

Classification of radial solutions to the Emden–Fowler equation on the hyperbolic space

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Abstract We study the Emden–Fowler equation $-\Delta u = |u|^{p-1}u$ on the hyperbolic space \mathbb{H}^n . We are interested in radial solutions, namely solutions depending only on the geodesic distance from a given point. The critical exponent for such equation is $p = (n+2)/(n-2)$ as in the Euclidean setting, but the properties of the solutions show striking differences with the Euclidean case. While the papers (Bhakta and Sandeep, Poincaré Sobolev equations in the hyperbolic space, 2011; Mancini and Sandeep, Ann Sci Norm Sup Pisa Cl Sci 7(5):635–671, 2008) consider finite energy solutions, we shall deal here with infinite energy solutions and we determine the exact asymptotic behavior of wide classes of finite and infinite energy solutions.

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1 Introduction

In this paper we consider the following nonlinear elliptic equation

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{H}^n, \quad (1.1)$$

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on the simplest example of manifold with negative curvature, the hyperbolic space \mathbb{H}^n , in dimension $n \geq 3$. Δ is the Laplace–Beltrami operator on \mathbb{H}^n and we take $p > 0$. When posed in the Euclidean space \mathbb{R}^n this equation is known as the Emden–Fowler equation and the study goes back to Lane [22], Emden [13], Fowler [14], Chandrasekhar [9] and others, and plays an important role in Astrophysics. Attention was focused on the existence and description of radial solutions. There is a host of later important contributions to the subject; among them we must mention the famous paper by Joseph and Lundgren [21] where a complete classification of radial solutions is done. For more general nonlinear elliptic equations in the Euclidean space \mathbb{R}^n we refer to [4, 16–18, 24, 25, 27].

The existence of radial solutions of the Emden–Fowler equation can be addressed in the hyperbolic space in the setting of radial functions provided we define a function to be radial if it depends on the Riemannian distance r from a pole o . We recall that several models can be used to describe \mathbb{H}^n in an explicit coordinate system. For instance, one may realize \mathbb{H}^n as an embedded hyperboloid in \mathbb{R}^{n+1} , endowed with the inherited metric. It is also possible to use one of the two Poincaré realizations, namely the ball model or the half–space model, in the sense that topologically one can identify \mathbb{H}^n with the unit ball in \mathbb{R}^n or with the upper half–space, each of which endowed with an appropriate metric with the property that the Riemannian distance from any given point to points approaching the topological boundary tends to $+\infty$. Another possible realization is the Klein model, see [2, 28] for a comprehensive discussion. Because of the structure of the isometry group of \mathbb{H}^n it is convenient to describe the hyperbolic space as a *model manifold*, see [19] for details. On such a manifold, a pole o is given and the metric has the form

$$ds^2 = dr^2 + f(r)^2 d\omega^2,$$

for an appropriate function f , where r is the Riemannian distance from the pole o and $d\omega^2$ denotes the canonical metric on the unit sphere. The hyperbolic space \mathbb{H}^n is obtained by making the choice $f(r) = \sinh r$. It is then known, see [11] and references therein, that the radial part of the Laplacian has the explicit expression, on radial functions u ,

$$\Delta_{rad}u(r) = u''(r) + (n - 1)(\coth r)u'(r) = \frac{1}{(\sinh r)^{n-1}} [(\sinh r)^{n-1}u'(r)]', \tag{1.2}$$

and that in such coordinates the volume element is $d\mu = (\sinh r)^{n-1} dr dv_{n-1}$, where dv_{n-1} is the volume element on the $(n - 1)$ -dimensional unit sphere \mathbb{S}^{n-1} .

Our aim in this paper is to classify the smooth radial solutions to (1.1), which satisfy the ODE

$$u''(r) + (n - 1)(\coth r)u'(r) + |u(r)|^{p-1}u(r) = 0 \quad \text{for } r > 0, \tag{1.3}$$

together with the initial conditions

$$u(0) = \alpha, \quad u'(0) = 0. \tag{1.4}$$

The mathematical study of this problem was initiated in [5, 23] for the slightly more general equation $\Delta u + \lambda u + |u|^{p-1}u = 0$ in the range $p \in (1, \frac{n+2}{n-2})$ and *energy solutions* are considered. An energy solution is a function in $H^1(\mathbb{H}^n)$, which is the natural space where variational methods can be successfully employed.

1.1 Results and methods

Here we study the whole class of radial solutions to (1.1)–(1.4) and consider all values of $p > 0$. The study of non-variational solutions is quite natural both in the supercritical case

$p \geq \frac{n+2}{n-2}$, where no radial solution belongs to the energy space, but also in the subcritical case $p \in (1, \frac{n+2}{n-2})$, where there exist infinitely many positive solutions to (1.1) which do not belong to the energy space, see [23].

We determine the intersection properties and the asymptotic behavior at infinity of all radial solutions. In the subcritical case we also prove that there are infinitely many sign-changing solutions of (1.1) not included in the energy space; we determine again their asymptotic behavior as well as the asymptotics of the (sign-changing) energy solutions. We also show that any sign-changing solution has finitely many oscillations, in striking contrast with the Euclidean behavior, see [27]. Finally, in the sublinear case $0 < p < 1$, we prove that no positive radial solution to (1.1) exists and that all sign-changing solutions exhibit infinitely many oscillations.

For the proof of our results we construct a generalized Pohožaev-type functional (notice the alternative spelling Pokhozhaev) [26] in the hyperbolic setting. This construction requires a delicate choice of the test functions involved: instead of powers of r we use particular combinations of the hyperbolic functions. This functional gives several different information in the three cases $p \geq \frac{n-2}{n+2}$, $1 < p < \frac{n-2}{n+2}$, and $0 < p < 1$. We refer to [3, 8, 16, 26] for information on the roles played by these exponents in \mathbb{R}^n . The next step consists in adapting the techniques developed by Ni–Serrin [24, 25] to this new framework. The decay rate of solutions to (1.3) follows by a careful reworking of the differential equation at hand combined with the Pohožaev-type functional.

The exact statements of our results are given in next section, at the end of which we shall also discuss briefly, for the sake of completeness, the linear case $p = 1$. In Sect. 3 we complement our results by further remarks and open problems.

In Sect. 4 we introduce the Pohožaev-type functional which turns out to be very powerful in the study of the qualitative behavior of solutions to (1.3) for any value of p . Section 5 is devoted to the proofs of the results in the supercritical case. Sections 6 and 7 deal, respectively, with the proofs of the results concerning positive solutions and sign-changing solutions in the subcritical case. Finally, in Sect. 8 we briefly deal with the sublinear case $p < 1$. In these sections we shall also provide numerical simulations and plots of the qualitative properties of solutions.

We conclude this introduction with two remarks:

- (1) The case of the hyperbolic space with curvature $-c^2$, $c > 0$ can be easily reduced to the case $c = 1$ that we treat here in detail. Actually, for $c \neq 1$ the radial solutions of (1.1) satisfy an ODE of the same form

$$u''(r) + (n - 1)c \coth(cr) u'(r) + |u(r)|^{p-1}u(r) = 0 \quad \text{for } r > 0. \tag{1.5}$$

The change of variables

$$\bar{u}(r) = c^q u(cr), \quad q = 2/(p - 1), \tag{1.6}$$

transforms solutions $\bar{u}(r)$ of (1.5) into solutions $u(r)$ of (1.3). Note that $c \coth(cr) \rightarrow 1/r$ as $c \rightarrow 0$, so we recover the Euclidean case in that limit. We will comment later on some consequences, see Remark 2.5 (iii).

- (2) We expect that the study of the elliptic problem (1.1) might be relevant for the study of the fine asymptotics of the solutions to the corresponding evolutionary problem $u_t = \Delta u^m$, with $1 > m = \frac{1}{p}$, as initiated in [6], in the spirit of the recent results given in [7]. This same elliptic problem has been recently considered in [10] in the study of existence and stability of finite energy solitons for the subcritical nonlinear Schrödinger equation. Although our results can be extended to the equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{H}^n, \tag{1.7}$$

where f satisfies suitable assumptions, in the present paper we limit ourselves to consider the particular case $f(u) = |u|^{p-1}u$.

2 Classification of radial solutions: statement of results

The results strongly depend on the exponent p in three different ranges. When $p \geq \frac{n+2}{n-2}$ we say that p is *supercritical*, when $1 < p < \frac{n+2}{n-2}$ we say that p is *subcritical*, when $0 < p < 1$ we say that p is *sublinear*.

2.1 The supercritical case

Our main result for this case (see Figs. 1, 2) is the following.

Theorem 2.1 *For any $p \geq \frac{n+2}{n-2}$ equation (1.1) admits infinitely many positive radial solutions $u = u(r)$ and infinitely many negative solutions. In fact, all radial solutions u to (1.1) with $u(0) > 0$, $u'(0) = 0$, are everywhere positive and decay polynomially at infinity with the following rate*

$$\lim_{r \rightarrow +\infty} r^{1/(p-1)}u(r) = c(n, p) := \left(\frac{n-1}{p-1}\right)^{1/(p-1)} \tag{2.1}$$

and

$$\lim_{r \rightarrow +\infty} \frac{u'(r)}{u(r)} = \lim_{r \rightarrow +\infty} \frac{u''(r)}{u'(r)} = 0.$$

On the other hand, for $u(0) < 0$, $u'(0) = 0$, the solutions are everywhere negative and decay polynomially with just the opposite limit $-c(n, p)$, in (2.1). In particular, any such solution u belongs to $L^q(\mu)$ only for $q = \infty$.

This result is qualitatively similar to the Euclidean case, but the power-law decay determined in (2.1) is different. We recall that solutions in the Euclidean case decay like $r^{-2/(p-1)}$, see [25, Theorem 2.2], except in the critical case in which the rate of decay is $r^{-(n-2)}$.

As a byproduct of the proof of Theorem 2.1 we obtain the following non-existence result for solutions to the Dirichlet problem in a ball:

Corollary 2.2 *If $p \geq \frac{n+2}{n-2}$, then for any radius $R > 0$, equation (1.1) admits no positive radial solution $u = u(r)$ satisfying $u(R) = 0$.*

This result may also be obtained by adapting the arguments in [29] valid for $p = \frac{n+2}{n-2}$ to the supercritical range $p \geq \frac{n+2}{n-2}$

2.2 The subcritical case

A first main novelty of this case is the existence of a positive global solution having fast decay at infinity (a hyperbolic nonlinear ground state). This type of solution has been obtained by Mancini–Sandeep [23, Theorems 1.3–1.4, Lemma 3.4] using variational methods in the space

$$H_r^1(\mathbb{H}^n) = \{u \in L^2(\mu); \nabla u \in L^2(\mu), u = u(r)\},$$

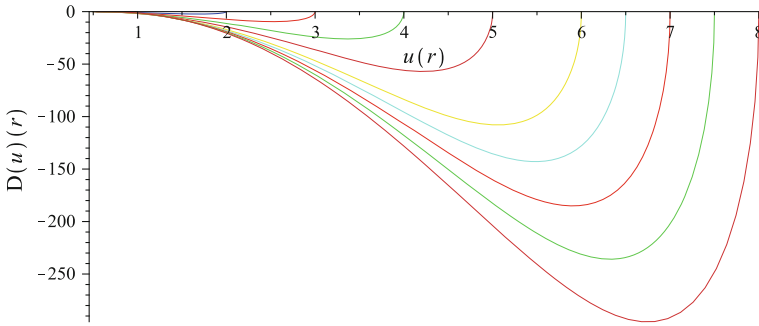


Fig. 1 Phase plot of some solutions when $d = 3, p = 6$ (supercritical case)

where μ is the Riemannian measure, ∇ is the Riemannian gradient and r is the Riemannian distance from a given pole o . We state their result for convenience.

