon(s)\,ds = \frac{1}{2t}\int_0^t ds \int_0^1 |\nabla u^{\epsilon,s}(r)|^2 \phi_t^\epsilon(r;s) r dr\ ,$$

where we have applied Lemma 10 to the family $D_{\epsilon,t}$. Put

$$\delta(r;t) := \sup_{0<s<t} \big| |\nabla u^{\epsilon,s}(r)|^2 - |\nabla u_D(r)|^2 \big|\ .$$

By (17) we know that $\phi_t^\epsilon \geq 0$ so that we obtain for all $r \in (0,1)$:

$$\frac{1}{t}\left|\int_0^t \left[|\nabla u^{\epsilon,s}(r)|^2 - |\nabla u_D(r)|^2\right]\phi_t^\epsilon(r;s)\,r\,ds\right| \leq \frac{r\,\delta(r;t)}{t}\int_0^t \phi_t^\epsilon(r;s)ds = r\,\delta(r;t)\psi^\epsilon(r;t)$$

where

$$\psi^\epsilon(r;t) := \frac{\phi^\epsilon(r;t)}{t} \ .$$

By Lemma 11, for all $r \in (0,1)$, $\delta(r;t) \to 0$ as $t \to 0^+$. Moreover, by construction ψ^ϵ is bounded by 1 and

$$\frac{1}{t}\int_0^t |\nabla u_D(r)|^2 \phi^\epsilon_t(r;s) ds = |\nabla u_D(r)|^2 \psi^\epsilon(r;t) \qquad \forall r \in [0,1) \ .$$

Then, by Fubini's Theorem and Fatou's Lemma, we obtain for all $\epsilon > 0$:

$$\liminf_{t\to 0^+} \frac{C(t)-C(0)}{t} \geq \frac{1}{2}\int_0^1 |\nabla u_D(r)|^2 \big(\liminf_{t\to 0^+} \psi^\epsilon(r;t)\big)\, r\, dr \ . \tag{19}$$

Assume for a moment to know that

$$\liminf_{t\to 0^+} \psi^\epsilon(r;t) \geq 1-r \qquad \text{for a.e. } r \in (0,1) \ . \tag{20}$$

Then, by (19) and Lemma 13, we get

$$\liminf_{t\to 0^+} \frac{C(t)-C(0)}{t} \geq \frac{1}{2}\int_0^1 |\nabla u_D(r)|^2 (1-r)\, r\, dr = \frac{2}{\pi^2}\int_0^1 \frac{r}{1+r}\, dr = \frac{2}{\pi^2}(1-\log 2) \ ,$$

and statement (i) of Proposition 1 is proved.
We now go back to the proof of (20). We set

$$\psi(r;t) := \frac{\phi(r;t)}{t} \ . \tag{21}$$

By construction we have $0 \leq \psi \leq 1$, $\psi_r(r;t) = \frac{1}{t}\phi_r(r;t) \leq 0$, and

$$\int_0^1 |\psi_r(r;t)|\, dr = -\int_0^1 \psi_r(r;t)\, dr = \frac{\phi(t;0)-\phi(t;1)}{t} \leq 1.$$

It follows that, for any subsequence $\{t_k\}$ decreasing to zero, there exists a subsequence $\{t^l\} := \{t_{k_l}\}$ such that, for $p \in [1,\infty)$, $\psi(\cdot;t^l)$ converges in $L^p(0,1)$ (and pointwise a.e.) to some function $g \in L^p(0,1)$. Like $\psi(\cdot;t)$, the function g is nonnegative, nonincreasing and concave on $[0,1]$. Thanks to the L^p-convergence of $\psi(\cdot;t^l)$ to g, the sequence $\psi^\epsilon(r;t^l) = \frac{\phi^\epsilon(r;t^l)}{t^l}$ converges uniformly on $[0,1]$ to the solution g^ϵ of

$$g^\epsilon - \epsilon (g^\epsilon)'' = g \ , \quad g^\epsilon(0) = 1 \ , \quad g^\epsilon(1) = 0 \ . \tag{22}$$

By concavity of g^ϵ, we have

$$\lim_l \psi^\epsilon(r;t^l) = g^\epsilon(r) \geq 1-r \qquad \forall\, r \in [0,1] \ . \tag{23}$$

By the arbitrariness of the initial sequence $\{t_k\}$, we deduce that (20) holds true, and the proof of statement (i) is achieved.

