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PERIODIC MOTIONS OF AN INFINITE LATTICE OF PARTICLES WITH NEAREST NEIGHBOR INTERACTION

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1. INTRODUCTION

We consider one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction; the state of our system at time t is represented by a sequence $q(t) = \{q_i(t)\}$, $i \in \mathbb{Z}$, where $q_i(t)$ is the state of the i-th particle.

Let Φ_i denote the potential of the interaction between the i-th and the $(i+1)$-th particle, then the equation governing the state of $q_i(t)$ reads

$$\ddot{q}_i = \Phi'_{i-1}(q_{i-1} - q_i) - \Phi'_i(q_i - q_{i+1}). \tag{1}$$

The assumptions required on $\Phi_i : \mathbb{R} \to \mathbb{R}$ will be given in Section 2. If we define the potential of the system $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ by $\Phi(q) := \sum_{i \in \mathbb{Z}} \Phi_i(q_i - q_{i+1})$, then equation (1) becomes

$$\ddot{q} = -\Phi'(q). \tag{2}$$

Our main purpose is to prove, under suitable assumptions on the potential Φ, the existence of a T-periodic nonconstant solution of equation (2) for T large enough. This solution will be obtained as a critical point of a suitable functional.

We use the Lions' concentration-compactness lemma [1] in order to prove that the solution is nonzero, and in fact we give a detailed picture of the behavior of the Palais–Smale sequences.

We point out that the existence of a nontrivial periodic solution of finite energy is quite surprising, because one could expect that an infinite lattice of particles interacting with a nonlinear force tends to spread its energy.

A pioneering work on lattices is the famous Fermi, Pasta and Ulam numerical experiment [2]: they tried to numerically test a conjecture, and in fact they gave start to the fruitful branch of perturbation techniques. The conjecture was that, even if in a chain of particles with nearest neighbor interactions of linear type the energy of each normal mode is constant, it is enough to introduce a small perturbation in order to destroy such stability and spread the energy among all the modes. On the contrary, they obtained the opposite result, i.e. they saw that if the perturbation is small enough, the energy does not spread.

Years later, Toda [3] proved that a chain of particles with exponential interaction potential is integrable, and therefore it does not spread the energy at all.

In a recent paper, Ruf and Srikanth [4] proved with variational techniques that a finite chain of particles with a Toda type potential admits periodic solutions. This is the first step to show that the system is not ergodic; it would be interesting to prove that the solution they found is stable in some sense, in order to show that there is a region in the phase space where the motion of the system does not lead to a spread of energy.

We are not trying to achieve this task here, instead our work extends in some sense Ruf–Srikanth result to an infinite dimensional system.

2. VARIATIONAL SETTING

We work in the following Hilbert space

$$H := \left\{ q : S^1 \to \mathbb{R}^\infty;\ \int_0^T q_0(t) = 0, \|q\|^2 := \sum_{i \in \mathbb{Z}} \int_0^T \left[(\dot{q}_i(t))^2 + (q_i(t) - q_{i+1}(t))^2 \right] \mathrm{d}t < \infty \right\}$$

and we consider the functional $J : H \to \mathbb{R}$ defined by

$$J(q) := \frac{1}{2} \int_0^T |\dot{q}(t)|^2 \,\mathrm{d}t - \int_0^T \Phi(q(t)) \,\mathrm{d}t.$$

Assume that $\forall i \in \mathbb{Z}$

(i) $\Phi_i(t) = -\alpha_i t^2 + V_i(t)$, $\alpha_i > 0$

(ii) $V_i'(t)t \geq (2 + \delta)V_i(t) \geq 0$, $\forall t \in \mathbb{R}$, for a suitable $\delta > 0$

(iii) $\lim_{|t| \to +\infty} V_i(t) = +\infty$

(iv) $V_i \in C^{1,1}_{\mathrm{loc}}$

(v) $\exists m \in \mathbb{N}$ such that $\Phi_{i+m} \equiv \Phi_i$.

Condition (v) is a spatial periodicity which is required in order to apply a modified version of Lions' concentration-compactness lemma.

Note that conditions (i), (ii), (iii) and (iv) imply that $\forall i \in \mathbb{Z}$:

— $\Phi_i(0) = 0$;

— V_i is superquadratic at the origin and at infinity;

— Φ_i has a strong local maximum in 0;

— Φ_i admits at least two local minima;

— denote by ϑ_{i_j} the nonzero stationary points of Φ_i; then $\max_j \Phi_i(\vartheta_{i_j}) < 0$.

These remarks will be used to obtain a mountain pass critical point, to exclude the vanishing case in the application of Lions' lemma and to exclude the trivial constant solution. The main result we prove is the following theorem.

THEOREM 1. Assume that (i), (ii), (iii), (iv) and (v) hold. Then $\exists \bar{T} > 0$ such that if $T > \bar{T}$, system (2) admits a nonconstant T-periodic solution of finite energy.

