Some results on p-Laplace equations with a critical growth term

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Abstract

Equations involving the p-Laplacian with a term in the critical growth range are considered: existence results are obtained under minimal assumptions on the lower order perturbation. The problem is studied by means of variational methods: in particular, a problem with linking geometry is treated thanks to the orthogonalization technique introduced in [13].

AMS subject classification: 35J20, 35J70

1 Introduction

In this paper we study the following p-Laplacian degenerate elliptic equation:

$$\begin{cases} -\Delta_p u = g(x,u) + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary $\partial\Omega$, $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent of the imbedding $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and $g(\cdot, s)$ is a subcritical term. We consider the case $1 < p < n$; when $n = p$ the Sobolev imbedding is into Orlicz spaces [21] and related results may be found in [1].

Recently, much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brezis and Nirenberg [5], where the case $p = 2$ is considered. For arbitrary values of $p > 1$, we recall in particular the results obtained in [10, 11], where the existence of a nontrivial solution is proved for equations with a homogeneous subcritical term and constant coefficient, i.e. $g(x,u) = \lambda|u|^{q-2}u$ for some q and $\lambda \in \mathbb{R}$; more general results with nonconstant coefficient are obtained in [4, 9, 14, 15].

We consider the Banach space $W := W_0^{1,p}(\Omega)$, normed by

$$\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p} \quad ;$$

we denote by $\|\cdot\|_p$ the L^p norm. Let λ_1 be the first generalized eigenvalue of $-\Delta_p$ relative to the homogeneous Dirichlet problem in Ω, i.e. the smallest value of λ for which the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a nontrivial solution. It is well-known [2, 16] that λ_1 is positive and simple, and the corresponding eigenfunction does not change sign; moreover, the operator $-\Delta_p$ admits a sequence of eigenvalues diverging to $+\infty$, see [10].
Define the functional $J : W \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega G(x,u)dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx \ ,$$

where $G(x,s) = \int_0^s g(x,t)dt$; if g is continuous, then $J \in C^1(W,\mathbb{R})$ and the critical points of the functional J correspond to weak solutions of equation (1). However, standard variational arguments do not apply because the imbedding $W \subset L^{p^*}(\Omega)$ is not compact and the functional J does not satisfy the Palais-Smale condition (PS condition). And indeed, for equations with critical growth, nontrivial solutions may not exist; if Ω is starshaped and $\lambda \leq 0$, then the following equation admits in W only the trivial solution $u \equiv 0$, see [14, 18]:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{2}$$

When $p = 2$ one observes a strong dependence of the results on the dimension n of the space, in particular different existence results hold for $n = 3$, $n = 4$ and $n \geq 5$, see [5, 6, 13]. By allowing p to vary in $(1, +\infty)$ and having therefore another parameter to deal with, one can try to describe this "strange" behaviour. Figures 1 and 2 give a picture of the phenomenon we face with our method; we do not know if such phenomenon is a consequence of a limit of our techniques or if it indeed reflects an intrinsic property of the equation studied.
In a recent paper [13], an orthogonalization technique has been developed for the study of critical growth problems in semilinear elliptic equations; such technique is based on variational methods. To assure that the considered minimax levels lie in the range of compactness, suitable classes of approximating functions having disjont support with the Sobolev concentrating functions are constructed. We call Sobolev concentrating functions some truncation of the positive radial functions which achieve the best constant in Sobolev inequalities in $\mathbb{R}^n$, see [22]. In [5] such functions have been found responsible for the loss of compactness of the problem, see also [7].
In this paper, we prove existence results for (1) for all $n > p > 1$: our results are obtained under minimal assumptions on the subcritical term $g(x,u)$: in particular we do not require it to be neither homogeneous with respect to u nor positive. In general the assumptions on g are stricter in lower dimensions. We consider the cases where the functional J has a mountain pass geometry or a linking structure with or without resonance: roughly speaking, these three cases correspond respectively to

$$0 \leq \lim_{s\to 0^+} \frac{G(x,s)}{s^p} < \frac{\lambda_1}{p} \ , \qquad \lim_{s\to 0^+} \frac{G(x,s)}{s^p} = \frac{\lambda_1}{p} \ , \qquad \frac{\lambda_1}{p} < \lim_{s\to 0^+} \frac{G(x,s)}{s^p} \ .$$

To our knowledge no results for the $p-$Laplace equation have been found by linking techniques, even if in [8] some results on the subcritical case are obtained in a right neighborhood of the first eigenvalue via

a different method: as noted in [14], the linking structure case needs a deeper knowledge of the spectral properties of $-\Delta_p$; nevertheless, we prove in Section 3.3 the existence of a linking geometry in a right neighborhood of the first eigenvalue.
The proofs are performed by slight modifications of the techniques used in [13] for the semilinear case, see also [3, 12]; in particular, we use the orthogonalization technique introduced there to study the linking case. With this technique, the mountain pass case and the nonresonant linking case look similar: for these cases, Figure 1 summarizes the results relative to (2). In Figure 2 we illustrate the results relative to the more complicated case of resonant linking, that is, the case where $\lambda = \lambda_1$ in (2).

