Math. Nachr. **280,** No. 3, 242 – 255 (2007) / **DOI** 10.1002/mana.200410478

Some estimates for the torsional rigidity of composite rods

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Received 9 March 2004, revised 17 January 2005, accepted 4 April 2005
Published online 9 January 2007

Key words Torsional rigidity, web functions, nonconvex minimization problems
MSC (2000) 49K30, 52A10, 74B05

A well-known problem in elasticity consists in placing two linearly elastic materials (of different shear moduli) in a given plane domain Ω, so as to maximize the torsional rigidity of the resulting rod; moreover, the proportion of these materials is prescribed. Such a problem may not have a classical solution as the optimal design may contain homogenization regions, where the two materials are mixed in a microscopic scale. Then, the optimal torsional rigidity becomes difficult to compute. In this paper we give some different theoretical upper and lower bounds for the optimal torsional rigidity, and we compare them on some significant domains.

1 Introduction

A challenging problem in the theory of elasticity (see [2], [4], [13]–[16]) consists in maximizing the torsional rigidity (in brief: torsion) of an infinitely long elastic rod with given convex cross-section $\Omega \subset \mathbb{R}^2$, under the constraint that it is made of two different linearly elastic materials in fixed proportions. This problem may be tackled as an optimal design problem. Let μ_1^{-1} and μ_2^{-1} be the shear moduli of the two materials, say with $0 < \mu_1 < \mu_2$; for a given $\rho \in (0,1)$ the "stronger" material (with shear modulus μ_1^{-1}) should occupy a region $D \subset \Omega$ with $|D|/|\Omega| = \rho$. Here and in the sequel, $|\cdot|$ denotes the measure of a set: either its two dimensional Lebesgue measure or its one dimensional Hausdorff measure, depending on the context. The admissible configurations are then described by all functions μ in the class

$$\mathcal{M}(\Omega) := \left\{ \mu\colon \Omega \to \{\mu_1, \mu_2\} \text{ measurable } : \ \frac{1}{|\Omega|} \int_\Omega \mu(x)\, dx = \mu_1 \rho + \mu_2 (1-\rho) \right\}.$$

For a given $\mu \in \mathcal{M}(\Omega)$, the torsion of the corresponding rod is given by $\int_\Omega u_\mu \, dx$, where u_μ is the unique weak solution to the boundary value problem

$$\begin{cases} -\mathrm{div}(\mu(x)\nabla u) = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

The optimal design problem consists in determining the optimal torsion $\mathcal{T}(\Omega)$ when μ varies in the class of admissible configurations $\mathcal{M}(\Omega)$, i.e.,

$$\mathcal{T}(\Omega) := \sup \left\{ \int_\Omega u_\mu \, dx \ : \ \mu \in \mathcal{M}(\Omega) \right\}. \tag{1.1}$$

Alternatively, $\mathcal{T}(\Omega)$ can be evaluated by a "double" minimization procedure, that is

$$\mathcal{T}(\Omega) = -2 \inf \left\{ J(u,\mu) \ : \ (u,\mu) \in H_0^1(\Omega) \times \mathcal{M}(\Omega) \right\}, \tag{1.2}$$

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the functional J being defined by

$$J(u,\mu) := \int_\Omega \left(\frac{\mu(x)}{2} |\nabla u|^2 - u \right) dx. \tag{1.3}$$

One should not expect the existence of an optimal $\mu \in \mathcal{M}(\Omega)$ in the maximization problem (1.1). Actually, if the integral constraint satisfied by all admissible μ is incorporated in the functional J through a Lagrange multiplier, and the infimum with respect to μ is evaluated point-wise, the resulting functional (which only depends on u) turns out to be of the kind $\int_\Omega \big(j(|\nabla u|) - u \big)\, dx$ with j non convex, see [13] and [15]. Thus, only a minimizer for the relaxed problem, obtained by replacing j with its convexification j^{**}, can be guaranteed. If the Euclidean norm of the gradient of the solution to the relaxed problem falls into the region where $j^{**} \neq j$, this means that minimizing sequences develop an oscillatory behaviour, and that the original problem (1.1) does not admit a maximizer. From a physical point of view, it amounts to saying that the optimal design may be a composite, layered material obtained by mixing the two initial materials on a microscopic scale. Murat–Tartar [20] have shown that this homogenization phenomenon "always" happens. More precisely, if problem (1.1) admits a maximizer μ whose corresponding region D is simply connected with ∂D of class C^1, then necessarily Ω is a disk: in this case, the optimal design consists in a central disk made of the weaker material and the complementary annulus made of the stronger material.

In view of these considerations, the optimal design is not easy to manufacture and the optimal torsion $\mathcal{T}(\Omega)$ is not simple to compute, even by numerical methods. One is then led to find theoretical estimates. This is precisely the main purpose of the present paper.

A first natural attempt in this direction relies on symmetrization. Inspired by previous results in [1] and [24], in Theorem 3.1 we prove that

$$\mathcal{T}(\Omega) \le \mathcal{T}(\Omega^*), \tag{1.4}$$

where Ω^* is the disk of radius $R = \sqrt{|\Omega|/\pi}$; the torsion $\mathcal{T}(\Omega^*)$ is easily written in terms of the data $|\Omega|, \mu_1, \mu_2$, and ρ. However, if the domain Ω is "thin" (i.e., very different from a ball), it is not reasonable to expect the upper bound (1.4) to be satisfactory.

In order to find alternative estimates, one may start from the physical intuition that the stronger material should occupy the region closer to the boundary $\partial\Omega$. This guess is confirmed by the numerical experiments in [13] and [15] where it is shown that, when Ω is a square, the weaker material tends indeed to concentrate near the center, while the stronger material is placed near $\partial\Omega$. Thus, it seems reasonable to seek an estimate involving the so-called *inner parallel sets* Ω_t, introduced by Steiner in [23] (see also [21]) and defined by

$$\Omega_t := \{x \in \Omega \,:\, d_{\partial\Omega}(x) > t\}, \tag{1.5}$$

where $d_{\partial\Omega}$ is the distance function from $\partial\Omega$. In Theorem 3.2, we obtain a second different upper bound for $\mathcal{T}(\Omega)$ of the kind

$$\mathcal{T}(\Omega) \le \Lambda(\Omega), \tag{1.6}$$

where $\Lambda(\Omega)$ depends on the Lebesgue measure of the family of the parallel sets $\{\Omega_t\}_t$. For many domains, $\Lambda(\Omega)$ can be computed explicitly; for instance, when Ω is the tangential body of a ball, it can be written simply in terms of the data $|\Omega|, \mu_1, \mu_2$, and ρ (see Corollary 5.1). The proof of (1.6) is quite involved and the basic idea is the following: we first minorize $J(\cdot,\mu)$ for a fixed μ, roughly by neglecting the component of ∇u tangential to $\partial\Omega_t$; second, we proceed with the minimization process with respect to μ, which requires several fine tools of convex geometry.

In Section 5 we compare the upper bounds in (1.4) and (1.6), and we show that, depending on the domain Ω, we may have both $\Lambda(\Omega) < \mathcal{T}(\Omega^*)$ and the converse inequality.

