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FINITE TIME BLOW-UP AND GLOBAL SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS WITH INITIAL DATA AT HIGH ENERGY LEVEL

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Abstract. For a class of semilinear parabolic equations on a bounded domain Ω, we analyze the behavior of the solutions when the initial data varies in the phase space $H_0^1(\Omega)$. We obtain both global solutions and finite time blow-up solutions. Our main tools are the comparison principle and variational methods. Particular attention is paid to initial data at high energy level; to this end, a basic new idea is to exploit the weak dissipativity (respectively antidissipativity) of the semiflow inside (respectively outside) the Nehari manifold.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Depending on suitable properties of the initial datum u_0, we are interested in both finite time blow-up solutions and solutions which exist globally in time of the following parabolic problem

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \Omega \times (0,T) \\ u(0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

where $u_0 \in H_0^1(\Omega)$, $T \in (0,\infty]$ and $1 < p < \frac{n+2}{n-2}$, understanding that $\frac{n+2}{n-2} = +\infty$ if $n = 2$.

Problem (1.1) has been studied by many authors and it appears a hard task to mention all of them. A strange fact about (1.1) is that the corresponding literature seems to be "partitioned" into equivalence classes, which

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is due to the fact that (1.1) may be tackled with several different and apparently unlinked tools. In particular, we mention critical-point theory and the mountain pass theorem by Ambrosetti-Rabinowitz [3], the potential well theory which started with the paper by Tsutsumi [45] (see also [33]), semigroup theory for which the starting point seems the paper by Weissler [46], classical tools such as smoothing effects and comparison methods revisited in a new functional analytic framework as in the paper by Hoshino-Yamada [23] (see also previous work in the monograph by Henry [21]).

In order to better explain this partition, let us discuss the assumption $p < \frac{n+2}{n-2}$, which is none other than a compactness condition (compactness of the embedding $H^1_0(\Omega) \subset L^{p+1}(\Omega)$). From the critical-point theory point of view, it is a necessary and sufficient condition for the validity of the Palais-Smale condition. From a purely elliptic point of view it is a necessary and sufficient condition for the existence of smooth nontrivial stationary solutions of (1.1) independently of the geometry of the domain. From the evolution point of view, contrary to the subcritical case $p < \frac{n+2}{n-2}$, if $p = \frac{n+2}{n-2}$ it is not clear in which way the blow-up occurs, see [25, Remark 2.5]; moreover, in [5, Theorem 1.1] it is shown that critical and supercritical growth parabolic problems are not uniformly well posed in a suitable sense. Finally, contrary to what happens for $p < \frac{n+2}{n-2}$, when $p \geq \frac{n+2}{n-2}$ global unbounded solutions may exist [32]. These different interpretations of the very same assumption should give an idea of how far apart the corresponding points of view are.

One purpose of the present paper is to collect, complement, and partly improve some of the results which have been obtained with the just-mentioned tools. At some points, we also provide new proofs to already established results.

The second (and main) purpose of the present paper is explained by its title. Depending on the initial datum u_0, it was shown in [24, 33] that (1.1) admits both solutions which blow up in finite time and global solutions which converge to $u \equiv 0$ as time tends to infinity. What is meant by blow-up will be made precise in Definition 1 in Section 3. Let us introduce the sets

$$\mathcal{B} = \{u_0 \in H^1_0(\Omega) : \text{the solution } u = u(t) \text{ of (1.1) blows up in finite time}\}$$

$$\mathcal{G} = \{u_0 \in H^1_0(\Omega) : \text{the solution } u = u(t) \text{ of (1.1) exists for all } t > 0\}$$

$$\mathcal{G}_0 = \{u_0 \in \mathcal{G} : u(t) \to 0 \text{ in } H^1_0(\Omega) \text{ as } t \to \infty\}\ .$$

