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On the role of energy convexity in the web function approximation

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Abstract. For a given $p > 1$ and an open bounded convex set $\Omega \subset \mathbb{R}^2$, we consider the minimization problem for the functional $J_p(u) = \int_\Omega (\frac{1}{p}|\nabla u|^p - u)$ over $W_0^{1,p}(\Omega)$. Since the energy of the unique minimizer u_p may not be computed explicitly, we restrict the minimization problem to the subspace of *web functions*, which depend only on the distance from the boundary $\partial\Omega$. In this case, a representation formula for the unique minimizer v_p is available. Hence the problem of estimating the error one makes when approximating $J_p(u_p)$ by $J_p(v_p)$ arises. When Ω varies among convex bounded sets in the plane, we find an optimal estimate for such error, and we show that it is decreasing and infinitesimal with p. As $p \to \infty$, we also prove that $u_p - v_p$

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converges to zero in $W_0^{1,m}(\Omega)$ for all $m < \infty$. These results reveal that the approximation of minima by means of web functions gains more and more precision as convexity in J_p increases.

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex domain. Following [12], we call *web functions* on Ω those functions u which have the same family of level lines as the distance function $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$ from the boundary of Ω. The class of web functions turns out to have relevant properties regarding the minimization of integral functionals J on Ω. In fact, if $\mathcal{K}_p(\Omega)$ denotes the subspace of web functions in $W_0^{1,p}(\Omega)$, and J has the form

$$J(u) = \int_\Omega h(|\nabla u|) - u, \quad u \in W_0^{1,p}(\Omega),$$

suitable assumptions on the integrand h (*not* involving convexity), guarantee the existence of a minimizer for J on $\mathcal{K}_p(\Omega)$ for some $p \geq 1$. Moreover, the energy of such minimizer admits a simple representation formula, see (4) below for the case $h(s) = s^p/p$. It has therefore some interest to ask whether the minimum value of J over $\mathcal{K}_p(\Omega)$ can be used to approximate efficiently the infimum value of J over $W_0^{1,p}(\Omega)$, when the explicit computation of the latter is not possible. To this end, as suggested in [8, 9], one is led to consider the quotient functional

$$\mathcal{E}(\Omega) = \frac{\min_{u \in \mathcal{K}_p(\Omega)} J(u)}{\inf_{u \in W_0^{1,p}(\Omega)} J(u)}.$$

Assuming that $\inf_{u \in W_0^{1,p}(\Omega)} J(u) < 0$ and $J(0) = 0$, the ratio $\mathcal{E}(\Omega)$ is well-defined, and its value falls into the closed interval $[0, 1]$. Of course, the closer $\mathcal{E}(\Omega)$ is to 1, the better is the approximation with web functions for the energy J. So, in order to determine a sharp lower bound for the error when the shape Ω varies, one has to consider the optimization problem

$$\inf\{\mathcal{E}(\Omega) \ : \ \Omega \subset \mathbb{R}^2, \ \Omega \text{ convex bounded}\}. \tag{1}$$

In our previous work [7], we studied the above problem for the quadratic energy

$$J_2(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - u\right)$$

and we proved that the corresponding value for the infimum in (1) is $3/4$, and it is not attained. In this paper we consider the same problem for the energy

$$J_p(u) = \int_\Omega \left(\frac{1}{p}|\nabla u|^p - u\right), \tag{2}$$

where p belongs to the interval $(1, +\infty)$. The Euler equation for the functional J_p on $W_0^{1,p}(\Omega)$ is

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian. It is worth noticing that, thanks to an integration by parts in (3), we get

$$\min_{u \in W_0^{1,p}(\Omega)} J_p(u) = J_p(u_p) = -\frac{p-1}{p}\int_\Omega |\nabla u_p|^p.$$

Thus the integral $\int_\Omega |\nabla u_p|^p$ equals $C^{1/(p-1)}$, where $C = C(\Omega)$ is the best constant for the Sobolev inequality $\|u\|_1^p \le C\|\nabla u\|_p^p$ on $W_0^{1,p}(\Omega)$, and the lower bound for $\mathcal{E}(\Omega)$ can also be used to obtain an upper bound for $C(\Omega)$ in terms of the minimum of J_p over $\mathcal{K}_p(\Omega)$. From the web function point of view, the interest in studying the case of a general exponent p is twofold: to understand the role of convexity in the approximation error $\mathcal{E}$ and to investigate the limit of such error as $p \to \infty$. It is known [2] that the solutions to (3) converge uniformly to the distance function $d_\Omega(x)$ as $p \to \infty$. Therefore, a better (larger) lower bound for $\mathcal{E}$ is expected when the convexity exponent p increases. Our results confirm this expectation. In fact we show that, when J_p is given by (2), the infimum in (1) has the value

