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On Positivity for the Biharmonic Operator under Steklov Boundary Conditions

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Abstract

The positivity-preserving property for the inverse of the biharmonic operator under Steklov boundary conditions is studied. It is shown that this property is quite sensitive to the parameter involved in the boundary condition. Moreover, positivity of the Steklov boundary value problem is linked with positivity under boundary conditions of Navier and Dirichlet type.

1. Introduction

Let Ω be a bounded and smooth domain in $\mathbb{R}^n$ ($n \geqq 2$) and consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 \text{ and } \Delta u = \alpha \frac{\partial u}{\partial \nu} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\alpha \in C(\partial\Omega)$, $f \in L^2(\Omega)$ and ν is the outside normal (we will also use $u_\nu = \frac{\partial u}{\partial \nu}$). As usual a domain means an open and connected subset. Elliptic problems with parameters in the boundary conditions are called *Steklov problems* from their first appearance in [29]. In the case of the biharmonic operator, these conditions were first considered by Kuttler & Sigillito [20] and Payne [25], who studied the isoperimetric properties of the first (constant) eigenvalue δ_1, see its variational characterization in formula (21) below. As pointed out by Kuttler [18, 19], δ_1 is the sharp constant for a priori estimates for the (second-order) Laplace equation under nonhomogeneous Dirichlet boundary conditions. More recently, the whole spectrum of the biharmonic Steklov problem was studied in [11]. We also refer to [4, 5] for some related nonlinear problems and for a first attempt to describe the positivity preserving property for (1). We are here interested precisely in which conditions on α guarantee that (1) is positivity preserving, meaning $f \geqq 0$ implies that $u \geqq 0$.

A model from elasticity. When Ω is a planar domain, problem (1) appears in the description of the deformation of a linear elastic hinged or supported plate. Its energy is defined by

$$E(u;\Omega) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 + (1-\sigma)\left(u_{xy}^2 - u_{xx}u_{yy}\right) + f\,u\right) dx. \tag{2}$$

Here f is the exterior force and u the bending of the plate; σ is the Poisson ratio, see for example [30, Chapter VI] or [10]. The Poisson ratio is defined by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with constants λ, μ depending on the material. Usually $\lambda \geqq 0$ and $\mu > 0$ hold true and hence $0 \leqq \sigma < \frac{1}{2}$. Some exotic materials have a negative Poisson ratio (see [21]). For metals the value σ lies around 0.3 (see [22, p.105]). For rubber $\mu \ll \lambda$ and σ is near 0.5.

Fixing the position of the plate on the boundary leads to the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$. Minimizing the energy E over this space gives the Euler equation

$$\int_\Omega \left(\Delta u \Delta v + (1-\sigma)\left(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}\right) + f\,v\right) dx = 0$$

for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Assuming $u \in H^4(\Omega)$ we may integrate by parts and find, setting $\nu = (\nu_1, \nu_2)$ the outward normal and using $v = 0$ on $\partial\Omega$, that

$$0 = \int_\Omega \left(\Delta^2 u - f\right) v \,dx + \int_{\partial\Omega} \left(\Delta u + (1-\sigma)\left(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\right)\right) v_\nu \,ds.$$

Note that the term $(1-\sigma)\left(u_{xy}^2 - u_{xx}u_{yy}\right)$ in (2) has no influence on the differential equation but does change one of the boundary conditions on none-straight boundary parts. Indeed one obtains $\Delta u + (1-\sigma)\left(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\right) = 0$ on $\partial\Omega$. Let us recall that for $u = 0$ on $\partial\Omega$ it holds that

$$\begin{aligned}
&\Delta u + (1-\sigma)\left(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\right) \\
&= \sigma\Delta u + (1-\sigma)\left(2u_{xy}\nu_1\nu_2 + u_{xx}\nu_1^2 + u_{yy}\nu_2^2\right) \\
&= \sigma\Delta u + (1-\sigma)\,u_{\nu\nu} = u_{\nu\nu} + \sigma\kappa u_\nu = \Delta u - (1-\sigma)\,\kappa u_\nu.
\end{aligned}$$

Here κ is the curvature of the boundary measured from inside, that is, positive where the boundary of the domain is convex. This implies that the physically relevant boundary value problem reads

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 \text{ and } \Delta u = (1-\sigma)\,\kappa u_\nu & \text{on } \partial\Omega. \end{cases} \tag{3}$$

A system approach. The fourth-order boundary vale problem in (1) can be rewritten as a system of two coupled second-order equations:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = -\alpha u_\nu & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

For $\alpha \geqq 0$ this system shares the properties of a cooperative system (see [23]) although here part of the coupling occurs through the boundary condition. This will allow us to use Krein–Rutman-type arguments to find a positive first “eigenvalue” $\delta_1 > 0$ such that if $0 \leqq \alpha < \delta_1$ then $f > 0$ implies $u, v > 0$. The upper bound will be sharp.

For $\alpha \ngeqq 0$ the system is generically not positivity preserving but nevertheless we will show that there is $\delta_c < 0$ such that for $\delta_c \leqq \alpha < \delta_1$ an $f > 0$ implies that $u > 0$ (but in general not $v > 0$). For the plate problem $\alpha = (1-\sigma)\kappa$ which is negative on concave boundary parts. Since $\delta_c < 0$ it means that there may exist *nonconvex domains* for which upward exterior forces f still guarantee positivity of the bending u for the hinged plate described by (3). The proof of $\delta_c < 0$ uses pointwise estimates for the Green function which need sufficiently smooth domains. For nonsmooth domains not only u_ν might not be well-defined but if the domain has a reentrant corner then by [24] one knows that the $H^1(\Omega) \times H^1(\Omega)$-solution of a system not necessarily coincides with the $H^2(\Omega)$-solution for the original fourth-order problem.

Set-up of the paper. In Section 2 the main results will be stated. The more elaborate proofs are presented in the following sections. These proofs combine tools from the Hilbert setting in Section 4, with tools in the Schauder setting in Section 5. The intricate estimates of the kernels involved, which are necessary for the Schauder setting, can be found in Section 6.

2. Main results

Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^2$ and consider the space

$$\mathcal{H}(\Omega) := H^2(\Omega) \cap H_0^1(\Omega).$$

Definition 2.1. For $f \in L^2(\Omega)$ we say that u is an $\mathcal{H}$-solution of (1) if $u \in \mathcal{H}(\Omega)$ and

$$\int_\Omega \Delta u \Delta v \, \mathrm{d}x - \int_{\partial\Omega} \alpha \, u_\nu v_\nu \, \mathrm{d}\sigma = \int_\Omega f v \, \mathrm{d}x \quad \text{for all } v \in \mathcal{H}(\Omega). \tag{5}$$

Note that $\mathcal{H}$-solutions are well defined for $\alpha \in C(\partial\Omega)$. For $u \in H^4(\Omega)$ one may integrate by parts to find indeed that an $\mathcal{H}$-solution of (5) satisfies the boundary value problem in (1).

Throughout the paper, we will use the following

Notation 2.2. *Let ϕ be a (continuous) function defined on the domain D.*

- $\phi > 0$ *means* $\phi(x) > 0$ *for all* $x \in D$.
- $\phi \ngeqq 0$ *means* $\phi(x) < 0$ *for some* $x \in D$.
- $\phi \gneqq 0$ *means* $\phi(x) \geqq 0$ *for all* $x \in D$ *and* $\phi \not\equiv 0$.
- $\phi^+ = \max(\phi, 0)$ *and* $\phi^- = \max(-\phi, 0)$.

For $\phi \in L^p(D)$ these estimates hold except for the usual almost everywhere. A function ϕ is described as being positive if $\phi \gneqq 0$. We say $\phi > \psi$ $(\ngeqq, \gneqq)$, whenever $\phi - \psi > 0$ $(\ngeqq, \gneqq)$ holds.

Our first statement describes existence, uniqueness and positivity of an $\mathcal{H}$-solution. A crucial role is played by a “first eigenvalue”:

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^2$. Then there are $\delta_1 := \delta_1(\Omega) \in (0, \infty)$ and $\delta_c := \delta_c(\Omega) \in [-\infty, 0)$ such that the following holds for a function $\alpha \in C(\partial\Omega)$.*

1. *If $\alpha \geqq \delta_1$ and if $0 \lneqq f \in L^2(\Omega)$ then* (1) *has no positive $\mathcal{H}$-solution.*
2. *If $\alpha = \delta_1$ then there exists a positive eigenfunction, that is, problem* (1) *admits a nontrivial $\mathcal{H}$-solution u_1 with $u_1 > 0$ in Ω for $f = 0$. Moreover, the function u_1 is, up to multiples, the unique solution of* (1) *with $f = 0$ and $\alpha = \delta_1$.*
3. *If $\alpha \lneqq \delta_1$ then for any $f \in L^2(\Omega)$ problem* (1) *admits a unique $\mathcal{H}$-solution u.*
   - (a) *If $\delta_c \leqq \alpha \lneqq \delta_1$ then $0 \lneqq f \in L^2(\Omega)$ implies $u \gneqq 0$ in Ω.*
   - (b) *If $\delta_c < \alpha \lneqq \delta_1$ then $0 \lneqq f \in L^2(\Omega)$ implies $u > 0$ in Ω.*
   - (c) *If $\alpha < \delta_c$ then there are $0 \lneqq f \in L^2(\Omega)$ with $0 \not\leq u$.*

Proof. The claim follows by taking $\beta = 1$ in Theorem 4.1. □

The result described in Theorem 2.3 quite closely resembles the structure for the resolvent of the biharmonic operator under Navier boundary conditions or for the biharmonic operator under Dirichlet boundary conditions in case the domain is a ball, see [15]. For all these problems the scheme is as follows:

$\exists f > 0$ with $u \not\geq 0$	$\forall f > 0: \exists u$ and $u \geq 0$		$\forall f > 0$ if $\exists u$ then $u \not\geq 0$
δ_c	0	δ_1	$\alpha \longrightarrow$

For Navier and Dirichlet boundary conditions it is known that the α for which $f \geqq 0$ implies $u \geqq 0$ is in fact an interval; this result is similar to that which we have obtained for (1).

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a smooth bounded domain and let $\alpha_i \in C(\partial\Omega)$ with $i = 1, 2$. Suppose that $\alpha_1 \leqq 0 \leqq \alpha_2$ are such that both for $\alpha = \alpha_1$ and $\alpha = \alpha_2$ we have the following: for all $f \in L^2(\Omega)$ there exists an $\mathcal{H}$-solution $u = u_i$ $(i = 1, 2)$ for* (1)*, and moreover,*

$$f \gneqq 0 \text{ implies } u \gneqq 0. \tag{6}$$

Then for any $\alpha \in C(\partial\Omega)$ satisfying $\alpha_1 \leqq \alpha \leqq \alpha_2$, and for each $f \in L^2(\Omega)$, a unique $\mathcal{H}$-solution of (1) *exists and* (6) *holds true.*

Proof. The result follows combining Lemma 5.11 and Theorem 4.1. □

A crucial difference with the biharmonic boundary value problems mentioned above however is that in those cases it holds that $\delta_c(\Omega) \in (-\infty, 0)$ while for problem (1) it might indeed happen that $\delta_c(\Omega) = -\infty$. Nevertheless, for general domains one cannot expect to have the positivity-preserving property for any negative α. This is stated in the next results which show that the limit situation where $\delta_c(\Omega) = -\infty$ is closely related with the positivity preserving property for the biharmonic Dirichlet problem

$$\begin{cases} \Delta^2 u = f \geqq 0 & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

To this end, let us recall that the positivity preserving property does not hold in general domains $\Omega \subset \mathbb{R}^n$ for (7). The unique solution $u \in H_0^2(\Omega)$ of (7) may not be positive if $f \gneqq 0$ and Ω have particular shapes, see for example [7, 12, 27]: these domains Ω fail to have the positivity preserving property under Dirichlet boundary conditions. On the other hand, the problem (7) is positivity preserving when Ω is a ball in any dimension [6], when Ω is some limaçon [9] or when Ω is a (planar) small perturbation of a disk [14, 26]. Note that (7) corresponds to the limit case $\alpha = -\infty$.

Our next statement establishes that positivity may be transmitted from the Steklov problem to the Dirichlet problem:

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^2$. If for every $m \in \mathbb{N}$ and $0 \lneqq f \in L^2(\Omega)$ the $\mathcal{H}$-solution of (1) with $\alpha = -m$ is positive, then for every $0 \lneqq f \in L^2(\Omega)$ the solution $u \in H_0^2(\Omega)$ of (7) satisfies $u \gneqq 0$.*

Proof. See Section 8. □

For this result we can only show a partial converse. Instead of just assuming positivity of the solution of (7) we need to assume that this solution is strongly positive, meaning that the solution lies strictly inside the appropriate positive cone. For a precise statement we first define the distance to the boundary:

$$d(x) := \min_{y \in \partial\Omega} |x - y| \tag{8}$$

and the kernel $Q(x, y)$ for problem (7), namely

$$u(x) = \int_\Omega Q(x, y) f(y)\, \mathrm{d}y$$

solves (7). Then, we have

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^{4,\gamma}$. If*

$$Q(x, y) \geqq c\, d(x)^2 d(y)^2, \tag{9}$$

then, for all $\alpha \in C(\partial\Omega)$ with $\alpha \lneqq \delta_1$ and $0 \lneqq f \in C(\overline{\Omega})$, the $\mathcal{H}$-solution u of (1) satisfies $u > 0$ in Ω.

