Stability analysis in some strongly prestressed rectangular plates

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Abstract

We consider an evolution plate equation aiming to model the motion of the deck of a periodically forced strongly prestressed suspension bridge. Using the prestress assumption, we show the appearance of multiple time-periodic uni-modal longitudinal solutions and we discuss their stability. Then, we investigate how these solutions exchange energy with a torsional mode. Although the problem is forced, we find a stability portrait similar to the one of the Mathieu equation. The techniques used rely on ODE analysis of stability and are complemented with numerical simulations.

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1 Introduction

A long narrow rectangular thin plate hinged at two opposite edges and free on the remaining two edges was used in [10] to model the deck of a suspension bridge which, at the short edges, is supported by the ground. If L denotes its length and 2ℓ denotes its width, a realistic assumption is that $2\ell \cong \frac{L}{75}$. After scaling, we may take $L = \pi$ so that, in the sequel, the plate we focus on will be

$$\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2 \ .$$

A suspension bridge is subject to external actions such as the vortex shedding induced by the wind, the traffic load, the synchronized step of pedestrians. These somehow periodic loads generate oscillations of the deck that can be of different kinds. In turn, the oscillations are characterized by means of the eigenfunctions of a suitable eigenvalue problem, see Section 2, where we classify the oscillations in *longitudinal* and *torsional*. The latter, that consist in a rotation of the deck around its main axis, are the most dangerous ones, see [23]: they were seen in several suspension bridges, see [16, §1.3,1.4] for a survey, and may lead to collapses as for the Tacoma Narrows Bridge [2, 22, 23]. One of the main issues in suspension bridges is thus their *stability*, a general word that may have several meanings. Here we are interested in both the asymptotic stability of periodic motions and the possibility that a longitudinal oscillation suddenly transforms into a torsional oscillation. In some recent papers [4, 6, 9], this topic was tackled for the plate equation introduced in [10]. These papers only consider the case of weakly prestressed plates, for which the equilibrium position is horizontal. But plates may also be strongly prestressed and the equilibrium position is no longer horizontal.

The purpose of the present paper is to analyze the stability of these positions in strongly prestressed plates under different points of view. In Section 2 we give some details on the physical model used to derive the PDE that reads

$$\begin{cases} u_{tt}+\delta u_t+\Delta^2 u+\left[P-S\int_\Omega u_x^2\right]u_{xx}=f & \text{in } \Omega\times(0,T)\\ u=u_{xx}=0 & \text{on } \{0,\pi\}\times[-\ell,\ell]\\ u_{yy}+\sigma u_{xx}=u_{yyy}+(2-\sigma)u_{xxy}=0 & \text{on } [0,\pi]\times\{-\ell,\ell\}\\ u(x,y,0)=u_0(x,y), \qquad u_t(x,y,0)=v_0(x,y) & \text{in } \Omega\,, \end{cases} \tag{1}$$

where $\delta>0$ is a damping parameter, $\sigma>0$ is the Poisson ratio, $P>0$ is the prestressing constant, and $S\int_\Omega u_x^2$ measures the geometric nonlinearity of the plate due to its longitudinal stretching. In Section 3 we prove the existence of periodic (in time) solutions of (1), providing some bounds for the ones concentrated on the first longitudinal mode, which we also determine explicitly in some particular cases. In Section 4, we first analyze the stability properties of these periodic solutions and then we study the torsional stability of bi-modal solutions of (1) having one longitudinal and one torsional component; for the latter, we use a notion of stability introduced in [13, 14], see Definition 2. Throughout the paper we mainly use ODE techniques, taking advantage of the results in [8, 17, 18, 20], and numerical simulations.

2 Strongly prestressed plates

In this section we explain the physical meaning of the parameters in (1). For the full dimensional form of (1) we refer to [6], here we mainly focus our attention on the parameter P.

The boundary conditions in (1) describe a rectangular plate which is hinged on the short edges and free on the long edges, as the deck of a suspension bridge. The constant σ is the Poisson ratio which is the negative ratio of transverse to axial strain: when a material is compressed in one direction, it tends to expand in the other two directions. The parameter σ is a measure of this effect, representing the fraction of expansion divided by the fraction of compression for small values of these changes. For metals, one has $\sigma\approx 0.3$, while for concrete σ is approximately between 0.1 and 0.2; since the deck of a bridge is a mixture of metal and concrete, we take

$$\sigma=0.2\,. \tag{2}$$

The length of the deck of a bridge (in the x-direction) is about 1km while the width is about 13m (4 lanes of 3m each plus 1m of separation). In order to maintain this proportion, we take

$$\ell=\frac{\pi}{150}\,. \tag{3}$$

For a partially hinged plate such as Ω, the buckling load only acts in the x-direction and therefore one obtains the term $\int_\Omega u_x^2$ as for a one-dimensional beam; see [19]. The nonlinear nonlocal term $(\int u_x^2)u_{xx}$ in (1) models the fact that the stiffness of the plate increases with the stretching energy: if the plate is stretched somewhere in the x-direction, then in all the other points of the plate this increases the resistance to further stretching. Prestressed models were introduced for beams by Woinowsky-Krieger [25] and, independently, by Burgreen [7]. A few years later they were extended to plates by Berger [5]. The parameter P is the prestressing constant: one has $P>0$ if the plate is compressed and $P<0$ if the plate is stretched in the x-direction.

In order to explain what we mean by *strongly prestressed*, we need to define the functional framework and to introduce the vibrating modes of the plate Ω. The phase (energy) space for (1) is

$$H^2_*(\Omega)=\{v\in H^2(\Omega);\ v=0 \text{ on } \{0,\pi\}\times[-\ell,\ell]\}\,,$$

with inner product defined by

$$(v,w)_{H^2_*} = \int_\Omega \Big(\Delta v\Delta w - (1-\sigma)\big(v_{xx}w_{yy}+v_{yy}w_{xx}-2v_{xy}w_{xy}\big)\Big) \qquad \forall v,w\in H^2_*(\Omega)\,.$$

This inner product defines a norm in $H^2_*(\Omega)$; see [10, Lemma 4.1]. We also denote by $(H^2_*(\Omega))'$ the dual space of $H^2_*(\Omega)$, with duality $\langle\cdot,\cdot\rangle$.

The vibrating modes of the plate Ω are the eigenfunctions of the problem

$$\begin{gathered}\Delta^2 u = \lambda u \qquad \text{in} \quad \Omega,\\ u = u_{xx} = 0 \text{ on } \{0,\pi\}\times[-\ell,\ell], \quad u_{yy}+\sigma u_{xx} = u_{yyy}+(2-\sigma)u_{xxy} = 0 \text{ on } [0,\pi]\times\{-\ell,\ell\}.\end{gathered} \tag{4}$$

The eigenvalues and eigenfunctions of (4) were fully determined in [10] and their properties were analyzed in several subsequent works [4, 6, 9]. We refer to these papers for all the details, here we only recall the following facts which will be useful for our purposes.

Proposition 1. *The set of eigenvalues of* (4) *may be ordered in an increasing sequence of strictly positive numbers diverging to* ∞ *and any eigenfunction belongs to* $C^\infty(\overline{\Omega})$*. The set of eigenfunctions of* (4) *is a complete system in* $H^2_*(\Omega)$*. Moreover:*
(i) for any $m\ge 1$*, there exists a unique eigenvalue* $\lambda=\mu_{m,1}\in((1-\sigma^2)m^4,m^4)$ *with corresponding eigenfunction*

$$\left[\big[\mu_{m,1}^{1/2}-(1-\sigma)m^2\big]\frac{\cosh\Big(y\sqrt{m^2+\mu_{m,1}^{1/2}}\Big)}{\cosh\Big(\ell\sqrt{m^2+\mu_{m,1}^{1/2}}\Big)}+\big[\mu_{m,1}^{1/2}+(1-\sigma)m^2\big]\frac{\cosh\Big(y\sqrt{m^2-\mu_{m,1}^{1/2}}\Big)}{\cosh\Big(\ell\sqrt{m^2-\mu_{m,1}^{1/2}}\Big)}\right]\sin(mx)\,;$$

(ii) for any $m\ge1$ *and any* $k\ge2$ *there exists a unique eigenvalue* $\lambda=\mu_{m,k}>m^4$ *satisfying* $\Big(m^2+\frac{\pi^2}{\ell^2}\left(k-\frac{3}{2}\right)^2\Big)^2<\mu_{m,k}<\Big(m^2+\frac{\pi^2}{\ell^2}\left(k-1\right)^2\Big)^2$ *and with corresponding eigenfunction*

$$\left[\big[\mu_{m,k}^{1/2}-(1-\sigma)m^2\big]\frac{\cosh\Big(y\sqrt{\mu_{m,k}^{1/2}+m^2}\Big)}{\cosh\Big(\ell\sqrt{\mu_{m,k}^{1/2}+m^2}\Big)}+\big[\mu_{m,k}^{1/2}+(1-\sigma)m^2\big]\frac{\cos\Big(y\sqrt{\mu_{m,k}^{1/2}-m^2}\Big)}{\cos\Big(\ell\sqrt{\mu_{m,k}^{1/2}-m^2}\Big)}\right]\sin(mx)\,;$$

(iii) for any $m\ge1$ *and any* $k\ge2$ *there exists a unique eigenvalue* $\lambda=\nu_{m,k}>m^4$ *with corresponding eigenfunctions*

$$\left[\big[\nu_{m,k}^{1/2}-(1-\sigma)m^2\big]\frac{\sinh\Big(y\sqrt{\nu_{m,k}^{1/2}+m^2}\Big)}{\sinh\Big(\ell\sqrt{\nu_{m,k}^{1/2}+m^2}\Big)}+\big[\nu_{m,k}^{1/2}+(1-\sigma)m^2\big]\frac{\sin\Big(y\sqrt{\nu_{m,k}^{1/2}-m^2}\Big)}{\sin\Big(\ell\sqrt{\nu_{m,k}^{1/2}-m^2}\Big)}\right]\sin(mx)\,;$$