With the aim of testing more accurately how sharp the upper bound $\Lambda(\Omega)$ is, we consider the class $\mathcal{W}(\Omega)$ of (measurable) functions which only depend on the distance $d_{\partial\Omega}$. We already dealt with this kind of functions in the previous papers [5]–[11], where we called them *web functions*. Here we consider the spaces

$$\mathcal{K}(\Omega) := H^1_0(\Omega) \cap \mathcal{W}(\Omega) \quad \text{and} \quad \mathcal{M}_d(\Omega) := \mathcal{M}(\Omega) \cap \mathcal{W}(\Omega),$$

and we denote by $\mathcal{N}(\Omega)$ the infimum in (1.2) when the admissible pairs (u,μ) vary over $\mathcal{K}(\Omega)\times\mathcal{M}_d(\Omega)$. This yields at once the lower bound

$$\mathcal{T}(\Omega)\geq\mathcal{N}(\Omega). \tag{1.7}$$

By exploiting the strong properties of web functions, in Theorem 4.1 we write $\mathcal{N}(\Omega)$ in a quite simple form in terms of the parallel sets. Notice in fact that the family of level sets of a function $u\in\mathcal{W}(\Omega)$ is precisely the family $\{\partial\Omega_t\}_t$, so there is a close link between web functions and parallel sets. In view of this link, it is clear that the quantities $\mathcal{N}(\Omega)$ and $\Lambda(\Omega)$ can be compared in a natural way to each other. By studying how close to 1 the ratio $\mathcal{E}(\Omega):=\mathcal{N}(\Omega)/\Lambda(\Omega)$ is, we test the precision of the upper bound $\Lambda(\Omega)$. This is done in Section 6, where we also compare the behaviour of the ratio $\mathcal{E}(\Omega)$ with the sharp bound we proved in [7] for the torsion problem with only one material.

The outline of the paper is as follows. In Section 2 we introduce and study the tools of convex geometry which are needed in order to state our estimates. In Section 3 we prove the two upper bounds (1.4) and (1.6), while in Section 4 we prove the lower bound (1.7). In Section 5 we compare the two upper bounds by testing them on simple classes of domains. Finally in Section 6 we estimate how sharp the upper bound (1.6) is.

2 Notation and preliminary lemmas of convex geometry

Let $\Omega\subset\mathbb{R}^2$ be a convex set. We denote by $d_{\partial\Omega}(x)$ the distance function from the boundary of Ω, and by R_Ω the inradius of Ω, that is, the supremum of $d_{\partial\Omega}(x)$ for $x\in\Omega$. Since the functionals $\mathcal{T}$, Λ and $\mathcal{N}$ are homogeneous of degree 4 under dilations, we restrict our attention to the class of domains

$$\mathcal{C}=\big\{\Omega\subset\mathbb{R}^2\ :\ R_\Omega=1,\ \Omega \text{ is bounded and convex}\big\}.$$

Let us stress that the choice $R_\Omega=1$ is made for convenience, and in particular all the results in this section continue to hold, with the suitable modifications, for domains with arbitrary inradius.

Some of our estimates will take a simpler form in the subclass

$$\widehat{\mathcal{C}}=\{\Omega\in\mathcal{C}\ :\ \Omega \text{ is a tangential body of a ball}\}.$$

We recall that a convex set $\Omega\subset\mathbb{R}^2$ is said to be a *tangential body* of a ball B, if through each boundary point of Ω there exists a support line to Ω that also supports B (see [22, p. 75]). Circumscribed polygons are the simplest domains of this kind. As a further example, one can take the "cap body" obtained as the union of a ball and a triangle, such that two of its sides have the common extreme lying outside the ball, and are tangent to the boundary of the ball at the other extreme.

Let $\Omega\in\mathcal{C}$. For $t\in[0,1]$, we consider the inner parallel sets defined by (1.5). In particular, one has $\Omega_0=\Omega$, and $\Omega_1=\emptyset$. Notice that $\partial\Omega_t$ is precisely the level line $\{d_{\partial\Omega}(x)=t\}$. We denote by $|\Omega_t|$ and $|\partial\Omega_t|$ respectively the Lebesgue measure of Ω_t and the one-dimensional Hausdorff measure of its boundary. As a direct consequence of the Brunn–Minkowski theorem, we have that the maps $t\mapsto|\partial\Omega_t|$ and $t\mapsto\sqrt{|\Omega_t|}$ are concave in $[0,1]$ for every $\Omega\in\mathcal{C}$ (see [3]). For $\Omega\in\widehat{\mathcal{C}}$, we have the following simple formulae (see [7, Section 2.2]):

$$|\Omega_t|=|\Omega|\,(1-t)^2 \quad\text{and}\quad |\partial\Omega_t|=2\,|\Omega|\,(1-t). \tag{2.1}$$

In the sequel, a major role will be played by the following functions, defined on $[0,1]$ with values into $\mathbb{R}^+$:

$$\psi_\Omega(t):=\frac{|\Omega_t|^2}{|\partial\Omega_t|},\quad \varphi_\Omega(t):=\int_t^1|\Omega_s|\,ds,\quad \Phi_\Omega(t):=\int_t^1\varphi_\Omega(s)\,ds. \tag{2.2}$$

(The function ψ_Ω is defined in $t=1$ by setting $\psi_\Omega(1)=0$.) Moreover, we will extensively use the quantities k and τ which depend on ρ and are implicitly defined by the equations

$$|\Omega_k|=\rho\,|\Omega| \tag{2.3}$$

and

$$|\Omega_\tau| = (1-\rho)\,|\Omega|\,. \tag{2.4}$$

For general $\Omega \in \mathcal{C}$, $k(\rho)$ and $\tau(\rho)$ cannot be determined explicitly in terms of ρ; however, we can say that they satisfy

$$k(0) = 1, \quad k(1) = 0, \quad \frac{dk}{d\rho} = -\frac{|\Omega|}{|\partial\Omega_k|},$$
$$\tau(0) = 0, \quad \tau(1) = 1, \quad \frac{d\tau}{d\rho} = \frac{|\Omega|}{|\partial\Omega_\tau|}.$$

In particular, we have $k = \tau$ if and only if $\rho = 1/2$ (same proportion of the two materials).

On the other hand, if we restrict our attention to the subclass $\widehat{\mathcal{C}}$ and we use (2.1), k, τ, and Φ_Ω read:

$$k(\rho) = 1 - \sqrt{\rho}, \quad \tau(\rho) = 1 - \sqrt{1-\rho}, \quad \Phi_\Omega(t) = \frac{|\Omega|}{12}(1-t)^4 \quad \text{for all} \quad \Omega \in \widehat{\mathcal{C}}. \tag{2.5}$$

Some useful properties of the above functions are collected in next lemma.