Theorem A ([23] Theorems 1.3–1.4, Lemma 3.4) *Let $1 < p < \frac{n+2}{n-2}$. There exists a unique function $U \in H_r^1(\mathbb{H}^n)$ which is a radial, smooth, positive and bounded solution to the Eq. 1.1. The function U is (radially) decreasing and there exists $\bar{c} > 0$ such that*

$$\lim_{r \rightarrow +\infty} e^{(n-1)r} U(r) = \bar{c}. \tag{2.2}$$

Of course, there exists a unique negative ground state which is given by $-U$. We use the ground state U in the classification of all radial solutions to (1.3)–(1.4). Without loss of generality we restrict ourselves to the case $u(0) = \alpha > 0$. One first class of radial solutions is given by the next result.

Theorem 2.3 *Let $1 < p < \frac{n+2}{n-2}$ and let U be the positive ground state described in Theorem A. Each local solution u to (1.3)–(1.4) satisfying*

$$0 < u(0) < U(0) \tag{2.3}$$

can be extended as a positive solution for $0 < r < \infty$, hence generating a positive radial solution to (1.1) on \mathbb{H}^n . Moreover, there exists a unique $r_0 > 0$ such that $u(r_0) = U(r_0)$ and the asymptotic behavior is given by

$$\lim_{r \rightarrow +\infty} r^{1/(p-1)} u(r) = c(n, p), \tag{2.4}$$

the constant of (2.1). None of these slow-decaying solutions belongs to the energy space.

The first part of the result is known from paper [23], and our contribution is the bound (2.4), which is exactly the same as in the supercritical case, and makes the solutions non-variational.

A second main difference with the supercritical case is the presence of sign-changing solutions, which we now discuss. Recall that in the Euclidean case, Pucci–Serrin [27, Theorem 15] show that all sign-changing solutions in \mathbb{R}^n have infinitely many zeros. We prove that this never happens in \mathbb{H}^n , namely any sign-changing radial solution to (1.1) has a finite number of zeros. Finally, we show that there exists infinitely many solutions of infinite energy.

Theorem 2.4 *Let p and U be as in Theorem 2.3. If u is a solution to (1.3)–(1.4) with $u(0) > U(0)$, then it is sign-changing. Moreover:*

- (i) the solution with initial data α vanishes for the first time at a finite point r_α and the map $\alpha \mapsto r_\alpha$ is a monotone decreasing 1-1 map from $(U(0), \infty)$ onto $(0, \infty)$;
- (ii) any radial sign-changing solution has finitely many zeros;
- (iii) there exist infinitely many radial sign-changing solutions $u \notin H^1(\mathbb{H}^n)$, having exactly one zero, and satisfying

$$\lim_{r \rightarrow +\infty} r^{1/(p-1)}u(r) = -c(n, p), \tag{2.5}$$

the constant of (2.1);

- (iv) for any integer $k \geq 1$ there exists infinitely many solutions to (1.3)–(1.4) having exactly k zeros, and satisfying either (2.4) when k is even or (2.5) when k is odd;
- (v) any radial sign-changing solution $u \in H^1(\mathbb{H}^n)$ satisfies (2.2) for some real constant \bar{c} .

Remark 2.5 (i) As a corollary of our results, we see that we can identify the solution U with the separatrix between the sign-changing class from the globally positive radial solutions in hyperbolic space. In particular all radial solutions u satisfying $u(0) > U(0)$ change sign.

- (ii) The L^∞ -norm $U(0)$ of the variational solution $U(r)$ is the optimal a priori bound for all positive radial and global solutions in the subcritical case. Sign-changing solutions have no a priori bound.
- (iii) When we work in the hyperbolic space with curvature $-c^2 \neq -1$ the rescaling stated in (1.6) implies that the ground state is $U_c(r) = c^{2/(p-1)}U(cr)$. Therefore the a priori bound is

$$M_c = \sup_{r \geq 0} U_c(r) = c^{2/(p-1)}U(0),$$

which goes to zero as $c \rightarrow 0$. This explains how the hyperbolic ground state disappears in the Euclidean limit.

- (iv) In fact, from our proof one sees that Item (iv) can be complemented with the statement that for any integer $k \geq 1$ there exists $\alpha_k > 0$ such that if $u(0) > \alpha_k$, then the solution to (1.3)–(1.4) has at least k zeros.

As for previous results, it was proved in [23, Proposition 4.4] that the corresponding Dirichlet problem admits a unique radial positive solution in any ball of finite radius. Moreover, Bhakta–Sandeep [5, Theorem 5.1] showed that there exist infinitely many sign-changing solutions to (1.1), which can be chosen to be radial and belonging to $H^1(\mathbb{H}^n)$ and that [5, Theorem 4.2] any solution in the energy space, not necessarily radial, satisfies the bound

$$|u(x)| \leq C e^{-\frac{n-2}{2}r} \tag{2.6}$$

where $r = \varrho(x, o)$, $o \in \mathbb{H}^n$ is the pole and ϱ is the Riemannian distance. Moreover, the proof of [5, Theorem 3.1] shows that for radial solutions the upper bound (2.6) can be improved from $\frac{n-2}{2}$ to $\frac{n-1}{2}$. We show that such solutions satisfy the stronger property (2.2) for a suitable real constant \bar{c} and we discuss their oscillation features.

2.3 The sublinear case

In this case we have no globally positive solutions at all. Moreover all sign-changing solutions have infinitely many zeros and “slow” decay at infinity in the sense that the bound $|u(r)| \leq C e^{-(n-1)r}$ cannot hold for all $r > 0$ and a suitable $C > 0$, contrary to (2.2).

Theorem 2.6 *Let $0 < p < 1$. Then there exists no positive radial solution to (1.1). All radial solutions to (1.1) change sign infinitely many times and*

$$\limsup_{r \rightarrow +\infty} e^{\frac{n-1}{p+1}r} u(r) > 0, \quad \liminf_{r \rightarrow +\infty} e^{\frac{n-1}{p+1}r} u(r) < 0. \tag{2.7}$$

2.4 Some comments on the linear case

Note that if $p \neq 1$ and u solves the equation $\Delta u + |u|^{p-1}u = 0$ on \mathbb{H}^n , then $v = c^{1/(1-p)}u$ solves the equation $\Delta v + c|v|^{p-1}v = 0$. But, of course, this simplifying trick does not apply when $p = 1$. For completeness and comparison with the nonlinear cases $p \neq 1$, we recall here some facts about the linear case $p = 1$.

It is well-known [2, 11] that the L^2 -spectrum of $-\Delta$ on \mathbb{H}^n coincides with the half-line $[\Lambda, +\infty)$ where $\Lambda = (n - 1)^2/4$. The equation

$$\Delta u + cu = 0 \quad \text{in } \mathbb{H}^n, \tag{2.8}$$

has a radial positive solution (generalized ground state) with exponential decay for $c = \Lambda$. In Sect. 3 we show that any such solution u is comparable to $(1 + r)e^{-\frac{n-1}{2}r}$ when $r \rightarrow +\infty$ so that, in particular, $u \notin L^2$.

If $0 < c < \Lambda$, then radial solutions to (2.8) with $u(0) > 0$ are positive and slowly decaying at infinity, this behavior bears some similarity with solutions corresponding to small initial data in the subcritical case, see Theorem 2.3. On the other hand, when $c > \Lambda$, it belongs to the L^2 -spectrum. In this case, radial solutions u to (2.8) change sign infinitely many times and $|u(r)| \leq C(1 + r)e^{-\frac{n-1}{2}r}$. This behavior has now to be compared with the sublinear case, see Theorem 2.6. Further details are given in Sect. 3.

3 Further remarks and open problems

3.1 Functional analysis on the hyperbolic space

\mathbb{H}^n is a non-compact manifold and, since the Ricci curvature is constant and negative and the space is simply connected, both the Sobolev and the Poincaré inequality hold. In other words, denoting by μ the Riemannian measure, we have both the inequalities

$$\left[\int_{\mathbb{H}^n} |u|^{2n/(n-2)} \, d\mu \right]^{(n-2)/n} \leq C_1 \int_{\mathbb{H}^n} |\nabla u|^2 \, d\mu \tag{3.1}$$

and

$$\int_{\mathbb{H}^n} u^2 \, d\mu \leq C_2 \int_{\mathbb{H}^n} |\nabla u|^2 \, d\mu \tag{3.2}$$

for all $u \in H^1(\mathbb{H}^n)$, the usual Sobolev space. For generalizations and sharp form of such inequalities see [5, 23]. These properties are important in the variational analysis.

3.2 Some explicit ground states

There are at least three explicit ground state solutions in the subcritical case $1 < p < (n + 2)/(n - 2)$. They are

$$U(r) = \frac{[n^2(n - 1)]^{n-1}}{(1 + \cosh r)^{n-1}} \quad \text{for } p = \frac{n}{n - 1}$$

$$U(r) = \frac{[n(n - 1)]^{(n-1)/2}}{(\cosh r)^{n-1}} \quad \text{for } p = \frac{n + 1}{n - 1}$$

$$U(r) = \left(\frac{n(n - 1)}{n + 1}\right)^{(n-1)/4} \frac{1}{\left((\cosh r)^2 - \frac{n}{n+1}\right)^{(n-1)/2}} \quad \text{for } p = \frac{n + 3}{n - 1}.$$

They solve (1.1) and have the announced exponential decay. The first two solutions have been already found in [23]. These explicit solutions provide extremals in the Sobolev-type inequalities

$$\|u\|_q \leq C \|\nabla u\|_2, \quad q = \frac{2n - 1}{n - 1}, \quad q = \frac{2n}{n - 1}, \quad q = \frac{2n + 2}{n - 1},$$

by making use of Theorems 2.1 and 5.1 in [23], see also [5]. These inequalities can be obtained by interpolating between the L^2 gap inequality (3.2) and the Sobolev inequality (3.1).