$$\mathcal{E}_p := \frac{2p-1}{p-1}\,\frac{1}{2^{p/(p-1)}}.$$

Again, such value is not attained, namely, there does not exist an optimal design. Note that $\mathcal{E}_p$ tends to 1 as $p \to \infty$. Actually we prove a stronger result about the asymptotic behaviour as $p \to \infty$: the difference between the solutions to (3) and the web minimizers of J_p tends to zero in the $W_0^{1,m}(\Omega)$-norm for all $m < \infty$. In order to obtain the sharp lower bound for $\mathcal{E}$, we follow the same line of our proof in [7]. Nevertheless, the quasilinear case of general p is far from being a straightforward extension of the linear case $p = 2$. In particular, the turning point is a p-dependent geometric-integral inequality for convex sets, see Proposition 4. In [7], it was stated just for polygons when $p = 2$. The proof for arbitrary p is more delicate as it is based directly on the Brunn-Minkowski Theorem, without involving the isoperimetric inequality for convex polygons. As a consequence, the validity of the statement is enlarged to the whole class of convex bodies. This makes the inequality under consideration more interesting from the point of view of possible applications. For instance, as a corollary of Proposition 4 applied with $p = 2$, it is easy to show that the disk supports the largest sandpile among all planar domains with a given area (see Remark 1 below). The contents are organized as follows. In Section 2 we state the main results: we establish the optimal lower bound for $\mathcal{E}$ in Theorem 1, and the asymptotic behaviour of solutions as $p \to \infty$ in Theorem 2. In Section 3, we study the extremal domains for the functional $\mathcal{E}$, by showing that $\mathcal{E}$ attains its maximum on a unique shape (disks) and exhibiting a minimizing sequence (made of isosceles triangles) which shows that the lower bound $\mathcal{E}_p$ is sharp. In Section 4, we give bounds for both the denominator and the numerator of $\mathcal{E}$ (the proofs are postponed respectively to Sections 6 and 7) and we show that, combined together, they entail Theorem 1. The proof of Theorem 2 is given in Section 5.

2 Main results

Before stating our results, some preliminaries are in order. Let $\mathcal{C}$ denote the class of all nonempty open bounded convex subsets of $\mathbb{R}^2$. For every $\Omega \in \mathcal{C}$, we denote by R_Ω its *inradius*, i.e. the supremum of the radii of the disks contained in Ω. Then, the level lines of web functions are none other than the boundary of the so-called *parallel sets* $\Omega_t := \{x \in \Omega : d_\Omega(x) \geq t\}$ for $t \in [0, R_\Omega]$. In terms of parallel sets, when J_p is given by (2), the representation formula [8, (25)] for the minimum of J_p over $\mathcal{K}_p(\Omega)$ reads

$$\min_{u \in \mathcal{K}_p(\Omega)} J_p(u) = -\frac{p-1}{p} \int_0^{R_\Omega} \left(\frac{|\Omega_t|^p}{|\partial \Omega_t|} \right)^{1/(p-1)} dt, \tag{4}$$

where $|\Omega_t|$ and $|\partial\Omega_t|$ indicate respectively the area and the perimeter of Ω_t. Formula (4) will be heavily exploited throughout the paper. It is also convenient to set

$$\mathcal{N}(\Omega) = -\frac{p}{p-1} \min_{u \in \mathcal{K}_p(\Omega)} J_p(u), \quad \mathcal{D}(\Omega) = -\frac{p}{p-1} \min_{u \in W_0^{1,p}(\Omega)} J_p(u),$$

so that both the numerator $\mathcal{N}$ and the denominator $\mathcal{D}$ of $\mathcal{E}$ are nonnegative. Note also that $\mathcal{N}$ and $\mathcal{D}$ are homogeneous, while the functional

$$\mathcal{E}(\Omega) = \frac{\mathcal{N}(\Omega)}{\mathcal{D}(\Omega)}$$

is invariant under dilations. We may now give the sharp lower bound for $\mathcal{E}$:

Theorem 1 *For all $\Omega \in \mathcal{C}$ we have*

$$\mathcal{E}(\Omega) > \inf_{\mathcal{C}} \mathcal{E} = \frac{2p-1}{p-1} \frac{1}{2^{p/(p-1)}} =: \mathcal{E}_p.$$

Note that the map $p \mapsto \mathcal{E}_p$ is strictly increasing on $(1, +\infty)$, and that

$$\lim_{p \to 1} \mathcal{E}_p = 0 \qquad \lim_{p \to \infty} \mathcal{E}_p = 1. \tag{5}$$

This tells us that more convexity in the functional gives more precision in the approximation with web functions. Clearly, the limiting behaviour (5) as $p \to \infty$ for the uniform lower bound implies the "pointwise" asymptotic

$$\lim_{p \to \infty} \mathcal{E}(\Omega) = 1 \quad \forall\, \Omega \in \mathcal{C}; \tag{6}$$

note however that the monotonicity of the map $p \mapsto \mathcal{E}_p$ does not imply the same property for each map $p \mapsto \mathcal{E}(\Omega)$ with fixed Ω. Investigating this property could be a further point of interest. On the other hand, we can make more precise how (6) occurs. By using a result in [2] (see also [13]), we prove

Theorem 2 *Let $\Omega \in \mathcal{C}$, and for all $p > 1$ denote by u_p and v_p the minimizers of J_p over $W_0^{1,p}(\Omega)$ and $\mathcal{K}_p(\Omega)$ respectively. Then, for all $m \in [1, +\infty)$,*

$$u_p \to d_\Omega \quad \textit{and} \quad v_p \to d_\Omega \quad \textit{in} \quad W_0^{1,m}(\Omega); \tag{7}$$

in particular, $\lim_{p\to\infty} \|u_p - v_p\|_{1,m} = 0$. Moreover,

$$\lim_{p\to\infty} J_p(u_p) = \lim_{p\to\infty} J_p(v_p) = \int_\Omega d_\Omega(x).$$