2 Statement of the results

We assume that the function g is subcritical in the following sense:

$$\begin{cases} g \ : \ \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function} \\ \forall \varepsilon > 0 \quad \exists a_\varepsilon \in L^{\frac{np}{n(p-1)+p}} \text{ such that } |g(x,s)| \le a_\varepsilon(x) + \varepsilon |s|^{\frac{n(p-1)+p}{n-p}} \text{ for a.e. } x \in \Omega \ , \ \forall s \in \mathbb{R} \ . \end{cases} \tag{3}$$

Other assumptions are imposed on the primitive $G(x,s) = \int_0^s g(x,t)dt$: the lower order perturbation $g(x,s)$ may change sign, provided that

$$G(x,s) \ge 0 \qquad \text{for a.e. } x \in \Omega \ , \quad \forall s \in \mathbb{R} \ . \tag{4}$$

2.1 The mountain pass case

Assume that there exist an open subset $\Omega_0 \subset \Omega$ and some constants $\sigma, \delta, \mu > 0$ and $b > a > 0$ such that

$$G(x,s) \le \frac{1}{p}(\lambda_1 - \sigma)|s|^p \qquad \text{for a.e. } x \in \Omega \ , \quad \forall \ |s| \le \delta \tag{5}$$

and

$$G(x,s) \ge \mu \quad \text{for a.e. } x \in \Omega_0 \ , \ \forall s \in [a,b] \ . \tag{6}$$

Under these assumptions we prove the following:

Theorem 1 *Assume (3), (4), (5) and (6).*
If $1 < p^2 < n$, then equation (1) admits a positive nontrivial solution.
If $n = p^2$ and μ in (6) is large enough, then equation (1) admits a positive nontrivial solution.

In particular, we obtain a result of [10]:

Corollary 1 *Let $1 < p^2 \le n$. Then equation (2) admits a positive nontrivial solution for all $\lambda \in (0, \lambda_1)$.*

If $p < n < p^2$ we are in the case of the critical dimensions of Pucci-Serrin [19]. In this case assumption (6) is no longer enough and we need a suitable behavior of G at infinity: more precisely, we require the existence of a nonempty open set $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{G(x,s)}{s^{\alpha_{np}}} = +\infty \qquad \text{uniformly in } \Omega_0 \ , \tag{7}$$

where $\alpha_{np} = \frac{p(np+p-2n)}{(p-1)(n-p)}$. Then we prove

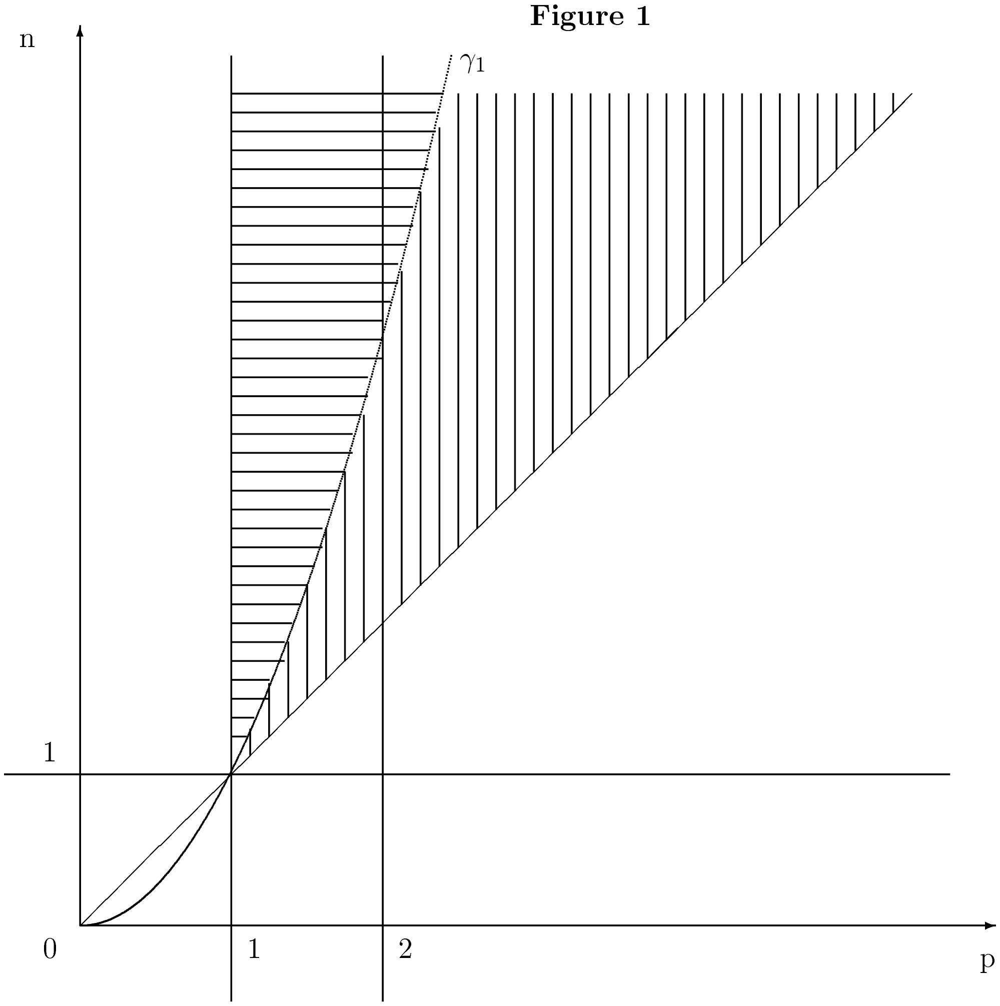
Theorem 2 *Let $1 < p < n < p^2$, assume (3), (4), (5) and (7). Then equation (1) admits a positive nontrivial solution.*

We remark that Theorems 1 and 2 generalize Theorem 3.3 in [11].