3.3 Asymptotic behavior in the linear case

If $c = \Lambda$, then by [2, 11] we know that (2.8) admits positive solutions u . Moreover, these solutions satisfy the upper bound $u(r) \leq C(1 + r)e^{-\frac{n-1}{2}r}$, see e.g. [1]. To show a similar lower bound we proceed using a strategy which inspires the one in the proof of Theorem 2.4. Any radial solution u to (2.8) with $c = \Lambda$ and $u(0) > 0$ satisfies the inequality

$$0 = u''(r) + (n - 1)(\coth r) u'(r) + \Lambda u(r) \leq u''(r) + (n - 1)u'(r) + \Lambda u(r)$$

as long as $u' \leq 0$. In turn, this happens as long as $u \geq 0$ since $[(\sinh r)^{n-1}u']' = -\Lambda u$ and $u'(0) = 0$. Therefore, as long as $u \geq 0$, we have

$$\left[r^2 \left(\frac{e^{\frac{n-1}{2}r}}{r} u(r) \right)' \right]' \geq 0. \tag{3.3}$$

Integrating this inequality on $[0, r]$ gives

$$\left(\frac{e^{\frac{n-1}{2}r}}{r} u(r) \right)' \geq -\frac{u(0)}{r^2}. \tag{3.4}$$

Since the derivative of the function $r \mapsto e^{(n-1)r/2}u(r)$ is positive at $r = 0$, we know that

$$\delta := e^{(n-1)\varepsilon/2}u(\varepsilon) - u(0) > 0$$

provided $\varepsilon > 0$ is sufficiently small. Choose one such ε and integrate (3.4) on $[\varepsilon, r]$ to get

$$\frac{e^{\frac{n-1}{2}r}}{r} u(r) \geq \frac{e^{\frac{n-1}{2}\varepsilon}}{\varepsilon} u(\varepsilon) - u(0) \left(\frac{1}{\varepsilon} - \frac{1}{r} \right) \geq \frac{\delta}{\varepsilon} \quad \forall r \geq \varepsilon.$$

Hence, u never vanishes and there exists $K > 0$ such that $u(r) \geq K(1+r)e^{-\frac{n-1}{2}r}$ for all $r \geq 0$.

If $0 < c < \Lambda$, we claim that for all $\varepsilon > 0$ and suitable constants $c_0, c_1(\varepsilon) > 0$ we have

$$c_0 e^{-\lambda_1 r} \leq u(r) \leq c_1 e^{-(\lambda_1 - \varepsilon)r}, \tag{3.5}$$

λ_1 being defined below. This means that radial solutions to (2.8) with $u(0) > 0$ are positive and slowly decaying at infinity. To prove this claim we set

$$\lambda_1 = \frac{n-1 - \sqrt{(n-1)^2 - 4c}}{2}, \quad \lambda_2 = \frac{n-1 + \sqrt{(n-1)^2 - 4c}}{2} \tag{3.6}$$

so that $\lambda_2 > \lambda_1 > 0$. Then, by arguing as for (3.3), we see that in any interval $[0, R]$ on which u is positive we get

$$\left[e^{(\lambda_2 - \lambda_1)r} (e^{\lambda_1 r} u(r))' \right]' \geq 0. \tag{3.7}$$

By integration we obtain $(e^{\lambda_1 r} u(r))' \geq \lambda_1 u(0) e^{(\lambda_1 - \lambda_2)r}$. Integrating again yields:

$$u(r) \geq \frac{u(0)}{\lambda_2 - \lambda_1} [\lambda_2 e^{-\lambda_1 r} - \lambda_1 e^{-\lambda_2 r}]. \tag{3.8}$$

This first shows that u never vanishes since the r.h.s. is always positive, and then that u is lower bounded at infinity by a multiple of $e^{-\lambda_1 r}$ with λ_1 given in (3.6), so the lower bound (3.5) is proven. As for the upper bound, integrating (3.7) once and taking the conditions at $r = 0$ into account gives, for all $r \geq 0$,

$$u'(r) + \lambda_1 u(r) \geq \lambda_1 u(0) e^{-\lambda_2 r} > 0.$$

By (3.8) we know that u is everywhere positive, hence we have that $\Theta(r) \geq -\lambda_1$ for all $r \geq 0$, where $\Theta(r) := u'(r)/u(r)$. Recall also that Θ is negative. We now compute

$$\Theta'(r) = \frac{u''(r)u(r) - u'(r)^2}{u(r)^2} = \frac{u''(r)}{u(r)} - \Theta(r)^2 = -(n-1)(\coth r)\Theta(r) - c - \Theta(r)^2. \tag{3.9}$$

Assume first that Θ has infinitely many stationary points r_m , so that $\Theta'(r_m) = 0$ and $r_m \rightarrow +\infty$ as $m \rightarrow +\infty$. Hence, using (3.9), we see that

$$\Theta(r_m) [(n-1)(\coth r_m) + \Theta(r_m)] + c = 0.$$

This implies that, as $m \rightarrow +\infty$, $\Theta(r_m)$ tends either to $-\lambda_1$ or to $-\lambda_2$. Since $\Theta(r) \in [-\lambda_1, 0]$, only the first possibility can hold and $\Theta(r) \downarrow -\lambda_1$ as $r \rightarrow +\infty$. If instead Θ has finitely many or no stationary points, then $\Theta(r)$ has a limit as $r \rightarrow +\infty$ and (3.9) shows that $\Theta'(r)$ as well has a limit, necessarily zero (recall that $\Theta(r) \in [-\lambda_1, 0]$). This corresponds to $(n-1)(\coth r)\Theta(r) + c + \Theta^2(r) \rightarrow 0$ as $r \rightarrow +\infty$, which again implies that $\Theta(r) \downarrow -\lambda_1$ as $r \rightarrow +\infty$. Hence this latter fact holds true in any case. In particular for all $\varepsilon > 0$ and all r sufficiently large we have $u'(r)/u(r) \leq -(\lambda_1 - \varepsilon)$, so that $u(r) \leq c_1 e^{-(\lambda_1 - \varepsilon)r}$ for all $r \geq 0$ and a suitable $c_1(\varepsilon) > 0$. This is precisely the upper bound in (3.5).

Finally we discuss the case $c > \Lambda$, so that c belongs to the L^2 -spectrum. As a straightforward application of [12, Theorem 2.1(b)], we see that solutions to (2.8) change sign infinitely many times. It is also known [1] that any eigenfunction u satisfies $|u(r)| \leq C(1+r)e^{-\frac{n-1}{2}r}$ where, by the above calculations, the r.h.s. describes the asymptotic behavior of any eigenfunction corresponding to the eigenvalue Λ .

3.4 The two-dimensional case

Since when $n = 2$ any exponent $p > 1$ is subcritical, Theorems 2.3 and 2.4 hold for any $p > 1$ when $n = 2$. Moreover, when $n = 2$, Theorem 2.6 holds with no modifications.

3.5 Numerics and open problems

- (1) Is it possible to get explicit bounds on $U(0)$, as given by Theorem 2.3?
- (2) In the subcritical case, numerical analysis seems to show that the number of zeros of a sign-changing solution increases as $u(0)$ increases, see Figs. 5, 6. Several natural questions then arise. Is it true that the number of zeros of u is nondecreasing as $u(0)$ increases? What is the asymptotic behavior as $k \rightarrow \infty$ of the shooting levels α_k where the solution to (1.3)–(1.4) switches from k to $k + 1$ zeros? We conjecture that the solutions corresponding to α_k have finite energy. In this respect, [5, Theorem 5.1] proves the existence of infinitely many radial sign-changing finite energy solutions to (1.1), whose energy is arbitrarily large.
- (3) Numerics shows that in the supercritical case and for large dimensions and p large the solutions are ordered and do not intersect. The corresponding result is well-known and quite interesting in the Euclidean setting, see [21]. This seems to require further investigation.

4 A Pohožaev-type functional

We are here interested in studying the behavior of local solutions to (1.3)–(1.4), namely solutions to the Cauchy problem

$$\begin{cases} u''(r) + (n - 1)(\coth r) u'(r) + |u(r)|^{p-1}u(r) = 0 & (r > 0) \\ u(0) = \alpha, \quad u'(0) = 0 \end{cases} \tag{4.1}$$

for some $\alpha > 0$. By arguing as in Proposition 1 in the Appendix of [25], one sees that (4.1) has a C^2 local solution. In fact, these solutions are global and vanish at infinity. This fact is known in the case $1 < p < \frac{n+2}{n-2}$ from [5, Theorem 4.1] in case of general solutions to (1.1). We give here a simpler proof in the case of radial solutions which works for any $p > 0$.

Lemma 4.1 *Let $p > 0$. For any $\alpha > 0$ the local solution $u = u(r)$ to (4.1) may be continued for all $r > 0$ and $\lim_{r \rightarrow +\infty} u(r) = 0$. Also in the non-Lipschitz case $p \in (0, 1)$ each solution intersects the r -axis transversally.*

Proof We introduce the Lyapunov functional

$$F(r) := \frac{1}{2}u'(r)^2 + \frac{1}{p+1}|u(r)|^{p+1} \tag{4.2}$$

and we show that F is decreasing. Indeed, by (1.3) we get

$$F'(r) = \left[u''(r) + |u(r)|^{p-1}u(r) \right] u'(r) = -(n - 1)(\coth r)u'(r)^2 \leq 0. \tag{4.3}$$

This implies in particular that both u and u' are bounded. A straightforward calculation using (4.3) shows that

$$\left[(\sinh r)^{2(n-1)} F(r) \right]' \geq 0$$

with strict inequality holding at least for r small. Hence $F(r) > 0$ for all $r \geq 0$ and in particular u may intersect the r -axis only transversally.