Proof of (ii). For a.e. $\eta \in (0,1)$, the increment of the surface area can be estimated as:

$$\begin{aligned} 0 < \frac{S(t)-S(0)}{2\pi t} &= \frac{1}{t}\int_0^1 [\sqrt{1+\phi_r^2}-1] r\, dr = \frac{1}{t}\int_0^1 \frac{\phi_r^2}{\sqrt{1+\phi_r^2}+1}\, r\, dr \\ &\leq \frac{-\phi_r(1-\eta;t)}{t}\int_0^{1-\eta} -\phi_r(r;t) dr + \frac{1}{t}\int_{1-\eta}^1 -\phi_r(r;t) dr \\ &\leq -\phi_r(1-\eta;t) + \psi(1-\eta;t). \end{aligned} \tag{24}$$

Now we notice that, by monotonicity and concavity of $r \mapsto \phi(r;t)$,

$$0 \le -\phi_r(1-\eta;t) \le \frac{\phi(1-\eta;t)}{\eta} \le \frac{t}{\eta} \ . \tag{25}$$

Then statement (ii) of Proposition 1 follows by letting t tend to 0 in (24) and recalling that $\|\psi\|_\infty = 1$.

Proof of (iii). Let $\{t_k\}$ be a sequence as in statement (iii), and let us choose a subsequence $\{t^l\} := \{t_{k_l}\}$ in the same way as in the proof of item (i), so that (23) holds. Then for such a subsequence we have (*cf.* (19))

$$\liminf_l \frac{C(t^l)-C(0)}{t^l} \ge \frac{1}{2}\int_0^1 |\nabla u_D(r)|^2 g^\epsilon(r)\, r\, dr = \frac{2}{\pi^2}\int_0^1 \frac{g^\epsilon(r)}{1-r^2}\, r\, dr \ . \tag{26}$$

We need now to vary ϵ and we notice that $g^\epsilon(r)$ increases to $g(r)$ as ϵ decreases to 0. Indeed, by differentiating (22) with respect to ϵ, we find that the derivative $v^\epsilon(r)$ of $g^\epsilon(r)$ with respect to ϵ is nonpositive since it satisfies

$$v^\epsilon - \epsilon (v^\epsilon)'' = (g^\epsilon)'' \le 0 \ , \quad v^\epsilon(0) = v^\epsilon(1) = 0 \ .$$

Therefore, by monotone convergence, by letting ϵ decrease to 0 in (26) we obtain

$$\liminf_l \frac{C(t^l)-C(0)}{t^l} \ge \frac{2}{\pi^2}\int_0^1 \frac{g(r)}{1-r^2}\, r\, dr \ . \tag{27}$$

To conclude, it is enough to show that the integral in the right hand side of (27) diverges, and to that aim the assumption on the initial sequence $\{t_k\}$ comes into play. In fact, we pass to the limit along $t = t^l \to 0$ in (24). By the choice of $\{t^l\}$, we have $\psi(1-\eta;t^l) \to g(1-\eta)$. Moreover, by (25), we have $\phi_r(1-\eta;t^l) \to 0$. Then, since by assumption there exists $\Lambda > 0$ such that $\frac{S(t_k)-S(0)}{2\pi t_k} \ge \Lambda > 0$, we infer that

$$g(1-\eta) \ge \Lambda \qquad \text{for a.e. } \eta \in (0,1) \ .$$

Clearly this implies that the right hand side of inequality (27) equals $+\infty$, and concludes the proof. □

Proof of Proposition 3. It is a particular case of the general situation of Proposition 1. Here, we have $\phi(r;t) = t\varphi(r)$, so that the function ψ defined in (21) is independent of t, as $\psi(r;t) = \varphi(r)$. By arbitrariness of η we infer from (24)-(25) that $S'(0) = 0$. □