Let S be the best constant of the Sobolev imbedding $W \subset L^{p^*}$, see [22]. In the case of the critical dimensions, we also prove a result about (2):

Theorem 3 *Let $\Lambda = S|\Omega|^{-p/n}$, assume $1 < p < n < p^2$ and $\lambda \in (\lambda_1 - \Lambda, \lambda_1)$. Then (2) admits a positive nontrivial solution.*

Figure 1



Note that the number of critical dimensions increases with p and tends to infinity as $p \to +\infty$. This is illustrated by Figure 1, where γ_1 is the curve of equation $n = p^2$ and the region below the curve corresponds to the critical dimensions.

2.2 The linking case

The first eigenvalue of $-\Delta_p$ can be obtained as

$$\lambda_1 = \min_{u \in W_0^{1,p} \cap B^1} \|u\|^p \ , \tag{8}$$

where $B^1 = \{u \in W_0^{1,p}; \|u\|_p = 1\}$. Now let $w \in W^{-1,p'}$, denote by $E_w^\perp$ the subspace of W orthogonal to w, i.e. $E_w^\perp = \{u \in W | (w,u) = 0\}$ where $(\ , \ )$ is the duality product between $W^{-1,p'}$ and W; let

$$\overline{\lambda} = \sup_{w \in W^{-1,p'}} \inf_{u \in E_w^\perp \cap B^1} \|u\|^p \ .$$

It is easy to observe that $\overline{\lambda} \leq \lambda_2$, where λ_2 is the second eigenvalue. If $p = 2$ then $\overline{\lambda} = \lambda_2$; if $p \neq 2$ it is not clear whether the equality holds or not, but in Lemma 2 below it is proved that $\overline{\lambda} > \lambda_1$.
We first deal with the case of non-resonance near the origin: assume that there exist $\delta, \sigma > 0$ such that

$$\begin{aligned} &\frac{1}{p}(\lambda_1 + \sigma)|s|^p \leq G(x,s) \leq \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \qquad \text{for a.e. } x \in \Omega \ , \quad \forall \ |s| \leq \delta \\ &G(x,s) \geq \frac{1}{p}(\lambda_1 + \sigma)|s|^p - \frac{1}{p^*}|s|^{p^*} \qquad \text{for a.e. } x \in \Omega \ , \quad \forall \ s \neq 0 \ . \end{aligned} \tag{9}$$

Note that both (5) and (9) imply $g(x,0) = 0$ for a.e. $x \in \Omega$ and $u \equiv 0$ is a solution of (1).

With the above assumptions we will prove

Theorem 4 *If $1 < p^2 \leq n$, assume (3), (4) and (9); if $1 < p < n < p^2$, assume (3), (4), (7) and (9). Then equation (1) admits a nontrivial solution.*

Finally, we deal with the more delicate case of resonance near the origin: assume that there exist $\delta > 0$ and $\sigma \in (0, 1/p^*)$ such that

$$\begin{aligned} &\frac{1}{p}\lambda_1|s|^p \leq G(x,s) \leq \frac{1}{p}(\overline{\lambda} - \sigma)|s|^p \qquad \text{for a.e. } x \in \Omega \ , \quad \forall \ |s| \leq \delta \\ &G(x,s) \geq \frac{1}{p}\lambda_1|s|^p - (\frac{1}{p^*} - \sigma)|s|^{p^*} \qquad \text{for a.e. } x \in \Omega \quad \forall \ s \in \mathbb{R} \ . \end{aligned} \tag{10}$$

Moreover, we need a condition at infinity on G for all n: more precisely, we require the existence of an open nonempty set $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{G(x,s)}{s^{\beta_{np}}} = +\infty \quad \text{uniformly in } \ \Omega_0 \ , \tag{11}$$

where $\beta_{np} := \frac{np(np+2p-2n)}{(n-p)(np+p-n)}$. We will prove the following

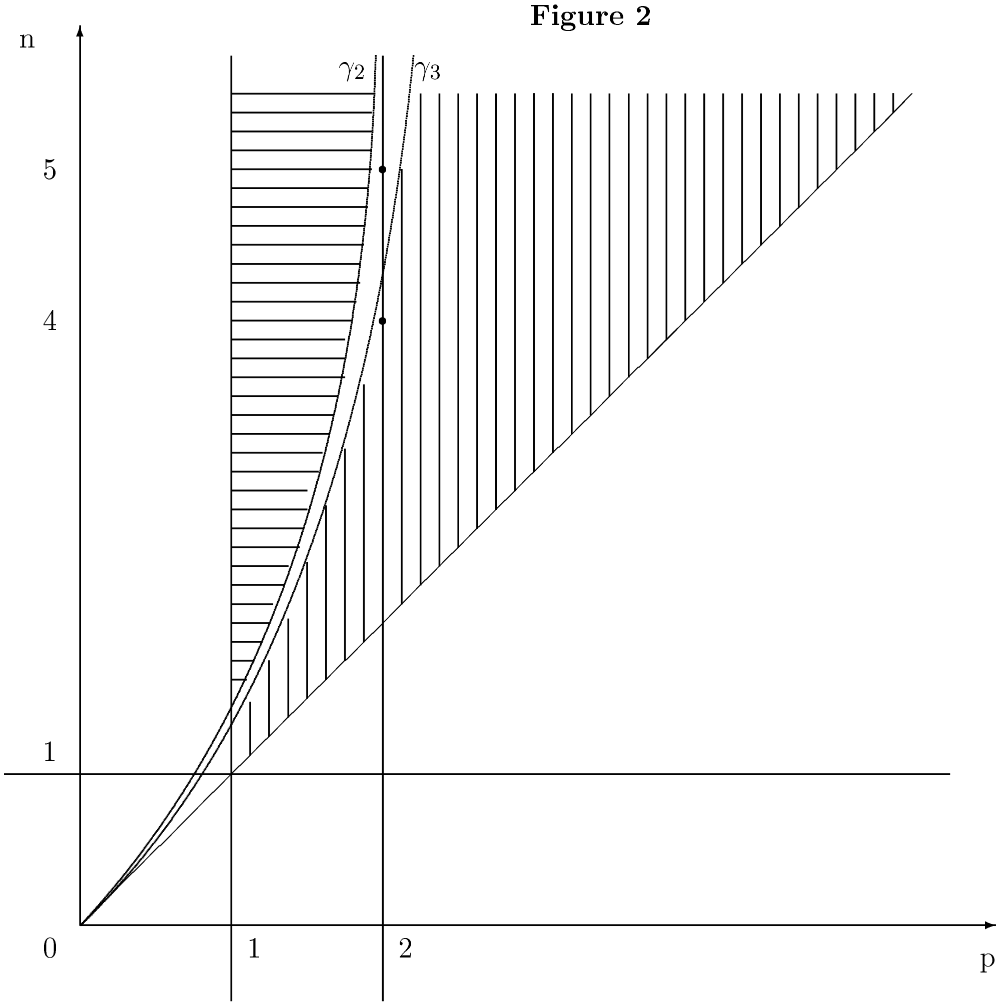
Theorem 5 *Let $n > p > 1$ and assume (3), (4), (10), (11). Then (1) admits a nontrivial solution.*