These conditions on the potentials Φ_i are physically meaningful because one can think to an infinite chain of parallel disks which are allowed to rotate around their axes, the variables $q_i(t)$ being the values of the angle of rotation (see [3]): the nearest neighbor disks are connected with a linear repulsive spring and a superlinear attractive spring so that they achieve at least an unstable equilibrium position, when the angles are equal, and two stable equilibrium positions, one in each direction of rotation.

As the solution that we obtain has a finite norm, all the disks, except a finite number, move in a neighborhood of their unstable equilibrium position, i.e. they never fall in the stable position where the potential attains its minimum, therefore almost all the energy remains contained in a finite region of the space.

Our first goal is to show that J is well defined on H and that $J \in C^1(H, \mathbb{R})$; notice that this result is not trivial as we deal with a series of integrals.

PROPOSITION 1. Assume (i), (ii), (iii) and (iv). Then $J \in C^1(H, \mathbb{R})$.

Proof. We first prove that the functional J is well defined for all $q \in H$. $H^1(S^1, \mathbb{R})$ is compactly imbedded in $L^\infty(S^1, \mathbb{R})$, hence $\forall q \in H$ and $\forall i \in \mathbb{Z}$

$$\|q_i - q_{i+1}\|_\infty \le c\big(\|\dot q_i - \dot q_{i+1}\|_2 + \|q_i - q_{i+1}\|_2\big) \le c\big(\|\dot q_i\|_2 + \|\dot q_{i+1}\|_2 + \|q_i - q_{i+1}\|_2\big),$$

which, since $(\alpha + \beta + \gamma)^2 \le 3(\alpha^2 + \beta^2 + \gamma^2)$ for all α, β, $\gamma \in \mathbb{R}$, yields

$$\sum_{i \in \mathbb{Z}} \|q_i - q_{i+1}\|_\infty^2 \le c\|q\|^2; \tag{3}$$

therefore $\forall \xi > 0$, for a finite number of indices only, we have $\|q_i - q_{i+1}\|_\infty > \xi$. On the other hand, because of property (i) and (ii), $\exists \xi_0 > 0$ such that $|\Phi_i(x)| \le cx^2 \, \forall x$, $|x| < \xi_0$. Let $K \subset \mathbb{Z}$ be the finite set of indices such that $\|q_i - q_{i+1}\|_\infty > \xi_0$; we have

$$|J(q)| \le c\|q\|^2 + \sum_{i \in K} \int_0^T |\Phi_i(q_i - q_{i+1})| < +\infty. \tag{4}$$

Next we prove that the Fréchet derivative of J exists and is continuous in H. We first check that

$$J'(q)[p] = \int_0^T (\dot q(t), \dot p(t))\,\mathrm{d}t - \int_0^T (\Phi'(q(t)), p(t))\,\mathrm{d}t; \tag{5}$$

the derivative of the quadratic part of the functional can be easily computed, therefore it is enough to prove that if $\|p\| \to 0$, then

$$\left| \sum_{i \in \mathbb{Z}} \int_0^T [V_i(q_i - q_{i+1} + p_i - p_{i+1}) - V_i(q_i - q_{i+1}) - V_i'(q_i - q_{i+1})(p_i - p_{i+1})] \right| = o(\|p\|).$$

Indeed, using the Lipschitz continuity of V_i' and (3) we get

$$\begin{aligned}
&\left| \sum_{i \in \mathbb{Z}} \int_0^T [V_i(q_i - q_{i+1} + p_i - p_{i+1}) - V_i(q_i - q_{i+1}) - V_i'(q_i - q_{i+1})(p_i - p_{i+1})] \right| \\
&\quad \le \sum_{i \in \mathbb{Z}} \int_0^T |V_i(q_i - q_{i+1} + p_i - p_{i+1}) - V_i(q_i - q_{i+1}) - V_i'(q_i - q_{i+1})(p_i - p_{i+1})| \\
&\quad \le c \sum_i \|p_i - p_{i+1}\|_\infty^2 \le c\|p\|^2 \text{ when } \|p\| \to 0;
\end{aligned}$$

this proves that (5) is a correct definition.