Suppose that u does not admit a limit as $r \rightarrow +\infty$. Since F is decreasing, u cannot oscillate while having a constant sign. Then u admits infinitely many negative minima and infinitely many positive maxima. Let $r_1^{(k)}$ be the sequence of zeros of u and $r_2^{(k)}$ be the sequence of the first maximum points of u after $r_1^{(k)}$. We then have:

$$\begin{aligned}
 |F(r_2^{(k)}) - F(r_1^{(k)})| &= \left| \int_{r_1^{(k)}}^{r_2^{(k)}} F'(r) \, dr \right| = (n - 1) \int_{r_1^{(k)}}^{r_2^{(k)}} (\coth r) u'(r)^2 \, dr \\
 &> (n - 1) \int_{r_1^{(k)}}^{r_2^{(k)}} u'(r)^2 \, dr = -(n - 1) \int_{r_1^{(k)}}^{r_2^{(k)}} u(r)u''(r) \, dr \\
 &= (n - 1) \int_{r_1^{(k)}}^{r_2^{(k)}} u(r) [(n - 1)(\coth r) u'(r) + u(r)^p] \, dr \\
 &> (n - 1)^2 \int_{r_1^{(k)}}^{r_2^{(k)}} u(r) u'(r) \, dr = \frac{(n - 1)^2}{2} u(r_2^{(k)})^2,
 \end{aligned}
 \tag{4.4}$$

where we used the positivity of u and u' in $[r_1^{(k)}, r_2^{(k)}]$ and the fact that $u(r_1^{(k)}) = u'(r_2^{(k)}) = 0$ for all k . As $F(r)$ is decreasing, it has a finite nonnegative limit. Therefore, the l.h.s. of (4.4) tends to zero as $k \rightarrow +\infty$. We conclude that $u(r_2^{(k)})$ tends to zero as well when $k \rightarrow +\infty$. Therefore u tends to zero on its maxima. Similar considerations hold for the minima, so that we can conclude that $u(r) \rightarrow 0$ as $r \rightarrow +\infty$ as claimed. \square

We now introduce the function

$$\varphi_n(r) = \int_0^r (\sinh s)^{n-1} \, ds.
 \tag{4.5}$$

We point out that the integral involving φ_n can be explicitly computed in terms of elementary functions, but since its form appears complicated, we leave it as in (4.5). For any $p > 0$ we also define the function

$$\psi_p(r) = \frac{p + 3}{2(p + 1)} (\sinh r)^{n-1} - (n - 1)\varphi_n(r) \coth r \quad (r > 0)
 \tag{4.6}$$

which is linked to local solutions to (1.3) by means of the following statement.

Lemma 4.2 *Let $p > 0$. For any local solution $u = u(r)$ to (4.1) let*

$$\Psi(r) := \varphi_n(r) \left(\frac{u'(r)^2}{2} + \frac{|u(r)|^{p+1}}{p + 1} \right) + (\sinh r)^{n-1} \frac{u(r)u'(r)}{p + 1}.
 \tag{4.7}$$

Then

$$\Psi(0) = 0 \quad \text{and} \quad \Psi'(r) = u'(r)^2 \psi_p(r).
 \tag{4.8}$$

Proof We use twice (1.3) to obtain

$$\begin{aligned} \Psi'(r) &= (\sinh r)^{n-1} \left(\frac{u'(r)^2}{2} + \frac{|u(r)|^{p+1}}{p+1} \right) + \varphi_n(r) \left(u''(r) + |u(r)|^{p-1} u(r) \right) u'(r) \\ &\quad + \frac{n-1}{p+1} (\sinh r)^{n-2} (\cosh r) u(r) u'(r) + \frac{(\sinh r)^{n-1}}{p+1} u'(r)^2 \\ &\quad + \frac{(\sinh r)^{n-1}}{p+1} u(r) u''(r) \\ &= (\sinh r)^{n-1} \left(\frac{u'(r)^2}{2} + \frac{|u(r)|^{p+1}}{p+1} \right) - (n-1) \varphi_n(r) (\coth r) u'(r)^2 \\ &\quad + \frac{(\sinh r)^{n-1}}{p+1} u'(r)^2 - \frac{(\sinh r)^{n-1}}{p+1} |u(r)|^{p+1} \\ &= \left(\frac{(\sinh r)^{n-1}}{2} - (n-1) \varphi_n(r) (\coth r) + \frac{(\sinh r)^{n-1}}{p+1} \right) u'(r)^2. \end{aligned}$$

Recalling (4.6), this proves the statement. □

In the supercritical case, the function Ψ is negative:

Lemma 4.3 *Assume $n \geq 3$ and $p \geq \frac{n+2}{n-2}$. For any local solution $u = u(r)$ to (4.1) the function Ψ defined in (4.7) satisfies $\Psi'(r) < 0$ for all $r > 0$. Hence, $\Psi(r) < 0$ for all $r > 0$.*

Proof In view of (4.8), the statement follows if we show that $\psi_p(r) < 0$ for all $r > 0$. In turn, since $\lim_{r \rightarrow 0} \psi_p(r) = 0$, it suffices to prove that $\psi'_p(r) < 0$ for all $r > 0$.

Some computations show that

$$\psi'_p(r) = \frac{n-1}{(\sinh r)^2} \left[\varphi_n(r) - \frac{p-1}{2(p+1)} (\sinh r)^n \cosh r \right] =: \frac{n-1}{(\sinh r)^2} h(r). \tag{4.9}$$

We are so led to determine the sign of h . Further computations show that

$$h'(r) = \frac{3p+1}{2(p+1)} (\sinh r)^{n-1} \left[1 - \left(1 + \frac{(n-2)p - (n+2)}{3p+1} \right) (\cosh r)^2 \right]. \tag{4.10}$$

In view of the assumption $p \geq \frac{n+2}{n-2}$, this shows that $h'(r) < 0$ for $r > 0$. Since $h(0) = 0$, this shows that $h(r) < 0$ for $r > 0$ so that $\psi'_p(r) < 0$ for all $r > 0$. □

In the subcritical case, we obtain a different statement

Lemma 4.4 *Assume $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$. For any local solution $u = u(r)$ to (4.1) the function Ψ defined in (4.7) admits a limit as $r \rightarrow +\infty$.*

Proof Let ψ_p be as in (4.6). Since $1 < p < \frac{n+2}{n-2}$, one sees that

$$\exists! R_{n,p} > 0 \quad \text{such that} \quad \psi_p(R_{n,p}) = 0 \tag{4.11}$$

and $\psi_p(r) > 0$ for $r < R_{n,p}$ whereas $\psi_p(r) < 0$ for $r > R_{n,p}$. Then (4.8) shows that $\Psi'(r) < 0$ for all $r > R_{n,p}$ so that $r \mapsto \Psi(r)$ is eventually decreasing and admits a limit as $r \rightarrow +\infty$. □

Remark 4.5 In the sublinear case $0 < p \leq 1$, (4.9) yields $\psi'_p(r) > 0$ for $r > 0$. Hence, $\psi_p(r) > 0$ and, by (4.8), one sees that $\Psi'(r) > 0$ for all $r > 0$. Finally, $\Psi(r) > 0$ and Ψ admits a positive limit (possibly $+\infty$) as $r \rightarrow +\infty$.

5 Supercritical case: proof of Theorem 2.1

We treat the case $p \geq (n + 2)/(n - 2)$. It is clearly sufficient to deal with radial solutions satisfying $u(0) > 0$. Notice that (1.3) may be rewritten as

$$\frac{1}{(\sinh r)^{n-1}} \left[(\sinh r)^{n-1} u'(r) \right]' = -|u(r)|^{p-1} u(r) \tag{5.1}$$

so that the map $r \mapsto (\sinh r)^{n-1} u'(r)$ is strictly decreasing as long as $u(r)$ remains positive. Since its value at $r = 0$ is 0, we infer that $u'(r) < 0$ as long as $u(r)$ remains positive and two cases may occur:

- (1) There exists $\rho > 0$ such that $u(r) > 0$ for $r \in [0, \rho)$, $u(\rho) = 0$, and $u'(\rho) < 0$;
- (2) $u(r) > 0$ for all $r \in [0, \infty)$.

If $\rho > 0$ as in case (1) exists, then $\Psi(\rho) > 0$, where Ψ in as (4.7), contradicting Lemma 4.3. This rules out case (1) and shows that case (2) occurs.

This, together with Lemma 4.1, shows that for any $\alpha > 0$ the Cauchy problem (4.1) admits a unique global positive solution which vanishes at infinity. This proves the first assertion in Theorem 2.1.

The rest of this section is devoted to the proof of the limit (2.1). This requires several intermediate results.

Lemma 5.1 *Assume that $p \geq \frac{n+2}{n-2}$ and let $u = u(r)$ be a solution to (4.1). Then there exist $C_0 > 0$ and $r_0 > 0$ such that*

$$u(r) > C_0 e^{-\frac{n-1}{2}r} \text{ for all } r > r_0.$$

Proof We use the function Ψ defined in (4.7). By Lemma 4.3 we see that there exists $\delta > 0$ and $r_0 > 0$ such that

$$\Psi(r) < -\delta \text{ for all } r > r_0.$$

Therefore,

$$(\sinh r)^{n-1} \frac{u(r)u'(r)}{p + 1} < -\delta \text{ for all } r > r_0$$

so that, for a suitable constant $C > 0$,

$$u(r)u'(r) < -C e^{(1-n)r} \text{ for all } r > r_0.$$

By integrating this inequality over $(r, +\infty)$ we get

$$-\frac{u(r)^2}{2} < -\frac{C}{n-1} e^{(1-n)r} \text{ for all } r > r_0$$

and the stated lower bound follows. □

Lemma 5.2 *There exist no strictly positive constants C, β such that the bound $u(r) \leq C e^{-\beta r}$ holds for all $r \geq 0$.*

Proof In the sequel, C will denote a positive constant which can change from line to line. Suppose by contradiction that, for suitable $C, \beta > 0$ the bound

$$u(r) \leq C e^{-\beta r} \tag{5.2}$$

is satisfied for all $r \geq 0$. Using (5.1) we get

$$\frac{1}{(\sinh r)^{n-1}} [(\sinh r)^{n-1} u'(r)]' = -u(r)^p \geq -C e^{-p\beta r}$$

and hence, using the inequality $\sinh r \leq e^r/2$,

$$[(\sinh r)^{n-1} u'(r)]' \geq -C e^{(n-1-p\beta)r}. \tag{5.3}$$

In order to reach a contradiction one can assume that β is small, in particular that $\beta < (n - 1)/p$. Let us assume this condition and integrate (5.3) between 0 and r . We get, for all $r > 0$:

$$(\sinh r)^{n-1} u'(r) \geq C \left(1 - e^{(n-1-p\beta)r}\right).$$

Therefore, for r sufficiently large, recalling that $n - 1 - p\beta > 0$:

$$u'(r) \geq C e^{-(n-1)r} \left(1 - e^{(n-1-p\beta)r}\right) \geq -C e^{-p\beta r}.$$

The latter inequality can be integrated between r and $+\infty$ so that, recalling that $u(r) \rightarrow 0$ as $r \rightarrow +\infty$, we have $-u(r) \geq -C e^{-p\beta r}$, or:

$$u(r) \leq C e^{-p\beta r},$$

first for sufficiently large r and then for all r since u is continuous. Therefore, compared with (5.2), we have proven a faster decay at infinity for u , since $p > 1$. The procedure can be iterated to prove that $u(r) \leq C e^{-p^k \beta r}$ for all positive integers k such that $p^{k-1} \beta < (n - 1)/p$ and for all $r \geq 0$. Therefore we conclude that the bound $u(r) \leq C_\varepsilon e^{-(n-1-\varepsilon)r}$ holds, given any positive ε , for a suitable constant C_ε and for all positive r . This contradicts Lemma 5.1. □

Lemma 5.3 *The equality*

$$\lim_{r \rightarrow +\infty} \frac{u'(r)}{u(r)} = 0$$

holds true.