Note that $\beta_{np} < p^*$ for all $n > p > 1$ and that $\beta_{np} > 0$ if $p > \frac{2n}{n+2}$. As a straightforward consequence of Theorem 5 we have:

Corollary 2 *Let $p > 1$ and let n satisfy $\frac{n^2}{n+1} > p^2$: then (2) admits a notrivial solution for $\lambda = \lambda_1$.*

Figure 2 summarizes the resonant case results: the curves γ_2 and γ_3 have equations respectively $p = \frac{2n}{n+2}$ and $\frac{n^2}{n+1} = p^2$. In the whole region $1 < p < n$ we obtain the existence of a nontrivial solution for equation (1) in the resonant case by assuming (11). In the region above the curve γ_2 the subcritical perturbation G does not need to blow up at infinity, and since $1 < p < 2$ the Laplacian is not concerned. Above the curve γ_3 we have solutions for equation (2) when $\lambda = \lambda_1$. Note that in the Laplacian case the existence of a solution is guaranteed only with $n \geq 5$ and not for $n = 4$ as stated in [6], see the remark in [13].

Figure 2



3 The variational characterization

3.1 The Palais-Smale sequences

Let $I \in C^1(W, \mathbb{R})$; we recall that a sequence $\{u_k\} \subset W$ is called a Palais-Smale sequence (PS for brevity) for I at level c if $I(u_k) \to c$ and $I'(u_k) \to 0$ in $W^{-1,p'}$, $p' = \frac{p}{p-1}$. The functional I is said to satisfy the PS condition at level c, if every PS sequence at level c is precompact.
The functional J here considered may not satisfy the PS condition; however, we state a crucial result which generalizes Lemma 2.3 in [11] and yields a sufficient condition for the existence of a nontrivial solution of (1).

Lemma 1 *Assume (3) and that there exists a PS sequence $\{u_k\} \subset W$ for J at level $c \in (0, \frac{S^{n/p}}{n})$. Then there exists $u \in W \setminus \{0\}$ such that $u_k \rightharpoonup u$ up to a subsequence and $J'(u) = 0$.*

Proof. Let $f(x,s) = g(x,s) + |s|^{p^*-2}s$ and $F(x,s) = \int_0^s f(x,t)dt$; since (3) holds, we have

$$\exists \vartheta \in (0, \frac{1}{p}) \quad \exists \bar{s} > 0 \qquad \text{such that} \qquad F(x,s) \leq \vartheta f(x,s)s \quad \text{for a.e. } x \in \Omega \quad \forall |s| \geq \bar{s} \ :$$

then $\{u_k\}$ is bounded (see [20]) and there exists u such that $u_k \rightharpoonup u$, up to a subsequence. Furthermore, $J'(u) = 0$ by weak continuity of J'.
Assume by contradiction that $u = 0$; as the term $g(x, u_k)u_k$ is subcritical, we infer from $J'(u_k)[u_k] = o(1)$ that

$$\|u_k\|^p - \|u_k\|_{p^*}^{p^*} = o(1) \ . \tag{12}$$

By the definition of S, for all $u \in W$ we have $\|u\|^p \geq S\ \|u\|_{p^*}^p$, hence

$$o(1) \geq \|u_k\|^p (1 - S^{-p^*/p} \|u_k\|^{p^*-p}) \ .$$

Now $\|u_k\| \not\to 0$ because $c > 0$; therefore, the last inequality implies $\|u_k\|^p \geq S^{n/p} + o(1)$ and by (12) we get

$$J(u_k) = \frac{1}{n}\|u_k\|^p + \frac{1}{p^*}(\|u_k\|^p - \|u_k\|_{p^*}^{p^*}) + o(1) \geq \frac{1}{n} S^{n/p} + o(1)$$

which contradicts the assumption $c < \frac{1}{n} S^{n/p}$. □

By Lemma 1 we can prove Theorems 1-5 by building a PS sequence for J at a level $c \in \left(0, \frac{S^{n/p}}{n}\right)$; for this reason we call $\frac{S^{n/p}}{n}$ the nontrivial threshold. We treat the mountain pass case and the linking case separately.