Proof of Proposition 4. Since the map $t \mapsto \sqrt{S(t)}$ is concave [2, §24], the map $t \mapsto \frac{S'(t)}{\sqrt{S(t)}}$ is nonincreasing and, in particular, admits a limit as $t \to 0^+$. Since $S(t)$ converges to $S(0)$ as $t \to 0^+$, it follows that there exists $\lim_{t\to 0^+} S'(t)$ $(= S'(0))$. Let us now show that $S'(0) \in (-4\pi, +\infty)$.
With no loss of generality we may assume that Ω contains the origin so that we may find two positive radii $r_1 < r_2$ such that $r_1 B \subseteq \Omega \subseteq r_2 B$, where B is the unit ball. The monotonicity of the surface measure on convex bodies yields

$$S\big[(1-t)D + tr_1 B\big] \le S(t) \le S\big[(1-t)D + tr_2 B\big] \ .$$

Then, if for any $r > 0$ we set $D_t^r := (1-t)D + trB$ and $f^r(t) := S(D_t^r)$, we are done provided the right derivative at $t = 0$ of the function $f^r(t)$ belongs to $(-4\pi, +\infty)$. Let us compute the explicit expression of $f^r(t)$. We notice that

$$D_t^r = (1-t+tr)D_\alpha^1 \ , \qquad \text{with } \alpha = \alpha(r,t) := \frac{tr}{1-t+tr} \ . \tag{28}$$

By the 2-homogeneity of S under dilations, we have

$$f^r(t) = S(D^r_t) = (1-t+tr)^2 S(D^1_\alpha) \ .$$

The surface area $S(D^1_\alpha)$ may be computed in many different ways. For instance, we notice that $S(D^1_\alpha) = 2S(\Gamma_\alpha)$, where Γ_α is the surface obtained by rotating of a 2π angle around the ordinate axis the graph γ_α of the function $g_\alpha : [0,1] \to \mathbb{R}$ defined by

$$g_\alpha(r) := \begin{cases} \alpha & \text{if } r \in [0, 1-\alpha] \\ \sqrt{\alpha^2 - \big(r - 1 + \alpha\big)^2} & \text{if } r \in [1-\alpha, 1] \ . \end{cases}$$

By Pappo-Guldino Theorem, $S(\Gamma_\alpha) = 2\pi x\, l(\gamma_\alpha)$ where x is the abscissa of the barycenter of γ_α, and $l(\gamma_\alpha)$ its length. The value of x can be computed as

$$x = \big(l(\gamma_\alpha)\big)^{-1}\Big[(1-\alpha)x_1 + \alpha\frac{\pi}{2}x_2\Big] \ ,$$

where x_1 and x_2 are the abscissae of the barycenters of the two curves $\gamma^1_\alpha := \{(r, g_\alpha(r)) \ : \ r \in [0, 1-\alpha]\}$ and $\gamma^2_\alpha := \{(r, g_\alpha(r)) \ : \ r \in [1-\alpha, 1]\}$. Since

$$x_1 = \frac{1-\alpha}{2} \qquad \text{and} \qquad x_2 = 1 - \alpha + \frac{\int_0^{\pi/2} \alpha^2 \cos\theta \, d\theta}{\alpha\frac{\pi}{2}} = 1 - \alpha + \frac{2\alpha}{\pi} \ ,$$

we obtain the explicit value of $S(D^1_\alpha)$ as

$$S(D^1_\alpha) = 4\pi x l(\gamma_\alpha) = 4\pi\Big[\frac{(1-\alpha)^2}{2} + \frac{\pi}{2}\alpha(1-\alpha) + \alpha^2\Big] = S(D)\Big[1 + (\pi-2)\alpha + (3-\pi)\alpha^2\Big] \ . \quad (29)$$

Recalling the definition of α in (28), we deduce that the explicit expression of $f^r(t)$ reads

$$f^r(t) = 2\pi\big[1 + (\pi r - 2)t + (2r^2 + 1 - \pi r)t^2\big] \ . \quad (30)$$

In particular, it follows that $(f^r)'(0) = 2\pi(\pi r - 2)$ which belongs to $(-4\pi, +\infty)$ for any $r > 0$.

Let us now prove that $C'(0) = +\infty$. Consider the half ball $B^+ = \{(x_1, x_2, x_3) \in B : x_3 > 0\}$ and, for any $r > 0$, set $U^r_t := (1-t)D + trB^+$. Notice that $U^r_t = (1-t+tr)U^1_\alpha$, with α as in (28). Then, if r is such that $rB \subseteq \Omega$, by monotonicity and homogeneity of capacity we have

$$C(t) \geq \mathrm{Cap}(U^r_t) = (1-t+tr)\mathrm{Cap}(U^1_\alpha) \ .$$

It follows that

$$\liminf_{t\to 0^+} \frac{C(t) - C(0)}{t} \geq \liminf_{t \to 0^+} \frac{(1-t+tr)\mathrm{Cap}(U^1_\alpha) - C(0)}{t}$$

which, by the definition of α in (28), implies

$$\liminf_{t\to 0^+} \frac{C(t) - C(0)}{t} \geq r \liminf_{\alpha\to 0^+} \frac{\mathrm{Cap}(U^1_\alpha) - C(0)}{\alpha} + (r-1)C(0) \ . \quad (31)$$