(iv) for any $m\ge1$ *satisfying* $\tanh(\sqrt{2}m\ell)<\Big(\frac{\sigma}{2-\sigma}\Big)^2\sqrt{2}m\ell$ *there exists a unique eigenvalue* $\lambda=\nu_{m,1}\in(\mu_{m,1},m^4)$ *with corresponding eigenfunction*

$$\left[\big[\nu_{m,1}^{1/2}-(1-\sigma)m^2\big]\frac{\sinh\Big(y\sqrt{m^2+\nu_{m,1}^{1/2}}\Big)}{\sinh\Big(\ell\sqrt{m^2+\nu_{m,1}^{1/2}}\Big)}+\big[\nu_{m,1}^{1/2}+(1-\sigma)m^2\big]\frac{\sinh\Big(y\sqrt{m^2-\nu_{m,1}^{1/2}}\Big)}{\sinh\Big(\ell\sqrt{m^2-\nu_{m,1}^{1/2}}\Big)}\right]\sin(mx)\,.$$

In fact, if the unique positive solution $s>0$ of the equation $\tanh(\sqrt{2}s\ell)=\Big(\frac{\sigma}{2-\sigma}\Big)^2\sqrt{2}s\ell$ is not an integer, then the only eigenvalues and eigenfunctions are the ones given in Proposition 1. Clearly, this condition has probability 0 to occur in general plates: if it occurs, there is an additional eigenvalue and eigenfunction, see [10]. When (2) and (3) hold (as in our case), this condition is not satisfied and no eigenvalues other than $(i)-(ii)-(iii)-(iv)$ exist.

Proposition 1 classifies the eigenfunctions in even and odd with respect to y: the former (cases $(i)-(ii)$) are called *longitudinal*, while the latter (cases $(iii)-(iv)$) are called *torsional*. Hence, as noticed in [6], longitudinal and torsional displacements have a simple characterization after introducing the subspaces of even and odd functions with respect to y:

$$H^2_{\mathcal{E}}(\Omega) := \{u \in H^2_*(\Omega) :\ u(x,-y) = u(x,y)\ \forall (x,y) \in \Omega\},$$
$$H^2_{\mathcal{O}}(\Omega) := \{u \in H^2_*(\Omega) :\ u(x,-y) = -u(x,y)\ \forall (x,y) \in \Omega\}.$$

The space $H^2_{\mathcal{E}}(\Omega)$ is spanned by the longitudinal eigenfunctions (classes (i) and (ii) in Proposition 1) whereas the space $H^2_{\mathcal{O}}(\Omega)$ is spanned by the torsional eigenfunctions (classes (iii) and (iv)). We have

$$H^2_{\mathcal{E}}(\Omega) \perp H^2_{\mathcal{O}}(\Omega)\,, \qquad H^2_*(\Omega) = H^2_{\mathcal{E}}(\Omega) \oplus H^2_{\mathcal{O}}(\Omega), \tag{5}$$

and these spaces allow to split the solutions of (1) in their longitudinal and torsional components.

If we assume (2) and (3), then the eigenvalues of (4) can be computed numerically, see [4, Table 1]. One finds that the least two eigenvalues are $\mu_{1,1}$ and $\mu_{2,1}$, which are both longitudinal and

$$\mu_{1,1} \approx 0.96\,, \qquad \mu_{2,1} \approx 15.37\,. \tag{6}$$

Moreover, the least 10 eigenvalues are all longitudinal, while the least torsional eigenvalues are the 11th and the 16th and are

$$\nu_{1,2} \approx 10943.25\,, \qquad \nu_{2,2} \approx 43785.56\,. \tag{7}$$

A weakly prestressed plate is obtained for $P \le \mu_{1,1}$, in which case the equilibrium position of the plate is horizontal. In strongly prestressed plates one has $P > \mu_{1,1}$ and the equilibrium position is no longer horizontal; see, e.g., Section 12.1 in [3].Although prestressing reinforces a plate, one should not abuse of it since complicated unstable equilibria may appear, see [3, Proposition 29]. For these reasons, throughout this paper we assume that

$$\mu_{1,1} < P < \mu_{2,1}\,, \tag{8}$$

where P is the prestressing constant and $\mu_{m,1}$ are the longitudinal eigenvalues, see Proposition 1. This situation is called "softening regime" in [17], to be compared with the "hardening regime" in which $P < \mu_{1,1}$. As mentioned therein, the case (8) is by far less explored.

3 Existence of periodic solutions for strongly prestressed plates

In order to state our results, we adopt the following conventions:

(C1) All the eigenfunctions in Proposition 1 are multiplied by a normalization constant so as to have $L^2(\Omega)$-norm equal to 1.

(C2) The L^2 normalized eigenfunctions will be denoted by $L_{m,k}$ (longitudinal, associated to $\mu_{m,k}$) and $T_{m,k}$ (torsional, associated to $\nu_{m,k}$).

(C3) The solutions of (1) are always intended in $C^0(\mathbb{R}_+, (H^2_*(\Omega))')$.

We then sort the eigenvalues provided by Proposition 1 into a unique sequence $\{\lambda_j\}_j$, accordingly ordering the corresponding eigenfunctions into the sequence $\{e_j\}_j$. Finally, to simplify the notations, throughout the paper we set

$$\xi := (x,y) \in \Omega\,.$$

3.1 One-mode solutions

We denote by $X_T \subset C^0(\mathbb{R}_+, L^2(\Omega))$ the subspace of functions $f = f(\xi, t)$ which are T-periodic in the t-variable and we define

$$Y_T = X_T \cap C^0(\mathbb{R}_+, H^2_*(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)) \cap C^2(\mathbb{R}_+, (H^2_*(\Omega))').$$

Then we state our first result on existence of periodic solutions.

Theorem 1. *Fix $T > 0$. Then, for every forcing term $f \in X_T$, there exists at least a T-periodic solution $u \in Y_T$ of (1). Moreover, if for a suitable set of indexes J it holds $\int_\Omega f(\xi,t)e_j(\xi)\,d\xi \equiv 0$ for every $j \in J$, then there exists a T-periodic solution u of (1) with $\int_\Omega u(\xi,t)e_j(\xi)\,d\xi \equiv 0$ for every $j \in J$.*

Proof. We proceed by following the steps of the proof of [6, Theorem 6], with some changes. We preliminarily notice that (1) is equivalent to the infinite-dimensional dynamical system

$$\ddot{h}_k(t) + \delta \dot{h}_k(t) + \lambda_k h_k(t) + m_k^2\Big[-P + S\sum_{j=1}^{\infty} m_j^2 h_j(t)^2\Big]h_k(t) = f_k(t) \qquad \text{for } k = 1, \dots, \tag{9}$$

where $u(\xi,t) = \sum_k h_k(t)e_k(\xi)$ and $f_k(t) = \int_\Omega f(\xi,t)e_k(\xi)\,d\xi$. We aim first to show the existence of a periodic solution for the approximation of (9) obtained by considering only the first n components of the forcing term; namely, we deal with system (9) after having set $f_k = 0$ for $k \geq n+1$. This is equivalent to look for a periodic solution u^n of (1), where f has been replaced by

$$f^n(\xi,t) = \sum_{k=1}^{n} f_k(t)e_k(\xi)\,,$$

and can be done as in [6]. We thus find a T-periodic solution of (9) in the form

$$u^n(\xi,t) := \sum_{k=1}^{n} h_k(t)e_k(\xi). \tag{10}$$

In order to prove an a priori bound for u^n, we introduce the energy

$$E(t) = \frac{1}{2}\|u^n_t(t)\|^2_{L^2} + \frac{1}{2}\|u^n(t)\|^2_{H^2_*} - \frac{P}{2}\|u^n_x(t)\|^2_{L^2} + \frac{S}{4}\|u^n_x(t)\|^4_{L^2} + \frac{\delta}{2}\int_\Omega u^n(\xi,t)u^n_t(\xi_t)\,d\xi \tag{11}$$

and we observe that, using (1), one has

$$\begin{aligned}\dot{E}(t) + \frac{\delta}{2}E(t) \;&=\; -\frac{\delta}{4}\|u^n_t(t)\|^2_{L^2} - \frac{\delta}{4}\|u^n(t)\|^2_{H^2_*} + \frac{\delta P}{4}\|u^n_x(t)\|^2_{L^2} - \frac{3S\delta}{8}\|u^n_x(t)\|^4_{L^2}\\ &\quad -\frac{\delta^2}{4}\int_\Omega u^n(\xi,t)u^n_t(\xi,t)\,d\xi + \int_\Omega f^n(\xi,t)\left(u^n_t(\xi,t) + \frac{\delta}{2}u^n(\xi,t)\right)d\xi\end{aligned}$$