3.2 The mountain pass case

Consider the cone of positive functions

$$\mathcal{C} := \{u \in W;\ u(x) \geq 0 \text{ for a.e. } x \in \Omega\} \ ;$$

assume that there exists $v \in \mathcal{C}$ such that $\max_{t \geq 0} J(tv) < \frac{S^{n/p}}{n}$ and note that, as the critical term is the leading term at $+\infty$, there exists $\tau_v > 0$ such that $J(\tau_v v) < 0$. Consider the class

$$\Gamma_v := \{\gamma \in C([0,1];\ W),\ \gamma(0) = 0,\ \gamma(1) = \tau_v v\}$$

and the infmax value

$$\alpha := \inf_{\gamma\in\Gamma_v} \max_{t\in[0,1]} J(\gamma(t)) \ .$$

If such v exists, we obtain a PS sequence of mountain-pass type for J at level $\alpha \in (0, \frac{S^{n/p}}{n})$ by standard variational methods, see [20]; by well-known arguments we can assume that the PS sequence is in $\mathcal{C}$. Therefore, Theorems 1, 2 and 3 are proved if we find a function $v \in \mathcal{C}$ satisfying the above stated requirement: a different choice of the function v is needed in Theorems 1, 2 and 3.

3.3 The linking case

The following lemma yields a sufficient condition for the linking geometry to hold:

Lemma 2 *For all $w \in W^{-1,p'}$ satisfying $(w, e_1) \neq 0$ there exists $\alpha_w > 0$ such that if $u \in E_w^\perp$, then $\|u\|^p - \lambda_1\|u\|_p^p \geq \alpha_w\|u\|^p$; therefore $\overline{\lambda} > \lambda_1$.*

Proof. Let $w \in W^{-1,p'}$, $(w, e_1) \neq 0$ and $E^1 = \text{span}\{e_1\}$: we have $W = E^1 \oplus E_w^\perp$ and there exists $c > 0$ such that $\|u - v\| \geq c$ for all $v \in E_w^\perp$ and for all $u \in E^1 \cap B^1$. If the statement is false, then there exists $\{u_k\} \subset E_w^\perp$ such that $\|u_k\| = 1$ and $\lambda_1\|u_k\|_p^p \to 1$; by rescaling and setting $v_k = u_k\|u_k\|_p^{-p}$, we have $\|v_k\|_p^p = 1$ and $\|v_k\| \to \lambda_1$. Then $\{v_k\}$ is a minimizing sequence for (8), hence $v_k \rightharpoonup v$ and $v_k \to v$ in L^p for some v. Finally $v_k \to v$ in $W_0^{1,p}$ as well because $\|v\| \geq \lambda_1 = \lim_k \|v_k\|$. But this leads to a contradiction, as $v \notin E^1$ and the first eigenvalue is simple. □

Denote by e_1 the positive eigenvector relative to λ_1 and such that $\|e_1\|_p = 1$; let Ω_0 be as in (7) or (11); without restrictions we may assume that $0 \in \Omega_0 \subset \Omega$. Let B_r denote the ball in $\mathbb{R}^n$ of radius r centered in 0. For all $m \in \mathbb{N}$ so large that $B_{2/m} \subset \Omega_0$, we define the functions $\zeta_m : \Omega \to \mathbb{R}$ by

$$\zeta_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m} \\ m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } x \in \Omega \setminus B_{2/m} \ . \end{cases}$$

Let $e^m := \zeta_m e_1$ be the "approximate first eigenfunction" and let $E^m := \text{span}\{e^m\}$ be the corresponding approximate eigenspace.

Lemma 3 *As $m \to \infty$ we have*

$$e^m \to e_1 \quad \textit{in } W \qquad \textit{and} \qquad \|e^m\|^p \leq \lambda_1 + \nu m^{p-n} \ ,$$

for a suitable $\nu > 0$.

Proof. We have

$$\|e^m - e_1\| = \|e_1\nabla\zeta_m + (\zeta_m - 1)\nabla e_1\|_p \leq \|e_1\nabla\zeta_m\|_p + \|(\zeta_m - 1)\nabla e_1\|_p \leq c(m^{p-n} + m^{-n}) \to 0 \ ;$$

therefore $e^m \to e_1$ and by the definition of e^m we also have $\|e^m\|^p \leq \lambda_1 + \nu m^{p-n}$. □

By the definition of $\overline{\lambda}$, for all $\delta > 0$ there exists $w \in W^{-1,p'}$ such that $\min_{u\in E_w^\perp\cap B^1} \|u\| \geq \overline{\lambda} - \delta$; we define $E_\delta^\perp := E_w^\perp$ for such w. We prove that if m is large, then the functional J has a linking geometry:

Lemma 4 *Assume (3), (4) and either (9) or (10); then there exist $\alpha, \delta, \rho > 0$ such that*

$$J(u) \geq \alpha \qquad \forall u \in \partial B_\rho \cap E_\delta^\perp \ .$$