By (30) and by symmetry, the right derivative at $\alpha = 0$ of the map $\alpha \mapsto S(U^1_\alpha)$ equals $\frac{1}{2}(f^1)'(0) = \pi(\pi - 2) > 0$. Then, by Proposition 1 (iii), we have

$$\lim_{\alpha\to 0^+} \frac{\mathrm{Cap}(U^1_\alpha) - C(0)}{\alpha} = +\infty \ .$$

Combined with (31), this implies $C'(0) = +\infty$. □

Remark 14. A different proof of the equality $C'(0) = +\infty$ in Proposition 4 could be given by exploiting Lemma 4.13 in [10]. We preferred to present the proof above, relying on Proposition 1, to be as possible self-contained. However, we point out that the approach based on Jerison estimate would allow to deal with perturbations of any convex planar set in place of the disk.

Proof of Theorem 5. Clearly, the statement follows if we have

$$\frac{d}{dt}\,\mathcal{E}(D_t)\Big|_{t=0^+} > 0\ . \tag{32}$$

Recalling that $\mathcal{E}(D_t) = \frac{C(t)}{\sqrt{\frac{S(t)}{4\pi}}}$, we see that (32) is satisfied if and only if

$$C'(0) > \frac{C(0)}{2S(0)}S'(0) = \frac{S'(0)}{2\pi^2}\ . \tag{33}$$

We check immediately that (33) holds for the family D_t considered in Proposition 4, since for such family there holds $C'(0) = +\infty$ and $S'(0) \in (-4\pi, +\infty)$.
For the families D_t considered in Propositions 1 and 3, since $C'(0)$ or $S'(0)$ may not exist, we argue by contradiction as follows: assume there exists a sequence $\{t_k\}$ decreasing to 0 such that $\mathcal{E}(D_{t_k}) \leq \mathcal{E}(D)$. This may be rewritten, for some $\lambda > 0$ close to $1/(2\pi^2)$ and independent of k, as:

$$\frac{C(t_k) - C(0)}{t_k} \leq \lambda \frac{S(t_k) - S(0)}{t_k}. \tag{34}$$

By statements (i) and (ii) of Proposition 1, we deduce

$$\frac{2}{\pi^2}(1 - \log 2) \leq \limsup_{k\to+\infty} \frac{C(t_k) - C(0)}{t_k} \leq \limsup_{k\to+\infty} \lambda \frac{S(t_k) - S(0)}{t_k} \leq 2\pi\lambda\ . \tag{35}$$

For the family D_t of Proposition 3, this leads immediately to a contradiction since $S'(0) = 0$. For the family D_t of Proposition 1, again we get a contradiction since by (35) and statement (iii) we know that

$$\limsup_{k\to+\infty} \frac{C(t_k) - C(0)}{t_k} = +\infty\ ,$$

against (35). □

Proof of Theorem 6. We first check that Theorem 5 continues to hold for a "two-sided parametric family" where D_t may be written as

$$D_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3\ :\ 0 \leq r < 1,\ -\phi_1(r;t) < x_3 < \phi_2(r;t)\}\ , \tag{36}$$

for two parametric families ϕ_i's satisfying assumptions (7) and (8). Indeed, let $C(t)$, $C_1(t)$ and $C_2(t)$ denote respectively the capacity of D_t, $D_t \cap \{x_3 \geq 0\}$, and $D_t \cap \{x_3 \leq 0\}$ (and similarly for surface areas). Assume by contradiction that (34) holds. By monotonicity of the capacity, we have for $i = 1, 2$

$$\frac{C(t_k) - C(0)}{t_k} \geq \frac{C_i(t_k) - C(0)}{t_k}\ .$$

On the other hand, we have

$$\frac{S(t) - S(0)}{t} = \frac{S_1(t) - S_1(0)}{t} + \frac{S_2(t) - S_2(0)}{t}\ .$$

Therefore, up to a subsequence, we may assume that, either for $i = 1$ or for $i = 2$,

$$\frac{C_i(t_k) - C(0)}{t_k} \le \frac{\lambda}{2} \frac{S_i(t_k) - S_i(0)}{t_k}.$$