To prove that J' is continuous we prove that for a sequence $q^{(n)} \to q$ we have

$$\sup_{\|p\|=1} \left| \int_0^T [\Phi'(q^{(n)}) - \Phi'(q)] \cdot p \right| \to 0.$$

Let $\varepsilon > 0$ and consider the finite set $I_\varepsilon \subset \mathbb{Z}$ such that for n large enough

$$\sum_{i \notin I_\varepsilon} \|(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1})\|_\infty^2 < \varepsilon^2;$$

let n be so large that

$$\sup_{\|p\|=1} \left| \sum_{i \in I_\varepsilon} \int_0^T \left[\Phi_i'(q_i^{(n)} - q_{i+1}^{(n)}) - \Phi_i'(q_i - q_{i+1}) \right] (p_i - p_{i+1}) \right| < \varepsilon.$$

Now, if $i \notin I_\varepsilon$, we use the Lipschitz continuity of Φ_i' to get

$$\sup_{\|p\|=1} \left| \sum_{i \notin I_\varepsilon} \int_0^T \left[\Phi_i'(q_i^{(n)} - q_{i+1}^{(n)}) - \Phi_i'(q_i - q_{i+1}) \right] (p_i - p_{i+1}) \right|$$

$$\leq c \sup_{\|p\|=1} \sum_{i \notin I_\varepsilon} \int_0^T \left| \left[(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1}) \right] (p_i - p_{i+1}) \right|$$

$$\leq c \left(\sum_{i \notin I_\varepsilon} \|(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1})\|_2^2 \right)^{1/2} < c\varepsilon. \quad \blacksquare$$

3. EXISTENCE OF A PALAIS–SMALE SEQUENCE

We prove that the functional J admits a Palais–Smale sequence, i.e. a sequence $q^{(n)} \in H$ such that $J(q^{(n)})$ is bounded and $J'(q^{(n)}) \to 0$.

THEOREM 2. Assume (i), (ii), (iii) and (iv). Then $\forall T > 0$, the functional J admits a Palais–Smale sequence $q^{(n)}$ which is bounded from below and above by two positive constants.

Proof. (1) $q(t) \equiv 0$ is a strict minimum of $J(q)$; indeed $J(0) = 0$ and by inequality (3) it follows that $\forall \varepsilon > 0$ there exists a neighborhood U of 0 in H such that $\|q_i - q_{i+1}\|_\infty \leq \varepsilon$, therefore, bearing in mind that $\Phi_i(t) < 0$ for $|t|$ small ($t \neq 0$), $q \in U \setminus \{0\} \Rightarrow J(q) > 0$.
(2) $\exists q(t) \in H$ such that $J(q) < 0$; to see this define $q^\sigma \in H$ by $q_i^\sigma \equiv 0$ if $i \neq 0$ and

$$q_0^\sigma(t) = \begin{cases} \sigma \sin\left(\dfrac{2\pi}{\eta T} t\right) & \text{if } 0 \leq t \leq \eta T \\ 0 & \text{if } \eta T \leq t \leq T \end{cases}$$

where $\eta \in (0, 1]$. A specific η will be required to prove proposition 2.

Because of assumption (i) and (ii), for σ large enough, say $\bar{\sigma}$, $J(q^{\bar{\sigma}}) < 0$. We now have all the hypotheses of the Mountain Pass Theorem [5] except the Palais–Smale condition. Let

$$b := \inf_{\gamma \in P} \max_{q \in \gamma} J(q) \tag{6}$$

where P is the class of continuous paths with end points $q \equiv 0$ and $q^{\bar{\sigma}}$; $\forall \varepsilon > 0 \exists \gamma_\varepsilon \in P$ such that

$$\max_{q \in \gamma_\varepsilon} J(q) \leq b + \varepsilon$$

and we obtain a Palais–Smale sequence $q^{(n)}$; indeed, if this is not the case $\exists \bar{\varepsilon} > 0 | J'(q^{(n)})| > \bar{\varepsilon}$ and therefore it is possible to deform continuously the path $\gamma_{\bar{\varepsilon}}$ into $\bar{\gamma}$ to obtain

$$\max_{q \in \bar{\gamma}} J(q) \leq b - \bar{\varepsilon}$$

which contradicts the definition of b.
(3) The sequence $q^{(n)}$ is bounded from above, indeed if $\varepsilon_n = \|J'(q^{(n)})\|$ and n is large enough so that $J(q^{(n)}) - b < \varepsilon/2$, then

$$\begin{aligned}
2b + \varepsilon + \varepsilon_n \|q^{(n)}\| &\geq 2J(q^{(n)}) - J'(q^{(n)})[q^{(n)}] \\
&= \sum_i \int_0^T \left[V_i'(q_i^{(n)} - q_{i+1}^{(n)})(q_i^{(n)} - q_{i+1}^{(n)}) - 2V_i(q_i^{(n)} - q_{i+1}^{(n)}) \right] \\
&\geq \frac{\delta}{2+\delta} \sum_i \int_0^T V_i'(q_i^{(n)} - q_{i+1}^{(n)})(q_i^{(n)} - q_{i+1}^{(n)}) \\
&\geq \frac{\delta}{2+\delta} \left[\int_0^T |\dot{q}^{(n)}|^2 + 2 \sum_i \alpha_i \int_0^T (q_i^{(n)} - q_{i+1}^{(n)})^2 - \varepsilon_n \|q^{(n)}\| \right]
\end{aligned}$$

we also have $\|q^{(n)}\|^2 \leq K[\int_0^T |\dot{q}^{(n)}|^2 + 2\sum_i \alpha_i \int_0^T (q_i^{(n)} - q_{i+1}^{(n)})^2]$; thus

$$2b + \varepsilon + 2\frac{\delta+1}{\delta+2} \varepsilon_n \|q^{(n)}\| \geq \frac{\delta}{(2+\delta)K} \|q^{(n)}\|^2,$$

hence the result.
(4) $\|q^{(n)}\| \geq c > 0$ uniformly in n because $J(q^{(n)}) \geq b > 0$ uniformly in n. ■