Proof. See Section 9. □

Remark 2.7. When Ω is a ball, the explicit formula of Boggio in [6] directly shows (9), which is even sharp for $n \leqq 3$. The positivity preserving property for the $\mathcal{H}$-solution in a ball can be found in Corollary 2.9.

We end this section with some explicit bounds for α, together with a discussion on what happens when $\alpha - \delta_1$ changes sign. On any smooth bounded domain Ω we may fix

$$h \in C(\overline{\Omega}) \cap C^2(\Omega) \text{ such that } \quad \Delta h = 0 \text{ in } \Omega \quad \text{and} \quad h \gneqq 0 \text{ on } \partial\Omega, \tag{10}$$

to find $h > 0$ in Ω, and solve

$$\begin{cases} -\Delta\psi = h & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

Then by the maximum principle we have $\psi > 0$ in Ω and by Hopf's boundary point lemma we find that $\psi_\nu < 0$ on $\partial\Omega$. We have

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ ($n \geqq 2$) be a bounded domain with $\partial\Omega \in C^2$. Let h and ψ be as in* (10)–(11) *and let δ_c be as in Theorem 2.3. Let $\alpha \in C(\partial\Omega)$ satisfy $\alpha \lneqq \frac{h}{-\psi_\nu}$ and let $0 \lneqq f \in L^2(\Omega)$. Then there exists a unique $\mathcal{H}$-solution u of* (1) *and moreover,*

1. *if $\delta_c \leqq \alpha$, then $u \gneqq 0$ in Ω;*
2. *if $\delta_c < \alpha$, then $u > 0$ in Ω.*

Proof. So $\beta := \frac{h}{-\psi_\nu}$ satisfies $0 \lneqq \beta \in C(\partial\Omega)$. Moreover, $\Delta^2\psi = 0$ in Ω and $\Delta\psi - \beta\psi_\nu = -h + h = 0 = \psi$ on $\partial\Omega$. Then we find by the second item of Theorem 4.1 that $\delta_{1,\beta} = 1$ and $u_{1,\beta} = c\psi$ for some $c > 0$. Here the notation is borrowed from that theorem. We now apply twice the third item of Theorem 4.1: β as above gives the upper bound for α, and $\beta = 1$ gives the lower bound δ_c, see also Theorems 2.3 and 2.4. □

In the unit ball of $\mathbb{R}^n$, the following holds:

Corollary 2.9. *Let B be the unit ball in $\mathbb{R}^n$ ($n \geqq 2$). Then, for all $0 \lneqq f \in L^2(B)$ and all $\alpha \in C(\partial B)$ such that $\alpha \lneqq n$, the $\mathcal{H}$-solution u of* (1) *satisfies $u > 0$ in Ω.*

Proof. By [4] we know that $\delta_1 = n$, where δ_1 is as in Theorem 2.3. We may also use Theorem 2.8 with $\psi(x) = 1 - |x|^2$ and $h(x) = 2n$. For the absence of a bound from below one notes that the estimates of [6] imply that (9) holds. Therefore, Theorem 2.6 applies. □

Remark 2.10. In Theorem 2.3 the bound δ_1 is absolute when considering constants. However, as will be seen in Theorem 4.1, positivity (or existence) is not necessarily lost when $\alpha(x) > \delta_1$ just for some $x \in \partial\Omega$ and $\alpha(x) < \delta_1$ for some other. Indeed, some compensation is possible as can be seen from the following example. For $\Omega = B$, we may take in (10)–(11) the function $\psi(x) = (1 + \frac{2}{n} + \varepsilon + x_1)\frac{1-|x|^2}{2n}$ where $\varepsilon > 0$ so that $h(x) = (1 + \frac{2}{n})(1 + x_1) + \varepsilon > 0$. Put

$$\beta(x) := -\frac{h(x)}{\psi_\nu(x)} = n + \frac{2x_1}{1 + \frac{2}{n} + \varepsilon + x_1} \quad \text{for all } x \in \partial B.$$

By Theorems 2.6–2.8, we infer that for all $\alpha \in C(\partial B)$ with $\alpha \lneqq \beta$ the $\mathcal{H}$-solution u of (1) is positive for $0 \lneqq f \in L^2(B)$. Notice that on part of the boundary $\beta > n$.

3. Regularity and further remarks

- We start this section by addressing the question when an $\mathcal{H}$-solution u is in fact classical. The Steklov boundary conditions in (1) satisfy the complementing conditions, see [4]. Therefore, standard elliptic regularity results apply directly to the fourth-order problem (1). That is, if $\partial\Omega \in C^4$ and $\alpha \in C^2(\partial\Omega)$, then the AGMON et al. [2] type estimates give us that $u \in W^{4,p}(\Omega)$ for all $p \in (1, \infty)$ and that for some $C = C_{\alpha,\Omega,p} > 0$:

$$\|u\|_{W^{4,p}(\Omega)} \leqq$$

$$\leqq C\left(\|u\|_{L^p(\Omega)} + \left\|\Delta^2 u\right\|_{L^p(\Omega)} + \|u\|_{W^{4-1/p,p}(\partial\Omega)} + \|\Delta u - \alpha u_\nu\|_{W^{2-1/p,p}(\partial\Omega)}\right) \tag{12}$$

whenever the right-hand side of (12) is bounded. For an $\mathcal{H}$-solution u one may formally integrate by parts to find

$$\int_\Omega \Delta^2 u\, v\, \mathrm{d}x + \int_{\partial\Omega} (\Delta u - \alpha\, u_\nu)\, v_\nu\, \mathrm{d}\sigma = \int_\Omega f v\, \mathrm{d}x \quad \text{for all } v \in \mathcal{H}(\Omega). \tag{13}$$

Here $\int_\Omega \Delta^2 u\, v\, dx$ denotes the pairing between $\Delta^2 u$ in the dual "negative" Sobolev space $\mathcal{H}(\Omega)'$ and $v \in \mathcal{H}(\Omega)$. Similarly, $\int_{\partial\Omega} \Delta u\, v_\nu\, d\sigma$ denotes the pairing between $\Delta u_{|\partial\Omega} \in H^{-1/2}(\partial\Omega)$ and $v_\nu \in H^{1/2}(\partial\Omega)$, these properties of $\Delta u_{|\partial\Omega}$ and v_ν following from $u, v \in \mathcal{H}(\Omega)$. Since (13) holds for all $v \in \mathcal{H}(\Omega)$ we find $\Delta^2 u = f \in L^2(\Omega)$ and $\Delta u = \alpha u_\nu \in H^{1/2}(\Omega)$. So, we have $u = 0$ and $\Delta u - \alpha u_\nu = 0$ on $\partial\Omega$ and (12) gives $\|u\|_{H^4(\Omega)} \leqq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})$. If $f \in L^p(\Omega)$ with $p \in (1, \infty)$ we may bootstrap the solution to $u \in W^{4,p}(\Omega)$ without extra conditions. However, if we start with $f \in C^\gamma(\overline{\Omega})$ and want to get $u \in C^{4,\gamma}(\overline{\Omega})$ we need $\partial\Omega \in C^{4,\gamma}$ and $\alpha \in C^{2,\gamma}(\partial\Omega)$.

- As already mentioned, Theorem 2.6 is not the exact converse of Theorem 2.5. Moreover, the sufficient condition (9) may not be easily verified. We give here a criterion which enables us to verify this condition. Consider the problem

$$\begin{cases} \Delta^2 u = \lambda u + f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

The criterion directly follows from the following statement:

Lemma 3.1 ([17, Lemma 2]). *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^{4,\gamma}$. If* (14) *is positivity preserving for some $\lambda < 0$, then for every $0 \lneqq f \in C(\overline{\Omega})$ there exists $c_f > 0$ such that the solution u of* (7) *satisfies*

$$u(x) \geqq c_f\, d(x)^2 \quad \textit{for all } x \in \Omega.$$

Proof. In [17, Lemma 2] the result has been proven for the first eigenfunction but one may notice that a similar result holds for all right-hand sides $0 \lneqq f \in C(\overline{\Omega})$. □

- Next, we give an **alternative proof of Corollary 2.9** in a smooth setting that makes no use of the machinery of the present paper. It highlights the link between the fourth-order problem (1) and a related second-order Steklov problem, see (19) below.

Proposition 3.2. *Let B be the unit ball in $\mathbb{R}^n$ $(n \geqq 2)$. If $u \in C^4(\overline{B})$, $0 \lneqq f \in C(\overline{B})$ and $\alpha \in C(\partial B)$ with $\alpha \lneqq n$ satisfy* (1), *then $u > 0$ in B.*

Proof. Inspired by [5], we consider the auxiliary (smooth) function ϕ defined by

$$\phi(x) = (|x|^2 - 1)\Delta u(x) - 4x \cdot \nabla u(x) - 2(n-4)u(x), \quad x \in \overline{B}.$$

Hence, since $x = \nu$ and $u = 0$ on ∂B, we have

$$\phi = -4u_\nu \quad \text{on } \partial B. \tag{15}$$

Moreover, for $x \in B$ we have

$$\nabla\phi = (2\Delta u)x + (|x|^2 - 1)\nabla\Delta u + 2(2-n)\nabla u - 4\langle D^2 u, x\rangle, \tag{16}$$

$$-\Delta\phi = (1 - |x|^2) f(x) \geqq 0, \tag{17}$$

where $D^2 u$ denotes the Hessian matrix of u. By (16) we find

$$\phi_\nu = 2\Delta u + 2(2-n)u_\nu - 4\langle D^2 u, \nu\rangle \cdot \nu \quad \text{on } \partial B.$$

Now, since $\langle D^2 u, \nu\rangle \cdot \nu = u_{\nu\nu}$ and by recalling that $u = 0$ on ∂B and using the expression of Δu on the boundary, the previous equation reads $\phi_\nu = -2\Delta u + 2nu_\nu$. Finally, taking into account the second boundary condition in (1), we obtain

$$\phi_\nu = 2(n-\alpha)u_\nu \quad \text{on } \partial B. \tag{18}$$

So combining (15), (17), and (18) we find that ϕ satisfies the boundary value problem

$$\begin{cases} -\Delta\phi = (1 - |x|^2) f \geqq 0 & \text{in } B, \\ \phi_\nu + \frac{1}{2}(n-\alpha)\phi = 0 & \text{on } \partial B. \end{cases} \tag{19}$$

As $\alpha \lneqq n$, by the maximum principle we deduce that $\phi > 0$ in $\overline{B}$ and hence by (15) that $u_\nu \leqq 0$ on ∂B. By the positivity preserving property in B under Dirichlet boundary conditions of [6], it follows that also $\Delta^2 u \geqq 0$ in B with $u = 0$ and $-u_\nu \geqq 0$ on ∂B implies that $u > 0$ in B. □

- We conclude this section with an open problem.

Problem. The basic tool in the proof of Theorem 2.6 is Lemma 9.1 below. It states that the solution of the Dirichlet problem (7) is smaller than the solution of the corresponding Navier problem [that is, problem (1) with $d = 0$]. This result is obtained under the crucial assumption that Dirichlet boundary conditions are positivity preserving for the biharmonic operator. We conjecture that Lemma 9.1 remains true without this assumption.

4. Proof of Theorem 2.3

In this section we prove a slightly more general version than Theorem 2.3:

Theorem 4.1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ *be a bounded domain with* $\partial\Omega \in C^2$ *and let* $0 \lneqq \beta \in C(\partial\Omega)$. *Then there are* $\delta_{1,\beta} := \delta_{1,\beta}(\Omega) \in (0, \infty)$ *and* $\delta_{c,\beta} := \delta_{c,\beta}(\Omega) \in [-\infty, 0)$ *such that the following holds for a function* $\alpha \in C(\partial\Omega)$.