Then, we may continue as done in the proof of Theorem 5 for the family of Proposition 1.
Next, we remark that by the assumptions and thanks to the invariance of $\mathcal{E}$ with respect to translations and dilations, we may assume that

$$D_n = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ 0 \le r < 1\,,\ -\varphi_n^1(r) < x_3 < \varphi_n^2(r)\},$$

where, for $i = 1, 2$, $\varphi_n^i : [0, 1] \to [0, t_n^i]$ are continuous, decreasing, concave, with

$$\varphi_n^i(0) = t_n^i > 0\ , \quad \varphi_n^i(1) = 0\ , \quad (\varphi_n^i)'(0^+) \le 0,$$

and $\max\{t_n^1, t_n^2\}$, which represents the Hausdorff distance from D_n to D, is a decreasing sequence converging to 0. Note that each $n \mapsto t_n^i$ is non-increasing.
We define $\phi_i : [0, 1]^2 \to [0, 1]$ as follows:

$$\phi_i(r;t) := \frac{t - t_{n+1}^i}{t_n^i - t_{n+1}^i} \varphi_n^i(r) + \frac{t_n^i - t}{t_n^i - t_{n+1}^i} \varphi_{n+1}^i(r) \qquad \forall (r, t) \in [0, 1] \times [t_{n+1}^i, t_n^i]\,,\ \forall n \ge 0\ .$$

One then checks that the two functions ϕ_i's satisfy the assumptions of Proposition 1. Finally, we conclude since we saw that Theorem 5 also holds for two-sided parametric families as in (36). □

5 Proofs of global results

Proof of Theorem 7. As in the proof of Proposition 4, we set for brevity $D_t^r := (1 - t)D + trB$, and we notice that

$$\mathrm{Cap}(D_t^r) = (1 - t + tr)\mathrm{Cap}(D_\alpha^1) \qquad \text{and} \qquad S(D_t^r) = (1 - t + tr)^2 S(D_\alpha^1)\ ,$$

so that $\mathcal{E}(D_t^r) = \mathcal{E}(D_\alpha^1)$, and it is enough to show that $\mathcal{E}(D_\alpha^1) > \mathcal{E}(D)$ for all $\alpha \in (0, 1]$. In order to estimate from below the quotient $\mathcal{E}(D_\alpha^1)$, one may compute exactly $S(D_\alpha^1)$ and then give a lower bound for $\mathrm{Cap}(D_\alpha^1)$.
For the explicit expression of $S(D_\alpha^1)$, see (29). Next, we enforce Borell's Brunn-Minkowski inequality for capacity [3] in order to obtain, for all $\alpha \in (0, 1]$, the following lower bound for $\mathrm{Cap}(D_\alpha^1)$:

$$\mathrm{Cap}(D_\alpha^1) \ge \mathrm{Cap}(D) + \big[\mathrm{Cap}(B) - \mathrm{Cap}(D)\big]\alpha = \mathrm{Cap}(D)\Big[1 + \Big(\frac{\mathrm{Cap}(B)}{\mathrm{Cap}(D)} - 1\Big)\alpha\Big]\ .$$

Since $\mathrm{Cap}(B) = 1$ and $\mathrm{Cap}(D) = 2/\pi$, we deduce that

$$\mathrm{Cap}(D_\alpha^1) \ge \mathrm{Cap}(D)\Big[1 + \big(\frac{\pi}{2} - 1\big)\alpha\Big]\ . \tag{37}$$

By combining (29) and (37) we obtain

$$\mathcal{E}(D_\alpha^1) \ge \mathcal{E}(D)\ \frac{1 + \big(\frac{\pi}{2} - 1\big)\alpha}{\sqrt{1 + (\pi - 2)\alpha + (3 - \pi)\alpha^2}} = \mathcal{E}(D)\ \sqrt{\frac{1 + (\pi - 2)\alpha + (\pi/2 - 1)^2\alpha^2}{1 + (\pi - 2)\alpha + (3 - \pi)\alpha^2}} > \mathcal{E}(D)$$

for all $\alpha \in (0, 1]$. □

Proof of Theorem 8. It is immediate to check that, if Ω is a ball, its potential satisfies (15). Viceversa, assume that a generic ellipsoid $E_{a,b,c}$ in $\mathcal{T} \cap \mathcal{K}_o$ satisfies (15). We have to show that necessarily such an ellipsoid is a ball. Since $\mathcal{E}$ is invariant under dilations, we may take with no loss of generality $a = 1$. For $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi)$, set