Note that the convergence in (7) may not be improved to the $W_0^{1,\infty}(\Omega)$ norm topology. To see this, it suffices to consider the case when Ω is a disk: for all $p > 1$, we have $u_p = v_p \in C^1(\overline{\Omega})$, whereas $d_\Omega \notin C^1(\overline{\Omega})$. The convergence in (7) may also be seen in a different, more geometric, fashion. For all p denote by $\Omega_{p,t}$ the superlevels of u_p:

$$\Omega_{p,t} := \{x \in \Omega \ : \ u_p(x) \geq t\}.$$

By [14, Theorem 2], $u_p^{(p-1)/p}$ is a concave function over Ω, hence the sets $\Omega_{p,t}$ are *convex*. Their limit as $p \to \infty$ can be identified to the parallel set Ω_t. Indeed, as a consequence of the $W_0^{1,m}$ convergence of u_p to d_Ω, we obtain the following corollary of Theorem 2. It enlightens the role of p in the web function approximation for the p-Laplace problems (3): larger p yield level sets of the solutions more similar to parallel sets.

Corollary 1 *Let $\Omega \in \mathcal{C}$. For all $t \geq 0$, we have $\lim_{p\to\infty} \Omega_{p,t} = \Omega_t$ in the Hausdorff topology.*

3 The maximal shape and a minimizing sequence

In this section, we investigate the existence of extremal domains for the functional $\mathcal{E}$. We fix $p > 1$ and for convenience we introduce the conjugate exponent to p, namely $q := \frac{p}{p-1}$. Then, we may rewrite $\mathcal{N}(\Omega)$ in terms of q as

$$\mathcal{N}(\Omega) = \int_0^{R_\Omega} \frac{|\Omega_t|^q}{|\partial\Omega_t|^{q-1}}\, dt, \tag{8}$$

and the statement of Theorem 1 reads

$$\mathcal{E}(\Omega) > \inf_{\mathcal{C}} \mathcal{E} = \frac{q+1}{2^q} \quad \forall \Omega \in \mathcal{C}.$$

We start proving that $\mathcal{E}$ attains its maximum on Ω if and only if Ω is a disk. In other words, the solution u_p to (3) is a web function if and only if Ω is a disk.

Proposition 1 *$\mathcal{E}(\Omega) = 1$ if and only if Ω is a disk.*

Proof. If Ω is a disk then the solution to (3) is radially symmetric, hence $\mathcal{E}(\Omega) = 1$. Assume now that $\mathcal{E}(\Omega) = 1$, namely that the solution u_p to (3) coincides with the minimizing web function v_p. If $\partial\Omega$ is smooth, the statement follows from the extension of a classical result by Serrin to the quasilinear case (see [10, Theorem 1.3] for $1 < p < 2$ and [3, Theorem 4] for arbitrary p). For general $\partial\Omega$, to prove the result we exploit the representation formula for the web minimizer, which gives

$$v_p(x) = \int_0^{d_\Omega(x)} \left(\frac{|\Omega_t|}{|\partial\Omega_t|} \right)^{q-1} dt \ =: \phi(d_\Omega(x)), \tag{9}$$

see [6, Theorem 3.1]. On the other hand, since $v_p = u_p$, it is well-known [11, 16] that $v_p \in C^{1,\alpha}(\Omega)$. Since $\phi'(t) > 0$ for every $t \in [0, R_\Omega)$, this is possible only if d_Ω is of class C^1 in the open set $\{x \in \Omega;\ 0 < d_\Omega(x) < R_\Omega\}$, or equivalently if the sets Ω_t for $t \in [0, R_\Omega)$ have C^1 boundary. Following the arguments of [1, Section I.1.4], we deduce that this regularity property of parallel sets is satisfied uniquely when the equality sign holds in the Bonnesen's inequality

$$|\partial\Omega| \geq \frac{|\Omega|}{R_\Omega} + \pi R_\Omega.$$

On the other hand, the extremal domains for this inequality consist of a rectangle ended by two half disks, that is, are of the form $\Omega = S + B_{R_\Omega}$, where S is a compact (possibly degenerate) line segment, and B_{R_Ω} is a disk of radius R_Ω. Finally, on sets of this shape a direct computation shows that the solution u_p to (3) can be a web function if and only if S is degenerate, that is, if and only if Ω is a disk. □

We say that Ω is a *tangential body to a disk D* if through each point of $\partial\Omega$ there exists a tangent line to Ω which is also tangent to D. For such domains Ω, the following simple characterization of $\mathcal{N}(\Omega)$ in terms of R_Ω holds.