1. *If* $\alpha \geqq \delta_{1,\beta}\beta$ *and if* $0 \lneqq f \in L^2(\Omega)$ *then* (1) *has no positive* $\mathcal{H}$*-solution.*
2. *If* $\alpha = \delta_{1,\beta}\beta$ *then there exists a positive eigenfunction, that is, problem* (1) *with* $f = 0$ *admits an* $\mathcal{H}$*-solution* $u_{1,\beta}$ *that satisfies* $u_{1,\beta} > 0$ *and* $-\Delta u_{1,\beta} > 0$ *in* Ω, $\frac{\partial}{\partial\nu} u_{1,\beta} < 0$ *on* $\partial\Omega$. *This eigenfunction* $u_{1,\beta}$ *is unique in the following ways. If* u *is an* $\mathcal{H}$*-solution of* (1) *for* $\alpha = \delta\beta$ *and* $f = 0$ *and*
   (a) *if* $\delta = \delta_{1,\beta}$, *then* $u = cu_{1,\beta}$ *for some* $c \in \mathbb{R}$;
   (b) *if* $u \gneqq 0$, *then* $\delta = \delta_{1,\beta}$ *and* $u = cu_{1,\beta}$ *for some* $c \in \mathbb{R}_+$.
3. *If* $\alpha \lneqq \delta_{1,\beta}\beta$, *then for any* $f \in L^2(\Omega)$ *problem* (1) *admits a unique* $\mathcal{H}$*-solution* u.
   (a) *If* $\delta_{c,\beta}\beta \leqq \alpha \lneqq \delta_{1,\beta}\beta$ *and if* $0 \lneqq f \in L^2(\Omega)$ *then* $u \gneqq 0$.
   (b) *If* $\delta_{c,\beta}\beta < \alpha \lneqq \delta_{1,\beta}\beta$ *and if* $0 \lneqq f \in L^2(\Omega)$ *then for some* $c_f > 0$ *it holds that* $u \geqq c_f\, d$ *with* d *as in* (8). *Furthermore, if* $\alpha(x_0) < 0$ *for some* $x_0 \in \partial\Omega$ *then* $-\Delta u \ngeq 0$ *in* Ω, *whereas if* $\alpha \geqq 0$ *then* $0 \lneqq f$ *implies* $-\Delta u \geqq 0$ *in* Ω.
   (c) *If* $\alpha < \delta_{c,\beta}\beta$ *then there are* $0 \lneqq f \in L^2(\Omega)$ *with* $0 \nleq u$.

Remark 4.2. In the theorem we compare α with multiples of a fixed function β. Of course one may read the theorem both with $\beta = \beta_a$ and with $\beta = \beta_b$, where β_a, β_b are two different functions. As a consequence, one will find that the statement in item 3(*a*) may start as *"If* $\delta_{c,\beta_a}\beta_a \leqq \alpha \lneqq \delta_{1,\beta_b}\beta_b$ *then ... "*.

Remark 4.3. We expect that, under the assumption in 3.(a), it will hold that $u > 0$ in Ω, and that for generic α and $\partial\Omega$ only the strong positivity in the sense that $u > cd$ will only break down at an isolated point. This breakdown is expected to occur for a Dirac-δ type source term. If it breaks down at an isolated point by such an isolated source term then by continuity arguments it will follow that for each fixed $0 \lneqq f \in C(\overline{\Omega})$ or $L^2(\Omega)$ a number $\delta^f_{c,\beta} < \delta_{c,\beta}$ will exist such that the corresponding solutions $u_{f,\delta,\beta}$ are strictly positive in Ω for all $\delta > \delta^f_{c,\beta}$.

The proof of Theorem 4.1 follows by combining the lemmas and the proposition below, as described at the end of the present section.

We assume in the sequel that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^2$ and $\alpha \in C(\partial\Omega)$. We fix $\beta \in C(\partial\Omega)$ such that $\beta \gneqq 0$ and set

$$J_\beta(u) = \left(\int_\Omega |\Delta u|^2 \,\mathrm{d}x\right) \left(\int_{\partial\Omega} \beta u_\nu^2 \,\mathrm{d}\sigma\right)^{-1} \quad \text{for} \quad \int_{\partial\Omega} \beta u_\nu^2 \,\mathrm{d}\sigma \neq 0 \tag{20}$$

and $J_\beta(u) = \infty$ otherwise.

The first statement of the next lemma in the case that $\beta = 1$ can be found in [4, Theorem 1].

Lemma 4.4. *Let* $0 \lneqq \beta \in C(\partial\Omega)$. *The minimum*

$$\delta_{1,\beta} = \delta_{1,\beta}(\Omega) := \inf_{u \in \mathcal{H}(\Omega)} J_\beta(u) \tag{21}$$

is achieved and hence $\delta_{1,\beta} > 0$. *Moreover, the following holds:*

1. *The minimizer* $u_{1,\beta}$ *of* (21) *is unique up to multiplication by constants. If we fix* $u_{1,\beta}$ *such that* $u_{1,\beta}(x_0) = 1$ *for some* $x_0 \in \Omega$, *then* $-\Delta u_{1,\beta} \geqq 0$ *in* Ω *so that* $\frac{\partial}{\partial \nu} u_{1,\beta} < 0$ *on* $\partial\Omega$ *and* $u_{1,\beta} \geqq cd$ *in* Ω;
2. *If* $\beta_1 \lneqq \beta_2$ *are as above, then* $\delta_{1,\beta_1} > \delta_{1,\beta_2}$.

Proof. For every $u \in \mathcal{H}(\Omega)$ the functional in (20) is strictly positive, possibly ∞. Since the linear map $H^2(\Omega) \to L^2(\partial\Omega)$ defined by $u \mapsto u_\nu$ is compact, there exists a minimizer and $\delta_{1,\beta} > 0$.

Let $u_1 \in \mathcal{H}(\Omega)$ be a minimizer for (21) and let $\tilde{u}_1$ be the unique solution in $\mathcal{H}(\Omega)$ of $-\Delta\tilde{u}_1 = |\Delta u_1|$. Then, by the maximum principle we infer that $|u_1| \leqq \tilde{u}_1$ in Ω and $|\frac{\partial}{\partial \nu} u_1| \leqq |\frac{\partial}{\partial \nu}\tilde{u}_1|$ on $\partial\Omega$. If Δu_1 changes sign then these inequalities are strict and imply $J_\beta(u_1) > J_\beta(\tilde{u}_1)$. So Δu_1 is of fixed sign, say $-\Delta u_1 \geqq 0$, so that the maximum principle implies $\frac{\partial}{\partial \nu} u_1 < 0$ on $\partial\Omega$ and $u_1 \geqq cd$ in Ω. Similarly, if u_1 and u_2 are two minimizers which are not multiples of each other then there is a linear combination which is a sign changing minimizer and one proceeds as above to find a contradiction.

Finally, let u_1 be the minimizer for β_1 and u_2 the one for β_2. Then, since $\left(\frac{\partial}{\partial \nu} u_1\right)^2 > 0$ on $\partial\Omega$, we find $\delta_{1,\beta_1} = J_{\beta_1}(u_1) > J_{\beta_2}(u_1) \geqq J_{\beta_2}(u_2) = \delta_{1,\beta_2}$. □

Lemma 4.5. *Assume that* $\alpha \geqq \delta_{1,\beta}\beta$ *and* $0 \lneqq f \in L^2(\Omega)$. *If there exists an* $\mathcal{H}$*-solution* u *of* (1) *it cannot be positive.*

Proof. Suppose that u is a positive solution. Hence, $u_\nu \leqq 0$ on $\partial\Omega$. Let $u_{1,\beta}$ be as in Lemma 4.4. By taking $v = u_{1,\beta}$ in (5) one obtains

$$\begin{aligned} 0 < \int_\Omega f\, u_{1,\beta}\, \mathrm{d}x &= \int_\Omega \Delta u\, \Delta u_{1,\beta}\, \mathrm{d}x - \int_{\partial\Omega} \alpha\, u_\nu \left(u_{1,\beta}\right)_\nu \mathrm{d}\sigma \\ &\leqq \int_\Omega \Delta u\, \Delta u_{1,\beta}\, \mathrm{d}x - \int_{\partial\Omega} \delta_{1,\beta}\, \beta\, u_\nu \left(u_{1,\beta}\right)_\nu \mathrm{d}\sigma = 0, \end{aligned}$$

a contradiction. The last equality follows by the fact that $u_{1,\beta}$ minimizes (21). □

Lemma 4.6. *Assume that* $\alpha \lneqq \delta_{1,\beta}\beta$. *Then for every* $f \in L^2(\Omega)$ *the system in* (1) *admits a unique* $\mathcal{H}$*-solution.*

Proof. On the space $\mathcal{H}(\Omega)$ we define the energy functional

$$I(u) := \tfrac{1}{2}\int_\Omega |\Delta u|^2\, \mathrm{d}x - \tfrac{1}{2}\int_{\partial\Omega} \alpha\, u_\nu^2\, \mathrm{d}\sigma - \int_\Omega f u\, \mathrm{d}x \quad u \in \mathcal{H}(\Omega). \tag{22}$$

Critical points of I are $\mathcal{H}$-solutions of (1) in the sense of Definition 2.1. We will show that for $\alpha \lneqq \delta_{1,\beta}\beta$ the functional I has a unique critical point.

If $\alpha < \delta_{1,\beta}\beta$ one sets

$$\varepsilon := \frac{\min\left\{\delta_{1,\beta}\beta(x) - \alpha(x);\ x \in \partial\Omega\right\}}{\max\left\{\delta_{1,\beta}\beta(x);\ x \in \partial\Omega\right\}} > 0, \tag{23}$$

and finds that $\alpha \leqq (1-\varepsilon)\delta_{1,\beta}\beta$. By the definition of $\delta_{1,\beta}$ we have for all $u \in \mathcal{H}(\Omega)$

$$\begin{aligned} &\int_\Omega |\Delta u|^2\,\mathrm{d}x - \int_{\partial\Omega} \alpha u_\nu^2\,\mathrm{d}\sigma \\ &\quad\geqq \varepsilon \int_\Omega |\Delta u|^2\,\mathrm{d}x + (1-\varepsilon)\left(\int_\Omega |\Delta u|^2\,\mathrm{d}x - \int_{\partial\Omega} \delta_{1,\beta}\beta u_\nu^2\,\mathrm{d}\sigma\right) \\ &\quad\geqq \varepsilon \int_\Omega |\Delta u|^2\,\mathrm{d}x, \end{aligned} \tag{24}$$

so that the functional I is coercive. Since it is also strictly convex, the functional I admits a unique critical point which is its global minimum over $\mathcal{H}(\Omega)$.

In order to deal with the case that $\alpha^+ \lneqq \delta_{1,\beta}\beta$, but $\alpha^+(x) = \delta_{1,\beta}\beta(x)$ for some $x \in \partial\Omega$, we set

$$\tilde\beta := \frac{1}{2}\left(\beta + \delta_{1,\beta}^{-1}\alpha^+\right).$$

Since $0 \lneqq \tilde\beta \lneqq \beta$ we find by Lemma 4.4 that $\delta_{1,\tilde\beta} > \delta_{1,\beta}$. Instead of (23) we set

$$\varepsilon := 1 - \delta_{1,\beta}/\delta_{1,\tilde\beta} > 0,$$

find for $x \in \partial\Omega$ that $\alpha \leqq \alpha^+ = \delta_{1,\beta}(2\tilde\beta - \beta) \leqq \delta_{1,\beta}\tilde\beta = (1-\varepsilon)\delta_{1,\tilde\beta}\tilde\beta$ and proceed by replacing all β in (24) with $\tilde\beta$.

If $\alpha^+ = \delta_{1,\beta}\beta$ and $\alpha^- \gneqq 0$, then one may not proceed as directly as before. However, instead of the functional in (20), one may use

$$J_\beta^{\alpha^-}(u) = \left(\int_\Omega |\Delta u|^2\,\mathrm{d}x + \int_{\partial\Omega} \alpha^- u_\nu^2\,\mathrm{d}\sigma\right)\left(\int_{\partial\Omega} \beta u_\nu^2\,\mathrm{d}\sigma\right)^{-1}.$$

Then, defining $\delta_{1,\beta}^{\alpha^-}$ for $J_\beta^{\alpha^-}$ as in (21), this minimum is assumed, say, by $u_{1,\beta}^{\alpha^-}$. Since

$$\delta_{1,\beta}^{\alpha^-} = J_\beta^{\alpha^-}(u_{1,\beta}^{\alpha^-}) \geqq J_\beta(u_{1,\beta}^{\alpha^-}) \geqq J_\beta(u_{1,\beta}) = \delta_{1,\beta},$$

with the last inequality strict if $u_{1,\beta}^{\alpha^-} \neq c\, u_{1,\beta}$ and with the first inequality strict if $u_{1,\beta}^{\alpha^-} = c\, u_{1,\beta}$ since $(u_{1,\beta})_\nu^2 > 0$, we find $\delta_{1,\beta}^{\alpha^-} > \delta_{1,\beta}$. So

$$\int_\Omega |\Delta u|^2\,\mathrm{d}x + \int_{\partial\Omega} \alpha^- u_\nu^2\,\mathrm{d}\sigma \geqq \delta_{1,\beta}^{\alpha^-} \int_{\partial\Omega} \beta u_\nu^2\,\mathrm{d}\sigma \quad \text{for all } u \in \mathcal{H}(\Omega)$$

and by setting

$$\varepsilon := 1 - \delta_{1,\beta}/\delta_{1,\beta}^{\alpha^-} > 0$$

we find the result that replaces (24). Indeed

$$\begin{aligned}\int_\Omega |\Delta u|^2\,\mathrm{d}x - \int_{\partial\Omega} \alpha u_\nu^2\,\mathrm{d}\sigma &= \int_\Omega |\Delta u|^2\,\mathrm{d}x + \int_{\partial\Omega} \alpha^- u_\nu^2\,\mathrm{d}\sigma - \int_{\partial\Omega} \delta_{1,\beta}\beta u_\nu^2\,\mathrm{d}\sigma \\ &\geqq \varepsilon \int_\Omega |\Delta u|^2\,\mathrm{d}x + (1-\varepsilon)\Bigg(\int_\Omega |\Delta u|^2\,\mathrm{d}x + \int_{\partial\Omega} \alpha^- u_\nu^2\,d\sigma \\ &\qquad - \int_{\partial\Omega} \delta_{1,\beta}^{\alpha^-}\beta u_\nu^2\,\mathrm{d}\sigma\Bigg) \\ &\geqq \varepsilon \int_\Omega |\Delta u|^2\,\mathrm{d}x.\end{aligned}$$

Hence I is coercive and strictly convex and we may conclude as before. □

Lemma 4.7. *Assume that* $\alpha \lneqq \delta_{1,\beta}\beta$. *If* $\alpha \ngeqq 0$, *then for any* $f \in L^2(\Omega)$ *the unique* $\mathcal{H}$*-solution* u *of* (1) *cannot satisfy* $-\Delta u \geqq 0$ *in* Ω. *If* $\alpha \geqq 0$ *and* $0 \lneqq f \in L^2(\Omega)$ *then the* $\mathcal{H}$*-solution* u *of* (1) *satisfies* $-\Delta u \geqq 0$ *in* Ω.