Clearly, $H^1_0(\Omega) = \mathcal{G} \cup \mathcal{B}$; our purpose is to characterize the sets $\mathcal{B}$, $\mathcal{G}$ and $\mathcal{G}_0$, that is, to determine for which initial data u_0 in the phase space $H^1_0(\Omega)$ the solution of (1.1) blows up and for which data u_0 the solution is globally defined. In this context, the energy functional

$$J : H_0^1(\Omega) \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1},$$

plays an essential role. Recall that J is strictly decreasing along nonconstant solutions of (1.1). The results in [24, 33] describe the behavior of solutions of (1.1) when u_0 has *low energy*, namely energy smaller than the mountain pass level

$$d = \min_{u\in H_0^1(\Omega)\setminus\{0\}} \max_{s\geq 0} J(su); \tag{1.2}$$

see Theorem 9 below. In contrast, we are here mainly interested in the behavior of solutions of (1.1) when u_0 has energy larger than d.

In Section 3, we recall the local solvability of (1.1): for all $u_0 \in H_0^1(\Omega)$ there exists a unique local solution $u = u(t)$ of (1.1). This solution may be continued as long as it remains bounded in $H_0^1(\Omega)$; see Theorems 3 and 4. Moreover, global solutions are bounded and, up to a subsequence, they converge to a stationary solution; in Theorem 6 we show that they have "small time oscillations" at infinity. On the other hand, if the solution u blows up in finite time, in Theorem 7 we show that the L^2 norm of u_t blows up at a higher rate when compared to the L^{p+1} and H_0^1 norms of the solution u. In some sense, this means that blow-up is characterized by large time oscillations of the solution u.

In Section 2, we recall the definition of the Nehari manifold $\mathcal{N}$ relative to the stationary problem associated to (1.1), and we establish two crucial properties of stationary solutions and of the manifold itself. A striking difference is obtained between the two cases $p \leq 1 + \frac{4}{n}$ and $p > 1 + \frac{4}{n}$.

In Theorem 9 we recall the already-mentioned result by Ikehata-Suzuki [24] which describes the evolution of (1.1) when the initial datum u_0 has energy below the mountain pass level d: if u_0 is taken inside $\mathcal{N}$, the solution vanishes as $t \to \infty$ ($u_0 \in \mathcal{G}_0$), whereas, if u_0 is taken outside $\mathcal{N}$, the solution blows up ($u_0 \in \mathcal{B}$). Corollary 4 complements this result by exhibiting a class of initial data in $\mathcal{G}_0$ whose energy is larger than d but smaller than $2d$.

The importance of the Nehari manifold $\mathcal{N}$ for the dynamics of (1.1) is given by its role as a borderline separating regions of weak dissipativity (respectively antidissipativity) for the corresponding semiflow; see Lemma 7 below. Using this observation, we obtain classes of initial data with arbitrarily high energy which lie in $\mathcal{G}_0$ (respectively $\mathcal{B}$), and such that the corresponding solutions *do not cross* $\mathcal{N}$. This last property is not shared by every solution. Indeed, we will also give examples of solutions which cross $\mathcal{N}$ before blowing up. On the other hand, it is still open whether a crossing can occur in the other direction; that is, whether initial data in $\mathcal{G}_0$ could give rise to solutions which cross $\mathcal{N}$.

From the topological point of view, it is interesting to note the nonobvious fact that $\mathcal{G}_0 = \mathrm{int}(\mathcal{G})$, as stated in Theorem 8 below. This implies that most of the nontrivial dynamics of (1.1) takes place on the "thin" set $\partial\mathcal{G}$; in particular, every nontrivial equilibrium is contained in this set. The topological picture becomes even clearer when only initial data in the cone $\mathbb{K}$ of nonnegative functions are considered. Here we complement a result by Lions [27] which, roughly speaking, states that there exists a "dividing line" between $\mathcal{G}$ and $\mathcal{B}$. More precisely, each half line starting from the origin $0 \in H_0^1(\Omega)$ and lying in $\mathbb{K}$ is divided in three parts: a segment close to $u \equiv 0$ which is included in $\mathcal{G}_0$, a point in $\partial\mathcal{G}$, and the remaining half line which is included in $\mathcal{B}$. We provide new estimates for the dividing line and for the decay rate at infinity of the blow-up time. We emphasize that our asymptotic estimate is very simple and only depends on the L^1 and the L^2 norm of the initial datum and not on its L^∞ norm as in previous works.