$$x_1 = \cos\theta \sin\varphi \ , \qquad x_2 = b \sin\theta \sin\varphi \ , \qquad x_3 = c\cos\varphi \ ,$$

and let $H_{1,b,c}(\theta, \varphi)$ denote the mean curvature of $\partial E_{1,b,c}$ in these coordinates. Its explicit expression reads:

$$\begin{aligned} H_{1,b,c}(\theta,\varphi) \; &= \frac{bc\Big[3(1+b^2)+2c^2+(1+b^2-2c^2)\cos(2\varphi)-2(1-b^2)\cos(2\theta)\sin^2\varphi\Big]}{8\Big[b^2\cos^2\varphi+c^2\big(b^2\cos^2\theta+\sin^2\theta\big)\sin^2\varphi\Big]^{3/2}} \\ &=: \frac{bc}{8}\Lambda_{b,c}(\theta,\varphi) \ . \end{aligned}$$

By [10] we know that the electrostatic potential $u_{1,b,c}$ of $E_{1,b,c}$ is given by

$$u_{1,b,c}(x) = \frac{F(\lambda(x))}{F(0)} \ ,$$

where

$$F(\lambda) = \int_\lambda^{+\infty} \frac{ds}{\sqrt{(s+1)(s+b^2)(s+c^2)}} \ ,$$

and the function $\lambda(x)$ is implicitly defined on the complement of $E_{1,b,c}$ by the equation

$$\frac{x_1^2}{1+\lambda(x)} + \frac{x_2^2}{b^2+\lambda(x)} + \frac{x_3^2}{c^2+\lambda(x)} = 1 \tag{38}$$

(note that $\lambda(x) = 0$ for $x \in \partial E_{1,b,c}$). Hence, for $x \in \partial E_{1,b,c}$, we have

$$\nabla u_{1,b,c}(x) = \frac{F'(0)}{F(0)} \nabla\lambda(x) \ .$$

By differentiating (38) we get

$$\nabla\lambda(x) = 2\left(x_1^2 + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{-1}\left(x_1 \ , \ \frac{x_2}{b^2} \ , \ \frac{x_3}{c^2}\right) \qquad \forall x \in \partial E_{1,b,c} \ ,$$

so that, for $x \in \partial E_{1,b,c}$, there holds

$$\begin{aligned} |\nabla u_{1,b,c}(x)|^{-2} \; &= \frac{1}{4}\Big(\frac{F(0)}{F'(0)}\Big)^2\Big(x_1^2 + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\Big) \\ &= \frac{1}{4}\Big(\frac{F(0)}{F'(0)}\Big)^2\Big(\cos^2\theta\sin^2\varphi + \frac{\sin^2\theta\sin^2\varphi}{b^2} + \frac{\cos^2\varphi}{c^2}\Big) \\ &=: \frac{1}{4}\Big(\frac{F(0)}{F'(0)}\Big)^2\Gamma_{b,c}(\theta,\varphi) \ . \end{aligned}$$

Now, if $E_{1,b,c}$ satisfies (15), there exists a positive constant K, depending on b and c but not on θ and φ, such that

$$\Lambda_{b,c}(\theta,\varphi) \cdot \Gamma_{b,c}(\theta,\varphi) = K \ .$$

Taking $\varphi = 0$, we obtain

$$K = K_1 := \frac{4(1+b^2)}{c^2 b^3} \ .$$

Taking $\varphi = \pi/2$, we obtain

$$K = \frac{2}{b^2 c^3} \cdot \frac{1 + b^2 + 2c^2 - (1-b^2)\cos(2\theta)}{(b^2 \cos^2\theta + \sin^2\theta)^{1/2}} \qquad \forall \theta \in [0, 2\pi) \ ;$$

in turn, choosing $\theta = 0$ and $\theta = \pi/2$, we deduce respectively

$$K = K_2 := \frac{4(b^2 + c^2)}{b^3 c^3} \qquad \text{and} \qquad K = K_3 := \frac{4(c^2+1)}{b^2 c^3} \ .$$

It is immediate to check that $K_2 = K_3$ if and only of $b = 1$ or $b = c^2$, whereas $K_1 = K_3$ if and only if $b = c$ or $b = c^{-1}$. Hence four cases may occur:

$$b = 1 \text{ and } b = c, \qquad b = 1 \text{ and } b = c^{-1}, \qquad b = c^2 \text{ and } b = c, \qquad b = c^2 \text{ and } b = c^{-1}$$