Proof. Assume that there exists $x_0 \in \partial\Omega$ such that $\alpha(x_0) < 0$. If the $\mathcal{H}$-solution u were superharmonic, then by Hopf's boundary Lemma we would have $u_\nu(x_0) < 0$. Using the second boundary condition in (1), we would then obtain $\Delta u(x_0) > 0$, a contradiction.

If $\alpha \geqq 0$ and $f \gneqq 0$, then as in the proof of Lemma 4.4 we define $\tilde{u}$ as the unique solution in $\mathcal{H}(\Omega)$ of $-\Delta\tilde{u} = |\Delta u|$ in Ω. Since $\tilde{u} > u$ or $\tilde{u} = u$ in Ω, and $|\tilde{u}_\nu| \geqq |u_\nu|$ on $\partial\Omega$, for $f \gneqq 0$ one finds that

$$I(\tilde{u}) - I(u) = -\frac{1}{2}\int_{\partial\Omega} \alpha\,\left(\tilde{u}_\nu^2 - u_\nu^2\right)\,\mathrm{d}\sigma - \int_\Omega f\;(\tilde{u} - u)\,\mathrm{d}x \leqq 0.$$

Equality occurs only when $\tilde{u} = u$. Since I is strictly convex there is at most one critical point which is a minimum. So $u = \tilde{u} > 0$ and $-\Delta u = -\Delta\tilde{u} = |\Delta u| \geqq 0$. □

Proposition 4.8. *There exists* $\delta_{c,\beta} := \delta_{c,\beta}(\Omega) \in [-\infty, 0)$ *such that the following holds for an* $\mathcal{H}$*-solution* u *of* (1).

1. *for* $\delta_{c,\beta}\beta \leqq \alpha \lneqq \delta_{1,\beta}\beta$ *it follows that if* $0 \lneqq f \in L^2(\Omega)$ *then* $u \gneqq 0$;
2. *for* $\delta_{c,\beta}\beta < \alpha \lneqq \delta_{1,\beta}\beta$ *it follows that if* $0 \lneqq f \in L^2(\Omega)$ *then* $u \geqq c_f\,d$ *for some* $c_f > 0$ (*depending on* f), d *being the distance function from* (8);
3. *for* $\alpha < \delta_{c,\beta}\beta$ *there are* $0 \lneqq f \in L^2(\Omega)$ *with* u *somewhere negative.*

The proof of Proposition 4.8 will be given in Section 7. It will use estimates for the kernels involved and for this reason it seems more suitable to employ a Schauder setting and to finish by approximation.

Proof of Theorem 4.1. The first item is a direct consequence of Lemma 4.5. In item 2, existence of an eigenfunction and the first uniqueness property (*a*) are consequences of the first item of Lemma 4.4. The second uniqueness property (*b*) follows from Lemma 4.6 for $\delta < \delta_{1,\beta}$ and from Lemma 4.5 for $\delta > \delta_{1,\beta}$. Existence and uniqueness in item 3 follow from Lemma 4.6. The sign results (*a*) and (*b*) follow from Lemma 4.7 and Proposition 4.8. □

5. The Schauder setting

5.1. On the operators in the Schauder setting

Consider the Green operator $\mathcal{G}$ and the Poisson kernel $\mathcal{K}$, that is, $w = \mathcal{G}f + \mathcal{K}g$ formally solves

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega. \end{cases}$$

Moreover, let $(\mathcal{P}w)(x) := -\nu \cdot \nabla w(x) = -w_\nu(x)$ for $x \in \partial\Omega$. We will fix the appropriate setting so that $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{P}$ are well-defined positive operators. Let d denote the distance to $\partial\Omega$ as defined in (8).

Notation 5.1. *Set*

$$C_d(\overline{\Omega}) = \left\{u \in C(\overline{\Omega});\ \text{there exists } w \in C(\overline{\Omega}) \text{ such that } u = \mathrm{d}w\right\}$$

with norm

$$\|u\|_{C_d(\overline{\Omega})} = \sup\left\{\frac{|u(x)|}{d(x)};\ x \in \Omega\right\}.$$

Set also $C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega});\ u = 0 \text{ on } \partial\Omega\right\}$ *so that* $C_d(\overline{\Omega}) \subsetneqq C_0(\overline{\Omega})$.

We consider the three above operators in the following setting:

$$\mathcal{G} : C(\overline{\Omega}) \to C_d(\overline{\Omega}), \quad \mathcal{K} : C(\partial\Omega) \to C(\overline{\Omega}), \quad \mathcal{P} : C_d(\overline{\Omega}) \to C(\partial\Omega).$$

The space $C_d(\overline{\Omega})$ is a Banach lattice, that is, a Banach space with the ordering such that $|u| \leqq |v|$ implies $\|u\|_{C_d(\overline{\Omega})} \leqq \|v\|_{C_d(\overline{\Omega})}$, see [3] or [23]. The positive cone

$$C_d(\overline{\Omega})^+ = \left\{u \in C_d(\overline{\Omega});\ u(x) \geqq 0 \text{ in } \overline{\Omega}\right\} \tag{25}$$

is solid (namely, it has nonempty interior) and reproducing (that is, every $w \in C_d(\overline{\Omega})$ can be written as $w = u - v$ for some $u, v \in C_d(\overline{\Omega})^+$). Similarly, we define $C(\partial\Omega)^+$ and $C(\overline{\Omega})^+$.

Note that the interiors of the cones in these Schauder-type spaces are as follows:

$$\begin{aligned} C(\partial\Omega)^{+,\circ} &= \left\{v \in C(\partial\Omega);\ v(x) \geqq c \text{ for some } c > 0\right\}, \\ C(\overline{\Omega})^{+,\circ} &= \left\{u \in C(\overline{\Omega});\ u(x) \geqq c \text{ for some } c > 0\right\}, \\ C_d(\overline{\Omega})^{+,\circ} &= \left\{u \in C_d(\overline{\Omega});\ u(x) \geqq c\, d(x) \text{ for some } c > 0\right\}. \end{aligned}$$

Definition 5.2. The operator $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is described as

- nonnegative, $\mathcal{F} \geqq 0$, when $g \in \mathcal{C}_1^+ \Rightarrow \mathcal{F}g \in \mathcal{C}_2^+$;
- strictly positive, $\mathcal{F} \gneqq 0$, when $g \in \mathcal{C}_1^+\backslash\{0\} \Rightarrow \mathcal{F}g \in \mathcal{C}_2^+\backslash\{0\}$;
- strongly positive, $\mathcal{F} > 0$, when $g \in \mathcal{C}_1^+\backslash\{0\} \Rightarrow \mathcal{F}g \in \mathcal{C}_2^{+,\circ}$.

If $\mathcal{F} \geqq 0$ and $\mathcal{F} \neq 0$, that is, for some $g \in \mathcal{C}_1^+$ we find $\mathcal{F}g \gneqq 0$, we call $\mathcal{F}$ positive. Similarly, two operators are ordered through $\geqq$ (respectively $\gneqq$ or $>$) whenever their difference is nonnegative (respectively strictly or strongly positive).

The main purpose of this section is to prove the following:

Proposition 5.3. *Suppose that $\partial\Omega \in C^2$ and $\alpha \in C(\partial\Omega)$. Let $\mathcal{G}$, $\mathcal{K}$, and $\mathcal{P}$ be defined as above. Then $\mathcal{G}\mathcal{K}\alpha\mathcal{P} : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ is a well-defined compact linear operator. If in addition $\alpha \gneqq 0$, then $\mathcal{G}\mathcal{K}\alpha\mathcal{P}$ is positive, even to the extent that*

$$u \in C_d(\overline{\Omega})^+ \quad \textit{implies either} \quad \mathcal{G}\mathcal{K}\alpha\mathcal{P}u = 0 \quad \textit{or} \quad \mathcal{G}\mathcal{K}\alpha\mathcal{P}u \in C_d(\overline{\Omega})^{+,\circ}. \tag{26}$$

Proposition 5.3 will be a consequence of the following three lemmas.

Lemma 5.4. *The operator $\mathcal{G} : C(\overline{\Omega}) \to C_d(\overline{\Omega})$ is a compact linear operator and it is strongly positive.*

Proof. Take $\gamma \in (0, 1)$, $p > n(1-\gamma)^{-1}$ and fix the imbeddings $I_1 : C(\overline{\Omega}) \to L^p(\Omega)$, $I_2 : W^{2,p}(\Omega) \to C^{1,\gamma}(\overline{\Omega})$, and $I_3 : C^{1,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega}) \to C_d(\overline{\Omega})$. Since $\partial\Omega \in C^2$, for every $p \in (1,\infty)$ there exists a bounded linear operator $\mathcal{G}_p : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $-\Delta\mathcal{G}_p f = f$ for all $f \in L^p(\Omega)$, see [13, Theorem 9.15 and Lemma 9.17]. If $\mathcal{I}_d$ denotes the imbedding $C_d(\overline{\Omega}) \to C(\overline{\Omega})$, then the Green operator from $C_d(\overline{\Omega})$ to $C_d(\overline{\Omega})$ should formally be denoted by $\mathcal{G}\mathcal{I}_d$, where $\mathcal{G} = I_3 I_2 \mathcal{G}_p I_1$. Note that the imbedding $I_1 : C(\overline{\Omega}) \to L^p(\Omega)$ is bounded and the imbedding $I_2 : W^{2,p}(\Omega) \to C^{1,\gamma}(\overline{\Omega})$ is compact, see [1, p.144]. Since $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C^{1,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega})$ and $I_3 : C^{1,\gamma}(\overline{\Omega}) \cap C_0(\overline{\Omega}) \to C_d(\overline{\Omega})$ is bounded, $\mathcal{G}$ is not only well defined but also compact. The strong maximum principle and Hopf's boundary point Lemma imply that $\mathcal{G}$ is strongly positive. □

Lemma 5.5. *The operator $\mathcal{K} : C(\partial\Omega) \to C(\overline{\Omega})$ is a strictly positive bounded linear operator.*

Proof. Since $\partial\Omega \in C^2$ and Ω is bounded, all boundary points are regular. According to [13, Theorem 2.14] the Dirichlet boundary value problem is solvable for arbitrary continuous boundary values by

$$(\mathcal{K}\phi)(x) = \sup\left\{v(x);\, v \leqq \phi \text{ on } \partial\Omega \text{ and } v \text{ subharmonic in } \Omega\right\}.$$

For $\phi \in C(\partial\Omega)$ one obtains $\mathcal{K}\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ and by the maximum principle

$$\sup_{x\in\Omega}(\mathcal{K}\phi)(x) = \max_{x\in\partial\Omega}\phi(x) \quad \text{and} \quad \inf_{x\in\Omega}(\mathcal{K}\phi)(x) = \min_{x\in\partial\Omega}\phi(x)$$

implying not only that $\|\mathcal{K}\phi\|_{L^\infty(\Omega)} = \|\phi\|_{L^\infty(\partial\Omega)}$, but also that $\mathcal{K}$ is strictly positive. □

Lemma 5.6. *The operator $\mathcal{P} : C_d(\overline{\Omega}) \to C(\partial\Omega)$ is a positive bounded linear operator.*

Proof. It follows at once from the fact that every function $u \in C_d(\overline{\Omega})$ can be written as $u = dw$ for some $w \in C(\overline{\Omega})$ and $\mathcal{P}dw = w_{|\partial\Omega}$. □

Proof of Proposition 5.3. Compactness, and positivity of $\mathcal{G}\mathcal{K}\alpha\mathcal{P}$ when $\alpha \geqq 0$, is an immediate consequence of the last three lemmata. By Lemma 5.4 it follows that $\mathcal{K}\alpha\mathcal{P}u \gneqq 0$ implies that $\mathcal{G}\mathcal{K}\alpha\mathcal{P}u \in C_d(\overline{\Omega})^{+,\circ}$. □

5.2. Relation between the Hilbert and Schauder settings

Let us now explain how we will make use of Proposition 5.3. Instead of (1) or (4) we consider an integral equation. Let again $\mathcal{I}_d : C_d(\overline{\Omega}) \to C(\overline{\Omega})$ denote the imbedding operator, then the system in (4) turns into

$$u = \mathcal{G}\mathcal{K}\alpha\mathcal{P}u + \mathcal{G}\mathcal{I}_d\mathcal{G}f. \tag{27}$$

Definition 5.7. For $f \in C(\overline{\Omega})$ we say that u is a $\mathcal{C}$-solution of (1) if $u \in C_d(\overline{\Omega})$ satisfies (27).