We like to mention further related work already at this point. Our study heavily relies on a priori bounds for global solutions of (1.1) which have recently been proved by Quittner [39, 37]. The novelty of these bounds consists in their validity for sign-changing initial data. For positive global solutions, stronger universal bounds are available; see [13, 40, 38]. For superlinear parabolic equations, the sets attracted by equilibria and their boundaries have also been studied in [27, 28, 36, 41, 1, 2]. Partial results in an abstract framework are already contained in [34].

Although we hope that our results shed some further light on the evolution of (1.1), we believe that much work has still to be done in order to reach a full understanding. In Section 3.5, we list some open problems.

This paper is organized as follows. In Section 2, we establish some properties of the stationary problem associated to (1.1). In Section 3 we state our main results concerning the classification of initial data of (1.1) and compare them to the existing literature. Section 4 is devoted to the proofs of the results. Finally, in the appendix we include a proof of the comparison principle which we use extensively throughout the paper.

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2. Preliminary results about the stationary problem

Without further mention, we assume throughout the paper that $\Omega \subset \mathbb{R}^n$ is an open bounded smooth domain and that $1 < p < \frac{n+2}{n-2}$. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$ and by $\|\cdot\|$ the Dirichlet norm in $H_0^1(\Omega)$. Let us also introduce the cone of nonnegative functions

$$\mathbb{K} = \{u \in H_0^1(\Omega) : u \ge 0 \text{ a.e. in } \Omega\}.$$

Finally, for any $u \in H_0^1(\Omega)$, we denote its positive part by

$$u^+(x) := \max\{u(x), 0\}$$

and its negative part by

$$u^-(x) := \min\{u(x), 0\}.$$

Stationary solutions of (1.1) solve the elliptic problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{2.1}$$

Problem (2.1) may be tackled with critical-point theory. Consider the energy functional J and the Nehari functional K defined by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}, \quad K(u) = \int_\Omega |\nabla u|^2 - \int_\Omega |u|^{p+1}.$$

Then, J and K are of class C^1 over $H_0^1(\Omega)$ and critical points of J are (weak) solutions of (2.1). By the Moser iteration scheme and elliptic regularity, any weak solution of (2.1) is in fact a smooth classical solution. In view of [3], since $p < \frac{n+2}{n-2}$, the functional J satisfies the Palais-Smale condition and (2.1) admits at least a positive solution (called mountain pass solution) whose energy d may be characterized by

$$d = \min_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{s \ge 0} J(su)\ . \tag{2.2}$$

The number d in (2.2) is called *mountain pass level* or *potential well depth*. Consider the best Sobolev constant for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$:

$$S_{p+1} \ = \ \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{p+1}^2}\ . \tag{2.3}$$

As $p < \frac{n+2}{n-2}$, the embedding is compact and the infimum in (2.3) is attained. Then, it is well known (see, e.g. [33, Section 3]) that any mountain pass solution u of (2.1) is a minimizer for (2.3) (i.e. it satisfies $\|u\|^2 = S_{p+1}\|u\|^2_{p+1}$) and that S_{p+1} is related to its energy:

$$d = \frac{p-1}{2(p+1)} S_{p+1}^{(p+1)/(p-1)} . \tag{2.4}$$

Clearly, (2.1) also admits a *negative* mountain pass solution. The uniqueness of positive solutions and/or of mountain pass solutions for (2.1) strongly depends on the geometry of the domain Ω. For instance, if $\Omega = B$ (the unit ball), (2.1) admits a unique positive solution and hence a unique (up to the sign) mountain pass solution; see [17, Lemma 2.3]. For uniqueness results in more general convex domains, see [18]. On the other hand, Dancer [10] exhibits domains Ω where (2.1) admits an arbitrarily large number of positive solutions.