Lemma 2.1 *For all $\Omega \in \mathcal{C}$, there holds:*

(i) *the function $t \mapsto \psi_\Omega(t)$ is Lipschitz continuous and strictly monotone decreasing on $[0,1]$;*

(ii) *for all $t \in [0,1)$,*

$$\psi_\Omega(t) \geq \frac{3}{2}\,\varphi_\Omega(t);$$

(iii) *for all $t \in [0,1)$,*

$$\frac{|\Omega_t|}{3}\,(1-t) \leq \varphi_\Omega(t) \leq \frac{4}{1-t}\,\Phi_\Omega(t).$$

Moreover, all the above inequalities turn into equalities whenever $\Omega \in \widehat{\mathcal{C}}$.

Proof. By the standard isoperimetric inequality the function ψ_Ω is continuous in $t = 1$ (recall that we have set $\psi_\Omega(1) = 0$). By the isoperimetric inequality proved in [6, Corollary 3.3], we have

$$\psi'_\Omega(t) \leq -\frac{3}{2}\,|\Omega_t| \quad \text{for all} \quad t \in [0,1),$$

with equality if $\Omega \in \widehat{\mathcal{C}}$. Moreover, $\psi'_\Omega(t) \geq -2\,|\Omega_t|$, hence we conclude that ψ_Ω is Lipschitz continuous and monotone decreasing on $[0,1]$. Since $\psi_\Omega(1) = 0$ we have that

$$\psi_\Omega(t) \geq \frac{3}{2}\int_t^1 |\Omega_s|\,ds = \frac{3}{2}\,\varphi_\Omega(t) \quad \text{for all} \quad t \in [0,1).$$

To prove statement (iii) we observe that, since the map $t \mapsto \sqrt{|\Omega_t|}$ is concave and vanishes at $t = 1$, we have

$$\sqrt{|\Omega_s|} \geq \sqrt{|\Omega_t|}\,\frac{1-s}{1-t} \quad \text{for all} \quad s \in (t,1],$$

again with equality if $\Omega \in \widehat{\mathcal{C}}$, see (2.1). Hence,

$$\varphi_\Omega(t) \geq \int_t^1 |\Omega_t|\,\frac{(1-s)^2}{(1-t)^2}\,ds = \frac{|\Omega_t|}{3}\,(1-t). \tag{2.6}$$

By (2.6) we get

$$\Phi_\Omega(t) \geq \int_t^1 \frac{|\Omega_s|}{3}\,(1-s)\,ds = [\text{by parts}] = \frac{1}{3}\,\varphi_\Omega(t)(1-t) - \frac{1}{3}\,\Phi_\Omega(t),$$

which yields (iii). □

Next we recall the definition of piercing function introduced in [7], which is a powerful tool especially for convex polygons. Given an arbitrary domain $\Omega \in \mathcal{C}$, for a.e. $y \in \partial\Omega$ the outer unit normal is well-defined and will be denoted by n_y. For a.e. $x \in \Omega$, the point $\Pi(x) \in \partial\Omega$ such that $|x - \Pi(x)| = d_{\partial\Omega}(x)$ is uniquely determined. Then we set:

$$\lambda_\Omega(y) := \sup\{k \geq 0 \, : \, \Pi(y - kn_y) = y\} \quad \text{for a.e.} \quad y \in \partial\Omega. \tag{2.7}$$

We clearly have $0 \leq \lambda_\Omega(y) \leq 1$ on $\partial\Omega$. We will also make use of the following extension of λ_Ω to points $x \in \Omega$:

$$\lambda_\Omega(x) = \lambda_\Omega(\Pi(x)) - d_{\partial\Omega}(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{2.8}$$

When Ω is a convex polygon, the function λ_Ω is Lipschitz continuous on $\overline{\Omega}$ [6, Lemma 4.3] (for related results see also [19]). Clearly, also Π is Lipschitzian, with

$$|\nabla\Pi| = 1 \quad \text{a.e. in} \quad \Omega. \tag{2.9}$$

Hence, applying the coarea formula (see e.g. [12, Chapter 2, Section 1.1.3, Theorem 2]) and recalling (2.8), when Ω is a convex polygon one has

$$|\Omega_t| = \int_{\partial\Omega_t} \lambda_\Omega(y)\, dy. \tag{2.10}$$

Again based on the coarea formula, Lemma 2.2 below shows how the line integral over $\partial\Omega$ of a function depending on λ_Ω may be replaced by an integral over Ω. In view of (2.8), the variable of integration may be indifferently $d_{\partial\Omega}(x)$ or $\lambda_\Omega(x)$; this depends on the orientation chosen for the segments starting from the set where Π is multivalued (that is, the set where $d_{\partial\Omega}$ is not differentiable) and meeting $\partial\Omega$ orthogonally.

Lemma 2.2 *Let $\Omega \in \mathcal{C}$ be a polygon, and let $g : [0,1] \to \mathbb{R}$ be a Lipschitz continuous function such that $g(0) = 0$. Then*

$$\int_{\partial\Omega} g(\lambda_\Omega(y))\, dy = \int_\Omega g'(d_{\partial\Omega}(x))\, dx = \int_\Omega g'(\lambda_\Omega(x))\, dx.$$

P r o o f. Since $g(s) = \int_0^s g'(\sigma)\, d\sigma$, we have

$$\int_{\partial\Omega} g(\lambda_\Omega(y))\, dy = \int_{\partial\Omega} \left[\int_0^{\lambda_\Omega(y)} g'(\sigma)\, d\sigma\right] dy = \int_\Omega g'(d_{\partial\Omega}(x))\, dx,$$

where the second equality follows from the coarea formula and (2.9).

On the other hand, $g(s) = \int_0^s g'(s - \sigma)\, d\sigma$, hence, using (2.8),

$$\int_{\partial\Omega} g(\lambda_\Omega(y))\, dy = \int_{\partial\Omega} \left[\int_0^{\lambda_\Omega(y)} g'(\lambda_\Omega(y) - \sigma)\, d\sigma\right] dy = \int_\Omega g'(\lambda_\Omega(x))\, dx,$$

and the proof is complete. □

Applying the above lemma, we may give integral representation formulae for the functions φ_Ω and Φ_Ω in terms of the piercing function λ_Ω:

Lemma 2.3 *Let $\Omega \in \mathcal{C}$ be a polygon. Then the following identities hold for every $t \in [0,1)$:*

$$\varphi_\Omega(t) = \int_{\Omega_t} \lambda_\Omega(x)\, dx, \quad \Phi_\Omega(t) = \frac{1}{2}\int_{\Omega_t} \lambda_\Omega(x)^2\, dx.$$

P r o o f. To obtain the first identity it is enough to observe that, since Ω is a polygon, we may apply (2.10) to obtain

$$\int_{\Omega_t} \lambda_\Omega(x)\, dx = \int_t^1 \left(\int_{\partial\Omega_s} \lambda_\Omega(y)\, dy\right) ds = \int_t^1 |\Omega_s|\, ds. \tag{2.11}$$