For continuous f, $\mathcal{C}$-solutions coincide with the $\mathcal{H}$-solutions from Definition 2.1:

Proposition 5.8. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqq 2)$ with $\partial\Omega \in C^2$ and let $\alpha \in C(\partial\Omega)$. If $f \in C(\overline{\Omega})$ then a $\mathcal{C}$-solution of* (1) *is also an $\mathcal{H}$-solution.*

Proof. If $f \in C(\overline{\Omega})$ and $u \in C_d(\overline{\Omega})$ then by (27) it follows that $w = \mathcal{K}\alpha\mathcal{P}u + \mathcal{I}_d\mathcal{G}f \in C(\overline{\Omega}) \subset L^2(\Omega)$ and hence $u = \mathcal{G}w \in \mathcal{H}(\Omega)$. Moreover, for such u and for any $v \in \mathcal{H}(\Omega)$ we have

$$\int_\Omega \Delta u \Delta v \,\mathrm{d}x = \int_\Omega (\mathcal{K}\alpha\mathcal{P}u + \mathcal{G}f)\,\Delta v \,\mathrm{d}x = \int_{\partial\Omega} \alpha u_\nu v_\nu \,\mathrm{d}\sigma_x + \int_\Omega f v \,\mathrm{d}x,$$

which is precisely (5). □

A theorem named after Krein–Rutman tells us that a strictly positive compact linear operator on a Banach lattice such as $\mathcal{G}\mathcal{K}\beta\mathcal{P} : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ has a spectral radius $r_\sigma(\mathcal{G}\mathcal{K}\beta\mathcal{P}) > 0$ that is an eigenvalue with a positive eigenfunction ϕ_1, and moreover this eigenvalue has multiplicity one and is the only one with a positive eigenfunction; see, for example, [3, Theorem 3.2]. This would supply us with an alternative proof of Theorem 4.1 in a $C(\overline{\Omega})$-setting when $\alpha \geqq 0$.

In the following subsection we deal with $\mathcal{C}$-solutions in order to provide the tools needed in Proposition 4.8 that will take care of the case where $\alpha \leqq 0$. The proof of this proposition is given in Section 7.

5.3. Sign-changing and negative weights

We first note that (possibly by changing its sign) the minimizer $u_{1,\beta}$ from Lemma 4.4 lies in $C_d(\overline{\Omega})^{+,\circ}$.

Lemma 5.9. *Let $\partial\Omega \in C^2$ and suppose that $\alpha \in C(\partial\Omega)$ is such that $\alpha \lneqq \delta_{1,\beta}\beta$. Then*

$$\mathcal{E}^\alpha_\mathcal{G} := (\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\,\mathcal{G}\mathcal{I}_d\mathcal{G} : C(\overline{\Omega}) \to C_d(\overline{\Omega}), \tag{28}$$

$$\mathcal{E}^\alpha_\mathcal{K} := (\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\,\mathcal{G}\mathcal{K} : C(\partial\Omega) \to C_d(\overline{\Omega}), \tag{29}$$

are well-defined operators. Moreover, the following holds;

- *For $f \in C(\overline{\Omega})$ the unique $\mathcal{C}$-solution of problem* (1) *is $u = \mathcal{E}^\alpha_\mathcal{G} f$.*

- $u_{1,\beta}$ *is a positive eigenfunction of* $\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P} : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ *with eigenvalue* 1. *Any other nonnegative eigenfunction* $\tilde{u}$ *of* $\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P}$ *satisfies* $\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P}\tilde{u}=0$ *on* $\partial\Omega$.

Remark 5.10. Notice that for $\alpha \lneqq \delta_{1,\beta}\beta$ we have a positive eigenfunction for $\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P}$, and hence also for $\mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)$, without assuming positivity of $\mathcal{E}^{\alpha}_{\mathcal{G}}$ or $\mathcal{E}^{\alpha}_{\mathcal{K}}$.

Proof. By Lemma 4.6 one finds for $\alpha \lneqq \delta_{1,\beta}\beta$ that $\mu = 1$ is not an eigenvalue of the (compact) operator $\mathcal{G}\mathcal{K}\alpha\mathcal{P}$. Therefore, the operator $(\mathcal{I}-\mathcal{G}\mathcal{K}\alpha\mathcal{P})$ is invertible in $L^2(\Omega)$ and hence in $C_d(\overline{\Omega})$.

- Equation (27) reads $u = (\mathcal{I}-\mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\,\mathcal{G}\mathcal{I}_d\mathcal{G}f$.
- One directly checks that $u_{1,\beta}$ is an eigenfunction of $\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P}$ with $\lambda = 1$ for all $\alpha \lneqq \delta_{1,\beta}\beta$. By Lemma 4.4, up to its multiples, it is the unique eigenfunction with $\lambda = 1$. Let $\tilde{u}$ be another nonnegative eigenfunction of $\mathcal{E}^{\alpha}_{\mathcal{K}}\left(\delta_{1,\beta}\beta-\alpha\right)\mathcal{P}$ corresponding to some eigenvalue $\lambda \neq 1$. One finds that $\lambda = 0$ if and only if $(\delta_{1,\beta}\beta-\alpha)\mathcal{P}\tilde{u}=0$. For $\lambda \neq 0$ it holds that

$$\tilde{u}-\mathcal{G}\mathcal{K}\delta_{1,\beta}\beta\mathcal{P}\tilde{u} = \left(\lambda^{-1}-1\right)\mathcal{G}\mathcal{K}(\delta_{1,\beta}\beta-\alpha)\mathcal{P}\tilde{u}. \tag{30}$$

We have $u_{1,\beta}, \tilde{u} \in \mathcal{H}(\Omega)$; this fact allows us to combine (30) with an argument similar as in Lemma 4.5 to find a contradiction in the case that $(\delta_{1,\beta}\beta-\alpha)\mathcal{P}\tilde{u} \gneqq 0$:

$$\begin{aligned}
0 &= \int_{\Omega}\Delta u_{1,\beta}\Delta\tilde{u}\,\mathrm{d}x - \int_{\partial\Omega}\delta_{1,\beta}\beta(u_{1,\beta})_{\nu}\tilde{u}_{\nu}\,\mathrm{d}\sigma \\
&= \int_{\Omega}\Delta u_{1,\beta}\Delta\left(\tilde{u}-\mathcal{G}\mathcal{K}\delta_{1,\beta}\beta\mathcal{P}\tilde{u}\right)\mathrm{d}x \\
&= \left(\lambda^{-1}-1\right)\int_{\Omega}\Delta u_{1,\beta}\,\mathcal{G}\mathcal{K}(\delta_{1,\beta}\beta-\alpha)\mathcal{P}\tilde{u}\,\mathrm{d}x \\
&= \left(1-\lambda^{-1}\right)\int_{\Omega}u_{1,\beta}\,\mathcal{K}(\delta_{1,\beta}\beta-\alpha)\mathcal{P}\tilde{u}\,\mathrm{d}x,
\end{aligned}$$

and this last expression has a sign if $\lambda \neq 1$. □

Lemma 5.11. *Let* $\partial\Omega \in C^2$ *and suppose that* $\alpha \in C(\partial\Omega)$ *is such that* $\alpha \lneqq \delta_{1,\beta}\beta$. *Let* $\mathcal{E}^{\alpha}_{\mathcal{G}}$ *and* $\mathcal{E}^{\alpha}_{\mathcal{K}}$ *be as in Lemma* 5.9 *and suppose that* $\mathcal{E}^{\alpha}_{\mathcal{G}}$ *is a positive operator.*

1. *Then* $\mathcal{E}^{\alpha}_{\mathcal{G}}$, $\mathcal{E}^{\alpha}_{\mathcal{K}}$, $\mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{G}}$, *and* $\mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{K}}$ *are strictly positive operators.*
2. *If* $\tilde{\alpha} \in C(\partial\Omega)$ *is such that* $\alpha \leqq \tilde{\alpha} \lneqq \delta_{1,\beta}\beta$, *then* $\mathcal{E}^{\tilde{\alpha}}_{\mathcal{G}} \geqq \mathcal{E}^{\alpha}_{\mathcal{G}}$, $\mathcal{E}^{\tilde{\alpha}}_{\mathcal{K}} \geqq \mathcal{E}^{\alpha}_{\mathcal{K}}$, $\mathcal{P}\mathcal{E}^{\tilde{\alpha}}_{\mathcal{G}} \geqq \mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{G}}$, *and* $\mathcal{P}\mathcal{E}^{\tilde{\alpha}}_{\mathcal{K}} \geqq \mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{K}}$.
3. *If* $\tilde{\alpha} \in C(\partial\Omega)$ *is such that* $\alpha < \tilde{\alpha} \lneqq \delta_{1,\beta}\beta$, *then* $\mathcal{E}^{\tilde{\alpha}}_{\mathcal{G}} > \mathcal{E}^{\alpha}_{\mathcal{G}}$, $\mathcal{E}^{\tilde{\alpha}}_{\mathcal{K}} > \mathcal{E}^{\alpha}_{\mathcal{K}}$, $\mathcal{P}\mathcal{E}^{\tilde{\alpha}}_{\mathcal{G}} > \mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{G}}$, *and* $\mathcal{P}\mathcal{E}^{\tilde{\alpha}}_{\mathcal{K}} > \mathcal{P}\mathcal{E}^{\alpha}_{\mathcal{K}}$.

Proof. In the following items we will assume that $0 \lneqq f \in C(\overline{\Omega})$ and $0 \lneqq \varphi \in C(\partial\Omega)$. Moreover, we will write $u_{\alpha} = \mathcal{E}^{\alpha}_{\mathcal{G}}f$ and $v_{\alpha} = \mathcal{E}^{\alpha}_{\mathcal{K}}\varphi$, so

$$(\mathcal{I}-\mathcal{G}\mathcal{K}\alpha\mathcal{P})\,u_{\alpha} = \mathcal{G}\mathcal{I}_d\mathcal{G}f \quad \text{and} \quad (\mathcal{I}-\mathcal{G}\mathcal{K}\alpha\mathcal{P})\,v_{\alpha} = \mathcal{G}\mathcal{K}\varphi \tag{31}$$

1. If $u_\alpha = \mathcal{E}_\mathcal{G}^\alpha f = 0$ for $f \gneqq 0$, then

$$u_\alpha = \mathcal{G}\mathcal{K}\alpha\mathcal{P}u_\alpha + \mathcal{G}\mathcal{I}_d\mathcal{G}f = \mathcal{G}\mathcal{I}_d\mathcal{G}f > 0 \tag{32}$$

by the maximum principle, a contradiction. So $\mathcal{E}_\mathcal{G}^\alpha$ positive implies that $\mathcal{E}_\mathcal{G}^\alpha$ is strictly positive. Since $K(x, y^*) = \lim_{t\downarrow 0} G(x, y^* - t\nu)/t$ for $x \in \Omega$, $y^* \in \partial\Omega$ and ν the exterior normal at y^*, we find that positivity of $\mathcal{E}_\mathcal{G}^\alpha$ implies that $\mathcal{E}_\mathcal{K}^\alpha$ is positive. We even have strict boundary positivity. Indeed, if $\mathcal{P}u_\alpha = 0$ then $u_\alpha = \mathcal{G}\mathcal{I}_d\mathcal{G}f$ and Hopf's boundary point lemma gives $\mathcal{P}u_\alpha > 0$, a contradiction. A similar argument holds for v_α. This proves the first set of claims.

2. Let $\alpha \leqq \tilde{\alpha} \lneqq \delta_{1,\beta}\beta$. We have

$$(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})\, u_{\tilde{\alpha}} = \mathcal{G}\mathcal{K}\,(\tilde{\alpha} - \alpha)\,\mathcal{P}u_{\tilde{\alpha}} + \mathcal{G}\mathcal{I}_d\mathcal{G}f$$

and, in turn, since $(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})$ is invertible in view of Lemma 5.9,

$$\left(\mathcal{I} - \mathcal{E}_\mathcal{K}^\alpha(\tilde{\alpha} - \alpha)\mathcal{P}\right) u_{\tilde{\alpha}} = u_\alpha. \tag{33}$$

For $\|\tilde{\alpha} - \alpha\|_{L^\infty(\partial\Omega)}$ small enough (say $\|\tilde{\alpha} - \alpha\|_{L^\infty(\partial\Omega)} < \varepsilon$) one may invert the operator in (33) and find an identity with a convergent series:

$$\mathcal{E}_\mathcal{G}^{\tilde{\alpha}} = \mathcal{E}_\mathcal{G}^\alpha + \sum_{k=1}^{\infty} \left(\mathcal{E}_\mathcal{K}^\alpha\,(\tilde{\alpha} - \alpha)\,\mathcal{P}\right)^k \mathcal{E}_\mathcal{G}^\alpha. \tag{34}$$

Since $\mathcal{E}_\mathcal{K}^\alpha\,(\tilde{\alpha} - \alpha)\,\mathcal{P} \geqq 0$ holds, one finds that $u_{\tilde{\alpha}} = \mathcal{E}_\mathcal{G}^{\tilde{\alpha}} f \geqq \mathcal{E}_\mathcal{G}^\alpha f = u_\alpha$. The series formula (34) holds for $\|\tilde{\alpha} - \alpha\|_{L^\infty(\partial\Omega)} < \varepsilon$. However, if $\|\tilde{\alpha} - \alpha\|_{L^\infty(\partial\Omega)} \geqq \varepsilon$, then the above argument can be repeated by considering some intermediate $\alpha := \alpha_0 \lneqq \alpha_1 \lneqq \cdots \lneqq \alpha_k := \tilde{\alpha}$ such that $\|\alpha_{i+1} - \alpha_i\|_{L^\infty(\partial\Omega)} < \varepsilon$ for all i. A similar reasoning applies to $v_{\tilde{\alpha}}$, v_α. This proves the second set of claims.