Proof. By (3) and either (9) or (10) we have $G(x,s) + \frac{1}{p^*}|s|^{p^*} \leq \frac{\overline{\lambda}-\sigma}{p} s^p + c|s|^{p^*}$, for all $s \in \mathbb{R}$; therefore if δ is small, then for all $u \in E_\delta^\perp$ we have

$$J(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\overline{\lambda}-\sigma}{p}\int_\Omega |u|^p - c\int_\Omega |u|^{p^*} \geq c_1\|u\|^p - c_2\|u\|_{p^*}^{p^*}$$

and the result follows by choosing ρ small enough. □

Consider the family of functions

$$u_\varepsilon^*(x) := \frac{c_n \varepsilon^{\frac{n-p}{p(p-1)}}}{(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{n-p}{p}}} \qquad (\varepsilon > 0) \ ,$$

which achieve the best Sobolev constant S in the imbedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ (see [17, 22]): take a positive cut-off function $\eta \in C_c^\infty(B_{1/m})$ such that $\eta \equiv 1$ in $B_{1/2m}$, $\eta \leq 1$ in $B_{1/m}$ and $\|\nabla\eta\|_\infty \leq 3m$. For all $\varepsilon > 0$ let

$$u_\varepsilon(x) := \eta(x) u_\varepsilon^*(x) \ , \tag{13}$$

then, as $\varepsilon m \to 0$, the following estimates hold (see [10, 11, 13]):

$$\|u_\varepsilon\|^p \leq S^{n/p} + c(\varepsilon m)^{\frac{n-p}{p-1}} \ , \qquad \|u_\varepsilon\|_{p^*}^{p^*} \geq S^{n/p} - c(\varepsilon m)^{\frac{n}{p-1}} \ . \tag{14}$$

Note that for all $\varepsilon > 0$ and $m \in \mathbb{N}$ we have

$$\mathrm{supp}(u_\varepsilon) \cap \mathrm{supp}(e^m) = \emptyset \ ; \tag{15}$$

consider the set Q_m^ε defined by:

$$Q_m^\varepsilon = \{u \in W |\ u = ae^m + bu_\varepsilon,\ |a| \leq R,\ 0 \leq b \leq R\} \ .$$

Note that ∂Q_m^ε and $\partial B_\rho \cap E^\perp$ link (see [20]) if $R > \rho$. Furthermore, by (15) and by the definition of e^m it follows that if R and m are large enough, then $J(u) \leq 0$ for all $u \in \partial Q_m^\varepsilon$. Hence the functional J satisfies all the assumptions of the linking theorem except for the PS condition.

Let $\Gamma := \{h \in C(Q_m^\varepsilon, W);\ h(u) = u\ ,\ \forall u \in \partial Q_m^\varepsilon\}$; by standard methods we obtain a PS sequence for J at level

$$c = \inf_{h\in\Gamma} \max_{u\in Q_m^\varepsilon} J(h(u)) \ .$$

Moreover, since the identity map $Id \in \Gamma$, we have

$$c \leq \max_{u \in Q_m^\varepsilon} J(u) \ .$$

We will prove Theorems 4 and 5 by showing that the PS sequence fulfills the assumptions of Lemma 1: more precisely, we prove that for ε small enough we have $\max_{u\in Q_m^\varepsilon} J(u) < \frac{1}{n}S^{n/p}$.

4 Proof of Theorem 1

We follow the ideas of [5, 11, 13] and consider the family of functions defined in (13): we claim that if ε is small enough, then

$$\max_{t\geq 0} J(tu_\varepsilon) < \frac{1}{n}S^{n/p} \ . \tag{16}$$

Arguing by contradiction, assume that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$J(t_\varepsilon u_\varepsilon) \geq \frac{1}{n}S^{n/p} \ . \tag{17}$$

As $\varepsilon \to 0$, the sequence $\{t_\varepsilon\}$ is upper and lower bounded by two positive constants and therefore it converges, up to a subsequence; indeed by (14), if $t_\varepsilon \to +\infty$ then $J(t_\varepsilon u_\varepsilon) \to -\infty$, while $t_\varepsilon \to 0$ implies $J(t_\varepsilon u_\varepsilon) \to 0$: in both cases we contradict (17).

Again by (14), as $\varepsilon \to 0$ we have

$$\frac{\|t_\varepsilon u_\varepsilon\|^p}{p} - \frac{\|t_\varepsilon u_\varepsilon\|_{p^*}^{p^*}}{p^*} \leq \frac{S^{n/p}}{n} + \left(t_\varepsilon^p - 1 - \frac{n-p}{n}(t_\varepsilon^{p^*} - 1)\right)\frac{S^{n/p}}{p} + O(\varepsilon^{(n-p)/(p-1)}) \leq \frac{S^{n/p}}{n} + O(\varepsilon^{(n-p)/(p-1)}) \tag{18}$$

the second inequality following from $\max\limits_{x\geq 0}\left\{x^p - 1 - \dfrac{n-p}{n}(x^{p^*} - 1)\right\} = 0$.

By a direct computation, there exist $c_2 > c_1 > 0$ such that, for ε small enough

$$c_1\varepsilon^{1/p} < |x| < c_2\varepsilon^{1/p} \quad \Longrightarrow \quad a < t_\varepsilon u_\varepsilon^*(x) < b \ ;$$

therefore, as $B_\varepsilon \subset \Omega_0$ for small ε, by (4) and (6) we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq c\mu \int_{c_1\varepsilon^{1/p}}^{c_2\varepsilon^{1/p}} r^{n-1}\ dr \ \geq c\mu\varepsilon^{n/p} \ . \tag{19}$$

If $n > p^2$, we infer that there exists a function $\tau = \tau(\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0}\tau(\varepsilon) = +\infty$ and such that for ε small enough we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \tau(\varepsilon)\cdot\varepsilon^{(n-p)/(p-1)} \ ;$$

if $n = p^2$, from (18) we obtain

$$J(t_\varepsilon u_\varepsilon) \leq \frac{S^{n/p}}{n} + O(\varepsilon^p) - c\mu\varepsilon^p < \frac{S^{n/p}}{n}$$

for suitable small ε and large μ. Hence, (17) cannot be true: if we choose ε small enough, by (18) and the estimate of $\int_\Omega G(x, t_\varepsilon u_\varepsilon)$ we have

$$J(t_\varepsilon u_\varepsilon) < \frac{1}{n}S^{n/p}$$

for all $n \geq p^2$. Hence, (16) holds and we obtained a PS sequence (in $\mathcal{C}$) for J at level $\alpha \in \left(0, \frac{S^{n/p}}{n}\right)$: its weak limit is positive, nontrivial and it solves (1) by Lemma 1.