It is also known [3, 6], that (2.1) admits infinitely many nodal (sign-changing) solutions of arbitrarily high energy. We remark that any nodal solution of (2.1) has energy larger than the double of the mountain pass level:

Theorem 1. *Let u be a nodal solution of* (2.1). *Then, $J(u) > 2d$.*

This observation seems to be known, but it is hardly mentioned in the standard literature. For the reader's convenience we give a proof in Section 4.1.

In the sequel, a crucial role is played by the *Nehari manifold* relative to J, namely

$$\mathcal{N} = \{w \in H^1_0(\Omega) \setminus \{0\} : K(w) = 0\} . \tag{2.5}$$

By studying the map $s \mapsto K(su)$ for $\|u\| = 1$, it is easy to show that each half line starting from the origin of $H^1_0(\Omega)$ intersects exactly once the manifold $\mathcal{N}$; see [33, Lemma 2.2]. Clearly, $\mathcal{N}$ separates the two unbounded sets

$$\mathcal{N}_+ = \{w \in H^1_0(\Omega) : K(w) > 0\} \quad \text{and} \quad \mathcal{N}_- = \{w \in H^1_0(\Omega) : K(w) < 0\} . \tag{2.6}$$

We also need to consider the (open) sublevels of J:

$$J^k = \{u \in H^1_0(\Omega) : J(u) < k\} .$$

It is readily seen that the mountain-pass level d defined in (2.2) may also be characterized as

$$d \;=\; \min_{u \in \mathcal{N}} \; J(u) . \tag{2.7}$$

Hence,

$$\mathcal{N}_a := \mathcal{N} \cap J^a \equiv \left\{u \in \mathcal{N} : \|u\| < \sqrt{\frac{2a(p+1)}{p-1}}\right\} \neq \varnothing \qquad \text{for all } a > d \;. \tag{2.8}$$

The above alternative characterization of d also shows that

$$\operatorname{dist}(0, \mathcal{N}) = \min_{u \in \mathcal{N}} \|u\| = \delta := \sqrt{\frac{2d(p+1)}{p-1}} > 0 \;. \tag{2.9}$$

We now define

$$\lambda_a = \inf\{\|u\|_2 : u \in \mathcal{N}_a\}, \qquad \Lambda_a = \sup\{\|u\|_2 : u \in \mathcal{N}_a\} \qquad \text{for all } a > d.$$

Clearly we have the following monotonicity properties

$$a \mapsto \lambda_a \quad \text{is nonincreasing ,} \qquad a \mapsto \Lambda_a \quad \text{is nondecreasing .} \tag{2.10}$$

We also put $\lambda_\infty = \inf\{\|u\|_2 : u \in \mathcal{N}\}$. In the first statement of the next result we do not assume that p is subcritical: there, $\mathcal{N}$ is intended as a subset of $H_0^1(\Omega) \cap L^{p+1}(\Omega)$.

Theorem 2. *Assume that $p > 1$. Then*
(i) if $p > 1 + \frac{4}{n}$, then $\lambda_\infty = 0$.
(ii) if $p \leq 1 + \frac{4}{n}$, then $\lambda_\infty > 0$.
(iii) if $p < \frac{n+2}{n-2}$ and $d < a < \infty$, then $0 < \lambda_a < \Lambda_a < \infty$.

Remark 1. The exponent $\overline{p} := 1 + \frac{4}{n}$ is the largest exponent p for which every weak solution of (1.1) is regular in the sense of [19, 20]. The exponent $\overline{p}$ is also the critical exponent for the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^n \;;$$

more precisely, it is the smallest value of p for which conservation of mass and energy does not imply a global bound in energy space; see the recent paper [4] and the references therein.