We recall that a functional is said to satisfy the Palais–Smale condition if every Palais–Smale sequence admits a convergent subsequence. In our case the Palais–Smale condition is not fulfilled, indeed consider a precompact Palais–Smale sequence $q^{(n)} \in H$ then the sequence $p^{(n)}$ defined by

$$P_i^{(n)}(t) = q_{i+n}^{(n)}(t) - \frac{1}{T} \int_0^T q_n^{(n)}$$

is also a Palais–Smale sequence, but no subsequences of $p^{(n)}$ converge in H. The reason of the failure of the Palais–Smale condition is some lack of compactness, i.e. we do not have the Sobolev compact imbedding $H^1 \subset L^p$ for our infinite lattice.

As already mentioned, in order to manage the lack of compactness of our problem, we will study the behavior of Palais–Smale sequences by means of Lions' concentration-compactness lemma in a version adapted to our needs.

LEMMA 1. (concentration-compactness) Let $\{u^{(n)}\}_{n \in \mathbb{N}}$ be a sequence of sequences, $u^{(n)} = \{u_i^{(n)}\}_{i \in \mathbb{Z}}, u_i \geq 0$, $\forall i \in \mathbb{Z}$ and $\forall n \in \mathbb{N}$, such that $\sum_i u_i^{(n)} = \lambda$, $\lambda > 0$, $\forall n \in \mathbb{N}$. Then there exists a subsequence (still denoted by $\{u^{(n)}\}$) such that one of the following properties occurs:

(a) *Concentration.* There exists a sequence $\{M_n\}_{n\in\mathbb{N}}$ of integers such that $\forall\varepsilon>0$ there exists $N_\varepsilon\in\mathbb{N}$ such that $\forall n\in\mathbb{N}$ we have

$$\sum_{i=M_n-N_\varepsilon}^{M_n+N_\varepsilon} u_i^{(n)} \ge \lambda-\varepsilon.$$

(b) *Vanishing*

$$\lim_{n\to\infty}\left[\sup_{i\in\mathbb{Z}} u_i^{(n)}\right]=0.$$

(c) *Dichotomy.* There exists $\alpha\in(0,\lambda)$ and a sequence $\{M_n\}_{n\in\mathbb{N}}$ of integers such that $\forall\varepsilon>0$ there exist two integers N_ε and N_ε' such that $\forall n\in\mathbb{N}$ we have

$$\left|\sum_{|i-M_n|\le N_\varepsilon} u_i^{(n)}-\alpha\right|<\varepsilon, \qquad \left|\sum_{|i-M_n|>N_\varepsilon'} u_i^{(n)}-(\lambda-\alpha)\right|<\varepsilon$$

and $N_\varepsilon'-N_\varepsilon\to+\infty$ when $\varepsilon\to 0$.

Proof. It suffices to apply lemma I.1 in [1] to the functions $\rho_n:\mathbb{R}\to\mathbb{R}_+$ defined by

$$\rho_n(x):=\sum_i u_i^{(n)}\chi_{[i,i+1)}(x),$$

where χ_A denotes the characteristic function of A. ■

Remark. The integers M_n can be chosen to be multiples of m; we need such a choice because the functional J is invariant under translation of indices by such integers.

4. EXISTENCE OF A NONTRIVIAL PERIODIC SOLUTION

In this section we use lemma 1 in order to build a nontrivial solution of system (2). We apply the lemma to the sequence $\{u_i^{(n)}\}_{n\in\mathbb{N}}$, where

$$u_i^{(n)}=\int_0^T(\dot q_i^{(n)}(t))^2\,dt+\int_0^T(q_i^{(n)}(t)-q_{i+1}^{(n)}(t))^2\,dt; \tag{7}$$

we first study the vanishing case.

LEMMA 2. If $q^{(n)}$ is a bounded vanishing Palais–Smale sequence in H, then $q^{(n)}\to 0$.