(see [6, Lemma 18] for further details). The embedding inequalities $\|v\|^2_{L^2} \leq \|v_x\|^2_{L^2}$ and $\lambda_1\|v_x\|^2_{L^2} \leq \|v\|^2_{H^2_*}$, together with the Young inequality, imply that

$$\dot{E}(t) + \frac{\delta}{2}E(t) \leq \frac{\delta}{2}\left(P - \lambda_1 + \frac{\delta^2}{4}\right)\|u^n_x(t)\|^2_{L^2} - \frac{3S\delta}{8}\|u^n_x(t)\|^4_{L^2} + \frac{1}{\delta}\|f^n(t)\|^2_{L^2} \leq \frac{\delta C}{2} + \frac{1}{\delta}\|f^n(t)\|^2_{L^2},$$

where $C = (P - \lambda_1 + \delta^2/4)^2/12S$. For every $t > 0$, it then follows that

$$E(t) \leq \mathrm{e}^{-\delta t/2}E(0) + \left(C + \frac{2f^2_\infty}{\delta^2}\right)(1 - \mathrm{e}^{-\delta t/2}),$$

where $f_\infty = \max_{0\le t\le T}\|f(t)\|^2_{L^2}$ (so that the upper bound on E is independent of n). Therefore, by letting $t\to\infty$, we deduce

$$\limsup_{t\to\infty} E(t) \le C + \frac{2f_\infty^2}{\delta^2}.$$

By the T-periodicity of f and u^n, this is also a global bound:

$$E(t) \le C + \frac{2f_\infty^2}{\delta^2} \qquad \forall t\in[0,T].$$

Using similar arguments as in [6, Lemmas 19 and 20] and the expression in (11), we deduce that $\|u^n(t)\|_{H^2_*}$ and $\|u^n_t(t)\|_{L^2}$ are bounded in $[0,T]$, independently of n. The equation

$$\langle u^n_{tt}, v\rangle + \delta(u^n_t, v)_{L^2} + (u^n, v)_{H^2_*} + \big[S\|u^n_x\|^2_{L^2} - P\big](u^n_x, v_x)_{L^2} = (f^n, v)_{L^2}\,, \tag{12}$$

for all $t\in[0,T]$ and all $v\in H^2_*(\Omega)$, then yields a uniform $(H^2_*)'$-bound on u^n_{tt}. Up to a subsequence, we can therefore pass to the limit in (12):

$$\begin{aligned}
&u^n \to u \text{ weakly * in } L^\infty([0,T], H^2_*(\Omega)),\\
&u^n_t \to u_t \text{ weakly * in } L^\infty([0,T], L^2(\Omega)),\\
&u^n_{tt} \to u_{tt} \text{ weakly * in } L^\infty([0,T], (H^2_*(\Omega))').
\end{aligned}$$

Hence, there exists a T-periodic solution u of equation (1) in the sense of $L^\infty([0,T],(H^2_*(\Omega))')$. To conclude the proof of the existence part of the statement, observe that the continuity properties of u follow from [24, Lemma 4.1] and from the fact that (1) is satisfied in $C^0(\mathbb{R}_+,(H^2(\Omega))')$.

The second part of the statement follows by noticing that, in view of the assumption on f, projecting (1) onto the eigenspace generated by $\{e_j\}_{j\in J}$ yields the ODE system

$$\ddot U_j(t) + \delta\dot U_j(t) + (\lambda_j - P)U_j(t) + S\Big(\sum_k k^2 U_k^2(t)\Big)U_j(t) = 0, \qquad j\in J, \tag{13}$$

where $u(\xi,t) = \sum_{k=1}^\infty U_k(t)e_k(\xi)$. System (13) has the solution $U_j\equiv 0$, $j\in J$; the other components U_k of the solution $u(\xi,t)$, $k\notin J$, can now be recovered by solving the infinite-dimensional system of ODEs satisfied by their Fourier coefficients, which has a T-periodic solution (this can be shown similarly as in the previous step of the proof). □

We now consider the case where the deck at rest is excited by a periodic external force acting only on the first longitudinal mode $L_{1,1} = e_1$, that is,

$$f(\xi,t) = A(t)L_{1,1}(\xi) \qquad \text{with} \quad A\in C(\mathbb{R}/T\mathbb{Z},\mathbb{R}) \tag{14}$$

for some minimal period $T>0$. From Theorem 1 we immediately infer:

Corollary 1. *Fix $T>0$ and let $f\in X_T$ be as in* (14)*. Then there exists at least a T-periodic solution $u\in Y_T$ of* (1) *satisfying $\int_\Omega u(\xi,t)e_j(\xi)\,d\xi\equiv 0$ for every $j\ge 2$.*

In fact, more can be said. Since $P>\mu_{1,1}$, assuming (14) we may have multiplicity of periodic solutions, as the next statement shows. By "subharmonic solution proportional to $L_{1,1}$" we mean a solution u of (1) in the form $u(\xi,t) = z(t)L_{1,1}(\xi)$ where, for some integer $k>1$, the time coefficient z is kT-periodic, kT being its minimal period.

Theorem 2. *Assume (8). Let f be as in (14), let $L_{1,1}^{\infty} = \|L_{1,1}\|_{\infty}$ and assume that*

$$\delta \geq \sqrt{8(P-\mu_{1,1})}\,, \qquad \|f\|_{L^\infty} < \frac{\delta\, L_{1,1}^{\infty}}{32}\sqrt{\frac{(P-\mu_{1,1})^3}{S(\delta^2+P-\mu_{1,1})}}\,. \tag{15}$$

Then (1) admits exactly three T-periodic solutions $\wp^1$, $\wp^2$, $\wp^3$ having the form

$$\wp^j(\xi,t) = z_j(t)L_{1,1}(\xi)\,, \quad j=1,2,3, \tag{16}$$

and has no subharmonic solution proportional to $L_{1,1}$.

Moreover, one of the three periodic solutions found (which we name $\wp^1$) satisfies the bound

$$\max_{t\in[0,T]} |z_1(t)| < \sqrt{\frac{P-\mu_{1,1}}{3S}}, \tag{17}$$

while the other two fulfill the estimates

$$\max_{t\in[0,T]} \left| z_2(t) - \sqrt{\tfrac{P-\mu_{1,1}}{S}} \right| < \left[\sqrt{\tfrac{5}{3}}-1\right]\sqrt{\tfrac{P-\mu_{1,1}}{S}}, \quad \max_{t\in[0,T]} \left| z_3(t) + \sqrt{\tfrac{P-\mu_{1,1}}{S}} \right| < \left[\sqrt{\tfrac{5}{3}}-1\right]\sqrt{\tfrac{P-\mu_{1,1}}{S}}. \tag{18}$$

In particular, z_2 and z_3 have constant sign.

Proof. Periodic-in-time solutions (with period T) of (1) proportional to $L_{1,1}$ have the form $\wp(\xi,t) = z(t)L_{1,1}(\xi)$ for some $z \in C^2(\mathbb{R}/T\mathbb{Z},\mathbb{R})$. By replacing such $\wp$ into (1), we find that z solves the ODE

$$\ddot{z}(t) + \delta\dot{z}(t) + (\mu_{1,1}-P)z(t) + Sz(t)^3 = A(t)\,, \tag{19}$$

where $A \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R})$ is as in (14). We perform the change of variables

$$z(t) = \sqrt{\tfrac{P-\mu_{1,1}}{S}}\;\psi\Big(\sqrt{P-\mu_{1,1}}\,t\Big) \iff \psi(t) = \sqrt{\tfrac{S}{P-\mu_{1,1}}}\;z\Big(\tfrac{t}{\sqrt{P-\mu_{1,1}}}\Big) \tag{20}$$

so that ψ satisfies

$$\ddot{\psi}(t) + \varepsilon\dot{\psi}(t) - \psi(t) + \psi(t)^3 = g(t)\,, \tag{21}$$

where

$$\varepsilon = \frac{\delta}{\sqrt{P-\mu_{1,1}}}\,, \qquad g(t) = \sqrt{\frac{S}{(P-\mu_{1,1})^3}}\;A\left(\frac{t}{\sqrt{P-\mu_{1,1}}}\right)\,. \tag{22}$$

In this setting, the smallness assumptions (15) become

$$\varepsilon \geq 2\sqrt{2}\,, \qquad \|g\|_{L^\infty} < \frac{\varepsilon}{32\sqrt{1+\varepsilon^2}}\,. \tag{23}$$

Equation (21) has the form of (1) in [18] so that, in view of (23), we may apply the statements therein. By combining Corollary 1.3 and Corollary 1.4 in [18] we infer that:

– equation (21) admits exactly three periodic solutions ψ_1, ψ_2, ψ_3 having the same period as g;

– the same equation has no subharmonic solutions.

After undoing the change of variables (20) and after recalling (16), these two facts prove the first statement of the theorem.

Next, by combining Theorems 1.1 and 1.2 in [18], we infer that

$$\|\psi_1\|_{L^\infty} < \frac{1}{\sqrt{3}}, \quad \|\psi_2 - 1\|_{L^\infty} < \sqrt{\frac{5}{3}} - 1, \quad \|\psi_3 + 1\|_{L^\infty} < \sqrt{\frac{5}{3}} - 1.$$

Undoing the change of variables (20) then yields the bounds in the statement. □

We point out that the lower bound $(15)_1$ may be dropped, provided less explicit bounds on f hold, see [17].

3.2 Two explicit examples of periodic solutions

In this section, we provide two examples where one periodic solution of (1) can be written explicitly. To this end, we intensively exploit the Jacobi elliptic functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ and their properties [1].

We start with a result in case (15) is satisfied.

Proposition 2. *Assume that* $\delta \geq \sqrt{8(P-\mu_{1,1})}$ *and that*

$$0 < k^2 < \frac{2\sqrt{P-\mu_{1,1}}}{\sqrt{P-\mu_{1,1}} + 16\sqrt{2(\delta^2+P-\mu_{1,1})}}\,. \tag{24}$$

Also take

$$f(\xi,t) = -\frac{P-\mu_{1,1}}{\sqrt{S}}\,\frac{\delta\sqrt{2}\,k^2}{2-k^2}\,\mathrm{sn}\left(\sqrt{\frac{P-\mu_{1,1}}{2-k^2}}\,t,k\right)\,\mathrm{cn}\left(\sqrt{\frac{P-\mu_{1,1}}{2-k^2}}\,t,k\right)\,L_{1,1}(\xi)\,.$$

Then the assumptions of Theorem 2 are fulfilled and the function

$$\wp^2(\xi,t) = \sqrt{\frac{P-\mu_{1,1}}{S}}\,\sqrt{\frac{2}{2-k^2}}\,\mathrm{dn}\left(\sqrt{\frac{P-\mu_{1,1}}{2-k^2}}\,t,k\right)\,L_{1,1}(\xi)$$

is a periodic solution of (1) *satisfying the bound* (18).