3. Results about the parabolic problem

3.1. **Existence, uniqueness and behavior of solutions.** In this subsection, we recall a number of known facts which are the starting point for our analysis. We first consider the local solvability of (1.1):

Theorem 3. [8, 23, 47] *For all $u_0 \in H_0^1(\Omega)$ there exists $T \in (0, \infty]$ such that* (1.1) *admits a unique solution $u \in C^0([0,T]; H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$*

which becomes a classical solution for $t > 0$. Moreover, if $[0, T^*)$ denotes the maximal interval of continuation of u (as a classical solution), then

(i) if $T^* < \infty$ then $\|u(t)\|_q \to \infty$ as $t \to T^*$ for all $q \geq 1$ such that $\frac{n(p-1)}{2} < q \leq \infty$;

(ii) if $u_0 \geq 0$, $u_0 \not\equiv 0$, then $u(x,t) > 0$ in $\Omega \times (0, T^*)$ and $\frac{\partial}{\partial \nu} u(x,t) < 0$ on $\partial\Omega \times (0, T^*)$.

Existence, uniqueness, and regularity of the solutions are obtained in [23]; see also [24, Theorem 2.2]. For statement (i) see [47] and also [7]; the number $n(p-1)/2$ is sharp for this blow-up statement; see [14, 48]. Statement (ii) follows from the comparison principle: just take $v_0 \equiv 0$ in Proposition 1 in the appendix.

Remark 2. Since $p < \frac{n+2}{n-2}$, we also have $\frac{n(p-1)}{2} < \frac{2n}{n-2}$. Therefore, Theorem 3 (i) implies that $\|u(t)\| \to \infty$ as $t \to T^*$.

In the following, we let $T^*(u_0)$ denote the maximal existence time of the solution with initial condition $u_0 \in H_0^1(\Omega)$. Although not all the solutions are global, with an abuse of notation, in the sequel we denote by

$$S(t) \text{ the nonlinear semigroup associated to (1.1);}$$

therefore, instead of $u = u(t)$ we will also write $S(t)u_0$ for $t < T^*(u_0)$. The smoothing properties of this semigroup suggest that we consider the space

$$C_0^1(\Omega) := \{u \in C^1(\overline{\Omega}) \, : \, u = 0 \quad \text{on } \partial\Omega\} = C^1(\overline{\Omega}) \cap H_0^1(\Omega),$$

endowed with the standard norm $\|\cdot\|_{C^1}$ of $C^1(\overline{\Omega})$. The next result is concerned with some features of global solutions. If $T^*(u_0) = \infty$, we denote by

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{u(s) \, : \, s \geq t\}}$$

the ω-limit set of $u_0 \in H_0^1(\Omega)$. Here the closure is taken in $H_0^1(\Omega)$. Then, we have

Theorem 4. [37, 39] *Let $u_0 \in H_0^1(\Omega)$ be such that $T^*(u_0) = \infty$. Then:*

(i) The map $t \mapsto \|S(t)u_0\|_{C^1}$ is bounded on $[\delta, \infty)$ for all $\delta > 0$, and the bound depends only on $\|u_0\|$ and δ.

(ii) The trajectory $\{S(t)u_0 \, : \, t \geq \delta\}$ is relatively compact in $C_0^1(\Omega)$ for all $\delta > 0$.

(iii) The ω-limit set $\omega(u_0)$ is a nonempty compact and connected subset of $C_0^1(\Omega)$ which consists of solutions of (2.1).

Properties (i) and (ii) have been established by Quittner [37, 39]. Property (iii) follows in a standard way from (i) and (ii); see e.g. [42].