Proof We first derive an auxiliary inequality. Let φ_n be defined as in (4.5). We notice that by de l’Hôpital’s rule

$$\lim_{r \rightarrow \infty} \frac{\varphi_n(r)}{(\sinh r)^{n-1}} = \lim_{r \rightarrow \infty} \frac{(\sinh r)^{n-1}}{(n - 1)(\sinh r)^{n-2}(\cosh r)} = \frac{1}{n - 1}. \tag{5.4}$$

This proves that for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$\frac{\varphi_n(r)}{(\sinh r)^{n-1}} > \frac{1}{n - 1 + \varepsilon} \text{ for all } r > r_\varepsilon. \tag{5.5}$$

From Lemma 4.3 we know that the function Ψ defined in (4.7) is strictly negative, namely

$$\frac{\varphi_n(r)}{(\sinh r)^{n-1}} \left(\frac{u'(r)^2}{2} + \frac{u(r)^{p+1}}{p + 1} \right) + \frac{u(r)u'(r)}{p + 1} < 0 \text{ for all } r > 0.$$

In turn, by using (5.5) this shows that

$$\frac{1}{n - 1 + \varepsilon} \frac{u'(r)^2}{2} + \frac{u(r)u'(r)}{p + 1} < 0 \text{ for all } r > r_\varepsilon.$$

Dividing by $u'(r) < 0$ the latter inequality we obtain

$$u'(r) + \frac{2(n - 1 + \varepsilon)}{p + 1} u(r) > 0 \quad \text{for all } r > r_\varepsilon. \tag{5.6}$$

Set now $\Theta(r) := u'(r)/u(r)$, and notice that $\Theta(0) = 0$, $\Theta(r) < 0$ for all $r > 0$. We can rule out the possibility that

$$\limsup_{r \rightarrow +\infty} \Theta(r) < 0$$

since this would imply a uniform exponential upper bound for u , against Lemma 5.2. Therefore, either the statement is true or Θ has no limit. Should the latter possibility hold, Θ has infinitely many maxima and minima r_m , at which Θ' vanishes, with $r_m \rightarrow +\infty$ as $m \rightarrow +\infty$. This means that, at each r_m , $u''u - (u')^2 = 0$. Multiplying equation (1.3) by u , we thus get, for all m ,

$$u'(r_m) [(n - 1)u(r_m)(\coth r_m) + u'(r_m)] = -u(r_m)^{p+1}.$$

Recalling that u never vanishes shows that the quantity in the l.h.s. above never vanishes as well, so that we may rewrite the above equality as

$$u'(r_m) = -\frac{u(r_m)^{p+1}}{(n - 1)u(r_m)(\coth r_m) + u'(r_m)}. \tag{5.7}$$

Since $p > 1$, using the auxiliary inequality (5.6), we have that

$$\begin{aligned} (n - 1)u(r_m)(\coth r_m) + u'(r_m) &> u(r_m) \left[(n - 1)(\coth r_m) - 2\frac{n - 1 + \varepsilon}{p + 1} \right] \\ &= u(r_m)(n - 1) \left[(\coth r_m) - \frac{2}{p + 1} \frac{n - 1 + \varepsilon}{n - 1} \right] \\ &\geq \frac{u(r_m)}{\alpha} \end{aligned}$$

for a suitable positive constant α , provided $0 < \varepsilon < (p - 1)(n - 1)/2$ and m is sufficiently large, say $m \geq \bar{m}$. Putting this bound into (5.7) yields, for $m \geq \bar{m}$:

$$0 > u'(r_m) > -\alpha u(r_m)^p$$

which shows that, for the same set of indices,

$$0 > \frac{u'(r_m)}{u(r_m)} > -\alpha u(r_m)^{p-1}.$$

This and Lemma 4.1 imply that $u'(r_m)/u(r_m) \rightarrow 0$ as $m \rightarrow +\infty$. The definition of the sequence $\{r_m\}$ then shows that $u'(r)/u(r) \rightarrow 0$ as $r \rightarrow +\infty$ even if Θ has infinitely many stationary points. This concludes the proof. \square

We have now all the ingredients to prove (2.1) (see Fig. 2). By using an argument similar to the one in the proof of Lemma 5.3 we show that $\Theta_p(r) := u'(r)/u(r)^p$ has a limit as $r \rightarrow +\infty$. This would clearly hold if Θ_p has finitely many maxima and minima (or none at all). If there is instead a sequence $\{r_m\}$ of extremals, with $r_m \rightarrow +\infty$ as $m \rightarrow +\infty$, we would have, by computing explicitly derivatives and recalling that u never vanishes, $u''(r_m)u(r_m) - pu'(r_m)^2 = 0$. Using (1.3) this would imply that

$$u'(r_m) = -\frac{u(r_m)^{p+1}}{(n - 1)u(r_m)(\coth r_m) + pu'(r_m)}.$$

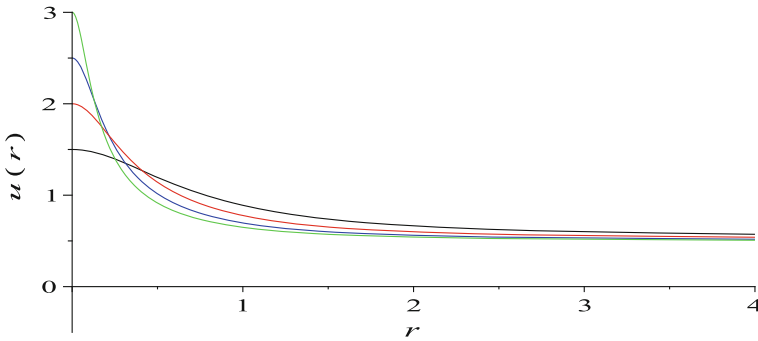


Fig. 2 Plot of some solutions when $d = 3, p = 6$ (supercritical case)

But we have proved in Lemma 5.3 that $u' = o(u)$ as $r \rightarrow +\infty$, in particular along the sequence $\{r_m\}$. Therefore we have $u'(r_m) \sim -u(r_m)^p/(n - 1)$ (where $f \sim g$ means that $f/g \rightarrow 1$), and in particular $\Theta_p(r_m) \rightarrow -1/(n - 1)$ as $m \rightarrow +\infty$. By the definition of the sequence $\{r_m\}$ this entails that $\Theta_p(r) \rightarrow -1/(n - 1)$ as $r \rightarrow +\infty$. In any case $\Theta_p(r)$ has a limit as $r \rightarrow +\infty$.

Using again (1.3), we may write

$$\frac{u''(r)}{u'(r)} + \frac{u(r)^p}{u'(r)} \rightarrow 1 - n \text{ as } r \rightarrow +\infty.$$

Since $\frac{u^p(r)}{u'(r)}$ has been shown to have a limit as $r \rightarrow +\infty$, $\frac{u''(r)}{u'(r)}$ has a limit as well. But then de l'Hôpital's rule and Lemma 5.3 imply that

$$\lim_{r \rightarrow +\infty} \frac{u''(r)}{u'(r)} = \lim_{r \rightarrow +\infty} \frac{u'(r)}{u(r)} = 0,$$

as claimed in the statement. Thus we have proved that

$$\lim_{r \rightarrow +\infty} \frac{u'(r)}{u(r)^p} = \frac{1}{1 - n}.$$

Therefore, for all $\varepsilon > 0$ there exists r_ε such that, for all $r \geq r_\varepsilon$:

$$\frac{p - 1}{n - 1} - \varepsilon \leq (u^{1-p})'(r) \leq \frac{p - 1}{n - 1} + \varepsilon.$$

By integration between r_ε and r we then get

$$\begin{aligned} \left[\left(\frac{p-1}{n-1} + \varepsilon \right) (r - r_\varepsilon) + u(r_\varepsilon)^{1-p} \right]^{-1/(p-1)} &\leq u(r) \\ &\leq \left[\left(\frac{p-1}{n-1} - \varepsilon \right) (r - r_\varepsilon) + u(r_\varepsilon)^{1-p} \right]^{-1/(p-1)}. \end{aligned}$$

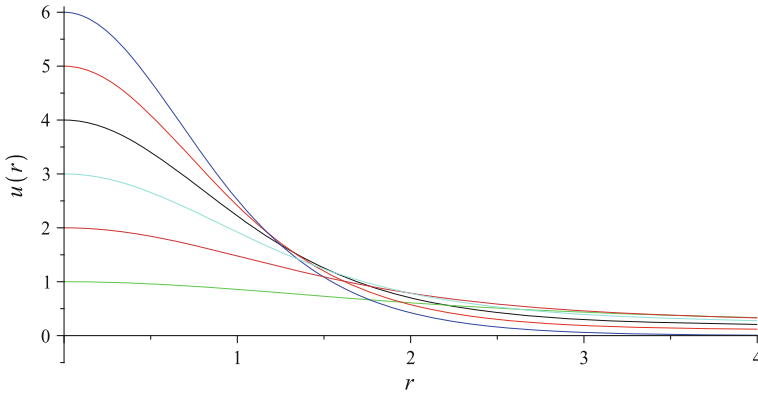


Fig. 3 Plot of some positive solutions when $d = 3, p = 2$ (subcritical case). The special exponentially decaying solution U corresponds to the *blue line* ($U(0) = 6$). (Color figure online)

Multiplying such inequalities by $r^{1/(p-1)}$ we obtain

$$\begin{aligned} \left(\frac{p-1}{n-1} + \varepsilon\right)^{-1/(p-1)} &\leq \liminf_{r \rightarrow +\infty} r^{1/(p-1)}u(r) \\ &\leq \limsup_{r \rightarrow +\infty} r^{1/(p-1)}u(r) \leq \left(\frac{p-1}{n-1} - \varepsilon\right)^{-1/(p-1)}. \end{aligned}$$

Finally, since this holds for all positive ε , the proof of (2.1) is complete. □

6 Subcritical case: proof of Theorem 2.3

Let u be a local solution to (4.1) satisfying (2.3). By [23, Corollary 4.6], there exists a unique $r_0 > 0$ such that $u(r_0) = U(r_0)$. Therefore, u is positive on $[0, +\infty)$ (Figs. 3, 4). Moreover, we have

Lemma 6.1 *Assume that $p > 1$, let u be a solution to (4.1) which is positive for all $r > 0$ and assume that there exist $C, \alpha > 0$ such that $u(r) \leq Ce^{-\alpha r}$ for all $r > 0$. Then also the bound*

$$u(r) \leq Ae^{-(n-1)r} \tag{6.1}$$

holds for a suitable positive constant A and for all $r > 0$.