Proof. Let $q^{(n)}$ be a vanishing Palais–Smale sequence; then for n large enough we have

$$\|q_i^{(n)}-q_{i+1}^{(n)}\|_\infty<\xi_0 \qquad \forall i\in\mathbb{Z}.$$

By the properties of the potential Φ_i, $\exists\rho>0$ such that $\forall t$, $|t|<\xi_0$, $\Phi_i'(t)t\le-\rho t^2$; thus

$$J'(q^{(n)})[q^{(n)}]=\sum_i\int_0^T\left[(\dot q_i^{(n)})^2-\Phi_i'(q_i^{(n)}-q_{i+1}^{(n)})(q_i^{(n)}-q_{i+1}^{(n)})\right]$$

$$\ge\sum_i\int_0^T\left[(\dot q_i^{(n)})^2+\rho(q_i^{(n)}-q_{i+1}^{(n)})^2\right]\ge c\|q^{(n)}\|^2$$

therefore $\|q^{(n)}\|\to 0$. ■

Let $\tilde{q}^{(n)} \in H$ be a Palais–Smale sequence. Suppose that either concentration or dichotomy holds; define $q_i^{(n)} = \tilde{q}_{i+M_n}^{(n)} + \beta \forall i, n$ where M_n is the sequence of integers arising from lemma 1 and β is a constant added in order to have $q^{(n)} \in H$.

We remark that the functional and the norm are invariant under such translations by assumption (v). By this procedure we obtain a sequence $q^{(n)} \in H$ such that, for n large enough, the norm (in case of concentration) or a fixed part of it (in case of dichotomy) is "concentrated" in the terms with small index i. In both cases the sequence is bounded, therefore up to a subsequence it converges weakly to a function $q \in H$. From now on when we deal with Palais–Smale sequences, we assume the sequences to be already translated by the previous procedure.

Next we deal with the concentration case.

LEMMA 3. If concentration holds for a bounded Palais–Smale sequence $\{q^{(n)}\}$, then up to translations and subsequences, $q^{(n)} \to q$ strongly in H; hence q is a nonzero solution of problem (2).

Proof. Let $\lambda_n = \Sigma_i u_i^{(n)}$. As $\lambda_n = \|q^{(n)}\|^2$, $0 < c_1 \le \lambda_n \le c_2$ (see theorem 2), hence (up to a subsequence) it converges to $\lambda > 0$.

Up to index translations by multiples of m we have

$$\forall \varepsilon > 0 \qquad \exists N_\varepsilon, \bar{n} \text{ such that for } n > \bar{n} \sum_{|i| \le N_\varepsilon} u_i^{(n)} \ge \lambda - \varepsilon. \tag{8}$$

We apply the same translations to the sequence $q^{(n)}$, keeping the same notation and we first prove that

$$\sum_i \|(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1})\|_2^2 \to 0. \tag{9}$$

Indeed let $\varepsilon > 0$; because of the compact imbedding $H^1(S^1, \mathbb{R}) \subset L^2(S^1, \mathbb{R})$ we have, up to a subsequence $\forall i \in \mathbb{Z} (q_i^{(n)} - q_{i+1}^{(n)}) \to (q_i - q_{i+1})$ in L^2 and therefore for n large enough

$$\sum_{|i| \le N_\varepsilon} \|(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1})\|_2^2 < \varepsilon.$$

Furthermore, because of (8) we have

$$\sum_{|i| > N_\varepsilon} \|(q_i^{(n)} - q_{i+1}^{(n)}) - (q_i - q_{i+1})\|_2^2 < \varepsilon$$

which together give (9).

Analogously one could prove that

$$\sum_i \int_0^T \Phi_i'(q_i^{(n)} - q_{i+1}^{(n)})(q_i^{(n)} - q_{i+1}^{(n)}) \to \sum_i \int_0^T \Phi_i'(q_i - q_{i+1})(q_i - q_{i+1})$$

which, bearing in mind that $q^{(n)}$ is Palais–Smale, implies

$$\left| \sum_i \int_0^T (\dot{q}_i^{(n)})^2 - \sum_i \int_0^T \Phi_i'(q_i - q_{i+1})(q_i - q_{i+1}) \right| = \varepsilon_n$$

with $\varepsilon_n \to 0$; finally

$$\int_0^T |\dot{q}^{(n)}|^2 \to \int_0^T |\dot{q}|^2$$

that is the weak convergence of $q^{(n)}$ in H is in fact strong.

One can conclude that $J'(q) = 0$ and $\|q\| = \lambda > 0$. ■

Finally we consider the dichotomy case.

By lemma 2 we know that $\exists \alpha \in (0, \lambda)$ such that up to a suitable translation $\forall \varepsilon > 0$ $\exists N_\varepsilon$, $N'_\varepsilon \in \mathbb{N}$ such that

$$\left| \sum_{|i| \le N_\varepsilon} u_i^{(n)} - \alpha \right| < \varepsilon \quad \text{and} \quad \left| \sum_{N_\varepsilon < |i| < N'_\varepsilon} u_i^{(n)} \right| < \varepsilon. \tag{10}$$

Let $\varepsilon_n > 0$ be any vanishing sequence and define $\forall i \in \mathbb{Z}$, $\forall n \in \mathbb{N}$

$$Q_i^{(n)}(t) = \begin{cases} T^{-1} \int_0^T q_{N_{\varepsilon_n}}^{(n)} & \text{if } i > N_{\varepsilon_n} \\ q_i^{(n)}(t) & \text{if } |i| \le N_{\varepsilon_n} \\ T^{-1} \int_0^T q_{-N_{\varepsilon_n}}^{(n)} & \text{if } < -N_{\varepsilon_n}, \end{cases} \tag{11}$$

then the following lemma holds.