Remark 3. Statement (iii) implies that, up to a subsequence, $\{u(t)\}$ converges to some stationary solution. In general, it cannot be improved with the statement that the *whole* trajectory converges; see [35] and references therein. However, if the nonlinearity is *analytic*, then it is known that in fact the statement holds true on the whole orbit; see [43] and [26]. Since $p < \frac{n+2}{n-2}$, the nonlinearity $|u|^{p-1}u$ in equation (1.1) is analytic only if $n = p = 3$ or $n = 2$ and p is any odd integer larger than 2. It has been pointed out to us by Dancer that the map $u \mapsto u^p$ is analytic for all $p \in (1, \frac{n+2}{n-2})$ when restricted to open subsets of the positive cone in a weighted space of continuous functions; see [11].

Definition 1. If case (ii) in Theorem 3 occurs, namely $T^* < \infty$, we say that for u_0 we have **blow-up** and that T^* is the **blow-up time**; we then write $u_0 \in \mathcal{B}$. If $T^*(u_0) = \infty$, we write $u_0 \in \mathcal{G}$. Finally, when $\omega(u_0) = \{0\}$ (i.e., $u(t) \to 0$ in $H^1_0(\Omega)$ as $t \to \infty$), we say that we have **vanishing** and we write $u_0 \in \mathcal{G}_0$.

We conclude this preliminary subsection with some continuity properties of T^* and $S(t)$:

Theorem 5. [39] *The function $T^* : H^1_0(\Omega) \to (0, \infty]$ is continuous. Moreover, for all $u_0 \in H^1_0(\Omega)$ and for all $t \in (0, T^*(u_0))$, the semigroup $S(t)$ maps an $H^1_0(\Omega)$ neighborhood of u_0 continuously into $C^1_0(\Omega)$.*

3.2. **Characterization of $\mathcal{G}$ and $\mathcal{B}$.** We start by showing that global solutions of (1.1) have small time oscillations while blow-up solutions have large time oscillations. First, we prove

Theorem 6. *Assume that $u_0 \in \mathcal{G}$. Then, for any $k > 0$ we have*

$$\lim_{t\to\infty} \|S(t+k)u_0 - S(t)u_0\|_{C^1} = 0 \ .$$

Then, we show that (in case of blow-up) the L^2 norm of u_t diverges at a higher rate when compared with the L^{p+1} and H^1_0 norms of the solution u:

Theorem 7. *Assume that $u_0 \in \mathcal{B}$ and let $u(t) = S(t)u_0$. Then,*

$$\liminf_{t\to T^*} \frac{\|u_t(t)\|_2}{\|u(t)\|^p_{p+1}} > 0 \ , \qquad \liminf_{t\to T^*} \frac{\|u_t(t)\|_2}{\|u(t)\|^{2p/(p+1)}} > 0 \ .$$

We now characterize topologically the sets $\mathcal{B}$ and $\mathcal{G}$.

Theorem 8. *The set $\mathcal{G}$ is closed in $H_0^1(\Omega)$. The set $\mathcal{B}$ is open in $H_0^1(\Omega)$. The sets $\mathcal{G}$, $\partial\mathcal{G}$, $\partial\mathcal{G}_0$ and $\mathcal{B}$ are invariant under the semiflow of* (1.1)*. Finally,* $\mathrm{int}(\mathcal{G}) = \mathcal{G}_0$.

The first three statements had first been noted by Cazenave-Lions [8, Lemma 9], but with respect to the topology of $C^0(\overline{\Omega})$ instead of $H_0^1(\Omega)$. The last statement (the most important for our purposes) is only known for convex C^2 nonlinearities; see [27, Theorem 2.1]. In equation (1.1) the nonlinearity is not convex and it is C^2 only if $p > 2$.