In any case we infer that $b = c = 1$, so that $E_{1,b,c}$ is the ball of radius 1. □

6 Appendix: first variation of capacity

In this section we derive a general formula to compute the derivative of capacity under domain perturbations. Within this paper, it is useful to prove Proposition 1. However, we believe it may be of interest in its own right.
Let Ω_t be a one-parameter family of convex sets obtained by perturbing a given domain $\Omega \in \mathcal{K}_o$ (*cf.* definition (11)). A classical formula dating back to Hadamard allows to compute the first variation of capacity, namely the derivative at $t = 0$ of the map $t \mapsto \mathrm{Cap}(\Omega_t)$: it holds when both the initial domain and the initial velocity of the deformation are smooth, see *e.g.* [7, 19]. The extension of such formula to the case of nonsmooth domains and deformations is quite delicate. In the nineties, Jerison [10, 11] has shown that the formula continues to hold for Minkowski sums, namely when $\Omega_t = \Omega + t\omega$ for some $\omega \in \mathcal{K}_o$. Moreover, Elcrat-Miller [6] obtained the analogous formula for Lipschitz domains when a suitable notion of relative capacity is considered. The following theorem is strictly related to those papers. Its proof is based on previous results by Henrot and Pierre [9] for p.d.e.'s stated on bounded domains.
Below, we denote by n_Ω the unit outer normal to $\partial\Omega$, and by u_Ω the potential of Ω according to (2). We recall from [5] that, since $\Omega \in \mathcal{K}_o$, the gradient $\nabla u_\Omega(y)$ has (non-tangential) limits as $y \to x \in \partial\Omega$ for $\mathcal{H}^2$-a.e. $x \in \partial\Omega$; moreover, denoting such limits by $\nabla u_\Omega(x)$, we have $|\nabla u_\Omega| \in L^2(\partial\Omega, d\mathcal{H}^2)$.

Theorem 15. *Let $\Omega \in \mathcal{K}_o$ and $\Omega_t = \Phi_t(\Omega)$, where $t \in [0,T) \mapsto \Phi_t \in W^{1,\infty}(\mathbb{R}^3)$ is differentiable at $t = 0$, with $\Phi_0(x) = x$ and $\frac{d}{dt}_{|t=0}\Phi_t(x) = V(x)$. Then*

$$\frac{d}{dt}\mathrm{Cap}(\Omega_t)_{|t=0} = \frac{1}{4\pi}\int_{\partial\Omega} V \cdot n_\Omega |\nabla u_\Omega|^2 d\mathcal{H}^2 \ .$$

Proof. We argue as in the proof of Théorème 5.3.2 in [9], the only difference being that here we work with p.d.e.'s on an unbounded domain. Let $\mathcal{W} := W^{1,\infty}(\mathbb{R}^3)$ be equipped with its natural norm. For θ small in $\mathcal{W}$, we denote

$$\Omega_\theta = (I + \theta)(\Omega), \quad u_\theta = u_{\Omega_\theta}, \quad v_\theta = u_\theta \circ (I + \theta).$$

We denote by $\mathcal{D}$ (resp. $\mathcal{D}_0$) the closure of the space of smooth functions compactly supported in $\mathbb{R}^3 \setminus \Omega$ (resp. in $\mathbb{R}^3 \setminus \overline{\Omega}$) with respect to the Dirichlet norm $u \to \int_{\mathbb{R}^3\setminus\Omega} |\nabla u|^2$.

Let us prove that $\mathcal{W} \ni \theta \mapsto v_\theta \in \mathcal{D}$ is differentiable near $\theta = 0$. We fix $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ on an open neighborhood of } \overline{\Omega}.$$

As in [9], since $\Delta(\psi - u_\theta) = \Delta\psi$ on Ω_θ, we check that $w_\theta = \psi \circ (I + \theta) - v_\theta$ is solution of

$$w_\theta \in \mathcal{D}_0, \quad -\nabla \cdot (A(\theta)\nabla w_\theta) = [f \circ (I + \theta)]J_\theta, \tag{39}$$

where

$$f = -\Delta\psi, \quad A(\theta) = J_\theta(I + D\theta)^{-1}(I +^t D\theta)^{-1}, \quad J_\theta = det\,(I + D\theta)\,.$$