By Lemma 2.2 (applied on Ω_s), using (2.11) and (2.10), we have

$$\begin{aligned}\int_{\Omega_t} \lambda_\Omega(x)^2\, dx &= \int_t^1 \left(\int_{\partial\Omega_s} \lambda_\Omega(y)^2\, dy \right) ds \\ &= \int_t^1 \left(\int_{\Omega_s} 2\lambda_\Omega(x)\, dx \right) ds \\ &= 2\int_t^1 \left(\int_s^1 |\Omega_\sigma|\, d\sigma \right) ds\,,\end{aligned}$$

and the proof is complete. □

3 Two upper bounds for the torsion

3.1 Upper bound via symmetrization

For any $\Omega \in \mathcal{C}$, we denote by Ω^* the disk having the same measure as Ω. Then we have

Theorem 3.1 *For every $\Omega \in \mathcal{C}$, there holds:*

$$\mathcal{T}(\Omega) \le \mathcal{T}(\Omega^*) = \frac{|\Omega|^2}{8\pi} \left[\frac{2\rho - \rho^2}{\mu_1} + \frac{(1-\rho)^2}{\mu_2} \right].$$

Proof. Let $\mu \in \mathcal{M}(\Omega)$ and $u \in W_0^{1,\infty}(\Omega)$. Then also $|u| \in W_0^{1,\infty}(\Omega)$ and

$$J(u,\mu) \ge J(|u|,\mu). \tag{3.1}$$

We may so restrict our attention to nonnegative $u \in W_0^{1,\infty}(\Omega)$.

For any $f \in L^1(\Omega)$ denote by f^* (resp. f_*) its spherical decreasing (resp. increasing) rearrangement on Ω^*. Then,

$$\int_\Omega \mu(x)\, |\nabla u(x)|^2\, dx \ge \int_{\Omega^*} \mu^*(y)\, |\nabla u|_*^2(y)\, dy. \tag{3.2}$$

Let $F_u(\vartheta) := |\{x \in \Omega\, :\, |\nabla u(x)| \le \vartheta\}|$ and $M := \|\nabla u\|_\infty$. Then, [1, Theorem 4] yields

$$\int_\Omega u \le \frac{1}{3\sqrt{\pi}} \int_0^M \left[|\Omega|^{3/2} - F_u(\vartheta)^{3/2} \right] d\vartheta. \tag{3.3}$$

The radius of the ball Ω^* is given by $R = \sqrt{\frac{|\Omega|}{\pi}}$; define over Ω^* the radial function

$$\overline{u}(y) = \int_{|y|}^R |\nabla u|_*(t)\, dt, \quad y \in \Omega^*,$$

so that $\overline{u}$ is a dome function (radially symmetric, non-increasing and concave) such that $|\nabla \overline{u}(y)| = |\nabla u|_*(y)$ for a.e. $y \in \Omega^*$. Therefore, we may apply again [1, Theorem 4] to obtain

$$\int_{\Omega^*} \overline{u} = \frac{1}{3\sqrt{\pi}} \int_0^M \left[|\Omega|^{3/2} - F_u(\vartheta)^{3/2} \right] d\vartheta. \tag{3.4}$$

Summarizing, by (3.1)–(3.4), we infer

$$J(u,\mu) \ge \int_{\Omega^*} \left[\frac{\mu^*(y)}{2}\, |\nabla \overline{u}(y)|^2 - \overline{u} \right] dy \ge \inf_{(\nu,v) \in \mathcal{M}(\Omega^*) \times H_0^1(\Omega^*)} \int_{\Omega^*} \left[\frac{\nu}{2}\, |\nabla v|^2 - v \right] = -\frac{1}{2}\, \mathcal{T}(\Omega^*)$$

for all $(\mu, u) \in \mathcal{M}(\Omega) \times W_0^{1,\infty}(\Omega)$. By a density argument we infer that

$$-2J(u,\mu) \le \mathcal{T}(\Omega^*) \quad \text{for all} \quad (\mu,u) \in \mathcal{M}(\Omega) \times H_0^1(\Omega)\,. \tag{3.5}$$

Now, the right-hand side of (3.5) equals

$$\frac{|\Omega|^2}{8\pi}\left[\frac{2\rho-\rho^2}{\mu_1}+\frac{(1-\rho)^2}{\mu_2}\right] \quad \text{for all} \quad \Omega\in\mathcal{C}.$$

This representation formula has been first proved by Voas–Yaniro in [24, Theorem 1.1]. Alternatively, it can be obtained noticing that $\mathcal{T}(\Omega^*)=\mathcal{N}(\Omega^*)$ and using (4.1) below $\big($taking care to apply it with the obvious modifications required by the fact that $R_{\Omega^*}=\sqrt{|\Omega|/\pi}\,\big)$. □

3.2 Upper bound via one-dimensional optimization

We now prove a different kind of upper bound for $\mathcal{T}(\Omega)$, in terms of the constant k and of the functions φ_Ω and Φ_Ω.

Theorem 3.2 *For every $\Omega\in\mathcal{C}$, there holds:*

$$\mathcal{T}(\Omega)\le\Lambda(\Omega):=\frac{2}{\mu_2}\,\Phi_\Omega(0)+\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\big\{2\Phi_\Omega(k)+k^2\,|\Omega_k|+2k\varphi_\Omega(k)\big\},$$

where k is defined in (2.3)*, and φ_Ω and Φ_Ω are defined in* (2.2)*.*

We first give some lemmas as preliminary steps, by restricting our attention to polygons.

Lemma 3.3 *For every polygon $\Omega\in\mathcal{C}$, there holds*

$$\mathcal{T}(\Omega)\le\max_{\mu\in\mathcal{M}(\Omega)}\int_{\partial\Omega}\left\{\int_0^{\lambda_\Omega(y)}\frac{(\lambda_\Omega(y)-t)^2}{\mu(y-tn_y)}\,dt\right\}dy. \tag{3.6}$$

P r o o f. We follow the approach developed in [7, Theorem 3] with several modifications.

Assume that Ω has N sides and denote them by $F_1,\dots,F_N$. For simplicity, for all $j=1,\dots,N$ we denote by F_j the open segment, namely the j-th side of Ω without its endpoints. The piercing function λ_Ω introduced in (2.7) is defined in every point of $\partial\Omega$ except for the N vertices. Moreover, $n_y\equiv n_j$ is a constant vector on F_j. We take a partition of Ω into N open subpolygons $P_1,\dots,P_N$ defined as follows:

$$P_j=\big\{y-tn_j\ :\ y\in F_j,\ 0<t<\lambda_\Omega(y)\big\}.$$

For our convenience, we also view each polygon P_j as the (open) epigraph Z_j of the function λ_Ω on F_j, namely

$$Z_j=\{(y,t)\ :\ y\in F_j,\ 0<t<\lambda_\Omega(y)\}.$$

We now fix $\mu\in\mathcal{M}(\Omega)$. For all $j\in\{1,\dots,N\}$ let

$$H^1_*\big(P_j\big):=\big\{v\in H^1\big(P_j\big)\ :\ v=0 \text{ on } F_j\big\},$$
$$H^1_*\big(Z_j\big):=\big\{v\in H^1\big(Z_j\big)\ :\ v(y,0)=0 \text{ for all } y\in F_j\big\},$$

and consider the functional

$$J_j(v)=\int_{P_j}\left(\frac{\mu(x)}{2}\,|\nabla v|^2-v\right)dx \quad \text{for all} \quad v\in H^1_*\big(P_j\big).$$