LEMMA 4. If $\{q^{(n)}\}$ is a bounded dichotomic Palais–Smale sequence and $Q^{(n)}$ is defined as in (11) then:

(a) $\|Q^{(n)}\| \to \alpha$

(b) $Q^{(n)} \to q$ in H (therefore $\|q\| = \alpha$)

(c) $J'(q) = 0$ (therefore q is a nontrivial solution of (2)).

Proof. (a) follows trivially by the definition and (10). (b) and (c) follow if we prove that $J'(Q^{(n)}) \to 0$, as we would be back to the concentration case and the result would follow from lemma 3.

To prove this, take any $p \in H$, $\|p\| = 1$ and $\forall n \in \mathbb{N}$ let $P^{(n)} \in H$ be defined by ($\bar{p}$ denotes the mean value of p over $[0, T]$)

$$P_i^{(n)} = \begin{cases} p_i & \text{if } |i| \le N_{\varepsilon_n} \\ \bar{p}_{N_{\varepsilon_n}} & \text{if } i > N_{\varepsilon_n} \\ \bar{p}_{-N_{\varepsilon_n}} & \text{if } i < -N_{\varepsilon_n}; \end{cases} \tag{12}$$

note that $|J'(q^{(n)})[P^{(n)}]| \to 0$ and that the following estimates hold

$$|J'(q^{(n)})[P^{(n)}] - J'(Q^{(n)})[P^{(n)}]|$$

$$= \left| \int_0^T \left[\Phi'_{N_{\varepsilon_n}}\left(q^{(n)}_{N_{\varepsilon_n}} - \bar{q}^{(n)}_{N_{\varepsilon_n}}\right) - \Phi'_{N_{\varepsilon_n}}\left(q^{(n)}_{N_{\varepsilon_n}} - q^{(n)}_{N_{\varepsilon_n}+1}\right)\right]\left(P_{N_{\varepsilon_n}} - \bar{p}_{N_{\varepsilon_n}}\right)\right.$$

$$\left. + \int_0^T \left[\Phi'_{-N_{\varepsilon_n}-1}\left(\bar{q}^{(n)}_{-N_{\varepsilon_n}} - q^{(n)}_{-N_{\varepsilon_n}}\right) - \Phi'_{-N_{\varepsilon_n}-1}\left(q^{(n)}_{-N_{\varepsilon_n}-1} - q^{(n)}_{-N_{\varepsilon_n}}\right)\right]\left(\bar{p}_{-N_{\varepsilon_n}} - p_{-N_{\varepsilon_n}}\right)\right|$$

$$\le c\left(\|q^{(n)}_{N_{\varepsilon_n}} - \bar{q}^{(n)}_{N_{\varepsilon_n}}\|_\infty + \|q^{(n)}_{N_{\varepsilon_n}} - q^{(n)}_{N_{\varepsilon_n}+1}\|_\infty + \|\bar{q}^{(n)}_{-N_{\varepsilon_n}} - q^{(n)}_{-N_{\varepsilon_n}}\|_\infty + \|q^{(n)}_{-N_{\varepsilon_n}-1} - q^{(n)}_{-N_{\varepsilon_n}}\|_\infty\right) \to 0$$

$$|J'(Q^{(n)})[p] - J'(Q^{(n)})[P^{(n)}]|$$

$$= \left| \int_0^T \Phi'_{N_{\varepsilon_n}}\left(q^{(n)}_{N_{\varepsilon_n}} - \bar{q}^{(n)}_{N_{\varepsilon_n}}\right)\left[(p_{N_{\varepsilon_n}} - p_{N_{\varepsilon_n}+1}) - \left(P_{N_{\varepsilon_n}} - \bar{p}_{N_{\varepsilon_n}}\right)\right]\right.$$

$$\left. + \int_0^T \Phi'_{-N_{\varepsilon_n}-1}\left(\bar{q}^{(n)}_{-N_{\varepsilon_n}} - q^{(n)}_{-N_{\varepsilon_n}}\right)\left[(p_{-N_{\varepsilon_n}-1} - p_{-N_{\varepsilon_n}}) - \left(\bar{p}_{-N_{\varepsilon_n}} - p_{-N_{\varepsilon_n}}\right)\right]\right|$$

$$\le c\left(\|q^{(n)}_{N_{\varepsilon_n}} - \bar{q}^{(n)}_{N_{\varepsilon_n}}\|_\infty + \|\bar{q}^{(n)}_{-N_{\varepsilon_n}} - q^{(n)}_{-N_{\varepsilon_n}}\|_\infty\right) \to 0;$$

therefore, $Q^{(n)}$ is a concentrated Palais–Smale sequence. As it is bounded, up to a subsequence it converges weakly to a function $Q \in H$, and by lemma 3 the convergence is in fact strong to a solution of problem (1). It is easy to see that $Q = q$, that is the strong limit of the truncated sequence is equal to the weak limit of the untruncated sequence. ■