Proposition 2 *Let $\Omega \in \mathcal{C}$ be a tangential body to a disk. Then we have*

$$\mathcal{N}(\Omega) = \frac{|\Omega|}{(q+2)2^{q-1}}\ {R_\Omega}^q.$$

Proof. By the density of polygons and the continuity of $\Omega \mapsto \mathcal{N}(\Omega)$ with respect to the Hausdorff topology [6, Section 6], it suffices to prove the statement when Ω is a circumscribed polygon. In that case, by the arguments in the proof of [7, Proposition 3], we know that $|\partial\Omega| = 2|\Omega|/R_\Omega$, and for all $t \in (0, R_\Omega)$

$$|\Omega_t| = \frac{|\Omega|}{{R_\Omega}^2}(R_\Omega - t)^2, \quad |\partial\Omega_t| = \frac{2|\Omega|}{{R_\Omega}^2}(R_\Omega - t).$$

Therefore,

$$\frac{|\Omega_t|^q}{|\partial\Omega_t|^{q-1}} = \frac{|\Omega|}{2^{q-1}{R_\Omega}^2}(R_\Omega - t)^{q+1}$$

so that the statement follows from (8) after integration over $(0, R_\Omega)$. □

We now determine an upper bound for the infimum of $\mathcal{E}$:

Proposition 3 *There exists a sequence of isosceles triangles* $\{T^h\}_h \subset \mathcal{C}$ *such that*

$$\limsup_{h\to+\infty} \mathcal{E}(T^h) \le \frac{q+1}{2^q}. \tag{10}$$

In particular, $\inf_{\Omega\in\mathcal{C}} \mathcal{E}(\Omega) \le (q+1)2^{-q}$.

Proof. For all integer $h \ge 1$ consider the isosceles triangle $T^h \in \mathcal{C}$ defined by

$$T^h = \left\{(x,y) \in \mathbb{R}^2;\ 0 < y < \frac{\pi}{h},\ \frac{h^2 y}{\pi} - h < x < h - \frac{h^2 y}{\pi}\right\}.$$

Taking into account that $R_{T^h} = \pi h(h^2 + \sqrt{h^4+\pi^2})^{-1}$, using Proposition 2 and letting $h \to \infty$, we obtain

$$\mathcal{N}(T^h) \approx \frac{\pi^{q+1}}{2^{2q-1}} \frac{1}{q+2} h^{-q} \quad \text{as } h \to \infty. \tag{11}$$

In order to find an asymptotic lower bound for $\mathcal{D}(T^h)$, we choose a particular function $v_h \in W_0^{1,p}(T^h)$. We take

$$v_h(x,y) = \frac{1}{q}\left[\left(\frac{\beta_h(x)}{2}\right)^q - \left|\frac{\beta_h(x)}{2} - y\right|^q\right],$$

where β_h is the function defining the two equal sides of T^h, namely

$$\beta_h(x) = \min\left\{\frac{\pi}{h^2}(h+x), \frac{\pi}{h^2}(h-x)\right\} \quad x \in [-h,h].$$

We have

$$\mathcal{D}(T^h) = -q \min_{u\in W_0^{1,p}(T^h)} J_p(u) \ge -qJ_p(v_h) \quad \forall h \ge 1.$$

By the symmetry of T^h and v^h we may restrict our attention to the (half) triangle $T_h = \{(x,y) \in T^h;\ x > 0\}$ and double all the integrals involved. In T_h we have

$$v_h(x,y) = \frac{1}{q}\left[\left(\frac{\pi(h-x)}{2h^2}\right)^q - \left|\frac{\pi(h-x)}{2h^2} - y\right|^q\right].$$

With the change of variables $x = hs$, $y = t/h$, T_h transforms into the fixed triangle

$$T = \left\{(s,t) \in \mathbb{R}^2;\ 0 < t < \pi,\ 0 < s < 1 - \frac{t}{\pi}\right\};$$

moreover, we have:

$$\begin{aligned}
v_h(s,t) &= h^{-q}\left\{\frac{1}{q}\left[\left(\frac{\pi}{2}\right)^q(1-s)^q-\left|\frac{\pi}{2}(1-s)-t\right|^q\right]\right\} =: h^{-q}\alpha_q(s,t)\\
\partial_x v_h(s,t) &= h^{-q-1}\left\{-\left(\frac{\pi}{2}\right)^q(1-s)^{q-1}+\frac{\pi}{2}\left|\frac{\pi}{2}(1-s)-t\right|^{q-2}\left(\frac{\pi}{2}(1-s)-t\right)\right\}\\
&=: h^{-q-1}\beta_q(s,t)\\
\partial_y v_h(s,t) &= h^{-q+1}\left\{\left|\frac{\pi}{2}(1-s)-t\right|^{q-2}\left(\frac{\pi}{2}(1-s)-t\right)\right\} =: h^{-q+1}\gamma_q(s,t)\ .
\end{aligned}$$

Hence, with the above change of variables, we obtain

$$\begin{aligned}
J_p(v_h) = 2\int_T &\left\{\frac{q-1}{q}[h^{2(-q-1)}\beta_q^2(s,t)+h^{2(-q+1)}\gamma_q^2(s,t)]^{q/(2(q-1))}\right.\\
&\left. -h^{-q}\alpha_q(s,t)\right\} ds\,dt\\
&\approx 2h^{-q}\int_T\left\{\frac{q-1}{q}|\gamma_q(s,t)|^{q/(q-1)}-\alpha_q(s,t)\right\} ds\,dt \quad \text{as } h\to\infty.
\end{aligned}$$

Some calculations give

$$\int_T \alpha_q(s,t)\,ds\,dt = \int_T |\gamma_q(s,t)|^{q/(q-1)}\,ds\,dt = \left(\frac{\pi}{2}\right)^{q+1}\frac{2}{(q+1)(q+2)}.$$

Then we get

$$\mathcal{D}(T^h) \geq -qJ_p(v_h) \approx \frac{\pi^{q+1}}{2^{q-1}}\frac{1}{(q+1)(q+2)}h^{-q} \quad \text{as} \quad h\to\infty.$$

Combining this asymptotic inequality with (11) proves (10) by letting $h\to\infty$. □

4 Proof of Theorem 1 (optimal lower bound)

In view of Proposition 3, the proof of Theorem 1 is complete if we show that

$$\mathcal{E}(\Omega) > \frac{q+1}{2^q} \quad \forall\Omega\in\mathcal{C}. \tag{12}$$

We first recall the definition of the piercing function λ, see [7] and previous work by Cellina [5].