5 Proof of Theorem 2

The proof follows the same steps as for Theorem 1, except for the estimate (19). By (7) there exists an increasing function τ such that $\lim_{x\to+\infty}\tau(x)=+\infty$ satisfying $G(x,s)\geq\tau(s)\cdot s^{\alpha_{np}}$ for a.e. $x\in\Omega_0$ and for all $s\geq 0$; hence,

$$\int_\Omega G(x,t_\varepsilon u_\varepsilon)\geq\tau(c\varepsilon^{(p-n)/p})\varepsilon^{\alpha_{np}(p-n)/p}\int_0^\varepsilon r^{n-1}dr\geq\tau(c\varepsilon^{(p-n)/p})\varepsilon^{(n-p)/(p-1)}\ , \tag{20}$$

where we have used the fact that $\min\limits_{|x|\leq\varepsilon} t_\varepsilon u_\varepsilon(x)\geq c\varepsilon^{(p-n)/p}$.

6 Proof of Theorem 3

We show that the PS sequence obtained by the mountain pass argument is at level below the nontrivial threshold, by choosing a different direction than in the previous proofs. We follow an idea of [7], see also [3]: let e_1 be the first positive eigenfunction of $-\Delta_p$ in Ω and let $u=te_1$ for some $t>0$; by Hölder inequality we obtain

$$J(u)=\frac{\lambda_1-\lambda}{p}\|u\|_p^p-\frac{n-p}{np}\|u\|_{p^*}^{p^*}\leq\frac{\lambda_1-\lambda}{p}|\Omega|^{p/n}\|u\|_{p^*}^p-\frac{n-p}{np}\|u\|_{p^*}^{p^*}\leq\frac{(\lambda_1-\lambda)^{n/p}}{n}|\Omega|\ ,$$

the last inequality following from

$$\forall a,b>0\qquad \max_{x\geq 0}(ax-bx^{n/(n-p)})=\frac{ap}{n}\left(\frac{a(n-p)}{nb}\right)^{(n-p)/p}\ .$$

Then, if $\lambda\in(\lambda_1-\Lambda,\lambda_1)$, we have

$$\max_{t\geq 0}J(te_1)<\frac{1}{n}S^{n/p}$$

and the existence of a solution follows as for Theorem 1.

7 Proof of Theorem 4

We first consider the case $n\geq p^2>1$.
Choose m large enough so that $\nu m^{p-n}<\sigma$, where ν is as in Lemma 3 and σ is as in (9). Then

$$\forall w\in E^m\qquad J(w)\leq 0\ . \tag{21}$$

We claim that there exists $\varepsilon>0$ such that

$$\max_{u\in Q_m^\varepsilon}J(u)<\frac{1}{n}S^{n/p}\ : \tag{22}$$

by contradiction assume that

$$\forall\varepsilon>0\qquad \max_{u\in Q_m^\varepsilon}J(u)\geq\frac{1}{n}S^{n/p}\ . \tag{23}$$

Note that the set $\{u\in Q_m^\varepsilon;\ J(u)\geq 0\}$ is compact and the supremum in (23) is attained. Therefore, for all $\varepsilon>0$ there exist $w_\varepsilon\in E^m$ and $t_\varepsilon\geq 0$ such that, for $v_\varepsilon:=w_\varepsilon+t_\varepsilon u_\varepsilon$, we have

$$J(v_\varepsilon)=\max_{u\in Q_m^\varepsilon}J(u)\geq\frac{1}{n}S^{n/p}\ ,$$

that is

$$\frac{1}{p}\|v_\varepsilon\|^p-\int_\Omega G(x,v_\varepsilon)-\frac{1}{p^*}\|v_\varepsilon\|_{p^*}^{p^*}\geq\frac{1}{n}S^{n/p}\ ,\quad\forall\,\varepsilon>0\ . \tag{24}$$

As in the proof of Theorem 1 we infer that t_ε is bounded between two positive constants. We estimate the lower order term $\int_\Omega G(x,t_\varepsilon u_\varepsilon)$:

Lemma 5 *There exists a function $\tau=\tau(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\tau(\varepsilon)=+\infty$ and such that for ε small enough we have*

$$\int_\Omega G(x,t_\varepsilon u_\varepsilon)\geq\tau(\varepsilon)\cdot\varepsilon^{(n-p)/(p-1)}\ .$$