Proof. By using the properties of the Jacobi functions [1], one can check that for all $k\in(0,1)$ and all $\varepsilon>0$ the function

$$\psi(t) = \sqrt{\frac{2}{2-k^2}}\,\mathrm{dn}\left(\frac{t}{\sqrt{2-k^2}},k\right)$$

solves equation (21) with

$$g(t) = -\frac{\varepsilon\sqrt{2}\,k^2}{2-k^2}\,\mathrm{sn}\left(\frac{t}{\sqrt{2-k^2}},k\right)\,\mathrm{cn}\left(\frac{t}{\sqrt{2-k^2}},k\right).$$

We thus have to show that the assumptions of Theorem 2 are fulfilled. To this end, we notice that the first bound in (23) follows from the lower bound on δ. On the other hand, since $\max_s|\mathrm{sn}(s,k)\,\mathrm{cn}(s,k)| = \frac{1}{2}$ for any $k\in(0,1)$ (see again [1]), we have

$$\|g\|_{L^\infty} = \frac{\varepsilon\,k^2}{(2-k^2)\sqrt{2}}\,.$$

Thus, g fulfills the second bound in (23) provided that

$$0 < k^2 < \frac{2}{1+16\sqrt{2(1+\varepsilon^2)}}\,,$$

which is equivalent to (24). Theorem 2 thus applies and the statement is proved. □

Notice that from [1] we also know that $\sqrt{1-k^2} \leq \mathrm{dn}(s,k) \leq 1$ for all $s\geq 0$ so that, from the constraints on k and ε,

$$\left.\begin{array}{l} \psi(t) \geq \sqrt{\frac{2(1-k^2)}{2-k^2}} > \sqrt{1-\frac{1}{16\sqrt{2(1+\varepsilon^2)}}} \geq \sqrt{1-\frac{1}{48\sqrt{2}}} \approx 0.993 \\ \psi(t) \leq \sqrt{\frac{2}{2-k^2}} < \sqrt{1+\frac{1}{16\sqrt{2(1+\varepsilon^2)}}} \leq \sqrt{1+\frac{1}{48\sqrt{2}}} \approx 1.007 \end{array}\right. \qquad \forall t>0\,,$$

improving here the bounds on the solution ψ (since $\sqrt{5/3}-1\approx 0.29$).

We now turn to a situation where (15) is not satisfied.

Proposition 3. *Assume that* $k \in (1/\sqrt{2}, 1)$ *and take*

$$f(\xi,t) = -\frac{P-\mu_{1,1}}{\sqrt{S}}\,\frac{\delta\sqrt{2}\,k}{2k^2-1}\ \mathrm{sn}\left(\sqrt{\frac{P-\mu_{1,1}}{2k^2-1}}\,t, k\right)\ \mathrm{dn}\left(\sqrt{\frac{P-\mu_{1,1}}{2k^2-1}}\,t, k\right)\ L_{1,1}(\xi)$$

(which does not satisfy (15)*). Then the function*

$$\wp(\xi,t) = \sqrt{\frac{P-\mu_{1,1}}{S}}\ \sqrt{\frac{2k^2}{2k^2-1}}\ \mathrm{cn}\left(\sqrt{\frac{P-\mu_{1,1}}{2k^2-1}}\,t, k\right)\ L_{1,1}(\xi)$$

is a periodic solution of (1)*.*

Proof. By using the properties of the Jacobi functions [1], one can check that for all $k \in (1/\sqrt{2}, 1)$ and all $\varepsilon > 0$ the function

$$\psi(t) = \sqrt{\frac{2k^2}{2k^2-1}}\ \mathrm{cn}\left(\frac{t}{\sqrt{2k^2-1}}, k\right)$$

solves equation (21) with

$$g(t) = -\frac{\varepsilon\sqrt{2}\,k}{2k^2-1}\ \mathrm{sn}\left(\frac{t}{\sqrt{2k^2-1}}, k\right)\ \mathrm{dn}\left(\frac{t}{\sqrt{2k^2-1}}, k\right).$$

Since $\max_s |\,\mathrm{sn}(s,k)\,\mathrm{dn}(s,k)| = \frac{1}{2k}$ for any $k \in (1/\sqrt{2}, 1)$ (see again [1]), we have

$$\|g\|_{L^\infty} = \frac{\varepsilon}{\sqrt{2}(2k^2-1)}.$$

It can now be checked that with no choice of k we can fulfill the second bound in (23). □

4 Stability of periodic solutions for strongly prestressed plates

In this section, we study the stability of periodic solutions of (1), both from a theoretical and from a numerical point of view.

4.1 Stability of 1-mode solutions

To state our next result, we denote by

$$\mathbb{P} = \{\wp^1, \wp^2, \wp^3\}$$

the set of periodic solutions of (1) given by Theorem 2.

Theorem 3. *Under the assumptions of Theorem 2, if* $u = u(\xi,t)$ *is any solution of* (1) *with initial conditions proportional to* $L_{1,1}$*, that is,*

$$u(\xi,0) = \alpha L_{1,1}(\xi) \qquad \text{and} \qquad u_t(\xi,0) = \beta L_{1,1}(\xi) \qquad \text{in } \Omega\,, \tag{25}$$

for some (possibly null) $\alpha, \beta \in \mathbb{R}$*, then there exists* $\wp \in \mathbb{P}$ *such that*

$$\lim_{t\to\infty}\left(\|u_t(t) - \wp_t(t)\|_{L^2} + \|u(t) - \wp(t)\|_{H^2_*}\right) = 0\,. \tag{26}$$

Proof. For a given couple $(\alpha,\beta)\in\mathbb{R}^2$, consider the solution $u=u(\xi,t)$ of (1) with initial conditions (25). By replacing into (1) and exploiting the uniqueness of the solution, one sees that u has the form

$$u(\xi,t)=U(t)L_{1,1}(\xi)$$

with $U\in C^2(\mathbb{R}_+)$ solving

$$\ddot{U}(t)+\delta\dot{U}(t)-(P-\mu_{1,1})U(t)+SU(t)^3=A(t),\qquad U(0)=\alpha,\quad \dot{U}(0)=\beta.$$

We then perform the same change of variables as (20) by setting

$$U(t)=\sqrt{\tfrac{P-\mu_{1,1}}{S}}\,V\Big(\sqrt{P-\mu_{1,1}}\,t\Big)\iff V(t)=\sqrt{\tfrac{S}{P-\mu_{1,1}}}\,U\Big(\tfrac{t}{\sqrt{P-\mu_{1,1}}}\Big)$$

so that V satisfies (21), that is,

$$\ddot{V}(t)+\varepsilon\dot{V}(t)-V(t)+V(t)^3=g(t)\,. \tag{27}$$

Since all the assumptions of Theorem 1.2 in [18] are satisfied (see above) we infer that any solution of (27), thereby also V, satisfies (for some $j=1,2,3$)

$$\lim_{t\to\infty}\Big(|\dot{V}(t)-\dot{z}_j(t)|+|V(t)-z_j(t)|\Big)=0\,.$$

After undoing the change of variables, this proves the desired statement. □

Theorem 3 states that if f has the form (14) and if the phase space is reduced to the first longitudinal mode, then the attractor for (1) is given by the set $\mathbb{P}$ of the three periodic solutions of (1) having the form (16). If in this framework it is nice to learn that there are no subharmonic solutions (called a "bad phenomenon" by Haraux [18]), on the other hand the multiplicity of periodic solutions yields an uncertainty of the behavior of all the other solutions. In fact, in [17] it is shown that Theorem 3 holds also in the *whole* phase space; nevertheless, since we are mainly interested in uni-modal solutions and in *quantitative* results, we preferred to state it in this precise form.

Theorem 3 also states that, if $P>\mu_{1,1}$, one cannot predict the long-time behavior of the solutions of (1) even for simple forces such as (14). Translated to the bridge model, this means that

in presence of strong prestressing, the effect of the wind depends on the initial position of the deck.

A fundamental problem is then to study the *asymptotic stability* of the three periodic solutions in $\mathbb{P}$, according to the following definition.