Next we note that the sets

$$\mathcal{S}_\pm := \{u \in C_0^1(\Omega) \,:\, \pm u > 0 \quad \text{in} \quad \Omega, \quad \pm\frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega\}$$

and

$$\mathcal{S}_n := \{u \in C_0^1(\Omega) \,:\, u(x) < 0 < u(y) \quad \text{for some points} \quad x, y \in \Omega\}$$

are open and disjoint in $C_0^1(\Omega)$. Moreover, by the Hopf boundary lemma, every nontrivial solution of (2.1) lies either in $\mathcal{S}_+$, in $\mathcal{S}_-$, or in $\mathcal{S}_n$. Hence, from Theorem 4 (*iii*) we deduce

Corollary 1. *Let $u_0 \in \partial\mathcal{G} = \mathcal{G} \setminus \mathcal{G}_0$. Then precisely one of the following three cases occurs:*

$$(a)\ \omega(u_0) \subset \mathcal{S}_+ \qquad (b)\ \omega(u_0) \subset \mathcal{S}_- \qquad (c)\ \omega(u_0) \subset \mathcal{S}_n\ .$$

As a straightforward consequence of Theorem 1 and Corollary 1, we obtain a further result in domains $\Omega \subset \mathbb{R}^n$ which admit a unique positive solution v for (2.1); see [17, Lemma 2.3] and [18] for some examples of such domains. More precisely, we find data $u_0 \in \partial\mathcal{G}$ for which $S(t)u_0$ converges to v as $t \to \infty$.

Corollary 2. *Assume that Ω is a domain admitting a unique positive solution v of* (2.1)*. Then,*

(*i*) *for all $u_0 \in \partial\mathcal{G} \cap \mathbb{K}$ we have $\|S(t)u_0 - v\|_{C^1} \to 0$ as $t \to \infty$.*

(*ii*) *for all $u_0 \in \partial\mathcal{G} \cap J^{2d}$ we have either $\|S(t)u_0 - v\|_{C^1} \to 0$ or $\|S(t)u_0 + v\|_{C^1} \to 0$ as $t \to \infty$.*

Part (i) is stated in [8, Remarque 14] for continuous initial data.

3.3. **Low energy initial data and applications of the comparison principle.** We recall here a result from [24]; see also previous work in [33, 45]. It classifies the evolution of solutions of (1.1) with initial datum u_0 having energy below the mountain pass level.

Theorem 9. [24] *We have $(J^d \cap \mathcal{N}_+) \subset \mathcal{G}_0$ and $(J^d \cap \mathcal{N}_-) \subset \mathcal{B}$.*

We give a short proof of Theorem 9 in Section 4.6. In particular, Theorem 9 and (2.9) yield

Corollary 3. *If* $\|u_0\| \le \sqrt{2d}$*, then* $u_0 \in \mathcal{G}_0$.

In view of Theorem 9, we are led to study the behavior of solutions of (1.1) whose initial datum u_0 has energy $J(u_0)$ larger than d. A first observation in this direction is given by:

Corollary 4. *If* $u_0^+, u_0^- \in J^d \cap \mathcal{N}_+$*, then* $u_0 \in \mathcal{G}_0$.

This follows from the fact that $S(t)u_0^+ \to 0$ and $S(t)u_0^- \to 0$ in $H_0^1(\Omega)$ by Theorem 9, hence $S(t)u_0 \to 0$ by the comparison principle; see Proposition 1 below. Note that if $u_0^+, u_0^- \in J^d \cap \mathcal{N}_+$, then $K(u_0) = K(u_0^+) + K(u_0^-) > 0$ and $J(u_0) = J(u_0^+) + J(u_0^-) < 2d$ so that $u_0 \in \mathcal{N}_+$ but $J(u_0)$ may be larger than the mountain pass level d.

Similarly, from Corollary 3 and the comparison principle we also immediately deduce

Corollary 5. *If* $\|u_0^+\| \le \sqrt{2d}$ *and* $\|u_0^-\| \le \sqrt{2d}$*, then* $u_0 \in \mathcal{G}_0$.

The next statement exploits the comparison between the initial datum u_0 with stationary solutions:

Theorem 10. *Let* v *be a nontrivial solution of* (2.1)*, and let* $u_0 \in H_0^1(\Omega)$, $u_0 \not\equiv \pm v$.