Proof Proceeding as in the proof of Lemma 5.2, we can improve the bound $u(r) \leq Ce^{-\alpha r}$ by showing that $u(r) \leq C_\varepsilon e^{-(n-1-\varepsilon)r}$ for any $\varepsilon > 0$. To arrive at the stated upper bound, we go back to the proof of that Lemma. It has been shown there that the bound $u(r) \leq Ce^{-\alpha r}$ implies that $[(\sinh r)^{n-1}u'(r)]' \geq -Ce^{(n-1-p\alpha)r}$, with no restriction on α and p needed up to that point. We can then take α sufficiently close to $n - 1$ so that $n - 1 - p\alpha < 0$. Integrating the latter differential inequality between 0 and r we get, say for all $r \geq 1$, $(\sinh r)^{n-1}u'(r) \geq -C$ for a suitable $C > 0$. We integrate again between r and $+\infty$ so that, recalling that $\lim_{r \rightarrow +\infty} u(r) = 0$, we have $-u(r) \geq -Ae^{-(n-1)r}$, or $u(r) \leq Ae^{-(n-1)r}$. □

By Theorem A we know that any u as in the statement of Theorem 2.3 cannot satisfy (6.1). In fact we shall now prove that (6.1) improves to (2.2). Notice indeed that from equation

(1.3) we learn that $r \mapsto (\sinh r)^{n-1}U'(r)$ is decreasing and admits a limit $\ell \in [-\infty, 0)$. If $\ell = -\infty$, de l'Hôpital's rule shows that $U(r)e^{(n-1)r} \rightarrow +\infty$ as $r \rightarrow +\infty$, in contradiction with (6.1). Therefore, there exists $\gamma > 0$ such that

$$\lim_{r \rightarrow +\infty} e^{(n-1)r}U'(r) = -\gamma. \tag{6.2}$$

Using again de l'Hôpital's rule yields that

$$\lim_{r \rightarrow +\infty} e^{(n-1)r}u(r) = \frac{\gamma}{n-1} \tag{6.3}$$

as claimed. Hence, Lemma 6.1 shows that also Lemma 5.2 holds.

According to Lemma 4.4, two cases may occur:

$$(i) \lim_{r \rightarrow +\infty} \Psi(r) < 0 \quad (ii) \lim_{r \rightarrow +\infty} \Psi(r) \geq 0.$$

If case (i) occurs, then we obtain again (5.6) provided r_ε is sufficiently large. Since Lemma 5.2 and (5.6) hold, we can proceed exactly as in the supercritical case $p \geq \frac{n+2}{n-2}$, see all what follows (5.6), and obtain (2.4).

If case (ii) occurs, then $\Psi(r) > 0$ for all $r > 0$, that is

$$\varphi_n(r) \left(\frac{u'(r)^2}{2} + \frac{u(r)^{p+1}}{p+1} \right) + (\sinh r)^{n-1} \frac{u(r)u'(r)}{p+1} > 0 \quad \text{for all } r > 0.$$

In turn, since $\varphi_n(r) < (\sinh r)^{n-1}/(n-1)$, we obtain

$$u'(r)^2 + \frac{2(n-1)}{p+1} u(r)u'(r) + \frac{2}{p+1} u(r)^{p+1} > 0 \quad \text{for all } r > 0.$$

We solve this as a second order inequality with respect to $u'(r)$. By Lemma 4.1 we see that the discriminant of this equation, namely

$$\frac{(n-1)^2}{(p+1)^2} u(r)^2 - \frac{2}{p+1} u(r)^{p+1},$$

is eventually positive, say for $r > r_0$ suitably large. By continuity, one of the following alternatives holds:

$$(a) \quad u'(r) < -\frac{n-1}{p+1} u(r) - \left(\frac{(n-1)^2}{(p+1)^2} u(r)^2 - \frac{2}{p+1} u(r)^{p+1} \right)^{1/2} \quad \text{for all } r > r_0,$$

$$(b) \quad u'(r) > -\frac{n-1}{p+1} u(r) + \left(\frac{(n-1)^2}{(p+1)^2} u(r)^2 - \frac{2}{p+1} u(r)^{p+1} \right)^{1/2} \quad \text{for all } r > r_0.$$

If case (a) holds, then

$$\frac{u'(r)}{u(r)} < -\frac{n-1}{p+1} \quad \text{for all } r > r_0.$$

By integration over (r_0, r) we obtain

$$\log \frac{u(r)}{u(r_0)} \leq -\frac{n-1}{p+1} (r - r_0) \quad \text{for all } r > r_0$$

so that $u(r) \leq ce^{-\frac{n-1}{p+1}r}$, in contradiction with Lemma 5.2.

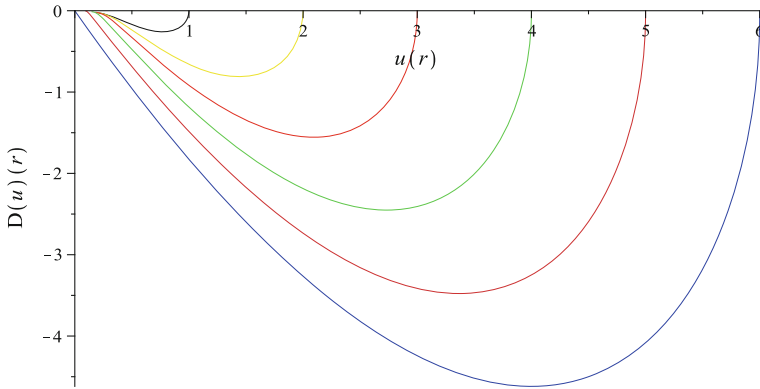


Fig. 4 Phase plot of some positive solutions when $d = 3, p = 2$ (subcritical case). The special exponentially decaying solution U corresponds to the *blue line* ($U(0) = 6$). (Color figure online)

Therefore, case (b) holds, namely

$$u'(r) > \frac{n-1}{p+1} u(r) \left[-1 + \left(1 - \frac{2(p+1)}{(n-1)^2} u(r)^{p-1} \right)^{1/2} \right] \text{ for all } r > r_0.$$

We may then exploit the inequality $\sqrt{1-\alpha} \geq 1-\alpha$ valid for all $\alpha \in [0, 1]$ to get

$$u'(r) > -\frac{2}{n-1} u(r)^p \text{ for all } r > r_0$$

where $r_0 \geq 0$ is such that $u(r) < 1$ for all $r > r_0$. By integrating this inequality over (r_0, r) we obtain

$$u(r) > \frac{c}{r^{1/(p-1)}} \text{ for all } r > r_1$$

for suitable $r_1 > r_0$ and some constant $c > 0$. The limit (2.4) may now be obtained as in the supercritical case, see Lemma 5.3 and the subsequent arguments.

7 Subcritical case: proof of Theorem 2.4

Consider the set P of initial values $u(0) > 0$ corresponding to positive solutions. By continuous dependence, P is closed as a subset of $(0, +\infty)$, hence its complement P^c in $(0, +\infty)$ is open. Moreover we have just proved that $(0, U(0)] \subseteq P$, see Theorem 2.3. As a first main result of this section we want to prove equality of such two sets.

The set P^c coincides with the set of $\alpha = u(0)$ such that the corresponding radial solution u to (4.1) changes sign. This set is contained in $(U(0), +\infty)$. Let us prove that it contains all the large values of α .

Lemma 7.1 *Let u be the solution to (4.1) with $1 < p < \frac{n+2}{n-2}$, corresponding to an initial datum $u(0) = \alpha$ with $\alpha > \alpha_0$ large. Then there exists a first point $r_0 > 0$ depending on α where $u(r_0) = 0$. Moreover, $r_0(\alpha) \rightarrow 0$ as $\alpha \rightarrow +\infty$ and u intersects U exactly once in $[0, r_0]$.*

Proof We use a blow-up method. Let u_λ be the solution of (4.1) with $u_\lambda(0) = \alpha = \lambda^{2/(p-1)}$. Define

$$v_\lambda(\lambda r) = u_\lambda(r)\lambda^{-\frac{2}{p-1}}, \quad \text{so that } v_\lambda(0) = 1. \tag{7.1}$$

Setting $s = \lambda r$, we obtain that v_λ satisfies the equation

$$v_\lambda''(s) + \frac{n-1}{s} \left(\coth \frac{s}{\lambda}\right) \frac{s}{\lambda} v_\lambda'(s) + |v_\lambda(s)|^{p-1} v_\lambda(s) = 0.$$

Let $S > 0$ and take the limit $\lambda \rightarrow \infty$, so that for $0 \leq s \leq S$ we have $\coth(s/\lambda) \rightarrow 1$ uniformly and the solution v_λ tends to the solution \bar{v} of the equation

$$\bar{v}''(s) + \frac{n-1}{s} \bar{v}'(s) + |\bar{v}(s)|^{p-1} \bar{v}(s) = 0, \quad \text{with } \bar{v}(0) = 1,$$

and the convergence of $v_\lambda \rightarrow \bar{v}$ holds in $C^1([0, S])$. Although the equation is singular at $s = 0$, the singularity is associated to the radial Euclidean Laplacian and there is a regular branch of solutions such that $v' = 0$ at $s = 0$. Such solutions depend smoothly on the coefficients and the data, and $0 < S < \infty$ is still to be chosen.

Choose S as bit larger than the first point S_0 for which $\bar{v}(S_0) = 0$ (we also know that $\bar{v}'(S) < 0$), hence, by the $C^1([0, S])$ -convergence we can prove that also $v_\lambda(s)$ crosses transversally the s -axis, in a point s_λ close to S_0 . Going back to the original variables, we have proved that $u_\lambda(r)$ crosses transversally the r -axis when the initial datum $u_\lambda(0) = \lambda^{2/(p-1)}$ is large, at a point r_0 which is approximatively S_0/λ . This also shows that the last claim holds true, indeed $r_0 = O(1/\lambda)$. As long as crossing of the axis is transversal r_0 is a C^1 function of α . □

Lemma 7.2 *P^c is connected so that there exists $A \geq U(0)$ such that if u is a radial solution to (1.1) satisfying $u(0) > A$ then it is sign-changing, whereas if $0 < u(0) \leq A$ then u is positive.*

Proof Suppose by contradiction that P^c contains a maximal connected interval (a, b) with $b > a > U(0)$. Then the radial solutions corresponding to $u(0) = a$ or $u(0) = b$ are everywhere positive. Continuity with respect to the initial datum then implies that the first zero of solutions corresponding both to initial data approaching a from above and b from below must tend to $+\infty$. This violates uniqueness of the solution to the Dirichlet problem on balls, as proved in [23, Proposition 4.4]. Therefore, P^c is connected as claimed. □

We want to prove that $A = U(0)$. To prove this fact we shall need several intermediate results.

For any two positive solutions u_α and u_β defined in some interval $(0, R)$ and satisfying $u_\alpha(0) = \alpha, u_\beta(0) = \beta$ we study the sign of the difference $w = u_\alpha - u_\beta$, which satisfies the equation

$$w'' + (\coth r)w' + b(r)w = 0, \quad b(r) = \frac{u_\alpha(r)^p - u_\beta(r)^p}{u_\alpha(r) - u_\beta(r)} = p\tilde{u}^{p-1}(r)$$

where $\tilde{u}(r)$ is an intermediate value between $u_\alpha(r)$ and $u_\beta(r)$. We have $b(r) > 0$.