For a.e. $y\in\partial\Omega$ the outer unit normal is well-defined and it will be denoted by $n(y)$. For a.e. $x\in\Omega$ the point $\Pi(x)\in\partial\Omega$ such that $|x-\Pi(x)|=d_\Omega(x)$ is uniquely determined. Then we set:

$$\lambda(y) = \sup\{k\geq 0;\ \Pi(y-kn(y))=y\} \quad \text{for a.e. } y\in\partial\Omega\ . \tag{13}$$

We clearly have $0 \leq \lambda(y) \leq R_\Omega$ on $\partial\Omega$. We also extend the definition of λ to points $x \in \Omega$:

$$\lambda(x) = \lambda(\Pi(x)) - |x - \Pi(x)| \quad \text{for a.e. } x \in \Omega. \tag{14}$$

When $\Omega \subset \mathbb{R}^2$ is a convex polygon, (14) enables us to write the measure of the parallel set Ω_t as

$$|\Omega_t| = \int_{\partial\Omega_t} \lambda(y)\, dy. \tag{15}$$

Moreover, we have the following upper bound for $\mathcal{D}(\Omega)$ in terms of the piercing function:

Theorem 3 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon and let u_p be the minimizer of J_p in $W_0^{1,p}(\Omega)$. Then there exists $\delta = \delta(\Omega)$ such that*

$$\mathcal{D}(\Omega) = \frac{1}{q+1} \int_{\partial\Omega} \lambda^{q+1}(y)\, dy \; - \delta(\Omega) \tag{16}$$

and

$$\delta(\Omega) \geq (q-1) \int_\Omega \left[|\nabla u_p(x)|^p - |\nabla u_p(x) \cdot n(\Pi(x))|^p \right] dx. \tag{17}$$

We also have a "dual" statement to Theorem 3, namely a lower bound for $\mathcal{N}(\Omega)$ in terms of the piercing function:

Theorem 4 *For all convex polygon $\Omega \subset \mathbb{R}^2$ we have*

$$\mathcal{N}(\Omega) \geq \frac{1}{2^q} \int_{\partial\Omega} \lambda^{q+1}(y)\, dy.$$

We postpone the proofs of Theorems 3 and 4 until Sections 6 and 7 respectively.

We are now in a position to prove inequality (12). If Ω is a disk, the inequality is obvious since $\mathcal{E}(\Omega) = 1$, see Proposition 1. If Ω is a polygon, (12) is a direct consequence of Theorems 3 and 4. If Ω is a bounded convex set other than a disk, we take an inner approximating sequence $\{P_h\}$ of polygons (that is, $P_h \subset \Omega$ for all h and $P_h \to \Omega$ in the Hausdorff topology as $h \to \infty$). By Theorems 3 and 4 we have

$$\mathcal{E}(P_h) = \frac{\mathcal{N}(P_h)}{\mathcal{D}(P_h)} \geq \mathcal{E}_p \, \frac{\mathcal{D}(P_h) + \delta(P_h)}{\mathcal{D}(P_h)} \quad \forall h \in \mathbb{N}.$$

Therefore, it is enough to show that (a subsequence of) $\{\delta(P_h)\}$ converges to a strictly positive constant C_Ω. We argue by contradiction, assuming that $\delta(P_h) \to 0$. Consider the distance functions $d_h = d_{P_h}$ and d_Ω, and let u^h and u_p be the minimizers of J_p over $W_0^{1,p}(P_h)$ and $W_0^{1,p}(\Omega)$. Extending them by zero on $\Omega \setminus P_h$, we may think of d_h and u^h as defined on the whole Ω. We have

$$\begin{aligned} 0 = \liminf_h \delta(P_h) &\geq (q-1) \int_\Omega \liminf_h \left[|\nabla u^h(x)|^p - |\nabla u^h(x) \cdot \nabla d_h(x)|^p \right] dx \\ &= (q-1) \int_\Omega \left[|\nabla u_p(x)|^p - |\nabla u_p(x) \cdot \nabla d_\Omega(x)|^p \right] dx \geq 0. \end{aligned} \tag{18}$$

The first inequality in (18) follows from (17) and Fatou's lemma. The subsequent equality is deduced from the a.e. convergence $\nabla d_h \to \nabla d$ and $\nabla u^h \to \nabla u_p$, which hold up to subsequences. For the former, see (28) in [7]. The latter can be deduced using the weak $W^{1,p}$-convergence of u^h to u_p (see for instance [4]) combined with the convergence $\|\nabla u^h\|_{L^p(\Omega)} \to \|\nabla u_p\|_{L^p(\Omega)}$ (deriving from the Euler equations (3) in Ω and P_h). Now we conclude from (18) that ∇u_p is parallel to ∇d_Ω a.e. in Ω, that is u_p is a web function. By Proposition 1, this contradicts the assumption that Ω is not a disk.