Note that

$$J_j(v)=\int_{F_j}\int_0^{\lambda_\Omega(y)}\left[\frac{\mu(y-tn_j)}{2}\big|\nabla v\big(y-tn_j\big)\big|^2-v\big(y-tn_j\big)\right]dt\,dy \qquad \text{for all} \quad v\in H^1_*\big(P_j\big). \tag{3.7}$$

For any $u\in C^1_c(\Omega)$ let u_j denote the restrictions of u to P_j ($j=1,\dots,N$) and set

$$u_j^*(y,t)=u_j\big(y-tn_j\big) \quad \text{for all} \quad (y,t)\in Z_j.$$

Since $u_j \in C^1 \cap H^1_*(P_j)$, we have $u_j^* \in C^1 \cap H^1_*(Z_j)$ and $\frac{\partial u_j^*}{\partial t} = -\nabla u_j \cdot n_j$ so that

$$\left|\frac{\partial u_j^*}{\partial t}(y,t)\right| \le \left|\nabla u_j\big(y - tn_j\big)\right| \quad \text{for all} \quad (y,t) \in Z_j .$$

Therefore we get

$$J_j\big(u_j\big) \ge I_j\big(u_j^*\big) \quad (j = 1, \ldots, N) \quad \text{for all} \quad u \in C^1_c(\Omega) \tag{3.8}$$

where

$$I_j(v) := \int_{F_j}\int_0^{\lambda_\Omega(y)} \left[\frac{\mu\big(y - tn_j\big)}{2}\left(\frac{\partial v}{\partial t}\right)^2 - v\right] dt\, dy \quad \text{for all} \quad v \in H^1_*\big(Z_j\big).$$

For all $y \in F_j$ define now the functional

$$I_y(g) = \int_0^{\lambda_\Omega(y)} \left[\frac{\mu\big(y - tn_j\big)}{2}\,|g'(t)|^2 - g(t)\right] dt,$$

that we wish to minimize over the space $\mathcal{G} := \big\{g \in H^1(0, \lambda_\Omega(y)),\ g(0) = 0\big\}$. We have

$$\min_{g\in\mathcal{G}} I_y(g) = -\frac{1}{2}\int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu(y - tn_j)}\, dt. \tag{3.9}$$

To see this, we solve the Euler equation corresponding to (3.9), that is, $\big[\mu\big(y - tn_j\big)g'(t)\big]' = -1$. This gives

$$g(t) = g_c(t) := \int_0^t \frac{c - s}{\mu(y - sn_j)}\, ds$$

for some $c \in \mathbb{R}$. By Fubini's Theorem we then have

$$I_y(g_c) = \frac{1}{2}\int_0^{\lambda_\Omega(y)} \frac{(c - t)^2 - 2(c - t)(\lambda_\Omega(y) - t)}{\mu\big(y - tn_j\big)}\, dt.$$

By differentiating with respect to c we see that $I_y(g_c)$ attains its minimum for $c = \lambda_\Omega(y)$ and (3.9) follows. By (3.9) we get at once that

$$I_j(v) \ge -\frac{1}{2}\int_{F_j}\int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu\big(y - tn_j\big)}\, dt\, dy \quad \text{for all} \quad v \in H^1_*\big(Z_j\big).$$

This, combined with (3.8), yields

$$\begin{aligned} J(u,\mu) &= \sum_{j=1}^N J_j\big(u_j\big) \ge \sum_{j=1}^N I_j\big(u_j^*\big) \ge -\frac{1}{2}\sum_{j=1}^N \int_{F_j}\int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu\big(y - tn_j\big)}\, dt\, dy \\ &= -\frac{1}{2}\int_{\partial\Omega}\int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu\big(y - tn_y\big)}\, dt\, dy \quad \text{for all} \quad u \in C^1_c(\Omega). \end{aligned}$$

The arbitrariness of $\mu \in \mathcal{M}(\Omega)$ and a density argument then imply

$$J(u,\mu) \ge -\frac{1}{2}\int_{\partial\Omega}\int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu\big(y - tn_y\big)}\, dt\, dy \quad \text{for all} \quad (u,\mu) \in H^1_0(\Omega) \times \mathcal{M}(\Omega).$$

This proves (3.6). □

Our next goal is to compute the maximum appearing in the right-hand side of (3.6). Since the function $t \mapsto (\lambda_\Omega(y) - t)^2$ is decreasing on $[0, \lambda_\Omega(y)]$, one expects this maximum to be achieved at a function μ such that, for every $y \in \partial\Omega$, the set $\{t \in [0, \lambda_\Omega(y)] \,:\, \mu(y - tn_y) = \mu_1\}$ is an interval $[0, \ell(y)]$, with $0 \le \ell(y) \le \lambda_\Omega(y)$. In view of this feeling, we are led to consider the class of functions

$$\mathcal{L} := \left\{ \ell : \partial\Omega \to \mathbb{R}^+ \,:\, \ell \text{ measurable}\,,\ \ell(y) \in [0, \lambda_\Omega(y)] \text{ for all } y \in \partial\Omega, \text{ and } \int_{\partial\Omega} \ell(y)\, dy = \rho\, |\Omega| \right\}.$$

Before computing the right-hand side of (3.6), we prove the following elementary result in the class $\mathcal{L}$.

Lemma 3.4 *For every polygon $\Omega \in \mathcal{C}$, the minimum problem*

$$\min \left\{ \int_{\partial\Omega} [\lambda_\Omega(y) - \ell(y)]^3\, dy \,:\, \ell \in \mathcal{L} \right\} \tag{3.10}$$

is solved by the function $\overline{\ell}(y) := \max\{0, \lambda_\Omega(y) - k\}$, where $k = k(\rho)$ is the quantity implicitly determined by Equation (2.3).

P r o o f. First notice that, by strict convexity, problem (3.10) admits a unique solution $\overline{\ell}$. Such function $\overline{\ell}$ must solve, for some $t \in \mathbb{R}$, the variational inequality

$$\frac{d}{d\varepsilon} F_t\big(\,\overline{\ell} + \varepsilon\big(l - \overline{\ell}\,\big)\big)\bigg|_{\varepsilon=0} \ge 0 \quad \text{for all} \quad l \in \mathcal{L}, \tag{3.11}$$

where

$$F_t(\ell) := \int_{\partial\Omega} [\lambda_\Omega(y) - \ell(y)]^3\, dy + t \left[\int_{\partial\Omega} \ell(y)\, dy - \rho\, |\Omega| \right].$$

By straightforward computations, (3.11) can be rewritten as

$$\int_{\partial\Omega} \big[-3\big(\lambda_\Omega - \overline{\ell}\,\big)^2 + t \big] \big(l - \overline{\ell}\,\big)\, dy \ge 0 \quad \text{for all} \quad l \in \mathcal{L}.$$

It is immediately checked that the above inequality is satisfied by the function $\overline{\ell}$, with k as in (2.3) (and $t = 3k^2$). □

We can now compute the right-hand side of (3.6).