3. Let us consider the sequence $\{\varphi_m\}_{m=0}^\infty \subset C_d(\overline{\Omega})$, defined by

$$\varphi_0 = \mathcal{E}_\mathcal{G}^\alpha f,$$
$$\varphi_{m+1} = \mathcal{E}_\mathcal{K}^\alpha \left(\delta_{1,\beta}\beta - \alpha\right) \mathcal{P}\varphi_m \quad \text{for } m \geqq 0.$$

Since $\mathcal{E}_\mathcal{G}^\alpha f \gneqq 0$, we find that $\varphi_m \geqq 0$ for all $m \geqq 0$. Moreover, since $\mathcal{E}_\mathcal{K}^\alpha \left(\delta_{1,\beta}\beta - \alpha\right)\mathcal{P}$ is compact, either

   i. there exists $m_0 > 0$ such that $\varphi_m \gneqq 0$ for $m < m_0$ and $\varphi_m = 0$ for all $m \geqq m_0$, or
   ii. $\varphi_m / \|\varphi_m\|_{C_d(\overline{\Omega})} \to \varphi_\infty$ where φ_∞ is a nonnegative eigenfunction (with $\lambda = 1$) of:

$$\mathcal{E}_\mathcal{K}^\alpha \left(\delta_{1,\beta}\beta - \alpha\right) \mathcal{P}\varphi_\infty = \lambda\varphi_\infty.$$

If $\mathcal{E}_\mathcal{K}^\alpha \left(\delta_{1,\beta}\beta - \alpha\right) \mathcal{P}\varphi_{m_0} = 0$ then we find by item 1. that $\left(\delta_{1,\beta}\beta - \alpha\right)\mathcal{P}\varphi_{m_0} = 0$ and hence $\mathcal{P}\varphi_{m_0} = 0$. We find a contradiction since as in the proof of the first item it follows that $\varphi_{m_0} = \mathcal{G}\mathcal{I}_d\mathcal{G}\varphi_{m_0 - 1}$ and $\mathcal{P}\varphi_{m_0} > 0$ holds by Hopf's boundary point lemma.

In the second case φ_∞ is a multiple of the unique positive eigenfunction $u_{1,\beta}$, see Lemma 5.9. So for sufficiently large m_1 there exist $c_2 > c_1 > 0$ such that

$$c_1 u_{1,\beta} \leqq \frac{\varphi_m}{\|\varphi_m\|_{C_d(\overline{\Omega})}} \leqq c_2 u_{1,\beta} \quad \text{for all } m \geqq m_1.$$

Now set

$$\begin{aligned} \psi_0 &= \mathcal{E}_{\mathcal{G}}^{\alpha} f, \\ \psi_{m+1} &= \mathcal{E}_{\mathcal{K}}^{\alpha} (\tilde{\alpha} - \alpha) \mathcal{P} \psi_m \quad \text{for } m \geqq 0. \end{aligned} \tag{35}$$

Since for some $\varepsilon > 0$ it holds that

$$\varepsilon \left(\delta_{1,\beta}\beta - \alpha\right) \leqq \tilde{\alpha} - \alpha \leqq \delta_{1,\beta}\beta - \alpha,$$

we obtain $\psi_m \geqq \varepsilon^m \varphi_m$ for all m and by (35)

$$\psi_m \geqq \varepsilon^m \varphi_m \geqq c_1 \varepsilon^m \|\varphi_m\|_{C_d(\overline{\Omega})} u_{1,\beta} \quad \text{for all } m \geqq m_1.$$

Then, from (34) it follows that there exists $c_3 > 0$ such that

$$\mathcal{E}_{\mathcal{G}}^{\tilde{\alpha}} f \geqq \mathcal{E}_{\mathcal{G}}^{\alpha} f + c_3 u_{1,\beta}.$$

In a similar way we proceed with $v_{\tilde{\alpha}}$ and v_α. □

With the result derived in Lemma 5.9 it will suffice to have positivity preserving for a negative $\alpha \in C(\partial\Omega)$ in order to ensure that this property will hold for any sign-changing $\tilde{\alpha}$ with $\alpha \leqq \tilde{\alpha} \lneqq \delta_{1,\beta}\beta$. So we may restrict ourselves to $\alpha \leqq 0$.

We now prove a crucial "comparison" statement in the case where $\mathcal{G}\mathcal{K}\alpha\mathcal{P}$ has a small spectral radius:

Lemma 5.12. *Let* $\partial\Omega \in C^2$ *and suppose that* $0 \geqq \alpha \in C(\partial\Omega)$ *is such that* $r_\sigma(\mathcal{G}\mathcal{K}\alpha\mathcal{P}) < 1$. *If there exists* $M > 0$ *such that*

$$\mathcal{G}\mathcal{K}\mathcal{P}\mathcal{G}\mathcal{I}_d\mathcal{G} \leqq M\, \mathcal{G}\mathcal{I}_d\mathcal{G}, \tag{36}$$

and if $\|\alpha\|_{L^\infty(\partial\Omega)} < M^{-1}$ *then* $\mathcal{E}_{\mathcal{G}}^{\alpha} > 0$.

Proof. Clearly, $\alpha = -\alpha^-$. Since $r_\sigma\left(\mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right) < 1$ the Equation (27) can be rewritten as a Neumann series

$$u = \left(\mathcal{I} + \mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right)^{-1} \mathcal{G}\mathcal{I}_d\mathcal{G} f = \sum_{k=0}^{\infty} \left(-\mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right)^k \mathcal{G}\mathcal{I}_d\mathcal{G} f,$$

which reads

$$u = \left(\sum_{k=0}^{\infty} \left(\mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right)^{2k}\right) \left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right) \mathcal{G}\mathcal{I}_d\mathcal{G} f \tag{37}$$

after joining the odd and even powers. Next, notice that in view of (37) it suffices to show that the operator $\left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^-\mathcal{P}\right) \mathcal{G}\mathcal{I}_d\mathcal{G}$ is strongly positive. This fact is a direct consequence of (36) and $\left\|\alpha^-\right\|_{L^\infty(\partial\Omega)} \leqq M^{-1}$. □

Lemma 5.12 guarantees the existence of strictly negative $\alpha \in C(\partial\Omega)$ for which (1) is positivity preserving provided one can show the existence of $M > 0$ such that (36) holds. We will prove the existence of such M in Proposition 7.1; to this end, we need fine estimates of the kernels related to $\mathcal{G}$ and $\mathcal{K}$. These are given in the next section.

Remark 5.13. The operator $\left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}\right) : C_d(\overline{\Omega}) \to C_d(\overline{\Omega})$ on its own cannot be expected to be positivity preserving. Indeed, although the identity $\mathcal{I}$ is strictly positive it is just pointwise positive. For example if we take $\Omega = \{|x| < 1\}$ and the positive function $f(x) = |x|^2 - |x|^4$, then $\mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}f > 0$ and

$$\left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}\right)(f)(0) = -\left(\mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}f\right)(0) < 0.$$

Even $\left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}\right)\mathcal{G} : C(\overline{\Omega}) \to C_d(\overline{\Omega})$ is not positivity preserving for any small $\alpha^{-} \gneqq 0$. A counterexample to positivity can be obtained by taking a sequence $\{f_m\}$ such that $\{d\, f_m\}$ converges to a Dirac delta distribution δ_y for some $y \in \partial\Omega$ with $\alpha^{-}(y) > 0$. In Section 6 we show that an additional $\mathcal{G}$ is sufficient in order to have for α^{-} small the positivity preserving property, in other words, that $\left(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha^{-}\mathcal{P}\right)\mathcal{G}\mathcal{I}_d\mathcal{G}$ is strongly positive.

6. Kernel estimates

In this section, we prove some new kernel estimates. Since they are of independent interest, we prove them under the slightly weaker assumption that $\partial\Omega \in C^{1,1}$. Indeed, for $C^{1,1}$-domains, the operators $\mathcal{G}$ and $\mathcal{K}$ defined in Section 5.1 can be represented by integral kernels which we denote by G and K, namely

$$(\mathcal{G}f)(x) = \int_{\Omega} G(x, y) f(y)\, \mathrm{d}y \quad \text{and} \quad (\mathcal{K}g)(x) = \int_{\partial\Omega} K(x, y) g(y)\, \mathrm{d}\sigma_y. \tag{38}$$

Moreover, it holds that

$$K(x, y) = \frac{-\partial}{\partial \nu_y} G(x, y) \quad \text{for all } (x, y) \in \Omega \times \partial\Omega. \tag{39}$$

We will estimate the kernels in (38) by using the following

Notation 6.1. *Let f, g be functions defined on the same domain D. We write $f \preceq g$ if there exists $c > 0$ such that $f(x) \leqq c g(x)$ for all $x \in D$. We write $f \simeq g$ if both $f \preceq g$ and $g \preceq f$.*

Based on several estimates due to Zhao [31, 32] (see also [28, 8]) one may show:

Proposition 6.2 ([16, Lemmas 3.1 and 3.2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^{1,1}$. Then the following uniform estimates hold for $(x, y) \in \Omega \times \Omega$:*

$$\text{if } n \geqq 5: \quad \int_\Omega G(x,z)G(z,y)\,\mathrm{d}z \simeq |x-y|^{4-n} \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right), \tag{40}$$

$$\text{if } n = 4: \quad \int_\Omega G(x,z)G(z,y)\,\mathrm{d}z \simeq \log\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right), \tag{41}$$

$$\text{if } n = 3: \quad \int_\Omega G(x,z)G(z,y)\,\mathrm{d}z \simeq \sqrt{d(x)d(y)} \min\left(1, \frac{\sqrt{d(x)d(y)}}{|x-y|}\right), \tag{42}$$

$$\text{if } n = 2: \quad \int_\Omega G(x,z)G(z,y)\,\mathrm{d}z \simeq d(x)d(y) \log\left(2 + \frac{1}{|x-y|^2 + d(x)d(y)}\right). \tag{43}$$

In order to use these estimates in our proofs, we also need the following geometric result:

Lemma 6.3. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^{1,1}$. For $x \in \Omega$ let $x^* \in \partial\Omega$ be any point such that $d(x) = |x - x^*|$.*

- *Then there exists $r_\Omega > 0$ such that for $x \in \Omega$ with $d(x) \leqq r_\Omega$ there is a unique $x^* \in \partial\Omega$.*
- *Then the following uniform estimates hold:*

$$\text{if } (x,y) \in \Omega \times \Omega: \ |x-y| \preceq d(x) + d(y) + \left|x^* - y^*\right|, \tag{44}$$

$$\text{if } (x,y) \in \Omega \times \Omega: \ \frac{d(x)}{d(x) + d(y) + |x^* - y^*|} \preceq \min\left(1, \frac{d(x)}{|x-y|}\right), \tag{45}$$

$$\text{if } (x,z) \in \Omega \times \partial\Omega: \ |x-z| \simeq d(x) + \left|x^* - z\right|. \tag{46}$$

And for $(x, y, z) \in \Omega \times \Omega \times \partial\Omega$:

$$\text{if } d(y) \leqq d(x) \text{ and } \left|x^* - y^*\right| \leqq d(x) + d(y) \text{ then } |x-z| \simeq d(x) + \left|y^* - z\right|. \tag{47}$$

Proof. Since $\partial\Omega \in C^{1,1}$ there exists $r_1 > 0$ such that Ω can be filled with balls of radius r_1. Set $r_\Omega = \frac{1}{2}r_1$. For $x \in \Omega$ with $d(x) \leqq r_\Omega$ there is a unique $x^* \in \partial\Omega$.

Estimate (44) is just the triangle inequality. Estimate (46) follows from the three inequalities

$$|x-z| \leqq \left|x - x^*\right| + \left|x^* - z\right| = d(x) + \left|x^* - z\right|,$$
$$d(x) \leqq |x-z| \text{ and } \left|x^* - z\right| \leqq \left|x^* - x\right| + |x-z| \leqq 2\,|x-z|.$$

In order to prove (47), we first remark that under the assumptions made we have $d(x) \geqq \frac{1}{2}|x^* - y^*|$. This yields the two inequalities

$$\begin{aligned} d(x) + |x^* - z| &\leqq d(x) + |x^* - y^*| + |y^* - z| \leqq 3d(x) + |y^* - z| \\ &\leqq 3(d(x) + |y^* - z|), \\ d(x) + |y^* - z| &\leqq d(x) + |x^* - y^*| + |x^* - z| \leqq 3d(x) + |x^* - z| \\ &\leqq 3(d(x) + |x^* - z|). \end{aligned}$$

In turn, these inequalities read $d(x) + |x^* - z| \simeq d(x) + |y^* - z|$. This, combined with (46), proves (47).