Lemma 7.3 *Let $\alpha_1 > \alpha_2 \geq \alpha_3 > \alpha_4 > 0$. Then the first intersection between u_{α_1} and u_{α_2} cannot take place after the first intersection between u_{α_3} and u_{α_4} .*

Proof Let $w_1 = u_{\alpha_1} - u_{\alpha_2}$ and $w_2 = u_{\alpha_3} - u_{\alpha_4}$. We have $w_1(0), w_2(0) > 0$ and $w'_1(0) = w'_2(0) = 0$. As long as there is no zero of w_1 and w_2 , say for $0 \leq r \leq r_1$, they satisfy

$$w''_1 + (\coth r) w'_1 + b_1(r) w_1 = 0, \quad w''_2 + (\coth r) w'_2 + b_2(r) w_2 = 0,$$

and the assumptions imply that $b_1 > b_2$ in $[0, r_1]$. A Sturm-type Theorem then implies that w_1 must vanish at least when w_2 vanishes for the first time, or before. In fact, the equation satisfied by the quotient $z = w_1/w_2$ is

$$z w'' + (2w'_2 + (\coth r)w_2) z' = -z(b_1 - b_2)w_2 \leq 0,$$

and moreover $z(0) > 0$ and $z'(0) = 0$. Integrating once this inequality it follows that $z'(r) \leq 0$, hence $z \leq z(0)$ for $0 < r < r_1$, which implies that z_2 cannot vanish before z_1 . \square

Given a continuous function $u = u(r)$ defined on a closed interval $I \subset \mathbb{R}$, the positive and negative sets of u are defined as follows:

$$\Omega_u^+ = \{r \in I : u(r) > 0\}, \quad \Omega_u^- = \{r \in I : u(r) < 0\}$$

A component of Ω_u^+ (or Ω_u^-) is a maximal open connected subset.

We define the number of sign changes of u in I as the number (finite or infinite) of connected components of $\{r : u(r) \neq 0\}$ minus one. This is also known briefly as the *lap number* of u in I , and is denoted by $Z(u, I)$. Alternatively, $Z(u, I)$ is the supremum of the integer numbers k such that there exist $k + 1$ points from I such that $r_0 < r_1 < \dots < r_k$, satisfying

$$u(r_j) \cdot u(r_{j+1}) < 0 \quad \forall j = 0, 1, \dots, k - 1.$$

We will apply the known facts of the theory of lap numbers (see [15,31]) to the difference of two solutions of the Emden–Fowler equation defined in an interval $I = [0, R] \subset [0, \infty)$. One of the solutions in this application will be U , the other one u_α for some $\alpha > 0, \alpha \neq U(0)$. We call

$$v_\alpha(r) = u_\alpha(r) - U(r)$$

Let I_α be the closure of the first connected component of $\Omega_{u_\alpha}^+$ and let $Z_\alpha := Z(v_\alpha, I_\alpha)$.

Lemma 7.4 *For $0 < \alpha < U(0)$ and $\alpha > A$ we have $Z_\alpha = 1$.*

- Proof* (i) For $\alpha < U(0)$ we know that $I_\alpha = [0, \infty)$ and the set where $v_\alpha \neq 0$ has two connected components, a negative one near $r = 0$ and a positive one for large r , separated by a unique zero r_α . Hence, $Z_\alpha = 1$ for all $\alpha \in (0, U(0))$.
- (ii) Let us now consider $\alpha > A$. We have proved in Lemma 7.1 that for all $\alpha \geq \alpha_0$ large enough, u_α crosses U transversally at some small r_α and after that v_α is negative until u_α becomes zero at some $R_\alpha > 0$, also small. This means that

$$Z_\alpha := Z(v_\alpha, [0, R_\alpha]) = 1 \quad \forall \alpha \geq \alpha_0.$$

- (iii) We prove now that Z_α does not change in value when we let $\alpha \downarrow A$. Indeed, for all $\alpha \in (A, \infty)$ v_α is positive at the beginning and negative at the end of I_α so that $Z_\alpha \geq 1$. The fact that $Z_\alpha < \infty$ is obtained by contradiction at a limit point of the set of zeros, since at that point v_α must have horizontal tangent, which contradicts the local uniqueness of solutions of the Cauchy problem (4.1).

Next, we note that an increase in the number of connected components as α decreases cannot take place near the ends of the defining interval. But in the middle of the interval an increase of Z_α means a new small interval of positivity and another one of negativity, hence

Z_α is always odd. When α reaches α_0 , defined as the supremum of those α for which $Z_\alpha > 1$, there exist a zero of v_α with horizontal tangent at any point which is limit of the connected component that is lost as $\alpha \rightarrow \alpha_0$. Again, this goes against the local uniqueness of solutions of the Cauchy problem associated to the differential equation considered. Therefore, $Z_\alpha = 1$ for all $\alpha > A$. \square

We can now prove

Lemma 7.5 *One has $A = U(0)$.*

Proof Assume that $A > U(0)$. The result that U_A has at least one intersection with U comes from Lemma 7.3, applied to $\alpha_1 = A, \alpha_2 = \alpha_3 = U(0)$ and $0 < \alpha_4 < U(0)$, since we know that the last two cross once at a finite distance, hence U_A must cross U before. The result that it does not have two or more intersections is proved by contradiction: if there is a later point at which $v_A(r_1) > 0$ using continuity of the solutions Z_α would be larger than 1 for $\alpha = A + \varepsilon$ near A . Hence, $Z(A) = 1$.

But then, since U_A cannot change sign, it will be less than U for all r sufficiently large, and this is a contradiction with the uniqueness of global positive fast-decaying solutions. \square

The just proved lemma shows that $\alpha = U(0)$ is the threshold between positive solutions and sign-changing solutions to (4.1). Let us now prove the four Items in Theorem 2.4.

Let $R(a)$ denote the first zero of radial solutions corresponding to $u(0) = a > U(0)$. By [23, Proposition 4.4] we know that the corresponding Dirichlet problem admits a unique radial positive solution in any ball of finite radius. This fact and the above results show that $a \mapsto R(a)$ is strictly decreasing with $\lim_{a \rightarrow U(0)} R(a) = +\infty$ and $\lim_{a \rightarrow +\infty} R(a) = 0$. This proves Item (i)

In order to prove Item (ii) we argue by contradiction assuming that u oscillates infinitely many times. By Lemma 4.1, for any $\varepsilon > 0$ there exists r_0 such that $|u(r)| < \varepsilon^{1/(p-1)}$ for all $r > r_0$. Let $r_1 > r_0$ be a point such that $u(r_1) = 0$, let $r_2 > r_1$ the first maximum of u after r_1 and $r_3 > r_2$ the next zero of u , so that $u > 0$ in $[r_2, r_3]$. For any $r \in (r_2, r_3]$ we also have $u'(r) < 0$ in view of (5.1). Since $\coth r > 1$ for all $r > 0$, from (1.3) we thus get

$$u'' + (n - 1)u' + \varepsilon u \geq 0 \quad \forall r \in [r_2, r_3].$$

We can and shall assume now that $\varepsilon < (n - 1)^2/4$. Put

$$\lambda_1 = \frac{n - 1 - \sqrt{(n - 1)^2 - 4\varepsilon}}{2}, \quad \lambda_2 = \frac{n - 1 + \sqrt{(n - 1)^2 - 4\varepsilon}}{2} \tag{7.2}$$

so that $\lambda_2 > \lambda_1 > 0$. In the new variable $s = r - r_2$ the differential inequality satisfied by $V(s) = u(r - r_2)$ reads

$$\left[e^{(\lambda_2 - \lambda_1)s} (e^{\lambda_1 s} V(s))' \right]' \geq 0,$$

with V satisfying the initial conditions $V(0) = B > 0, V'(0) = 0$. Integrating from 0 to s yields:

$$e^{(\lambda_2 - \lambda_1)s} (e^{\lambda_1 s} V(s))' \geq \lambda_1 V(0) + V'(0) = \lambda_1 B \tag{7.3}$$

or, equivalently, $(e^{\lambda_1 s} V)' \geq \lambda_1 B e^{(\lambda_1 - \lambda_2)s}$. Integrating again from 0 to s yields:

$$e^{\lambda_1 s} V(s) - B \geq \frac{\lambda_1 B}{\lambda_2 - \lambda_1} \left[1 - e^{-(\lambda_2 - \lambda_1)s} \right].$$

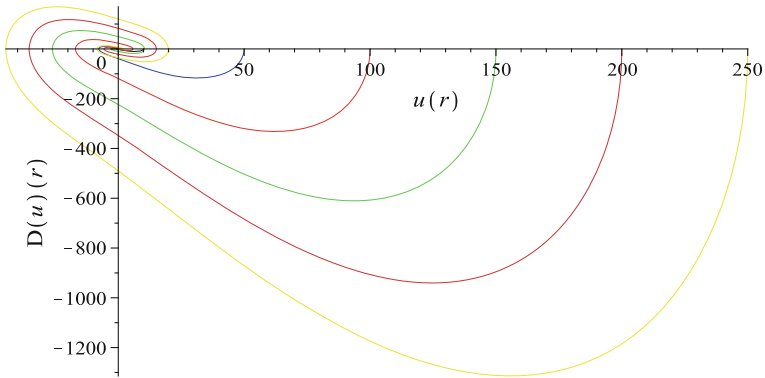


Fig. 5 Phase plot of some sign-changing solutions when $d = 3, p = 2$ (subcritical case)

This latter inequality can also be written as

$$V(s) \geq \frac{B}{\lambda_2 - \lambda_1} [\lambda_2 e^{-\lambda_1 s} - \lambda_1 e^{-\lambda_2 s}]$$

or, in the original variable r ,

$$u(r) \geq \frac{B}{\lambda_2 - \lambda_1} [\lambda_2 e^{-\lambda_1(r-r_2)} - \lambda_1 e^{-\lambda_2(r-r_2)}] \quad \forall r \in [r_2, r_3]. \tag{7.4}$$

But the r.h.s. of the latter inequality is positive since $\lambda_2 > \lambda_1$. This means that $u(r_3) > 0$, a contradiction. Therefore, $u(r)$ never vanishes for $r > r_2$ and hence its number of zeros is finite. This proves Item (ii).

In fact, the above arguments allow to prove the following stronger statement:

Lemma 7.6 *Let u be a sign-changing solution to (4.1) and assume that there exists $r_2 > 0$ such that*

$$u'(r_2) = 0 \quad \text{and} \quad \varepsilon := |u(r_2)|^{p-1} < \frac{(n-1)^2}{4}. \tag{7.5}$$

Then u does not vanish on $[r_2, +\infty)$ and there exists $C > 0$ such that

$$|u(r)| \geq C e^{-\lambda_1 r} \quad \forall r \geq r_2, \tag{7.6}$$

where λ_1 is as in (7.2).