Lemma 3.5 *For every polygon $\Omega \in \mathcal{C}$,*

$$\max_{\mu \in \mathcal{M}(\Omega)} \int_{\partial\Omega} \left\{ \int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu(y - tn_y)}\, dt \right\} dy = \Lambda(\Omega). \tag{3.12}$$

P r o o f. The maximum in the right-hand side of (3.6) is unchanged if it is computed for μ varying in $\mathcal{M}'(\Omega)$, the set $\mathcal{M}'(\Omega)$ being the class of functions $\mu \in \mathcal{M}(\Omega)$ such that, for every $y \in \partial\Omega$, the map $t \mapsto \mu\big(y - tn_y\big)$ is point-wise defined on $[0, \lambda(y)]$. Then, let us write $\mathcal{M}'(\Omega) = \bigcup \{\mathcal{M}_\ell \,:\, \ell \in \mathcal{L}\}$, where

$$\mathcal{M}_\ell := \left\{ \mu \in \mathcal{M}'(\Omega) \,:\, \big|\{t \in [0, \lambda_\Omega(y)] \,:\, \mu\big(y - tn_y\big) = \mu_1\}\big| = \ell(y) \text{ for all } y \in \partial\Omega \right\},$$

and let us first compute the maximum when the class of admissible functions is restricted to $\mathcal{M}_\ell$ for a fixed $\ell \in \mathcal{L}$. It is straightforward that a solution is the function $\overline{\mu}$ defined by

$$\overline{\mu}\big(y - tn_y\big) = \begin{cases} \mu_1 & \text{if} \quad t \in [0, \ell(y)], \\ \mu_2 & \text{if} \quad t \in (\ell(y), \lambda_\Omega(y)], \end{cases}$$

so that

$$\begin{aligned} &\max_{\mu \in \mathcal{M}_\ell(\Omega)} \int_{\partial\Omega} \left\{ \int_0^{\lambda_\Omega(y)} \frac{(\lambda_\Omega(y) - t)^2}{\mu\big(y - tn_y\big)}\, dt \right\} dy \\ &\quad = \frac{1}{3} \int_{\partial\Omega} \left\{ \frac{1}{\mu_1}\, \lambda_\Omega(y)^3 - \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) (\lambda_\Omega(y) - \ell(y))^3 \right\} dy. \end{aligned} \tag{3.13}$$

Now, we maximize the right-hand side above when ℓ varies over the class $\mathcal{L}$. The optimal ℓ is the function $\overline{\ell}$ found in Lemma 3.4. By plugging it into (3.13), we obtain

$$\Lambda(\Omega) = \frac{1}{3}\left\{\frac{1}{\mu_1}\int_{\partial\Omega}\lambda_\Omega(y)^3\,dy - \left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\int_{\partial\Omega}(\min\{\lambda_\Omega(y),k\})^3\,dy\right\}. \tag{3.14}$$

Let us compute the first integral in (3.14). Using Lemmas 2.2 and 2.3 we have

$$\int_{\partial\Omega}\lambda_\Omega(y)^3\,dy = 3\int_\Omega \lambda_\Omega(x)^2\,dx = 6\,\Phi_\Omega(0). \tag{3.15}$$

The second integral in (3.14) can be computed in a similar way:

$$\begin{aligned}\int_{\partial\Omega}(\min\{\lambda_\Omega(y),k\})^3\,dy &= 3\int_{\Omega\setminus\Omega_k} d_{\partial\Omega}(x)^2\,dx\\ &= 3\int_\Omega d_{\partial\Omega}(x)^2\,dx - 3\int_{\Omega_k} d_{\partial\Omega}(x)^2\,dx\\ &= 3\int_\Omega d_{\partial\Omega}(x)^2\,dx - 3\int_{\Omega_k}[k+d_{\partial\Omega_k}(x)]^2\,dx\\ &= 6\,\Phi_\Omega(0) - 3\big[k^2\,|\Omega_k| + 2k\varphi_\Omega(k) + 2\,\Phi_\Omega(k)\big]\\ &= 6\,(\Phi_\Omega(0)-\Phi_\Omega(k)) - 3k^2\,|\Omega_k| - 6k\varphi_\Omega(k).\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) we obtain (3.12). □

We are now in a position to give the

Proof of Theorem 3.2. If $\Omega \in \mathcal{C}$ is a polygon, the required estimate follows immediately from Lemmas 3.3 and 3.5. For a general $\Omega \in \mathcal{C}$, consider a sequence $\{P_\varepsilon\} \subset \mathcal{C}$ of convex polygons such that $R_{P_\varepsilon} = 1$, $P_\varepsilon \supset \Omega$, and $P_\varepsilon \to \Omega$ in Hausdorff distance as $\varepsilon \to 0$. Then, as $\varepsilon \to 0$, $k(P_\varepsilon) \to k(\Omega)$, and ψ_{P_ε}, φ_{P_ε} and Φ_{P_ε} converge uniformly on $[0,1]$ respectively to ψ_Ω, φ_Ω and Φ_Ω; thus, $\Lambda(P_\varepsilon) \to \Lambda(\Omega)$. On the other hand, any pair $(u,\mu) \in H^1_0(\Omega)\times\mathcal{M}(\Omega)$ can be embedded into $H^1_0(P_\varepsilon)\times\mathcal{M}(P_\varepsilon)$ by extending u to 0 over $P_\varepsilon \setminus \Omega$, and by considering any $\mu_\varepsilon \in \mathcal{M}(P_\varepsilon)$ such that $\mu_\varepsilon = \mu$ in Ω. Hence, by letting $\varepsilon \to 0$, we obtain $\mathcal{T}(\Omega) \le \mathcal{T}(P_\varepsilon) \le \Lambda(P_\varepsilon) \to \Lambda(\Omega)$, and the proof is complete. □

4 A lower bound for the torsion by means of web functions

In this section we prove the following lower bound for $\mathcal{T}(\Omega)$, in terms of the constant τ and the function ψ_Ω:

Theorem 4.1 *For all $\Omega \in \mathcal{C}$ we have*

$$\mathcal{T}(\Omega) \ge \mathcal{N}(\Omega) := \frac{1}{\mu_1}\int_0^\tau \psi_\Omega(t)\,dt + \frac{1}{\mu_2}\int_\tau^1 \psi_\Omega(t)\,dt\,,$$

where the constant $\tau = \tau(\rho)$ is defined in (2.4) *and the function ψ_Ω is defined in* (2.2).