To prove (45), we distinguish two cases. If $|x-y| \leqq \frac{\max(d(x),d(y))}{2}$, then $\frac{1}{2}d(x) \leqq d(y) \leqq 2d(x)$ and $|x-y| \preceq d(x) \simeq d(y)$. It follows that

$$\frac{d(x)}{d(x)+d(y)+|x^*-y^*|} \preceq 1 \simeq \min\left(1, \frac{d(x)}{|x-y|}\right)$$

and a similar estimate with x and y interchanged. If $|x-y| \geqq \frac{\max(d(x),d(y))}{2}$, we use (44) to find that

$$\frac{d(x)}{d(x)+d(y)+|x^*-y^*|} \preceq \frac{d(x)}{|x-y|} \simeq \min\left(1, \frac{d(x)}{|x-y|}\right)$$

and a similar estimate with x and y interchanged. □

We are now ready to prove the new estimates which are needed for our purposes.

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^n (n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^{1,1}$. Then the following uniform estimates hold for $(x,z) \in \Omega \times \partial\Omega$:*

$$\int_\Omega G(x,\xi)K(\xi,z)\,\mathrm{d}\xi \simeq \begin{cases} d(x)\ |x-z|^{2-n} & \textit{for}\ \ n \geqq 3, \\ d(x)\ \log\left(2+\frac{1}{|x-z|^2}\right) & \textit{for}\ n=2. \end{cases}$$

Proof. Let

$$H(x,z) := \int_\Omega G(x,\xi)G(\xi,z)\,\mathrm{d}\xi \quad \text{for all}\ (x,z) \in \Omega \times \partial\Omega.$$

In view of (39), and since $H(x,z)=0$ for $z \in \partial\Omega$, we have

$$\int_\Omega G(x,\xi)K(\xi,z)\,\mathrm{d}\xi = \frac{-\partial}{\partial\nu_z}H(x,z) = \lim_{t\to 0}\frac{H(x,z-t\nu_z)}{t}. \tag{48}$$

Note also that if r_Ω is as in Lemma 6.3, then $d(z-t\nu_z)=t$ for all $z \in \partial\Omega$ and $t \leqq r_\Omega$. Hence, by (40) we obtain for $n \geqq 5$

$$\begin{aligned}\lim_{t\to 0}\frac{H(x,z-t\nu_z)}{t} &\simeq \lim_{t\to 0}\frac{|x-z+t\nu_z|^{4-n}\ \min\left(1, \frac{td(x)}{|x-z+t\nu_z|^2}\right)}{t} \\ &= d(x)\ |x-z|^{2-n}.\end{aligned}$$

For $n=4$, we use (41) to obtain

$$\lim_{t\to 0}\frac{H(x,z-t\nu_z)}{t} \simeq \lim_{t\to 0}\frac{\log\left(1+\frac{td(x)}{|x-z+t\nu_z|^2}\right)}{t} \simeq d(x)\ |x-z|^{-2}.$$

For $n=3$, we use (42) to obtain

$$\lim_{t\to 0}\frac{H(x,z-t\nu_z)}{t} \simeq \lim_{t\to 0}\frac{\sqrt{td(x)}\ \min\left(1, \frac{\sqrt{td(x)}}{|x-z+t\nu_z|}\right)}{t} = d(x)\ |x-z|^{-1}.$$

And finally for $n = 2$, we use (43) to obtain

$$\lim_{t\to 0} \frac{H(x, z - t\nu_z)}{t} \simeq \lim_{t\to 0} \frac{td(x) \log\left(2 + \frac{1}{|x-z+t\nu_z|^2+td(x)}\right)}{t}$$
$$= d(x)\ \log\left(2 + \frac{1}{|x - z|^2}\right).$$

By (48), the statement is so proved for any $n \geqq 2$. □

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^n$ ($n \geqq 2$) be a bounded domain with $\partial\Omega \in C^{1,1}$. Then the following uniform estimates hold for $(x, y) \in \Omega \times \Omega$:*

$$\int_\Omega \int_{\partial\Omega} \int_\Omega G(x, \xi) K(\xi, z) \frac{-\partial}{\partial \nu_z} G(z, w) G(w, y)\, d\xi\, d\sigma_z\, dw$$
$$\preceq \begin{cases} d(x)d(y)\ \ (d(x) + d(y) + |x^* - y^*|)^{2-n} & \text{for } n \geqq 3, \\ d(x)d(y)\ \log\left(2 + \frac{1}{d(x)+d(y)+|x^*-y^*|)}\right) & \text{for } n = 2, \end{cases} \tag{49}$$

respectively, for $(x, y) \in \Omega \times \partial\Omega$:

$$\int_\Omega \int_{\partial\Omega} \int_\Omega G(x, \xi) K(\xi, z) \frac{-\partial}{\partial \nu_z} G(z, w) K(w, y)\, d\xi\, d\sigma_z\, dw$$
$$\preceq \begin{cases} d(x)\ |x - y|^{2-n} & \text{for } n \geqq 3, \\ d(x) \log\left(2 + \frac{1}{|x-y|)}\right) & \text{for } n = 2. \end{cases} \tag{50}$$

Proof. Setting

$$R(x, y) := \int_\Omega \int_{\partial\Omega} \int_\Omega G(x, \xi) K(\xi, z) \frac{-\partial}{\partial \nu_z} G(z, w) G(w, y)\, d\xi\, d\sigma_z\, dw,$$

and using (39) and the estimates from Lemma 6.4, the following holds:

$$R(x, y) \preceq d(x)d(y) \int_{\partial\Omega} |x - z|^{2-n}\, |z - y|^{2-n}\ d\sigma_z \qquad \text{if } n \geqq 3,$$
$$R(x, y) \preceq d(x)d(y) \int_{\partial\Omega} \log\left(2 + \frac{1}{|x - z|^2}\right) \log\left(2 + \frac{1}{|y - z|^2}\right) d\sigma_z \quad \text{if } n = 2.$$

Let r_Ω be as in Lemma 6.3. We distinguish three cases, according to the positions of $x, y \in \Omega$.

- Case 1: $\max(d(x), d(y)) \geqq r_\Omega$.

By symmetry we may assume that $d(y) \geqq r_\Omega$ and find for $n \geqq 3$ that

$$\int_{\partial\Omega} |x - z|^{2-n}\, |z - y|^{2-n}\ d\sigma_z \preceq \int_{\partial\Omega} |x - z|^{2-n}\ d\sigma_z \preceq \int_0^1 \frac{r^{n-2}}{(d(x) + r)^{n-2}}\, dr \preceq 1,$$

and for $n = 2$

$$\int_{\partial\Omega} \log\left(2 + \frac{1}{|x-z|^2}\right) \log\left(2 + \frac{1}{|z-y|^2}\right) d\sigma_z \preceq \int_0^1 \log\left(2 + \frac{1}{(d(x)+r)^2}\right) dr \preceq 1,$$

which imply (49) since $d(y) \geqq r_\Omega$.

- Case 2: $\max(d(x), d(y)) < r_\Omega$ and $|x^* - y^*| \geqq d(x) + d(y)$.

In this case, in view of Lemma 6.3, we have that (46) holds for both x and y. So, for $n \geqq 3$ we have

$$\int_{\partial\Omega} |x-z|^{2-n}\,|z-y|^{2-n}\,\mathrm{d}\sigma_z \preceq \int_{\partial\Omega} \frac{1}{(d(x)+|x^*-z|)^{n-2}}\,\frac{1}{(d(y)+|y^*-z|)^{n-2}}\,\mathrm{d}\sigma_z.$$

We split this integral as $I_x + I_y$ where I_x is the integral over

$$\partial\Omega_x = \left\{z \in \partial\Omega;\ \left|x^* - z\right| \leqq \left|y^* - z\right|\right\}$$

and I_y over $\partial\Omega_y = \partial\Omega \backslash \partial\Omega_x$. Over $\partial\Omega_x$ we have

$$|x^*-z| + |x^*-y^*| \leqq |x^*-z| + |x^*-z| + |y^*-z| \leqq 3|y^*-z|.$$

Hence, we find

$$\begin{aligned} I_x &\preceq \int_{\partial\Omega_x} \frac{1}{(d(x)+|x^*-z|)^{n-2}}\,\frac{1}{(d(y)+|x^*-z|+|x^*-y^*|)^{n-2}}\,\mathrm{d}\sigma_z \\ &\preceq \frac{1}{|x^*-y^*|^{n-2}} \int_0^1 \frac{r^{n-2}}{(d(x)+r)^{n-2}}\,\mathrm{d}r \preceq \left|x^*-y^*\right|^{2-n} \\ &\preceq \left(d(x)+d(y)+\left|x^*-y^*\right|\right)^{2-n} \end{aligned}$$

where, in the last estimate, we used $|x^* - y^*| \geqq d(x) + d(y)$.

Similarly, for $n = 2$ we find

$$\begin{aligned} I_x &\preceq \int_{\partial\Omega_x} \log\left(2 + \frac{1}{d(x)+|x^*-z|}\right) \\ &\quad \times \log\left(2 + \frac{1}{d(y)+|x^*-z|+|x^*-y^*|}\right)\mathrm{d}\sigma_z \\ &\preceq \log\left(2+\frac{1}{d(y)+|x^*-y^*|}\right)\int_0^1 \log\left(2+\frac{1}{d(x)+r}\right)\mathrm{d}r \\ &\preceq \log\left(2+\frac{1}{d(y)+|x^*-y^*|}\right) \preceq \log\left(2+\frac{1}{d(x)+d(y)+|x^*-y^*|}\right). \end{aligned}$$

Analogous estimates hold for I_y. All together these estimates prove (49) in Case 2.

- Case 3: $\max(d(x), d(y)) < r_\Omega$ and $|x^* - y^*| \leqq d(x) + d(y)$.

By symmetry, we may assume that $d(y) \leqq d(x)$. Then, we may use both (46) and (47). So, for $n \geqq 3$ we find

$$\begin{aligned}
&\int_{\partial\Omega} |x-z|^{2-n}\,|z-y|^{2-n}\,\mathrm{d}\sigma_z \\
&\preceq \int_{\partial\Omega} \frac{1}{(d(x)+|y^*-z|)^{n-2}}\,\frac{1}{(d(y)+|y^*-z|)^{n-2}}\,\mathrm{d}\sigma_z \\
&\preceq \int_0^1 \frac{r^{n-2}}{(d(x)+r)^{n-2}}\,\frac{1}{(d(y)+r)^{n-2}}\,\mathrm{d}r \\
&\preceq \frac{1}{d(x)^{n-2}} \preceq \left(d(x)+d(y)+\left|x^*-y^*\right|\right)^{2-n},
\end{aligned}$$

and for $n = 2$

$$\begin{aligned}
&\int_{\partial\Omega} \log\left(2+\frac{1}{|x-z|}\right)\log\left(2+\frac{1}{|y-z|}\right)\mathrm{d}\sigma_z \\
&\preceq \int_{\partial\Omega} \log\left(2+\frac{1}{d(x)+|y^*-z|}\right)\log\left(2+\frac{1}{d(y)+|y^*-z|}\right)\mathrm{d}\sigma_z \\
&\preceq \int_0^1 \log\left(2+\frac{1}{d(x)+r}\right)\log\left(2+\frac{1}{d(y)+r}\right)\mathrm{d}r \\
&\preceq \log\left(2+\frac{1}{d(x)}\right) \preceq \log\left(2+\frac{1}{d(x)+d(y)+|x^*-y^*|}\right).
\end{aligned}$$

This proves (49) in Case 3.

For the estimates in (50) one divides the estimates in (49) by $d(y)$, takes the limit for $d(y) \to 0$, and uses (46), namely that $d(x) + |x^* - y| \simeq |x - y|$ for $y \in \partial\Omega$. □

7. Proof of Proposition 4.8

We first use the kernel estimates of Section 6 to prove:

Proposition 7.1. *Let Ω be a bounded domain with $\partial\Omega \in C^{1,1}$. Then there exists a constant $M_\Omega > 0$ such that*

$$\mathcal{GKPGI}_d\mathcal{G} \leqq M_\Omega \mathcal{GI}_d\mathcal{G} \quad \text{and} \quad \mathcal{GKPGK} \leqq M_\Omega \mathcal{GIK}.$$

Proof. We know that the integral kernel R that corresponds to $\mathcal{GKPGI}_d\mathcal{G}$ satisfies the estimates in Lemma 6.5. By Proposition 6.2 we know estimates from below for $\mathcal{GI}_d\mathcal{G}$. We have to compare these estimates. To this end, we use the following trivial fact

$$\min(1,\alpha)\min(1,\beta) \leqq \min(1,\alpha\beta) \quad \text{for all} \quad \alpha,\beta \geqq 0,$$

combined with (45) and (44). Considering the different dimensions separately we then have the following. For $n \geqq 5$,

$$\left(d(x)+d(y)+\left|x^*-y^*\right|\right)^{2-n} d(x)d(y) \preceq |x-y|^{4-n}\,\min\left(1,\frac{d(x)d(y)}{|x-y|^2}\right).$$

This, combined with Lemma 6.5 and (40), proves the statement for $n \geqq 5$.