Definition 1. *We say that $\wp\in\mathbb{P}$ is (Lyapunov)* **stable** *if for every $\zeta>0$ there exists $\eta>0$ such that*

$$\|u_t(0)-\wp_t(0)\|_{L^2}+\|u(0)-\wp(0)\|_{H^2_*}<\eta\implies\|u_t(t)-\wp_t(t)\|_{L^2}+\|u(t)-\wp(t)\|_{H^2_*}<\zeta \tag{28}$$

for every $t>0$, where u solves (1) *with initial conditions such as* (25)*; we say that $\wp\in\mathbb{P}$ is* **unstable** *if it is not stable.*
We say that $\wp\in\mathbb{P}$ is **asymptotically stable** *if it is stable and there exists $\eta>0$ such that*

$$\|u_t(0)-\wp_t(0)\|_{L^2}+\|u(0)-\wp(0)\|_{H^2_*}<\eta\implies\lim_{t\to\infty}\Big(\|u_t(t)-\wp_t(t)\|_{L^2}+\|u(t)-\wp(t)\|_{H^2_*}\Big)=0\,,$$

where u solves (1) *with initial conditions such as* (25)*.*

To characterize the stability of a periodic solution, it will be useful to remark that

$$\text{if} \quad |c| < \tfrac{2}{3\sqrt{3}} \quad \text{then} \quad \rho^3 - \rho = c \quad \text{has three solutions} \quad -\rho_*(-c) < \rho_0(c) < \rho_*(c), \tag{29}$$

with $\rho_* > 0$ and increasing, $\rho_0(0) = 0$ and $c\,\rho_0(c) < 0$ if $c \neq 0$. We also set

$$\bar{\rho}_\delta = \rho_*\left(\tfrac{\delta}{32\sqrt{P-\mu_{1,1}+\delta^2}}\right), \quad T(\delta, f) = \pi\sqrt{\tfrac{12(P-\mu_{1,1})}{12(P-\mu_{1,1})\rho_*\left(\sqrt{\frac{S}{(P-\mu_{1,1})^3}}\|f\|_{L^\infty}\right)-\delta^2-4(P-\mu_{1,1})}},$$

$$R_0 = \rho_0\left(-\sqrt{\tfrac{S}{(P-\mu_{1,1})^3}}\tfrac{\|f\|_{L^\infty}}{L^\infty_{1,1}}\right), \quad R_- = -\rho_*\left(-\sqrt{\tfrac{S}{(P-\mu_{1,1})^3}}\tfrac{\|f\|_{L^\infty}}{L^\infty_{1,1}}\right), \quad R_+ = \rho_*\left(\sqrt{\tfrac{S}{(P-\mu_{1,1})^3}}\tfrac{\|f\|_{L^\infty}}{L^\infty_{1,1}}\right),$$

where the notation $L^\infty_{1,1}$ has been introduced in the statement of Theorem 2. Then the following stability results hold.

Theorem 4. *Let f be as in* (14) *and assume* (8) *and* (15)*. Then the following properties hold for the periodic solutions $\wp^j \in \mathbb{P}$ ($j = 1, 2, 3$) defined in* (16)*:*

1) $z_1(t)$ fulfills the bound

$$\|z_1\|_{L^\infty} < R_0 \tag{30}$$

and $\wp^1$ is unstable;

2) if one of the two following facts occurs

$$\sqrt{\frac{\delta^2}{12(P-\mu_{1,1})} + \frac{1}{3}} < \bar{\rho}_\delta \quad \textit{and} \quad T \le T(\delta, f) \qquad \textit{or} \qquad \sqrt{\frac{\delta^2}{12(P-\mu_{1,1})} + \frac{1}{3}} \ge \bar{\rho}_\delta, \tag{31}$$

then for every $t \in [0, T]$ one has

$$-R_+ < z_2(t) < R_- < 0 \qquad \textit{and} \qquad 0 < -R_- < z_3(t) < R_+ \tag{32}$$

and $\wp^2$ and $\wp^3$ are asymptotically stable.

Before proving this result, we briefly make some comments and state a more readable corollary. Conditions (31) should be read as assumptions of "smallness" and "largeness" of the damping, respectively; thus, Theorem 4 essentially says that the asymptotic stability of the two periodic solutions $\wp^2$, $\wp^3$ is obtained either for "small" damping (fulfilling however (15)) provided that the period of f is "small", or for sufficiently "large" damping independently of the period of f. Notice that $R_0 \to 0$ and $R_\pm \to \pm 1$ for $\|f\|_{L^\infty} \to 0$, so that the intervals in (30) and (32) shrink to a constant for $f \to 0$; moreover, as functions of $\|f\|_{L^\infty}$, R_- and R_+ are increasing, while R_0 (which is negative) is decreasing. Since by (15) we have $\|f\|_{L^\infty} < \frac{L^\infty_{1,1}}{32}\sqrt{\frac{(P-\mu_{1,1})^3}{S}}$, we derive the following statement.

Corollary 2. *Under the assumptions of Theorem 4, it is*

$$\|z_1\|_{L^\infty} < \rho_0\left(\frac{-1}{32}\right) \approx 0.0312806.$$

Moreover, if

$$\left(\sqrt{8(P-\mu_{1,1})} \le\right) \delta < \sqrt{8.18234(P-\mu_{1,1})} \quad \textit{and} \quad T < 25.4232 \qquad \textit{or} \qquad \delta \ge \sqrt{8.18234(P-\mu_{1,1})}$$

then, for every $t \in [0, T]$,

$$-1.01527 \approx -\rho_*\left(\frac{1}{32}\right) < z_2(t) < -\rho_*\left(-\frac{1}{32}\right) \approx -0.983993.$$

$$0.983993 \approx \rho_*\left(-\frac{1}{32}\right) < z_3(t) < \rho_*\left(\frac{1}{32}\right) \approx 1.01527,$$

The last part of this statement is obtained by taking the worst cases $\delta = \sqrt{8(P-\mu_{1,1})}$ and $\|f\|_{L^\infty} = \frac{L^\infty_{1,1}}{32}\sqrt{\frac{(P-\mu_{1,1})^3}{S}}$ to compute $T(\delta, f)$. To cover all the possible cases for the values of δ, it would remain to prove the asymptotic stability of $\wp^2$ and $\wp^3$ in case $\delta < \sqrt{8.18234(P-\mu_{1,1})}$ and $T \geq 25.4232$, range in which the technique which we here exploit does not work. We believe, also in view of numerical simulations, that stability is reached also herein, but filling in this gap seems nontrivial from a theoretical point of view.

To prove Theorem 4, we rely on lower and upper solutions techniques. We recall that by "lower solution" (resp., "upper solution") of (21) one means a function α_- (resp., α_+) such that

$$\ddot{\alpha}_- + c\dot{\alpha}_- + \alpha_-^3 - \alpha_- \gneq g(t), \qquad (\text{resp., } \ddot{\alpha}_+ + c\dot{\alpha}_+ + \alpha_+^3 - \alpha_+ \lneq g(t)).$$

Here the symbol $\gneq$ means: $\geq$ for all t and $>$ for some t (similarly for $\lneq$). Since we will merely use *constant* lower-upper solutions, we further say that constant functions $\alpha_-(t) \equiv \alpha_-$ and $\alpha_+(t) \equiv \alpha_+$ are, respectively, "lower solution" and "upper solution" of (21) if

$$\alpha_+^3 - \alpha_+ \lneq g(t) \lneq \alpha_-^3 - \alpha_- \qquad \forall t. \tag{33}$$

In the proof we will exploit two results in [8] and [20], which we briefly recall here for the reader's convenience, in versions adapted to equation (21) and for constant lower-upper solutions.

Proposition 4. *[8, Proposition 3.1] Let $\varepsilon \geq 0$, let g be T-periodic ($T > 0$), and assume that there exist constants $\alpha_- < \alpha_+$ such that* (33) *holds. Then* (21) *has at least one unstable T-periodic solution ψ such that*

$$\alpha_- \leq \psi(t) \leq \alpha_+ \quad \forall t \in [0, T], \tag{34}$$

provided that the number of periodic solutions ψ of (21) *fulfilling* (34) *is finite.*

Proposition 5. *[20, Theorem 1.2] Let $\varepsilon > 0$, let g be T-periodic ($T > 0$), and assume that there exist constants $\alpha_- > \alpha_+$ such that* (33) *holds. Moreover, assume that*

$$\max\{\alpha_-^2, \alpha_+^2\} \leq \frac{\pi^2}{3T^2} + \frac{1}{3} + \frac{\varepsilon^2}{12}. \tag{35}$$

Then (21) *has at least one asymptotically stable T-periodic solution ψ such that*

$$\alpha_+ \leq \psi(t) \leq \alpha_- \quad \forall t \in [0, T], \tag{36}$$

provided that the number of periodic solutions ψ of (21) *fulfilling* (36) *is finite.*

We emphasize the different assumptions $\alpha_- \lesseqgtr \alpha_+$ in these two statements. In fact, Proposition 4 is obtained by combining [8, Proposition 3.1] with [21, Lemma 3.2], see also [20, Theorem 1.1]. We are now ready to give the proof.

Proof of Theorem 4. After performing the change of variables (20), we are led to deal with equation (21), for which there exist exactly three periodic solutions ψ_j ($j = 1, 2, 3$) in view of Theorem 2. We thus know that the number of periodic solutions of (21) is finite.

Let us first prove Item 1). Recalling (15) and (22), we have that $\|g\|_{L^\infty} < 1/32$ and thus, by (29), there exist three solutions of the equations $\rho^3 - \rho = \pm\|g\|_{L^\infty}$. Then, also with the help of the left picture in Figure 1, we know that $\alpha_0 \equiv \rho_0(\|g\|_{L^\infty}) < 0$ and $\alpha^0 \equiv \rho_0(-\|g\|_{L^\infty}) = -\alpha_0$ are constant lower and upper solutions of (21) satisfying the assumptions of Proposition 4. Thus, there exists at least one unstable periodic solution ψ of (21) such that $\alpha_0 \leq \psi(t) \leq \alpha^0$. By undoing the change of variables (20), this corresponds to one of the three z_j's found in Theorem 2. Taking into account (17), it is necessarily z_1, which is therefore unstable (and thus $\wp^1$ is unstable, recall (16)) and satisfies the bound (30), completing the proof of Item 1).