Since $\mathcal{K}(\Omega)\times\mathcal{M}_d(\Omega) \subset H^1_0(\Omega)\times\mathcal{M}(\Omega)$, we clearly have $\mathcal{T}(\Omega) \ge -2\min_{\mathcal{K}(\Omega)\times\mathcal{M}_d(\Omega)} J(u,\mu)$. Therefore, Theorem 4.1 follows immediately once we prove

Lemma 4.2 *For all $\Omega \in \mathcal{C}$ we have*

$$-2\min_{\mathcal{K}(\Omega)\times\mathcal{M}_d(\Omega)} J(u,\mu) = \frac{1}{\mu_1}\int_0^\tau \psi_\Omega(t)\,dt + \frac{1}{\mu_2}\int_\tau^1 \psi_\Omega(t)\,dt. \tag{4.1}$$

Proof. We first fix $\mu \in \mathcal{M}_d(\Omega)$ and compute $F(\mu) := \min\{J(u,\mu) \,:\, u \in \mathcal{K}(\Omega)\}$. Since all the functions involved are web functions, by the coarea formula this problem is equivalent to evaluate

$$\min \int_0^1 |\partial\Omega_t|\left(\frac{\mu(t)}{2}\,|y'(t)|^2 - y(t)\right)dt, \tag{4.2}$$

where the unknown function y varies over the space

$$K = \{y \in AC_{\text{loc}}[0,1) \,:\, y(0) = 0,\ t \mapsto |\partial\Omega_t|\,[y'(t)]^2 \in L^1(0,1)\},$$

and with the convention $\mu(t) := \mu(x)$ if $t = d_{\partial\Omega}(x)$. (Here $AC_{\text{loc}}[0,1)$ denotes the set of measurable functions $y\colon [0,1] \to \mathbb{R}$ such that y is absolutely continuous on $[0,r]$ for every $0 < r < 1$.) Integrating by parts the term in $y(t)$ (see [5, Lemma 5.6]), the minimum problem (4.2) can be rewritten as

$$\min_{y\in K} \int_0^1 \left\{\frac{1}{2}\,|\partial\Omega_t|\,\mu(t)y'(t)^2 - |\Omega_t|\,y'(t)\right\} dt.$$

A minimizer satisfies the Euler equation, that is, $|\partial\Omega_t|\,\mu(t)y'(t) = |\Omega_t|$ for a.e. $t \in [0,1]$, and therefore it reads

$$y(t) = \int_0^t \frac{|\Omega_s|}{|\partial\Omega_s|\,\mu(s)}\,ds.$$

This yields

$$F(\mu) = -\frac{1}{2}\int_0^1 \frac{1}{\mu(t)}\frac{|\Omega_t|^2}{|\partial\Omega_t|}\,dt.$$

We now have to minimize F over $\mathcal{M}_d(\Omega)$. This is straightforward. Indeed, by Lemma 2.1 (i) the minimum of F on $\mathcal{M}_d(\Omega)$ is attained by the function

$$\overline{\mu}(t) = \begin{cases} \mu_1 & \text{if} \quad t \in [0,\tau], \\ \mu_2 & \text{if} \quad t \in (\tau,1], \end{cases}$$

where τ is determined by (2.4). Then (4.1) follows. □

5 Which upper bound is better?

In order to compare the two upper bounds for $\mathcal{T}(\Omega)$ obtained in Section 3, we focus our attention on two simple classes of domains. First we show that, within $\widehat{\mathcal{C}}$, the estimates simplify according to the following corollary:

Corollary 5.1 *For all $\Omega \in \widehat{\mathcal{C}}$, we have:*

$$\mathcal{T}(\Omega) \le \frac{|\Omega|}{8}\min\left\{\frac{4}{3}\left[\frac{1}{\mu_2} + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\rho\big(6 - 8\sqrt{\rho} + 3\rho\big)\right],\ \frac{|\Omega|}{\pi}\ \left[\frac{2\rho-\rho^2}{\mu_1} + \frac{(1-\rho)^2}{\mu_2}\right]\right\}.$$

P r o o f. By using (2.1)–(2.3) and (2.5), one can compute:

$$\Lambda(\Omega) = \frac{|\Omega|}{6}\left[\frac{1}{\mu_2} + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\rho\big(6 - 8\sqrt{\rho} + 3\rho\big)\right].$$

Then the result follows by Theorems 3.1 and 3.2. □

Notice that the minimum in the above corollary can be attained alternatively at one of the two quantities in brace. Indeed, it is clear that, as $|\Omega| \to +\infty$, the first term is smaller; on the other hand, the second one is smaller for $|\Omega| \to \pi$ and $\rho \to 0$.

The second simple class we consider is the one of rectangles. For $a \ge 1$, let $Q_a := (-1,1) \times (-a,a)$. The constants $k(\rho)$ and $\tau(\rho)$ are given by

$$k(\rho) = \frac{a+1-\sqrt{(a+1)^2 - 4a(1-\rho)}}{2}\,, \qquad \tau(\rho) = \frac{a+1-\sqrt{(a+1)^2 - 4a\rho}}{2}\,.$$

Moreover,

$$\psi_{Q_a}(t) = \frac{4(a-t)^2(1-t)^2}{1+a-2t}\,.$$

Then, $\mathcal{N}(Q_a)$, $\Lambda(Q_a)$, and $\mathcal{T}(Q_a^*)$ can be explicitly computed by using respectively (4.1), the definition of $\Lambda(\Omega)$ in Theorem 3.2, and the expression of $\mathcal{T}(\Omega^*)$ in Theorem 3.1. Lengthy but straightforward computations show that, for any ρ, μ_1 and μ_2, there holds

$$\lim_{a\to+\infty}\frac{\mathcal{N}(Q_a)}{\Lambda(Q_a)}=1 \quad \text{and} \quad \lim_{a\to+\infty}\frac{\mathcal{T}(Q_a)}{\mathcal{T}(Q_a^*)}\leq\lim_{a\to+\infty}\frac{\Lambda(Q_a)}{\mathcal{T}(Q_a^*)}=0.$$

Therefore, as $a\to+\infty$, the estimate provided by Theorem 3.2 becomes sharp, while the symmetrization estimate provided by Theorem 3.1 becomes useless. In particular, if $a\gg 1$, say $a\geq\overline{a}(\rho,\mu_1,\mu_2)$, the estimate of Theorem 3.2 improves the estimate of Theorem 3.1.

When $\rho=1/2$ and $\mu_1/\mu_2=1/2$, Figure 5 shows the plots of $\mathcal{N}(Q_a)/\Lambda(Q_a)$ for $a\in[1,3]$ and $a\in[1,40]$, whereas Figure 2 shows the plot of $\Lambda(Q_a)/\mathcal{T}(Q_a^*)$ for $a\in[1,6]$; in particular, we have $\overline{a}(1/2,\mu,\mu)\approx 1.55$.