For $n = 4$ we argue as for $n = 5$ to find

$$\begin{aligned}&\left(d(x) + d(y) + \left|x^* - y^*\right|\right)^{-2} d(x)d(y)\\&\quad\preceq \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) \preceq \log\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right).\end{aligned}$$

This, combined with Lemma 6.5 and (41), proves the statement for $n = 4$.

For $n = 3$ we have

$$\begin{aligned}&\left(d(x) + d(y) + \left|x^* - y^*\right|\right)^{-1} d(x)d(y)\\&\preceq \sqrt{d(x)d(y)\min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)} = \sqrt{d(x)d(y)}\min\left(1, \frac{\sqrt{d(x)d(y)}}{|x-y|}\right).\end{aligned}$$

This, combined with Lemma 6.5 and (42), proves the statement for $n = 3$.

For $n = 2$, by using (44) we find as a variation of (45) that

$$\log\left(2 + \frac{1}{d(x) + d(y) + |x^* - y^*|}\right) \preceq \log\left(2 + \frac{1}{|x-y|^2 + d(x)d(y)}\right).$$

This, combined with Lemma 6.5 and (43), proves the statement for $n = 2$. □

We can now prove the '$\mathcal{C}$-version' of Proposition 4.8:

Lemma 7.2. *There exists $\delta_{c,\beta} := \delta_{c,\beta}(\Omega) \in [-\infty, 0)$ such that the following holds for a $\mathcal{C}$-solution u of* (1).

1. *for $\delta_{c,\beta}\beta \leqq \alpha \lneqq \delta_{1,\beta}\beta$ it follows that if $0 \lneqq f \in C(\overline{\Omega})$ then $u \gneqq 0$;*
2. *for $\delta_{c,\beta}\beta < \alpha \lneqq \delta_{1,\beta}\beta$ it follows that if $0 \lneqq f \in C(\overline{\Omega})$ then $u \geqq c_f\, d$ for some $c_f > 0$ depending on f;*
3. *for $\alpha < \delta_{c,\beta}\beta$ there are $0 \lneqq f \in C(\overline{\Omega})$ with u somewhere negative.*

Proof. Let M_Ω be as in Proposition 7.1 and $\delta := -\left(M_\Omega \max_{x\in\partial B} \beta(x)\right)^{-1} < 0$. Then, by Lemmas 5.9 and 5.12 we infer that

$$\text{if } \delta\beta \leqq \alpha \lneqq \delta_{1,\beta}\beta \text{ and } f \gneqq 0 \quad \text{then} \quad u \gneqq 0 \text{ in } \Omega, \tag{51}$$

where u is the unique $\mathcal{C}$-solution of (1). Let $\delta_{c,\beta}$ be the (negative) infimum of all such δ which satisfy (51). Then, nonnegativity of the solution follows.

Moreover, if $\delta_{c,\beta}\beta < \alpha$, then Lemma 5.11 yields the existence of c_f as in the second statement.

In order to prove the the third statement of the lemma, we argue for contradiction. Assume that $\alpha < \delta_{c,\beta}$ and that for any $0 \lneqq f \in C(\overline{\Omega})$ the unique $\mathcal{C}$-solution u is positive. Then, we would contradict the above definition of $\delta_{c,\beta}$. □

Using a density argument we can finally give the

Proof of Proposition 4.8. Let $\delta_{c,\beta}\beta \leqq \alpha \lneqq \delta_{1,\beta}\beta$ and let $0 \lneqq f \in L^2(\Omega)$. Let $u \in \mathcal{H}$ be the unique $\mathcal{H}$-solution of (1), according to Lemma 4.6. Let $f_0 \in C(\overline{\Omega})$ be such that $0 \lneqq f_0 \lneqq f$ and let u_0 denote the unique $\mathcal{C}$-solution of (1) corresponding to f_0, according to Lemma 5.9. Since $(\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\mathcal{G}\mathcal{I}_d\mathcal{G}$ is a positive operator, we have

$$0 \lneqq f_0 \lneqq f \quad \Longrightarrow \quad (\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\mathcal{G}\mathcal{I}_d\mathcal{G} f \geqq (\mathcal{I} - \mathcal{G}\mathcal{K}\alpha\mathcal{P})^{-1}\mathcal{G}\mathcal{I}_d\mathcal{G} f_0 \quad \text{in } \Omega.$$

Hence, $u(x) \geqq u_0(x) \gneqq 0$ in view of Lemma 5.11, proving the first statement in Proposition 4.8.

If $\delta_{c,\beta}\beta < \alpha \lneqq \delta_{1,\beta}\beta$ and $0 \lneqq f \in L^2(\Omega)$, then the same arguments as above show that $u(x) \geqq u_0(x) \geqq c_{f_0} d$ for some $c_{f_0} > 0$ depending on f_0 and, therefore, also on f. The third statement follows directly from the third statement in Lemma 7.2 combined with Proposition 5.8. □

8. Proof of Theorem 2.5

Let us first recall the two boundary value problems addressed in the statement:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \left(\Delta + m\frac{\partial}{\partial\nu}\right) u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

For all $m > 0$ let $u_m \in \mathcal{H}(\Omega)$ be the unique $\mathcal{H}$-solution of the problem on the left in (52). Then, according to (5) we have

$$\int_\Omega \Delta u_m \Delta\phi \,\mathrm{d}x + m\int_{\partial\Omega} \frac{\partial u_m}{\partial\nu}\frac{\partial\phi}{\partial\nu}\,\mathrm{d}\sigma_x = \int_\Omega f\phi\,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{H}(\Omega). \tag{53}$$

Taking $\phi = u_m$ in (53) and using Hölder and Poincaré inequalities, gives (for all $m > 0$)

$$\begin{aligned} \|\Delta u_m\|^2_{L^2(\Omega)} &\leqq \|\Delta u_m\|^2_{L^2(\Omega)} + m\int_{\partial\Omega}\left|\frac{\partial u_m}{\partial\nu}\right|^2 \\ &= \int_\Omega f u_m \,\mathrm{d}x \leqq c\|f\|_{L^2(\Omega)}\|\Delta u_m\|_{L^2(\Omega)}. \end{aligned} \tag{54}$$

Inequality (54) shows that the sequence $\{u_m\}$ is bounded in $H^2(\Omega)$ so that, up to a subsequence, we have

$$u_m \rightharpoonup \overline{u} \quad \text{in } H^2(\Omega) \quad \text{as } m \to \infty \tag{55}$$

for some $\overline{u} \in \mathcal{H}(\Omega)$. Once boundedness is established, if we let $m \to \infty$ then (54) also tells us that

$$\frac{\partial u_m}{\partial\nu} \to 0 \quad \text{in } L^2(\partial\Omega) \quad \text{as } m \to \infty.$$

Therefore, $\overline{u} \in H^2_0(\Omega)$. Now take any function $\phi \in H^2_0(\Omega)$ in (53) and let $m \to \infty$: by (55) we obtain

$$\int_\Omega \Delta\overline{u}\Delta\phi = \int_\Omega f\phi \quad \text{for all } \phi \in H^2_0(\Omega).$$

Hence, $\overline{u}$ is the unique solution of the corresponding Dirichlet problem (7). Since (55) also implies that, up to a subsequence, $u_m(x) \to \overline{u}(x)$ for almost everywhere $x \in \Omega$, one finds that $\overline{u} \gneqq 0$.

9. Proof of Theorem 2.6

Throughout this section we denote by $\mathcal{Q}$ be the solution operator for (7): $u = \mathcal{Q}f$. We first compare these solutions with the solutions of the corresponding Navier problem. The next statement extends the comparison result of Lemma 5.11 to the limit case where $\tilde{\alpha} = 0$ and $\alpha = -\infty$:

Lemma 9.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ be a bounded domain with $\partial\Omega \in C^{4,\gamma}$. If $\mathcal{Q} > 0$, then $\mathcal{G}\mathcal{I}_d\mathcal{G} > \mathcal{Q}$. It even holds true that for each $0 \lneqq f \in C(\overline{\Omega})$ there exists $c_f > 0$ such that*

$$(\mathcal{G}\mathcal{I}_d\mathcal{G}f)(x) \geqq (\mathcal{Q}f)(x) + c_f\, d(x) \quad \textit{for all } x \in \Omega.$$

Proof. Since we assumed that $\partial\Omega \in C^{4,\gamma}$, the functions $\mathcal{G}\mathcal{I}_d\mathcal{G}f$ and $\mathcal{Q}f$ are in $W^{4,p}(\Omega)$ for all $p \in (1,\infty)$ and hence in $C^3(\overline{\Omega})$. The function $w = \mathcal{G}f + \Delta\mathcal{Q}f$ satisfies $-\Delta w = 0$ in Ω and $w = \Delta\mathcal{Q}f$ on $\partial\Omega$. Since $\mathcal{Q}f = \frac{\partial}{\partial\nu}\mathcal{Q}f = 0$ on $\partial\Omega$ and $\mathcal{Q}f \geqq 0$ in Ω we have $\Delta\mathcal{Q}f = \frac{\partial^2}{\partial\nu^2}\mathcal{Q}f \geqq 0$ on $\partial\Omega$. The maximum principle for harmonic functions implies that $w \geqq 0$ in Ω.

Next we set $v = \mathcal{G}\mathcal{I}_d\mathcal{G}f - \mathcal{Q}f$ and find $-\Delta v = w$ in Ω and $v = 0$ on $\partial\Omega$. Again, by the maximum principle we find $v \geqq 0$ in Ω. Moreover, by Hopf's boundary point Lemma either there exists $c > 0$ with $v(x) \geqq c d(x)$ for all $x \in \Omega$ or $v = 0$. Since $\frac{-\partial}{\partial\nu}\mathcal{G}\mathcal{I}_d\mathcal{G}f > 0$ and $\frac{-\partial}{\partial\nu}\mathcal{Q}f = 0$ on $\partial\Omega$ we have $v \neq 0$. □

Proof of Theorem 2.6. Let $0 \lneqq f \in C(\overline{\Omega})$ and note that (9) readily implies

$$\frac{\partial^2}{\partial\nu^2}\mathcal{Q}f(x) > 0 \quad \text{for all } x \in \partial\Omega. \tag{56}$$

Let $t \geqq 0$; then, by Lemma 5.9, the following problem admits a unique $\mathcal{C}$-solution which we denote by u^t:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 \text{ and } \Delta u = -t u_\nu & \text{on } \partial\Omega. \end{cases}$$

By Lemma 9.1 we find that $u^0(x) \geqq (\mathcal{Q}f)(x) + c_f d(x)$. Since all "eigenvalues" of (1) are strictly positive (see [11]), the map $t \mapsto u^t : [0,\infty) \to C_d(\overline{\Omega})$ is continuous. Using elliptic regularity we also have that $t \mapsto u^t : [0,\infty) \to C^2(\overline{\Omega})$ is continuous. Let t_0 be the supremum of the numbers t such that $u^t \geqq \mathcal{Q}f$ for all $t < t_0$. Theorem 2.6 follows if we show that $t_0 = +\infty$. For contradiction, assume that $t_0 < +\infty$. Then, we find that $w := u^{t_0} - \mathcal{Q}f$ satisfies

$$\begin{cases} \Delta^2 w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ -w_\nu = -u_\nu^{t_0} =: \psi \geqq 0 & \text{on } \partial\Omega. \end{cases}$$

Since t_0 is maximal we have that either $w(x_0) = 0$ for some $x_0 \in \Omega$ or $\psi(x_0) = 0$ for some $x_0 \in \partial\Omega$. If $x_0 \in \Omega$ we find a contradiction since

$$w(x) = \int_{\partial\Omega} K_1(x,z)\psi(z)\,\mathrm{d}\sigma_z$$

with $K_1(x,z) = \Delta_z Q(x,z) = \frac{\partial^2}{\partial \nu_z^2} Q(x,z) > c\, d(x)^2$ for $z \in \partial\Omega$ and $x \in \Omega$. So we find that $\psi(x_0) = 0$ for some $x_0 \in \partial\Omega$. Hence, using the well-known expression of Δu^{t_0} on $\partial\Omega$ and the fact that $u^{t_0} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial \nu^2} u^{t_0}(x_0) &= \Delta u^{t_0}(x_0) - (n-1)\kappa(x_0) u_\nu^{t_0}(x_0) \\ &= -\left(t_0 + (n-1)\kappa(x_0)\right) u_\nu^{t_0}(x_0) = 0, \end{aligned} \tag{57}$$

where $\kappa(x_0)$ denotes the mean curvature of $\partial\Omega$ at x_0. Since $w(x_0) = w_\nu(x_0) = 0$ and using both (57) and (56), at x_0 we have that

$$0 \leqq \frac{\partial^2}{\partial \nu^2} w(x_0) = \frac{\partial^2}{\partial \nu^2} u^{t_0}(x_0) - \frac{\partial^2}{\partial \nu^2} \mathcal{Q} f(x_0) < 0,$$

a contradiction. □

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