Proof This result was proved above in the case where r_2 is a relative maximum (at positive level) for u . The case where r_2 is a relative minimum (at negative level) for u can be treated similarly. Indeed, suppose that r_2 is a local minimum of u such that $\varepsilon := |u(r_2)|^{p-1} < \frac{(n-1)^2}{4}$. For contradiction, assume that there exists $r_3 > r_2$ such that $u(r_3) = 0$. We can use the fact that u is negative in $[r_2, r_3)$ and that u' is positive in $(r_2, r_3]$ to conclude that

$$u''(r) + (n-1)u'(r) + \varepsilon u(r) \leq 0 \quad \forall r \in [r_2, r_3].$$

Proceeding exactly as before we can conclude that $u(r_3) < 0$, a contradiction. Hence u remains negative for all $r > r_2$. Moreover, for a suitable $B > 0$, the bound

$$u(r) \leq -\frac{B}{\lambda_2 - \lambda_1} \left[\lambda_2 e^{-\lambda_1(r-r_2)} - \lambda_1 e^{-\lambda_2(r-r_2)} \right] \quad \forall r \in [r_2, r_3]$$

holds true. This also proves (7.6). □

We are now ready to prove Item (iii) (see Figs. 5, 6). For any $\delta > 0$ consider the radial solution u to (4.1) with initial data $\alpha = U(0) + \delta$ and let $R(\alpha)$ denote the first zero of u . By C^1 continuous dependence w.r.t. to initial data, we see that

$$R(\alpha) \rightarrow +\infty \quad \text{and} \quad u'(R(\alpha)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

The Lyapunov functional F defined in (4.2) satisfies $F(R(\alpha)) = u'(R(\alpha))^2/2$. Since F is decreasing this implies that, if we denote by r_2 the first minimum of u after $R(\alpha)$, $|u(r_2)| < [(p + 1)u'(R(\alpha))^2/2]^{1/(p+1)} \rightarrow 0$ as $\delta \rightarrow 0$. Provided δ is sufficiently small, the above steps then imply that (7.5) holds. Hence, Lemma 7.6 shows that for any such δ (7.6) holds. In view of (7.2) we know that $\lambda_1 \rightarrow 0$ as $\delta \rightarrow 0$. This proves that for all $\delta > 0$ sufficiently small the radial solutions corresponding to $u(0) = \alpha = U(0) + \delta$ do not belong to $L^2(\mathbb{H}^n)$, hence they do not belong to $H^1(\mathbb{H}^n)$ as well. Moreover, again by Lemma 7.6, these solutions only have one zero.

In order to complete the proof of Item (iii), we still have to prove (2.5). To this end, consider one of the solutions just found, namely a radial sign-changing solution u to (4.1) passing from positive to negative at some r_0 , being the unique zero of u . Since $|u(r)| < \varepsilon$ sufficiently small for all $r > r_0$, we may proceed as in the proof of (7.3). Setting $s = r - r_1$ and $V(s) = u(r - r_1)$ with $r_1 > r_0$ the first (unique) minimum of u , we find that

$$e^{(\lambda_2 - \lambda_1)s} \left(e^{\lambda_1 s} V(s) \right)' \leq \lambda_1 V(0) + V'(0) = -\lambda_1 B$$

for $B = -V(0) > 0$. This inequality implies that

$$V'(s) \leq -\lambda_1 V(s) - B e^{-\lambda_2 s} < -\lambda_1 V(s). \tag{7.7}$$

We now go back to the proof of Lemma 5.3. It was proved there that $\Theta(r) := u'(r)/u(r) \rightarrow 0$ as $r \rightarrow +\infty$ in the supercritical case and for positive solutions only. The proof was based on the bound (5.6). A careful investigation of the proof of Lemma 5.3 shows that it relies on the fact that $\frac{2(n-1+\varepsilon)}{p+1} < n - 1$ for ε small. One can check that Lemma 5.3 still holds for negative solutions satisfying $u'(r) \leq -v u(r)$ for all r sufficiently large and a $v < n - 1$. In the present situation, (7.7) shows that $u'(r) \leq -\lambda_1 u(r)$ for all $r \geq r_1$. Since λ_1 is given by (7.2) we see that $\lambda_1 < n - 1$ provided ε is small. Therefore, Lemma 5.3 applies and $u'(r) = o(u(r))$ as $r \rightarrow +\infty$. Since the argument outlined just after the end of the proof of Lemma 5.3 depends exactly on the fact that $u'(r) = o(u(r))$ as $r \rightarrow +\infty$, with the same arguments used there we conclude that (2.5) holds. This completes the proof of Item (iii).

In order to prove Item (iv) we remark that Lemma 7.6 shows that *zeros of u_α may enter from infinity one by one as $\alpha = u_\alpha(0)$ increases*. A further zero may only enter when the last critical point of u_α violates (7.5). So, since for $u_\alpha(0) < U(0)$ we have no zeros at all, it suffices to prove that the number of zeros of u_α can be arbitrarily large. To see this, we make use of the same notations in the proof of Lemma 7.1. Since \bar{v} has infinitely many zeros in view of [27, Theorem 15], we know that for any integer k there exists $S_k > 0$ such that \bar{v} has exactly k zeros in $[0, S_k]$ and $\bar{v}(S_k) \neq 0$. The convergence $v_\lambda \rightarrow \bar{v}$ in $C^1([0, S_k])$ shows that also v_λ has exactly k zeros in $[0, S_k]$ provided λ is sufficiently large. In turn, by (7.1) also u_λ has exactly k zeros in the interval $r \in [0, S_k/\lambda]$. Therefore, u_λ has at least k zeros, provided λ is large enough. The last part of the statement follows as in the proof of Item (iii).

For the proof of Item (v) we consider radial solutions u in $H^1(\mathbb{H}^n)$ and notice that, by the proof of [5, Theorem 3.1], it follows that $|u|$ satisfies a uniform exponential upper bound.

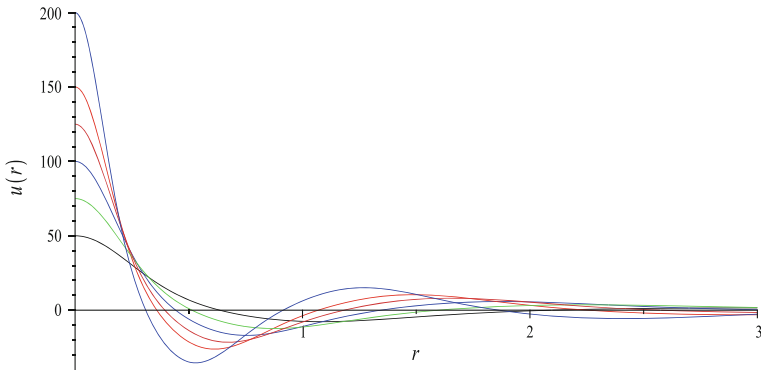


Fig. 6 Plot of some sign-changing solutions when $d = 3, p = 2$ (subcritical case)

Both the proof of Lemma 6.1 and the proof of Lemma 5.2 can be repeated with no change besides integrating over (r_1, r) instead of over $(0, r)$, where r_1 is the last stationary point of u . Then there exists $c > 0$ such that $|u(r)| \leq c e^{-(n-1)r}$ for all r . Passing from this to (2.2) can be done as in the comments just after the proof of Lemma 6.1. This completes the proof of Item (v) and of Theorem 2.4. □

8 Sublinear case: proof of Theorem 2.6

For contradiction, assume that there exists a local solution u to (4.1) which can be extended to a positive solution on $[0, +\infty)$. By (5.1) we have

$$-\left[(\sinh r)^{n-1} u'(r)\right]' = u(r)^p (\sinh r)^{n-1} \quad \text{for all } r > 0. \tag{8.1}$$

By integrating this inequality and taking into account that u is decreasing we obtain

$$-(\sinh r)^{n-1} u'(r) = \int_0^r u(s)^p (\sinh s)^{n-1} ds \geq u(r)^p \varphi_n(r) \quad \text{for all } r > 0. \tag{8.2}$$

where φ_n is as defined in (4.5). Hence,

$$-u'(r)u(r)^{-p} \geq \frac{\varphi_n(r)}{(\sinh r)^{n-1}} \quad \text{for all } r > 0.$$

By integrating over $(0, r)$ and recalling that $0 < p < 1$ we infer

$$-u(r)^{1-p} + u(0)^{1-p} \geq (1-p) \int_0^r \frac{\varphi_n(t)}{(\sinh t)^{n-1}} dt.$$

By letting $r \rightarrow +\infty$ and by recalling (5.4) we see that the l.h.s. tends to $u(0)^{1-p}$ whereas the r.h.s. tends to $+\infty$, contradiction.

The same argument can be used to show that any solution admits infinitely many oscillations. Assume for contradiction that u admits a local maximum at some r_0 and remains positive for all $r \geq r_0$. Then integrate (8.2) over (r_0, r) to reach the very same contradiction.

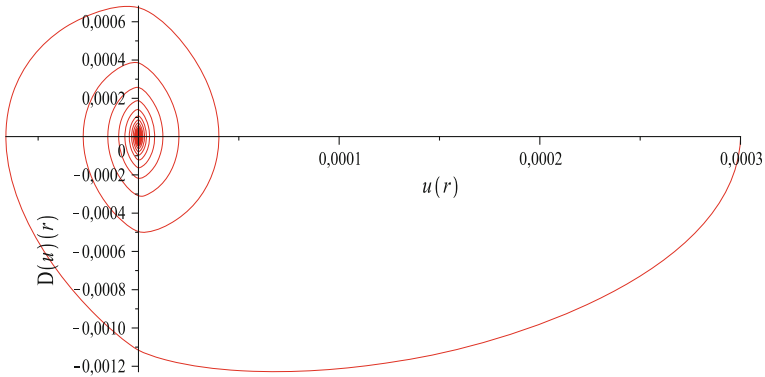


Fig. 7 Phase plot of one sign-changing solution when $d = 3, p = \frac{1}{2}$ (sublinear case)

Similarly, one obtains a contradiction if u admits a local minimum at some r_0 and remains negative for all $r \geq r_0$. Hence, any local solution to (4.1) admits infinitely many oscillations.

For the proof of (2.7), by Remark 4.5 we see that any solution u satisfies $\Psi(r) > \delta$ for some $\delta > 0$, provided $r \geq r_\delta > 0$. Hence, in any of the critical points $\rho \geq r_\delta$ of u , we have

$$\Psi(\rho) = \varphi_n(\rho) \frac{|u(\rho)|^{p+1}}{p+1} > \delta$$

so that

$$|u(\rho)| > \left(\frac{\delta(p+1)}{\varphi_n(\rho)} \right)^{1/(p+1)} \geq ce^{-\frac{n-1}{p+1}\rho}$$

and (2.7) follow (see Fig. 7).

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