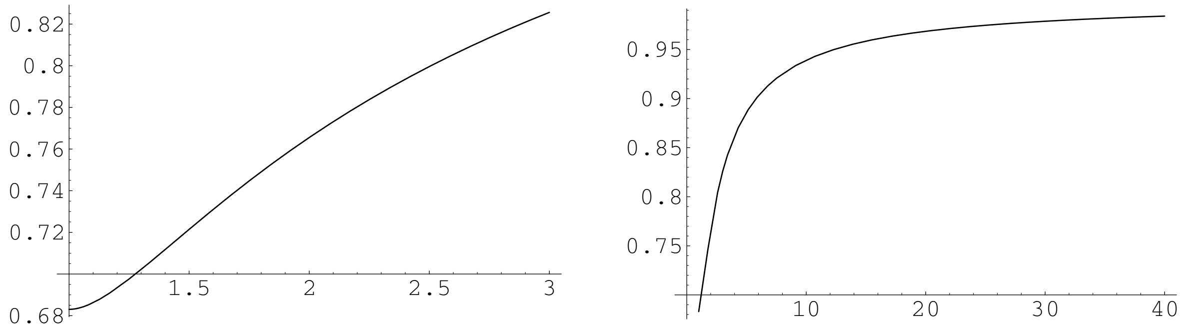

Fig. 1 The plots of $\mathcal{N}(Q_a)/\Lambda(Q_a)$, when $\rho=\mu_1/\mu_2=1/2$, for $a\in[1,3]$ and for $a\in[1,40]$

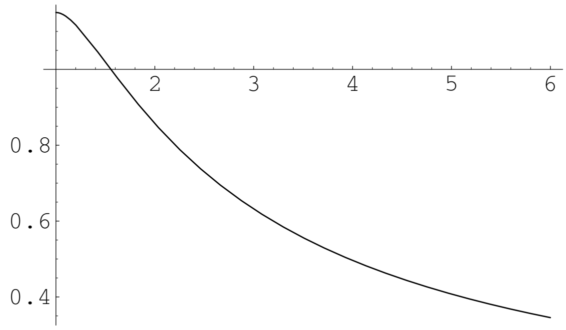

Fig. 2 The plot of $\Lambda(Q_a)/\mathcal{T}(Q_a^*)$, when $\rho=\mu_1/\mu_2=1/2$, for $a\in[1,6]$

6 An estimate of the estimates

In view of the results of Section 5, it is reasonable to wonder how sharp are the estimates given in Theorems 3.2 and 4.1. We are so led to estimate from below the quotient

$$\mathcal{E}(\Omega):=\frac{\mathcal{N}(\Omega)}{\Lambda(\Omega)}.$$

In order to obtain such estimate, a lower bound for $\mathcal{N}(\Omega)$ and an upper bound for $\Lambda(\Omega)$ are needed, preferably written in terms of similar quantities. To combine these bounds in a nice form, we introduce the parameter

$$\theta:=\frac{\mu_1}{\mu_2}\in(0,1]$$

and the function

$$\xi_\Omega(t) := \frac{\Phi_\Omega(t)}{\Phi_\Omega(0)}, \quad t \in [0,1].$$

Then we prove

Theorem 6.1 *For all $\Omega \in \mathcal{C}$, if the constants k and τ are defined as in (2.3) and (2.4), we have:*

$$\mathcal{E}(\Omega) \geq \frac{3}{4}\,(1-(1-\theta)\xi_\Omega(\tau))\left(\theta + (1-\theta)\frac{1+2k+3k^2}{(1-k)^2}\,\xi_\Omega(k)\right)^{-1}, \tag{6.1}$$

with equality for $\Omega \in \widehat{\mathcal{C}}$.

P r o o f. By (4.1) and Lemma 2.1 (ii), we obtain

$$\mathcal{N}(\Omega) \geq \frac{3}{2\mu_1}\int_0^\tau \varphi_\Omega(t)\,dt + \frac{3}{2\mu_2}\int_\tau^1 \varphi_\Omega(t)\,dt,$$

with equality for $\Omega \in \widehat{\mathcal{C}}$. Then, recalling the definition of the function Φ_Ω, we get:

$$\mathcal{N}(\Omega) \geq \frac{3}{2\mu_1}\Phi_\Omega(0) - \frac{3}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\Phi_\Omega(\tau)\,. \tag{6.2}$$

We now estimate the term $k^2\,|\Omega_k| + 2k\varphi_\Omega(k)$ appearing in the definition of $\Lambda(\Omega)$. By using Lemma 2.1 (iii) we obtain

$$k^2\,|\Omega_k| + 2k\varphi_\Omega(k) \leq \frac{k(2+k)}{1-k}\,\varphi_\Omega(k) \leq \frac{4k(2+k)}{(1-k)^2}\,\Phi_\Omega(k)\,,$$

with equality for $\Omega \in \widehat{\mathcal{C}}$. Then,

$$\Lambda(\Omega) \leq \frac{2}{\mu_2}\,\Phi_\Omega(0) + 2\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\left[1 + \frac{2k(2+k)}{(1-k)^2}\right]\Phi_\Omega(k) \quad \text{for all} \quad \Omega \in \mathcal{C}. \tag{6.3}$$

The lower bound (6.1) follows now combining (6.2) and (6.3). □

The statement of Theorem 6.1 deserves several comments.

We first point out that within the subclass $\widehat{\mathcal{C}}$ not only (6.1) becomes an equality, but it may be further simplified. Indeed, thanks to (2.5), we get

$$\mathcal{E}(\Omega) = E(\rho,\theta) := \frac{3}{4}\,\frac{1-(1-\theta)(1-\rho)^2}{\theta + (1-\theta)\rho\left(6 - 8\sqrt{\rho} + 3\rho\right)} \quad \text{for all} \quad \Omega \in \widehat{\mathcal{C}}. \tag{6.4}$$

This inequality should be compared with [7, Theorem 1], where we dealt with the optimal shape problem

$$\inf\left\{\frac{\mathcal{N}(\Omega)}{\mathcal{T}(\Omega)}\ :\ \Omega \in \mathcal{C}\right\}$$

in the limit situation $\theta = 1$ (case of one material). In that case we found that the value of the infimum is $3/4$ and that it is not attained. In this respect, notice that $\frac{\partial E}{\partial \theta} \geq 0$, so that $E(\rho,\theta) \leq E(\rho,1) = 3/4$ for all ρ, θ.

Finally, let us analyze the estimate (6.1) in some "extremal cases".

(i) Case of one material. If $\theta = 1$, that is when only one material is present, Theorem 6.1 reduces to the above mentioned uniform estimate proved in [7] for the torsion.

(ii) Case of very different rigidities. If $\theta \to 0^+$, that is, when the rigidities of the two materials are very different from each other, (6.1) behaves in opposite ways in two limit cases $\rho \to 1^-$ and $\rho \to 0^+$.

(ii.a) Case of very different rigidities with prevalence of the strong material. If $\rho \to 1$, we have $k \to 0$ and $\tau \to 1$, so that $\xi_\Omega(k) \to 1$ and $\xi_\Omega(\tau) \to 0$; therefore, the r.h.s. of (6.1) tends to $3/4$.

(ii.b) Case of very different rigidities with prevalence of the soft material. If $\rho \to 0$, we have $k \to 1$ and $\tau \to 0$, so that $\xi_\Omega(k) \to 0$ and $\xi_\Omega(\tau) \to 1$. Moreover, using De L'Hospital rule, one has $\lim_{k\to 1} \frac{\xi_\Omega(k)}{(k-1)^2} > 0$. Therefore, the r.h.s. of (6.1) tends to 0, and the estimates $\mathcal{N}(\Omega) \leq \mathcal{T}(\Omega) \leq \Lambda(\Omega)$ are not meaningful.

Acknowledgements The authors are grateful to Andrea Cianchi for raising their attention to the paper [1] and for fruitful discussions.

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