5 Proof of Theorem 2 (asymptotic behaviour)

The first part of (7) (the convergence $u_p \to d_\Omega$) is just Proposition 2.1 in [2]. In order to prove the convergence $v_p \to d_\Omega$, recall the explicit form (9) of v_p. Then, for all $m \geq 1$, we have

$$\int_\Omega |\nabla d_\Omega - \nabla v_p|^m = \int_\Omega \left[1 - \left(\frac{|\Omega_{d_\Omega(x)}|}{|\partial \Omega_{d_\Omega(x)}|} \right)^{1/(p-1)} \right]^m |\nabla d_\Omega(x)|^m \to 0 \quad \text{as } p \to \infty,$$

and (7) follows. Finally, the limit

$$\lim_{p\to\infty} J_p(u_p) = \lim_{p\to\infty} J_p(v_p) = \int_\Omega d_\Omega(x)$$

follows from (7) by direct computation.

6 Proof of Theorem 3 (upper estimate of $\mathcal{D}$)

We follow the same line of proof adopted for the linear case in [7]. We first show (16). Assume that Ω has N sides and denote them by $F_1, \ldots, F_N$. For simplicity, for all $j = 1, \ldots, N$ we denote by F_j the *open* segment, namely the j-th side of Ω *without* its endpoints. Note that the function λ introduced in (13) is defined in every point of $\partial\Omega$ except for the N vertices. Moreover, $n(y) \equiv n_j$ is a constant vector on F_j. We take a partition of Ω into N open subpolygons $P_1, \ldots, P_N$ defined as follows:

$$P_j = \{ y - t n_j ;\ y \in F_j,\ 0 < t < \lambda(y) \}.$$

Each polygon P_j may also be seen as the (open) epigraph Z_j of the function λ on F_j, namely

$$Z_j = \{(y,t);\ y \in F_j,\ 0 < t < \lambda(y)\}.$$

For all $j \in \{1,\dots,N\}$ let

$$\begin{aligned} W^{1,p}_*(P_j) &:= \{v \in W^{1,p}(P_j);\ v = 0 \text{ on } F_j\}, \\ W^{1,p}_*(Z_j) &:= \{v \in W^{1,p}(Z_j);\ v(y,0) = 0\ \forall y \in F_j\} \end{aligned}$$

and consider the functional

$$H_j(v) = \int_{P_j} \left(\frac{1}{p}|\nabla v|^p - v\right) \quad \forall v \in W^{1,p}_*(P_j).$$

Note that

$$H_j(v) = \int_{F_j}\int_0^{\lambda(y)} \left[\frac{1}{p}|\nabla v(y - tn_j)|^p - v(y - tn_j)\right] dt\, dy \quad \forall v \in W^{1,p}_*(P_j). \quad (19)$$

Let u_p be the minimizer of J_p. Let also u^j denote the restrictions of u_p to P_j $(j = 1,\dots,N)$ and set

$$w^j(y,t) = u^j(y - tn_j) \quad \forall (y,t) \in Z_j. \quad (20)$$

Since $u^j \in C^1 \cap W^{1,p}_*(P_j)$, we have $w^j \in W^{1,p}_*(Z_j)$ and $\dfrac{\partial w^j}{\partial t} = -\nabla u^j \cdot n_j$ so that

$$\left|\frac{\partial w^j}{\partial t}(y,t)\right| \le |\nabla u^j(y - tn_j)| \quad \forall (y,t) \in Z_j. \quad (21)$$

Therefore we get

$$H_j(u^j) \ge I_j(w^j) \quad (j = 1,\dots,N) \quad (22)$$

where

$$I_j(v) := \int_{F_j}\int_0^{\lambda(y)} \left[\frac{1}{p}\left(\frac{\partial v}{\partial t}\right)^p - v\right] dt\, dy \quad \forall v \in W^{1,p}_*(Z_j).$$

On the other hand, at each fixed $y \in F_j$, we have

$$\begin{aligned} &\min\left\{\int_0^{\lambda(y)} \left[\frac{1}{p}|g'(t)|^p - g(t)\right] dt\,;\ g \in W^{1,p}(0,\lambda(y)), g(0) = 0\right\} \\ &\quad = -\frac{1}{q(q+1)}\lambda^{q+1}(y). \end{aligned}$$

Therefore the minimum of I_j on $W^{1,p}_*(Z_j)$, which is attained by the function

$$w(y,t) = \frac{1}{q}\left(\lambda^q(y) - [\lambda(y) - t]^q\right),$$

may be evaluated as

$$\min\{I_j(v);\ v \in W^{1,p}_*(Z_j)\} = -\frac{1}{q(q+1)} \int_{F_j} \lambda^{q+1}(y)\, dy. \tag{23}$$

Then, by (22) and (23) we have

$$\begin{aligned} J_p(u_p) &= \sum_{j=1}^N H_j(u^j) \geq \sum_{j=1}^N I_j(w^j) \\ &\geq -\frac{1}{q(q+1)} \sum_{j=1}^N \int_{F_j} \lambda^{q+1}(y)\, dy \\ &= -\frac{1}{q(q+1)} \int_{\partial\Omega} \lambda^{q+1}(y)\, dy. \end{aligned} \tag{24}$$

This yields

$$\mathcal{D}(\Omega) = -q\, J_p(u_p) \leq \frac{1}{q+1} \int_{\partial\Omega} \lambda^{q+1}(y)\, dy. \tag{25}$$