5. PROOF OF THE MAIN THEOREM

The proof of theorem 1 follows by the previous theorems and lemmas: indeed theorem 2 supplies a Palais–Smale sequence, which is nonvanishing by lemma 2; we obtain a nonzero solution q by lemma 3 if concentration holds or by lemma 4 if dichotomy holds. We can guarantee that this solution is nonconstant for T large enough using the following proposition:

PROPOSITION 2. There exists $\bar{T} > 0$ such that $\forall T > \bar{T}$ any T-periodic solution of q of problem (2) at level b (as given in (6)) is nonconstant.

Proof. We first show that if q is a constant solution, then $\forall i \in \mathbb{Z} \Phi'_i(q_i - q_{i+1}) = 0$, that is no pair of particles undergoes any force. To prove this suppose the converse to be true, that is $\forall i \in \mathbb{Z}$ $\dot{q}_i(t) \equiv 0$ and $\exists i \in \mathbb{Z}$ such that $\phi'_i(q_i - q_{i+1}) = \gamma \neq 0$. Then, as q_i satisfies (1), we have $\Phi'_i(q_i - q_{i+1}) = \gamma$ $\forall i \in \mathbb{Z}$, and this is impossible because $(q_i - q_{i+1}) \to 0$ for $i \to \pm\infty$ and $\Phi'_i(t) \to 0$ for $t \to 0$.

Therefore, if q is a nonzero constant solution, $J(q) = -\sum_i \Phi_i(\vartheta_i)$ where the sum is extended only to a finite number of indices and ϑ_i is a stationary point for Φ_i.

Define

$$\bar{d} = \max_i \left[\max_j \left(\Phi_i(\vartheta_{i_j}) \right) \right] < 0$$

and

$$\underline{d} = \min_i \left[\min_j \left(\Phi_i(\vartheta_{i_j}) \right) \right] < 0,$$

where we remind that $\{\vartheta_{i_j}\}$ are the stationary points of Φ_i; the result will follow if we prove that $b < -T\bar{d}$.

Consider the path $\Pi := \{q^\sigma \in P, \sigma \in [0, \bar{\sigma}]\}$, where q^σ was defined in the proof of theorem 2; since

$$\int_0^T \Phi(q^\sigma) \geq 2\eta T \underline{d},$$

then, for a suitable choice of η and for T large we have

$$J(q^\sigma) \leq \frac{2\bar{\sigma}^2\pi^2}{\eta^2 T} - 2\eta T\underline{d} < -T\bar{d},$$

and the proof follows. ■

Finally the energy of the solution is finite by the same argument which leads to (4).

6. THE STRUCTURE OF A PALAIS–SMALE SEQUENCE

We investigate the structure of a Palais–Smale sequence by following the same procedure as in [6], which applies an idea of [7].

LEMMA 5. Let $q^{(n)}$ and q be as in Section 4 and let $\tilde{q}^{(n)} = q^{(n)} - q$; then:
(1) $J(\tilde{q}^{(n)}) + J(q^{(n)}) \to J(q)$;
(2) $J'(\tilde{q}^{(n)}) \to 0$;
(3) $\|\tilde{q}^{(n)}\| - \|q^{(n)}\| \to \|q\|$.

Proof. (1) Note that

$$\begin{aligned} J(q^{(n)}) &= J(\tilde{q}^{(n)} + q) \\ &= J(\tilde{q}^{(n)}) + J(q) \\ &\quad + \sum_i \int_0^T \left[\dot{\tilde{q}}_i^{(n)} \dot{q}_i - \Phi_i\left(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)} + q_i - q_{i+1}\right) - \Phi_i\left(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)}\right) - \Phi_i(q_i - q_{i+1}) \right]. \end{aligned}$$