Now, we consider the mapping

$$F : (\theta, w) \in \mathcal{W} \times \mathcal{D}_0 \to -\nabla \cdot (A(\theta)\nabla w) - [f \circ (I + \theta]J_\theta \in \mathcal{D}_0'.$$

We check as in [9] that F is $\mathcal{C}^1$ (it is even $\mathcal{C}^\infty$). Moreover, $D_w F(0, w_0) = -\Delta$ is an isomorphism from $\mathcal{D}_0$ into its dual space $\mathcal{D}_0'$ (Lax-Milgram's Theorem). By the implicit function Theorem, and by uniqueness for problem (39), it follows that $\theta \in \mathcal{W} \to w_\theta \in \mathcal{D}_0$ is $\mathcal{C}^1$ around $\theta = 0$ (and even $\mathcal{C}^\infty$). The same holds for $\theta \in \mathcal{W} \to v_\theta = \psi \circ (I + \theta) - w_\theta \in \mathcal{D}$ (recall that $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ and see e.g. Lemme 5.3.9 in [9]).
Now, if $\Omega_t = \Phi_t(\Omega)$ is as in the statement, it follows by composition that $t \mapsto v_t := v_{\Phi_t(\Omega)}$ is differentiable at $t = 0$ with values in $\mathcal{D}_0$. This implies that $t \mapsto u_t := u_{\Omega_t} = v_{\Phi_t(\Omega)} \circ \Phi(t)^{-1}$ is differentiable at $t = 0$, at least with values in $L^2_{loc}(\mathbb{R}^3)$. Moreover, if we denote u', v' the derivatives at $t = 0$ of u_t, v_t, we have (by the chain rule) $u' = v' - \nabla u_\Omega \cdot V$. Thus, u' is solution of

$$\Delta u' = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \qquad u' + \nabla u_\Omega \cdot V \in \mathcal{D}_0. \tag{40}$$

To prove the existence and to compute the derivative of $t \mapsto c(t) := 4\pi\mathrm{Cap}(\Omega_t) = \int_{\mathbb{R}^3\setminus\Omega_t} |\nabla u_t|^2$ at $t = 0$, we may use that, when t is small enough so that $\partial\Omega_t \subset \mathrm{Support}\,(\psi)$:

$$\int_{\mathbb{R}^3\setminus\Omega_t} |\nabla u_t|^2 = \int_{\partial\Omega_t} -\frac{\partial u_t}{\partial n_t} = \int_{\partial\Omega_t} -\psi\frac{\partial u_t}{\partial n_t} = -\int_{\mathbb{R}^3\setminus\Omega_t} u_t\Delta\psi = -\int_{\mathbb{R}^3} u_t\Delta\psi. \tag{41}$$

We obtain the differentiability of $c(t)$ at $t = 0$ and

$$c'(0) = -\int_{\mathbb{R}^3} u'\Delta\psi = -\int_{\mathbb{R}^3\setminus\Omega} u'\Delta\psi = \int_{\mathbb{R}^3\setminus\Omega} u'\Delta(u_\Omega - \psi).$$

So far, we did not use the regularity of the boundary of Ω (even the relations (41) could be justified by approximation of Ω_t by regular domains). Now we will use the fact that $\nabla u_\Omega(y)$ has (non-tangential) limits (denoted by $\nabla u(x)$) as $y \to x \in \partial\Omega$ for $\mathcal{H}^2$-a.e. $x \in \partial\Omega$, with $|\nabla u_\Omega| \in L^2(\partial\Omega, d\mathcal{H}^2)$. Recall also that $\mathcal{H}^2$-a.e. on $\partial\Omega$, $\nabla u_\Omega = (\nabla u_\Omega \cdot n_\Omega)n_\Omega$. Using this with (40), we obtain that u' has a trace in $L^2(\partial\Omega, d\mathcal{H}^2)$ and

$$u'\frac{\partial u_\Omega}{\partial n_\Omega} = -(V \cdot n_\Omega)|\nabla u_\Omega|^2 \;\; \mathcal{H}^2 - a.e. \; on \; \partial\Omega.$$

Together with the behaviour at ∞, integration by parts yields

$$\int_{\mathbb{R}^3\setminus\Omega} u'\Delta(u_\Omega - \psi) = -\int_{\partial\Omega} u'\frac{\partial(u_\Omega - \psi)}{\partial n_\Omega} = \int_{\partial\Omega} (V \cdot n_\Omega)|\nabla u_\Omega|^2 \, d\mathcal{H}^2.$$

□

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