To complete the proof of (16), it remains to show that the inequality in (25) is strict. We may have equality in (25) only if we have equalities in (21) for all $j = 1, \ldots, N$. But this is equivalent to $u_p \in \mathcal{K}_p(\Omega)$ (i.e. u_p web function), and in turn, to $\mathcal{E}(\Omega) = 1$, a contradiction in view of Proposition 1. We now prove (17). By using (24), (19) and (20) we obtain

$$\begin{aligned} \delta(\Omega) &= \frac{1}{q+1} \int_{\partial\Omega} \lambda^{q+1} - \mathcal{D}(\Omega) \\ &= q \left[J_p(u_p) + \frac{1}{q(q+1)} \int_{\partial\Omega} \lambda^{q+1} \right] \geq q \sum_j [H_j(u^j) - I_j(w^j)] \\ &= q \sum_j \int_{F_j} \int_0^{\lambda(y)} \left[\frac{1}{p} |\nabla u^j(y - tn_j)|^p - u^j(y - tn_j) \right. \\ &\quad \left. - \frac{1}{p} \left| \frac{\partial w^j}{\partial t}(y,t) \right|^p + w^j(y,t) \right] dt\, dy \\ &= (q-1) \sum_j \int_{F_j} \int_0^{\lambda(y)} [|\nabla u^j(y - tn_j)|^p - |\nabla u^j(y - tn_j) \cdot n_j|^p] dt\, dy \\ &= (q-1) \int_\Omega [|\nabla u_p(x)|^p - |\nabla u_p(x) \cdot n(\Pi(x))|^p]\ dx \end{aligned}$$

and (17) follows.

7 Proof of Theorem 4 (lower estimate of $\mathcal{N}$)

The main tool for the proof of Theorem 4 is the following geometric inequality for convex domains Ω, involving the family of its parallel sets. We believe it might have an independent interest in the framework of convex bodies (see Remark 1 below).

Proposition 4 *For every $\Omega \in \mathcal{C}$ and every $q \in [1, \infty)$ the following inequality holds:*

$$\left(\frac{|\Omega|}{|\partial\Omega|} \right)^q \geq \frac{q(q+1)}{2^q} \int_0^{R_\Omega} \frac{|\partial\Omega_t|}{|\partial\Omega|} t^{q-1}\, dt. \tag{26}$$

If $q > 1$, the equality sign holds if and only if Ω is a tangential body to a disk.

Proof. Let us define the function

$$\alpha(s) := \frac{|\partial\Omega_{sR_\Omega}|}{|\partial\Omega|}, \quad s \in [0,1].$$

With a linear change of variables, the inequality (26) can be rewritten as

$$\left(2\int_0^1 \alpha(s)\, ds \right)^q \geq q(q+1) \int_0^1 \alpha(s) s^{q-1}\, ds. \tag{27}$$

The function α is non-negative, strictly monotone decreasing and, by the Brunn-Minkowski Theorem (see [15, Thm. 6.4.3]), it is concave down on $[0,1]$. Hence the function

$$\beta(s) := \alpha(s) - (1-s), \quad s \in [0,1],$$

is non-negative, concave down, and satisfies $\beta(0) = 0$. If $\beta(s) \equiv 0$, then (27) is trivially satisfied; so assume that $\beta(s) \not\equiv 0$. Then, from the concavity of β, we infer that

$$\beta'(s) \leq \frac{\beta(s)}{s} \qquad \text{for a.e.} \quad s \in (0,1). \tag{28}$$

In terms of the function β, the inequality (27) now becomes

$$\left(2\int_0^1 \beta(s)\, ds + 1 \right)^q - 1 \geq q(q+1) \int_0^1 \beta(s) s^{q-1}\, ds. \tag{29}$$

Let us define the following transform of the function β:

$$F(\gamma) := \int_0^1 \beta(s) s^{\gamma-1}\, ds, \quad \gamma \geq 1.$$

As $\beta(s) \not\equiv 0$, we know that $F(\gamma) > 0$. Since $\beta \in AC[0,1]$, by an integration by parts and (28) we obtain the following estimate on the derivative of F:

$$\begin{aligned} F'(\gamma) &= \int_0^1 (\log s)\beta(s)s^{\gamma-1}\,ds \\ &= -\frac{1}{\gamma}\int_0^1 \beta(s)s^{\gamma-1}\,ds - \frac{1}{\gamma}\int_0^1 (\log s)\beta'(s)s^{\gamma}\,ds \\ &\le -\frac{1}{\gamma}F(\gamma) - \frac{1}{\gamma}F'(\gamma), \end{aligned}$$

hence

$$F'(\gamma) \le -\frac{1}{\gamma+1}F(\gamma), \quad \gamma \ge 1.$$

Since $F(1) > 0$, such differential inequality implies

$$F(\gamma) \le \frac{2F(1)}{\gamma+1}\ , \quad \gamma \ge 1. \tag{30}$$

Finally, from (30) we obtain

$$\begin{aligned} \left(2\int_0^1 \beta(s)\,ds + 1\right)^q - 1 &= (2F(1)+1)^q - 1 \ge 2qF(1) \ge q(q+1)F(q) \\ &= q(q+1)\int_0^1 \beta(s)s^{q-1}\,ds, \end{aligned} \tag{31}$$

so that (29) follows and the first part of the proposition is proved. In order to study the equality case in (26), it is enough to observe that, from (31), it holds if and only if $F(1) = 0$, that is if and only if $\beta \equiv 0$. In turn, this is equivalent to $\alpha(s) = 1 - s$, which holds if and only if Ω is a tangential body to a disk (see [15, Lemma 3.1.10]). $\square$