Proof. If ε is small enough, there exist $c_1>0$ such that $t_\varepsilon u^*_\varepsilon(x)\in(0,\delta)$ for all x satisfying $|x|\geq c_1\varepsilon^{(p-1)/p^2}$; moreover, if $x\in B_{1/2m}$, then $u_\varepsilon(x)=u^*_\varepsilon(x)$. Finally, by (9) and (13) we obtain

$$\begin{aligned}\int_\Omega G(x,t_\varepsilon u_\varepsilon)&\geq c\int_{c_1\varepsilon^{1/p}}^{1/2m}\frac{\varepsilon^{(n-p)/(p-1)}}{[\varepsilon^{p/(p-1)}+r^{p/(p-1)}]^{(n-p)}}r^{n-1}\,dr\\&\geq c\varepsilon^{(n-p)/(p-1)}\int_{c_1\varepsilon^{1/p}}^{1/2m}r^{(p^2-n-p+1)/(p-1)}\,dr\end{aligned}$$

and therefore

$$\int_\Omega G(x,t_\varepsilon u_\varepsilon)\geq c\varepsilon^{(n-p)/(p-1)}\begin{cases}\varepsilon^{(p^2-n)/(p-1)} & \text{if } n>p^2\\ |\log\varepsilon| & \text{if } n=p^2\end{cases}$$

and the function $\tau(\varepsilon)$ is then given. □

If (24) held, by (15), (18), (21) and Lemma 5 we would get

$$J(v_\varepsilon)\leq J(t_\varepsilon u_\varepsilon)\leq\frac{S^{n/p}}{n}+(c-\tau(\varepsilon))\varepsilon^{(n-p)/(p-1)}\ ;$$

and choosing ε small we would contradict (24). Therefore (22) holds.
The case $p<n<p^2$ follows by arguing as above and by taking into account the estimate (20).

8 Proof of Theorem 5

The proof of Theorem 5 follows the same lines as that of Theorem 4, although some modifications are necessary and we need to take m large enough. We set

$$\varepsilon(m)=m^{-(n-p-np)/p} \tag{25}$$

and we remark that $\varepsilon(m)=o(1/m)$ as $m\to\infty$, therefore the estimates (14) still hold. To keep in mind the dependence of ε on m we denote v^m,u^m,w^m instead of $v_\varepsilon,u_\varepsilon,w_\varepsilon$.
As for Theorem 4 we show that (22) holds for some $m\in\mathbb{N}$: if not, (23) is true and for all m large there exist $v^m\in Q_m$ (here we omit the superscript ε), $t_m\geq 0$ and $w^m\in E^m$ such that

$$\frac{1}{p}\|v^m\|^p-\int_\Omega G(x,v^m)-\frac{1}{p^*}\|v^m\|_{p^*}^{p^*}\geq\frac{1}{n}S^{n/p}\qquad\forall\ m\in M \tag{26}$$

where $v^m=t_mu^m+w^m$. If (26) holds, then the sequences $\{w^m\}$ and $\{t_m\}$ satisfy again

$$c_2\geq t_m\geq c_1>0\qquad\text{and}\qquad\|w^m\|\leq c\ . \tag{27}$$

Lemma 6 *Assume (25) and let $m \to \infty$; then, there exists a real function ϕ such that*

$$\lim_{x\to+\infty} \phi(x) = +\infty \qquad \text{and} \qquad \int_\Omega G(x, t_m u^m) \geq m^{n(p-n)/p}\phi(m) \ .$$

Proof. By (11) there exists an increasing function τ such that $\lim_{x\to+\infty} \tau(x) = +\infty$ satisfying $G(x, s) \geq \tau(s) \cdot s^{\beta_{np}}$ for a.e. $x \in \Omega_0$ and for all $s \geq 0$; hence,

$$\int_\Omega G(x, t_m u^m) \geq \tau(c\varepsilon^{(p-n)/p})\varepsilon^{\beta_{np}(p-n)/p} \int_0^\varepsilon r^{n-1} dr \geq \tau(c\varepsilon^{(p-n)/p})\varepsilon^{n(n-p)/(np+p-n)} =: m^{n(p-n)/p}\phi(m) \ ,$$

where we have used (25) and the fact that $\lim_{\varepsilon\to 0} \min_{|x|\leq\varepsilon} t_\varepsilon u_\varepsilon(x) \to +\infty$. □

By (14), (25) and Lemma 6 we infer that there exists a constant $C > 0$ such that if m is large enough we have

$$J(t_m u^m) \leq \frac{1}{n} S^{n/p} - Cm^{n(p-n)/p}\psi(m) \ , \tag{28}$$

where ψ satisfies $\lim_{x\to\infty} \psi(x) = +\infty$.
By (10), Lemma 3 and the equivalence of the norms in E^m we get, for large m,

$$J(w^m) \leq \frac{1}{p}\|w^m\|^p - \frac{\lambda_1}{p}\|w^m\|_p^p - \sigma\|w^m\|_{p^*}^{p^*} \leq c_1\|w^m\|_p^p \cdot m^{p-n} - c_2\|w^m\|_p^{p^*} \ . \tag{29}$$

Consider the function $h(x) = c_1 m^{p-n} \cdot x^p - c_2 \cdot x^{np/(n-p)}$: its derivative vanishes for $x = cm^{-(n-p)^2/p^2}$ and therefore, since (27) holds, we have $h(\|w^m\|_p) \leq cm^{n(p-n)/p}$; then by (29) we obtain the existence of a constant $C > 0$ such that if m is large enough we have

$$J(w^m) \leq Cm^{n(p-n)/p} \ .$$

Finally, taking into account (15) and (28) we obtain

$$J(v^m) = J(t_m u^m) + J(w^m) \leq \frac{1}{n} S^{n/p} - Cm^{n(p-n)/p} (\psi(m) - 1) < \frac{1}{n} S^{n/p} \ ,$$

for m sufficiently large: this contradicts (26) and the proof of Theorem 5 is complete.

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