As for Item 2), we notice that $\alpha^- \equiv \rho_*(\|g\|_{L^\infty})$ and $\alpha^+ \equiv \rho_*(-\|g\|_{L^\infty})$ are constant lower and upper solutions of (21) such that $\alpha^- > \alpha^+$, see the right picture in Figure 1. The same is fulfilled by $\alpha_- \equiv -\rho_*(-\|g\|_{L^\infty})$ and $\alpha_+ \equiv -\rho_*(\|g\|_{L^\infty})$. In view of (35), for $\wp^2$ to be asymptotically stable it is sufficient that

$$\rho_*(\|g\|_{L^\infty})^2 \leq \frac{\pi^2}{T^2} + \frac{1}{3} + \frac{\varepsilon^2}{12}.$$

This reads as

$$T \leq \frac{\pi}{\sqrt{\rho_*(\|g\|_{L^\infty})^2 - \frac{1}{3} - \frac{\varepsilon^2}{12}}}$$

as long as $\rho_*(\|g\|_{L^\infty})^2 > \frac{1}{3} + \frac{\varepsilon^2}{12}$, corresponding to $(31)_1$ in view of (22). If instead $\rho_*(\|g\|_{L^\infty})^2 \leq \frac{1}{3} + \frac{\varepsilon^2}{12}$, then T can be arbitrary and this is the situation described in $(31)_2$, yielding the asymptotic stability of $\wp^2$. One can reason analogously on $\wp^3$. Again by undoing the change of variables (20), we finally obtain the bounds in (32), concluding the proof. □

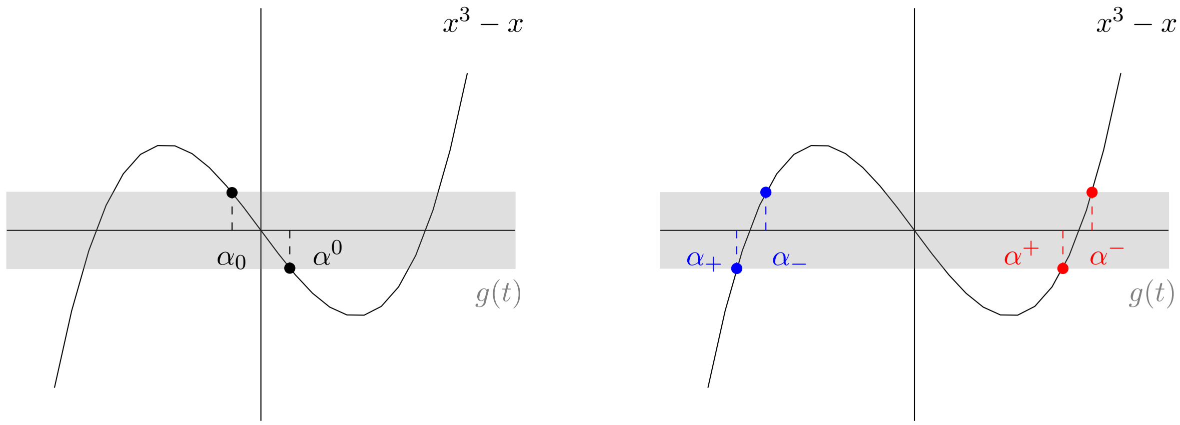

Figure 1: The choice of the lower and upper solutions in the proof of the bounds and of the stability properties of $\wp^1$ (left) and $\wp^2, \wp^3$ (right).

We make some complementary comments on Theorem 4. The difficulty regarding the bounds on $\wp^2$ and $\wp^3$ essentially lies in the fact that, while in presence of a couple of "well-ordered" lower and upper solutions α_- and α_+ (namely, satisfying $\alpha_- \leq \alpha_+$), there exists a periodic solution $\psi(t)$ such that $\alpha_- \leq \psi(t) \leq \alpha_+$, the same is not guaranteed in presence of reversed-order lower and upper solutions. The stability issue is even more complicated. On the other hand, according to [20, Remark 3.2], there is a significant number of situations in which the periodic solutions of (21) satisfying (34) or (36) are in finite number; in particular, we can partially extend Theorem 4 also to cases where (15) is not fulfilled. Roughly speaking, the local maximum and the local minimum of $h(\rho) = \rho^3 - \rho$, found in correspondence of $\rho = \mp\sqrt{\frac{1}{3}}$, play here a central role.

Proposition 6. *Assume that* $\|f\|_{L^\infty} < \frac{2\delta\, L^\infty_{1,1}}{3\sqrt{3}} \sqrt{\frac{(P-\mu_{1,1})^3}{S(\delta^2+P-\mu_{1,1})}}$. *Then, there exists a unique unstable periodic solution of* (19) *which has the same period as* f *and satisfies the bound* (30)*.*

This follows from [20, Remark 3.2], by noticing that we can find constant lower and upper solutions α_0, α^0 such that $\alpha_0 = -\alpha^0$ and $3\alpha_0^2 < 1$, provided that $\|g\|_{L^\infty} < \frac{2}{3\sqrt{3}}$. Indeed, we can take as lower and upper solutions the local maximum and the local minimum of h, respectively. Notice that, except for the stability part, the statement corresponds to the first part of [18, Theorem 1.1].

Also for the two asymptotically stable solutions it is possible to prove a similar statement; we here limit ourselves to give a "generalization" of Corollary 2, obtained taking a slightly larger value of the damping δ.

Proposition 7. *Assume that* $\|f\|_{L^\infty} < \frac{2\delta\, L^\infty_{1,1}}{3\sqrt{3}} \sqrt{\frac{(P-\mu_{1,1})^3}{S(\delta^2+P-\mu_{1,1})}}$ *and* $\delta \geq \sqrt{12(P-\mu_{1,1})}$. *Then, there exist two asymptotically stable periodic solutions of* (19) *having the same period as* f *and satisfying the bounds in* (32).

The assumptions of Proposition 7 guarantee, in fact, that it is possible to choose (constant) lower and upper solutions α_-, α_+ both greater than $1/\sqrt{3}$ (or less than $-1/\sqrt{3}$) and such that $\alpha_+ \leq \alpha_-$, so that $3s^2-1 \gneqq 0$ for every $s \in [\alpha_+, \alpha_-]$, entering the condition given in [20, Remark 3.2]. In this setting, what is not guaranteed is that only three periodic solutions exist; in principle, more complicated pictures for the solvability of (21) may appear. Concerning the existence and the localization of the periodic solutions, the reader may compare Proposition 7 with [18, Theorem 1.1], where in presence of a slightly stronger bound on f it is possible to relax the assumption on δ by taking it as in (15).

Finally, using once more the results in [18] it is possible to determine a subset of the basin of attraction for $\wp^2, \wp^3$, as the following statement shows. The proof comes directly from [18, Lemma 6.1].

Proposition 8. *Assume* (15). *Then, for any* $\gamma < \frac{1}{16}\sqrt{\frac{P-\mu_{1,1}}{S}}$ *there exists* $c > 0$ *such that if*

$$\left| z(t_0) - \sqrt{\frac{P-\mu_{1,1}}{S}} \right| \leq \gamma, \qquad |z'(t_0)| \leq c, \tag{37}$$

then the solution of (1) *in the form* (16) *converges to* $\wp^2$. *Under the same assumptions, if the first condition in* (37) *is replaced by*

$$\left| z(t_0) + \sqrt{\frac{P-\mu_{1,1}}{S}} \right| \leq \gamma,$$

then the solution of (1) *in the form* (16) *converges to* $\wp^3$.

4.2 Some numerical considerations on torsional instability

We analyze here the stability of the plate modeled by (1) from a different point of view. In order to introduce our characterization of *torsional instability*, we recall that the vortex shedding around the deck of a bridge generates a periodic lift force f which starts the longitudinal oscillations of the structure. The frequency of f determines which longitudinal mode is prevailing [13, 14] (that is, which mode captures almost all the energy from the vortices). We analyze the situation where the prevailing mode is $L_{1,1}$. After some transition time T_W, related to the so-called Wagner effect, the longitudinal oscillation is maintained in amplitude by a somehow perfect equilibrium between the input of energy from the wind and the structural dissipation. This corresponds to one of the periodic solutions in $\mathbb{P}$ found in Theorem 2. We are thus led to investigate whether "nearly periodic" longitudinal oscillations suddenly transform into (wide) torsional oscillations as in several bridges; to this end, we focus on the behavior of the special class of solutions of (1) having the form

$$w(\xi,t) = U(t)L_{1,1}(\xi) + V(t)T_{m,2}(\xi), \tag{38}$$

namely possessing only two nontrivial modes, one longitudinal and one torsional. To obtain such solutions we associate to (1) the initial conditions

$$w(\xi,0) = U_0L_{1,1}(\xi) + V_0T_{m,2}(\xi)\,, \qquad w_t(\xi,0) = U_1L_{1,1}(\xi) + V_1T_{m,2}(\xi)\,. \tag{39}$$

Assuming again that f has the form (14), if we replace (38) into (1) we find that (U,V) solves the nonlinear system

$$\begin{cases} \ddot{U}(t) + \delta\dot{U}(t) + (\mu_{1,1}-P)U(t) + S[U(t)^2 + m^2V(t)^2]U(t) = A(t) \\ \ddot{V}(t) + \delta\dot{V}(t) + (\nu_{m,2}-P)V(t) + S[U(t)^2 + m^2V(t)^2]V(t) = 0\,, \end{cases} \tag{40}$$

while the initial conditions (39) become

$$U(0) = U_0\,, \quad \dot{U}(0) = U_1\,, \quad V(0) = V_0\,, \quad \dot{V}(0) = V_1\,. \tag{41}$$

If $V_0 = V_1 = 0$ then $w(\xi, t)$ is proportional to $L_{1,1}$, see Corollary 1. Moreover, harking back to Theorem 2, if we assume (15) we know that w will converge to some $\wp \in \mathbb{P}$ as $t \to \infty$. The ideal situation would be that w coincides with one $\wp \in \mathbb{P}$, with the explicit form of $\wp$ being available; however, this is the case only in very particular situations, see Section 3.2. The natural question is then to investigate whether a suitable "smallness condition" on V_0 and V_1 implies that the V-component of w in (38) remains small for all t, possibly converging to 0 at infinity: this would mean that the three periodic solutions in $\mathbb{P}$ also attract solutions of (1) with two modes such as (38). A full answer to this question appears far from being trivial, since already at a linear level the available stability criteria are very few and fairly involved; cf. [26]. Furthermore, a linear analysis may not reflect the real behavior of structures, since linear instability does not necessarily correspond to the onset of large torsional oscillations. Therefore, we limit ourselves to the numerical simulations shown in Figures 2-4, for which we fix $m = 2$ (thus considering only the *second* torsional mode) and we use the approximations (6) and (7) of the eigenvalues $\mu_{1,1}$ and $\nu_{2,2}$. Moreover, we set

$$\delta = 6,\ P = 5,\ S = 100,\ f(\xi, t) = 0.005\cos(9t), \tag{42}$$

integrating (40) on the time interval $[0, 20]$. Notice that (15) is here fulfilled. For Figure 2, we take $U_0 = 0.2$ (in view of (18)) and $V_0 = 0.01$; we see that $V(t)$ converges very rapidly to 0 and $U(t)$ converges to z_2 (using the notation of Theorem 2), suggesting that $\wp^2$ is asymptotically stable.