Take $\varepsilon > 0$, then $\exists N_\varepsilon \in \mathbb{N}$ as in lemma 1; we first show that

$$\sum_{|i| \leq N\varepsilon} \int_0^T |\dot{\tilde{q}}_i^{(n)} \dot{q}_i - \Phi_i\left(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)} + q_i - q_{i+1}\right) + \Phi_i\left(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)}\right) + \Phi_i(q_i - q_{i+1})| < c\varepsilon \quad (13)$$

for n large enough, indeed $\dot{\tilde{q}}_i^{(n)} \to 0$ in L^2 for all i, hence the first term vanishes; furthermore $\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)} \to 0$ in L^2, thus (13) follows by the regularity of Φ_i. Next, we show that also

$$\sum_{|i|>N_\varepsilon} \int_0^T |\dot{\tilde{q}}_i^{(n)}\dot{q}_i - \Phi_i\big(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)} + q_i - q_{i+1}\big) + \Phi_i\big(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)}\big) + \Phi_i(q_i - q_{i+1})| < c\varepsilon^{1/2}. \quad (14)$$

Indeed, since

$$\sum_{|i|>N_\varepsilon} \int_0^T \Big[(\dot{q}_i)^2 + (q_i - q_{i+1})^2\Big] < \varepsilon \quad (15)$$

by lemmas 1 and 3, Hölder inequality yields

$$\sum_{|i|>N_\varepsilon} \int_0^T |\dot{\tilde{q}}_i^{(n)}\dot{q}_i| < \left\{\sum_{|i|>N_\varepsilon} \int_0^T \big(\dot{\tilde{q}}_i^{(n)}\big)^2\right\}^{1/2} \left\{\sum_{|i|>N_\varepsilon} \int_0^T (\dot{q}_i)^2\right\}^{1/2} < c\varepsilon^{1/2}.$$

Finally, to estimate

$$\sum_{|i|>N_\varepsilon} \int_0^T |\Phi_i\big(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)} + q_i - q_{i+1}\big) - \Phi_i\big(\tilde{q}_i^{(n)} - \tilde{q}_{i+1}^{(n)}\big) - \Phi_i(q_i - q_{i+1})|$$

we use (15) and the same procedure of proposition 1 to obtain (14).

(2) In a similar way we can prove that $J'(\tilde{q}^{(n)}) - J'(q^{(n)}) \to J'(q)$, that is $J'(\tilde{q}^{(n)}) \to 0$.

(3) It is a trivial consequence of the very definition of dichotomy. ∎

Next lemma collects and highlights some previous results.

LEMMA 6. There exist γ, $\nu > 0$ such that if q is a nonzero critical point of J at level b, then

(I) $b \geq \gamma$.

(II) $\|q\| \leq \left(\dfrac{Kb(2+\delta)}{\delta}\right)^{1/2}$, where $K = \max(2, (1/\alpha_1), \ldots, (1/\alpha_m))$.

(III) $\|q\| > \nu$.

Proof. The proof of (II) follows the lines of point (3) in theorem 2. (III) follows taking into account that 0 is a local minimum for J. (I) follows from (II) and (III). ∎

Remark. In lemma 5 we prove that if we have a dichotomic Palais–Smale sequence $q^{(n)}$ and we subtract its weak limit we obtain another Palais–Smale sequence to which we can apply again lemma 1; note that:

(1) Vanishing cannot occur because of lemma 2 and by definition of dichotomy.

(2) If concentration occurs we apply again lemma 3 and therefore $q^{(n)}$ converges in H to a nonconstant critical point of J.

(3) If dichotomy occurs we repeat the whole algorithm. Note that the above algorithm can be iterated at most $\mu = [(bK(2+\delta))/(\nu^2\delta)]$ times because of (I) in lemma 6 and (1) in lemma 5.

This algorithm explains the structure of a Palais–Smale sequence $q^{(n)}$: we can conclude that, up to a subsequence, $q^{(n)}$ consists of at most μ "bumps" where almost all the energy (the norm) of the system is concentrated. These bumps move away from each other for increasing

n; if we "follow" any bump by a suitable translation, the sequence converges weakly to a nonconstant critical point of the functional. If we flat all bumps but one in the sequence and we follow the remaining bump, then we obtain a concentrate Palais–Smale sequence and the convergence is strong.

We can state more precisely these observations: define the translation of $q \in H$ by an integer $k \in \mathbb{Z}$ setting $\tau(k)\{q_i\} = \{q_{i+k} + \sigma_k\}$: the following theorem summarizes the result we obtained on Palais–Smale sequences.

THEOREM 3. Assume (i), (ii), (iii), (iv) and (v); let $q^{(n)} \in H$ be a Palais–Smale sequence for J. Then there exist l $(1 \le l \le \mu)$ nonconstant critical points $q^i \in H$ $(i = 1, \dots, l)$, and l sequences of integers $k_n^i (n \in \mathbb{N})$, such that

$$\left\| q^{(n)} - \sum_{i=1,l} \tau(k_n^i) q^i \right\| \to 0,$$

$$\sum_{i=1,l} J(q^i) = b,$$

and for $i \neq j$

$$|k_n^i - k_n^j| \to \infty.$$

Remark. In the previous result l denotes the number of bumps, but this does not mean that there exist l different critical points; in fact two bumps may have the same strong limit.

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