Remark 1 From the classical isoperimetric inequality we have that

$$|\Omega|^{(q-1)/2} \le \left(\frac{|\partial\Omega|}{2\sqrt{\pi}}\right)^{q-1}. \tag{32}$$

From the identity $\int_\Omega d_\Omega^{q-1}(x)\,dx = \int_0^{R_\Omega} |\partial\Omega_t|\, t^{q-1}\,dt$, combining (26) and (32) we obtain the following inequality:

$$\int_\Omega d_\Omega^{q-1}(x)\,dx \le \frac{2}{q(q+1)\pi^{(q-1)/2}}\,|\Omega|^{(q+1)/2}, \quad \forall\Omega \in \mathcal{C},\ q \in [1,+\infty), \tag{33}$$

where the equality sign holds if and only if Ω is a disk. For $q = 2$ the last inequality reads

$$\int_\Omega d_\Omega(x)\,dx \le \frac{1}{3\sqrt{\pi}}|\Omega|^{3/2}.$$

As an application, this special case of inequality (33) can be used in order to prove that among all plane regions of a given area, the circle can support the largest sandpile (see for example [1, p. 8]).

Corollary 2 *Let $\Omega \subset \mathbb{R}^2$ be a convex polygon. Then, for every $q \in [1, \infty)$ the function*

$$\psi_q(t) := \frac{|\Omega_t|^q}{|\partial\Omega_t|^{q-1}} - \frac{q+1}{2^q} \int_{\partial\Omega_t} \lambda^q(z)\, dz \quad t \in [0, R_\Omega] \tag{34}$$

is nonnegative.

Proof. Since Ω is a polygon, we may apply Fubini's Theorem to obtain

$$\int_{\partial\Omega_t} \lambda^q(z)\, dz = q \int_t^{R_\Omega} (s-t)^{q-1} |\partial\Omega_s|\, ds.$$

Then, with a linear change of variables, (34) follows from Proposition 4 applied to the convex domains Ω_t. $\square$

Remark 2 If $q \in \mathbb{N}$, the proof of Corollary 2 does not require Proposition 4. Indeed, by (15) we have $\psi_1(t) \equiv 0$. Then, making use of the isoperimetric inequality for convex polygons in a similar way as in the proof of [7, Lemma 2], one sees that $\psi_q \geq 0$ implies $\psi_{q+1} \geq 0$. Therefore, Corollary 2 follows arguing by induction on $q \in \mathbb{N}$.

We are now ready to prove Theorem 4. Let $v_p \in \mathcal{K}_p(\Omega)$ be the (unique) minimizing web function, i.e.

$$J_p(v_p) = \min_{u \in \mathcal{K}_p(\Omega)} J_p(u).$$

By [6, Theorem 3.1] we have

$$v_p(x) = \int_0^{d_\Omega(x)} \nu^{q-1}(t)\, dt, \quad \nu(t) := \frac{|\Omega_t|}{|\partial\Omega_t|}.$$

Then, since $\mathcal{N}(\Omega) = -qJ_p(v_p)$, using the coarea formula and an integration by parts we infer

$$\begin{aligned} \mathcal{N}(\Omega) &= -q \int_{\partial\Omega} \int_0^{\lambda(y)} \left[\frac{\nu^q(t)}{p} - \int_0^t \nu^{q-1}(s) ds \right] dt\, dy \\ &= -q \int_{\partial\Omega} \int_0^{\lambda(y)} \left[\frac{\nu^q(t)}{p} - (\lambda(y) - t)\nu^{q-1}(t) \right] dt\, dy \\ &= \frac{1}{2^q} \int_{\partial\Omega} \lambda^{q+1}(y)\, dy + \Delta(\Omega) \end{aligned} \tag{35}$$

where

$$\Delta(\Omega) := \int_{\partial\Omega} \int_0^{\lambda(y)} \left[-\frac{q+1}{2^q}[\lambda(y) - t]^q - (q-1)\nu^q(t) + q[\lambda(y) - t]\nu^{q-1}(t) \right] dt\, dy.$$

By Fubini's Theorem and recalling that (14) defines λ in the whole Ω, we may rewrite $\Delta(\Omega)$ as

$$\Delta(\Omega) = \int_0^{R_\Omega} \int_{\partial\Omega_t} \left[-\frac{q+1}{2^q}\lambda^q(z) - (q-1)\nu^q(t) + q\lambda(z)\nu^{q-1}(t) \right] dz\, dt.$$

Finally, by (15), we have that $\nu(t)$ is the integral mean value of λ in $\partial\Omega_t$, and the above equation becomes

$$\Delta(\Omega) = \int_0^{R_\Omega} \left[\frac{|\Omega_t|^q}{|\partial\Omega_t|^{q-1}} - \frac{q+1}{2^q} \int_{\partial\Omega_t} \lambda^q \right] dt = \int_0^{R_\Omega} \psi_q(t)\, dt\ .$$

This, combined with (35) and Corollary 2, proves Theorem 4.

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