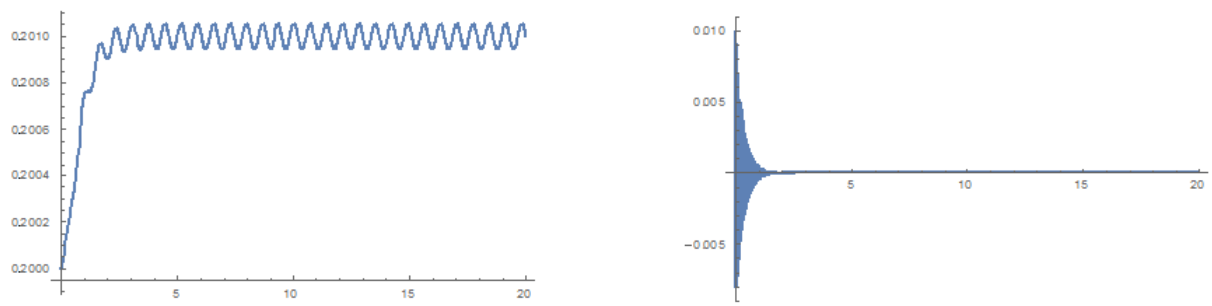

Figure 2: The components $U(t)$ (left) and $V(t)$ (right) in (38), with (42) and $U_0 = 0.2$, $V_0 = 0.01$.

In Figure 3, we depict the solution of (40) with the same choices as in (42) and $U_0 = 0.2$, $V_0 = 2$; here again $V(t) \to 0$ for t large, but, due to the action of the V-component, U may converge to z_3 even if it starts near z_2. On increasing of V_0, one can come back to the situation of Figure 2, where the U component converges to $\wp^2$ (Figure 4). It thus seems that, in the assumptions of Theorem 2 (in particular, in presence of a large damping), the periodic solutions $\wp^2$, $\wp^3$ indeed attract the solutions of (40), but the law according to which they "choose" to converge to $\wp^2$ or to $\wp^3$ appears unclear. Our feeling is that there exist alternating intervals of initial data for which the solution converges either to $\wp^2$ or $\wp^3$ and, as the total energy increases, the number of back-and-forth across a neighborhood of the saddle point of the energy function, also increases.

The notion of torsional instability we deal with was introduced in [14] (see also [13]), and we here recall it.

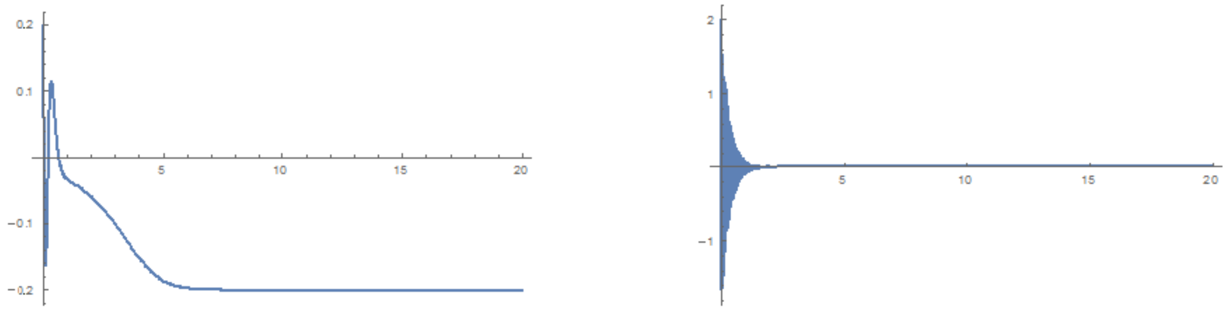

Figure 3: The components $U(t)$ (left) and $V(t)$ (right) in (38), with (42) and $U_0 = 0.2$, $V_0 = 2$.

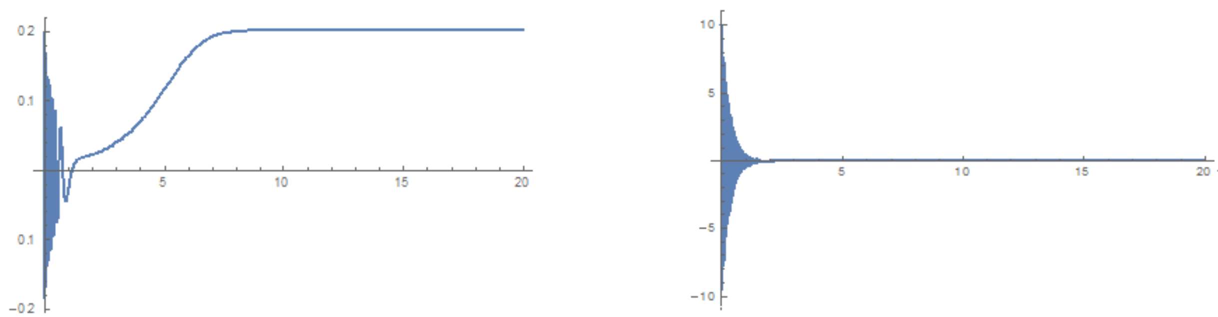

Figure 4: The components $U(t)$ (left) and $V(t)$ (right) in (38), with (42) and $U_0 = 0.2$, $V_0 = 10$.

Definition 2. *Let $T_W > 0$. We say that a weak solution u of* (1) *having the form* (38) *is* unstable *before time $T > 2T_W$ if there exists a time instant τ with $2T_W < \tau < T$ such that*

$$\frac{\|V\|_{L^\infty(0,\tau)}}{\|V\|_{L^\infty(0,\tau/2)}} > 10. \tag{43}$$

We say that u is stable *until time T if* (43) *is not fulfilled for any $\tau \in (2T_W, T)$.*

This definition extends in a quantitative way the classical linear instability to a nonlinear context, since it requires an "exponential-like" behavior of $V(t)$ in order to fulfill condition (43); for more details, see [13, 14]. Such a growth condition thus allows to highlight *abrupt changes* in the nature of the oscillations, from longitudinal to torsional, being more in line with the behavior of real structures. Our aim is to check it in relation to the values of the parameters appearing in (1). We proceed numerically and we first notice that, setting $(U(t), V(t)) = \alpha(\mathbf{U}(t), \mathbf{V}(t))$ for $\alpha > 0$, leads to the system

$$\begin{cases} \ddot{\mathbf{U}}(t) + \delta\dot{\mathbf{U}}(t) + (\mu_{1,1} - P)\mathbf{U}(t) + \hat{S}[\mathbf{U}(t)^2 + m^2\mathbf{V}(t)^2]\mathbf{U}(t) = \hat{A}(t) \\ \ddot{\mathbf{V}}(t) + \delta\dot{\mathbf{V}}(t) + (\nu_{m,2} - P)\mathbf{V}(t) + \hat{S}[\mathbf{U}(t)^2 + m^2\mathbf{V}(t)^2]\mathbf{V}(t) = 0\,, \end{cases}$$

where

$$\hat{S} = S\alpha^2, \qquad \|\hat{A}\|_{L^\infty} = \frac{\|A\|_{L^\infty}}{\alpha}.$$

Hence, we may scale (40) fixing either S or $\|A\|_{L^\infty}$. We fix $S = 100$ and we check condition (43) on varying of the parameters δ and $\|A\|_{L^\infty}$. Moreover, wishing to highlight the relationships between the time-*frequency* of the forcing term f and instability as well (cf. [6]), we fix $f(\xi,t) = M\cos(\nu t)L_{1,1}(\xi)$, so that $\|A\|_{L^\infty} = M$ and variations in the time-frequency of f correspond to variations of the parameter ν. Finally, as we want to reproduce the dynamics of the plate giving more emphasis to the action of the external forcing (in line with the previous discussion), we fix small initial conditions both on the longitudinal and on the torsional component, by setting $U_0 = V_0 = 0.01$ and $U_1 = V_1 = 0$ in (41). This position, which agrees with the fact that torsional oscillations are initially very small, allows not to undergo the influence of the initial width of longitudinal oscillations in spotting instability, enabling to better identify the role of the parameters in the dynamics of the structure.

We show some numerical results whose response in terms of stability is analyzed with the use of the same algorithm as in [14, Section 5.4]. In Figure 5, we show the plot of the V-component of the solutions of (40) on the time interval $[0,T] := [0,16]$ for

$$\delta = 0.01,\ P = 5,\ S = 100,\ \nu = 9,\ M = 5\cdot 10^5 \div 7\cdot 10^5, \tag{44}$$

where the increase step in the amplitude M of the forcing term is equal to 2000. Such components are all plotted within the range $[-6,6]$ as for the y-axis, in order to highlight the huge differences between solutions with different initial amplitudes. By use of the above mentioned algorithm, we checked that

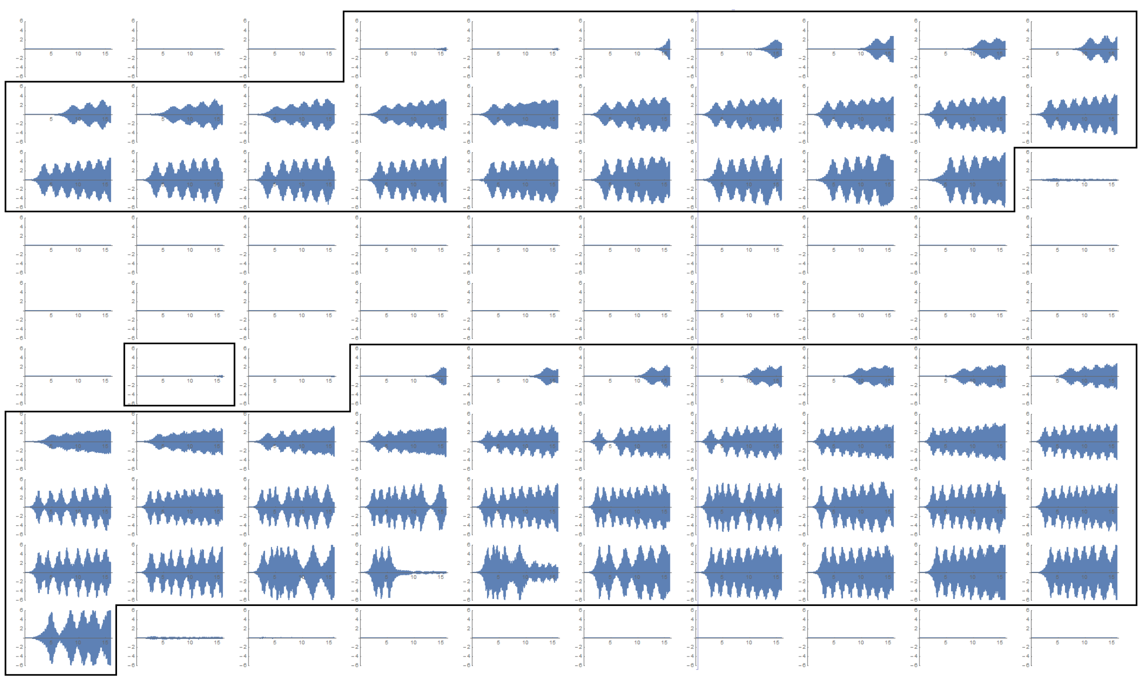

Figure 5: The V-component of the solution of (40) for values of the parameters given by (44).

the corresponding solution u of (1) in the form (38) is unstable for

$$M \in \mathcal{M}_1 = [508000, 558000],\ M = 604000,\ M \in \mathcal{M}_2 = [608000, 682000]$$

(the corresponding pictures being framed by black rectangle-like polygons in Figure 5) while it is stable otherwise. Moreover, the time τ in correspondence of which instability is observed has a quite neat trend: it starts being close to $T = 16$ in correspondence of the left endpoints of $\mathcal{M}_1$ and $\mathcal{M}_2$, then it reaches a (small) minimum while proceeding inside such intervals, before growing again until stability is acquired once more.

In Figure 6, we let the frequency vary, taking

$$\delta = 0.01,\ P = 5,\ S = 100,\ M = 5 \cdot 10^5,\ \nu = 0 \div 19.8, \tag{45}$$

with ν growing each time with step 0.2. Again, the situations where instability is spotted are framed by black boxes. Notice that for $\nu = 0$, corresponding to the time-constant forcing term $f(\xi, t) = ML_{1,1}(\xi)$, we are in an instability situation.

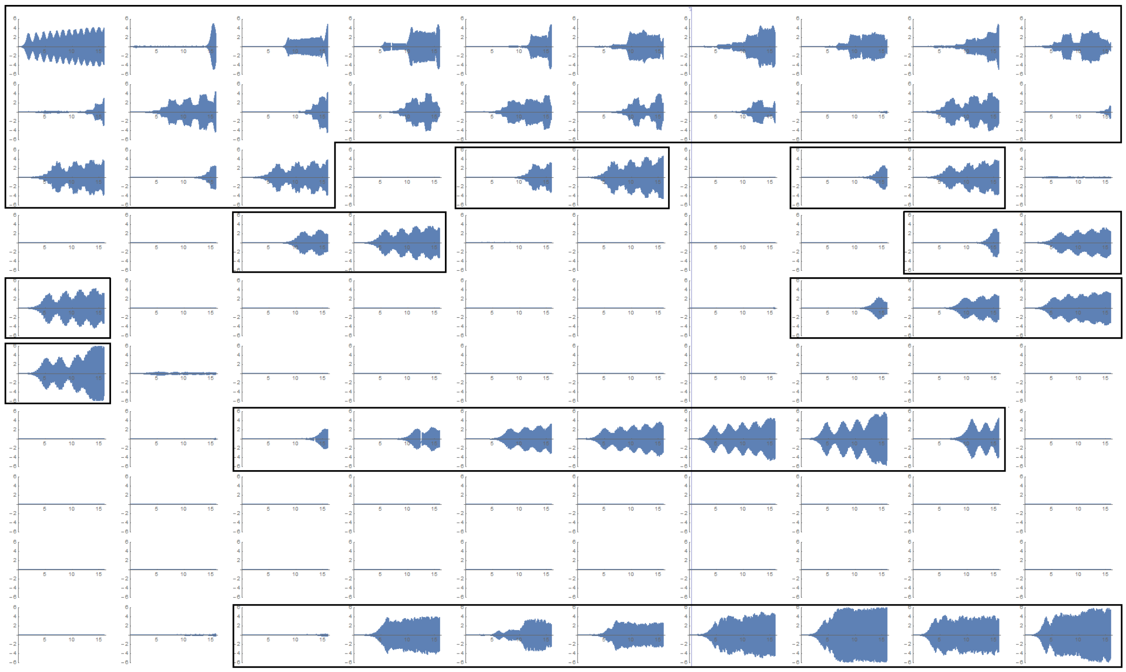

Figure 6: The V-component of the solution of (40) for values of the parameters given by (45).

Finally, as for the damping δ, we proceed with the choices

$$P = 5,\ S = 100,\ M = 5.5 \cdot 10^5,\ \nu = 9,\ \delta = 0 \div 0.495, \tag{46}$$

increasing δ with step 0.005 each time. Here we still find unstable solutions alternating with stable ones at the beginning (Figure 7), but then we are always in a stable regime.

According to all these figures, we conclude that, on varying of the parameters M and ν,

the stability portrait for system (40) is similar to the one of the Mathieu equation, where stability and instability regions alternate.

This alternation has somehow already been spotted (on growing of the initial datum) in Figures 2-4, where $\wp^2$ and $\wp^3$ seem to alternatively attract the solutions of (40) on varying of V_0. On the other hand, focusing on the role of the damping δ, we infer that

there exists a threshold $\delta_0 > 0$ such that, if $\delta \geq \delta_0$, then $V(t) \to 0$ for $t \to \infty$.

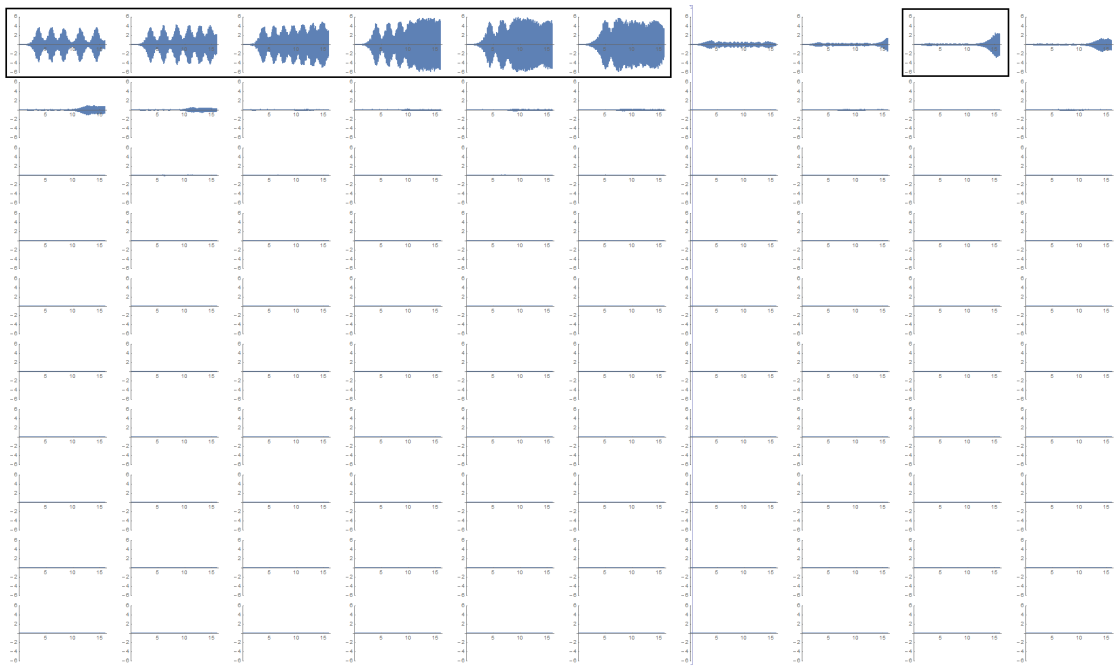

Figure 7: The V-component of the solution of (40) for values of the parameters given by (46).

An analytic proof of such claims appears tough and particularly challenging and will be the object of a future research.

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