(*i*) *If* $v^+ \neq 0$ *and* $u_0 \ge v$*, then* $u_0 \in \mathcal{B}$.
(*ii*) *If* $v^- \neq 0$ *and* $u_0 \le v$*, then* $u_0 \in \mathcal{B}$.
(*iii*) *If* $v > 0$ *and* $-v \le u_0 \le v$*, then* $u_0 \in \mathcal{G}_0$.

In the special case where $v > 0$, statements (*i*) and (*iii*) are essentially due to Fujita [15]; see also [12] for a different proof in this case.

By the comparison principle, vanishing and blow-up are simply characterized for initial data in the cone of nonnegative functions; in the next statement we complement a result by Lions [27]:

Theorem 11. *For all* $u_0 \in \mathbb{K} \setminus \{0\}$ *there exists* $\alpha^* = \alpha^*(u_0) > 0$ *such that:*

(*i*) *if* $0 \le \alpha < \alpha^*$*, then* $\alpha u_0 \in \mathcal{G}_0$.
(*ii*) $\alpha^* u_0 \in \partial\mathcal{G}$.
(*iii*) *if* $\alpha > \alpha^*$*, then* $\alpha u_0 \in \mathcal{B}$.

Moreover, if v *denotes a positive mountain pass solution of* (2.1) *and* S_{p+1} *is as in* (2.3)*, then*

$$\alpha^* \le \left(\frac{2^p}{p}\right)^{1/(p-1)} \left(\frac{p-1}{p+1} S_{p+1}^{(p+1)/(p-1)}\right)^{1/p} \|v\|_1^{(p-1)/p} \left(\int_\Omega u_0 v\right)^{-1}, \qquad (3.1)$$

the blow-up time $T_\alpha := T^*(\alpha u_0)$ *is decreasing with respect to* α *and*

$$\limsup_{\alpha\to\infty} \alpha^{p-1}T_\alpha \leq \frac{1}{p-1}\Big(\frac{\|u_0\|_1}{\|u_0\|_2^2}\Big)^{p-1}. \tag{3.2}$$

Theorem 11 complements [27, Theorem 2.1] with the new estimates (3.1) and (3.2). We obtain (3.1) and (3.2) thanks to the clever use of Young's inequality suggested by Mitidieri-Pohožaev [29]. This method was already used in [16] for a different class of semilinear parabolic problems. Both (3.1) and (3.2) follow from a general result (see Lemma 6 below) which may be easily used to obtain many other similar estimates. Let us recall that the standard test of (1.1) with the first eigenfunction e_1 of $-\Delta$ gives an upper bound for α^* in terms of $\int_\Omega u_0e_1$; in (3.1), the upper bound depends on $\int_\Omega u_0 v$. Finally, the asymptotic estimate $T_\alpha \leq C\alpha^{1-p}$ is probably sharp; see [31] for the optimal blow-up estimate of the corresponding Cauchy problem in $\mathbb{R}^n$ and for *bounded* initial data u_0; in such a case, the constant C depends on $\|u_0\|_\infty$ while in our case it merely depends on the ratio $\|u_0\|_1/\|u_0\|_2^2$.

3.4. **High energy initial data.** Recalling that $(J^d \cap \mathcal{N}_+) \subset \mathcal{G}_0$ by Theorem 9, one might ask whether $\mathcal{N}_+ \subset \mathcal{G}_0$. At first glance this appears reasonable since Lemma 7 below implies that, in a weak sense, (1.1) is dissipative in $\mathcal{N}_+$. However, the following result shows that initial data in $\mathcal{N}_+$ with high energy may generate both vanishing solutions and solutions which blow up.

Theorem 12. *For any* $M > 0$*, there exist* $u_M, v_M \in \mathcal{N}_+ \cap \mathbb{K} \cap C_0^1(\Omega)$ *with* $J(u_M), J(v_M) \geq M$ *and* $u_M \in \mathcal{G}_0$*,* $v_M \in \mathcal{B}$*.*

Such initial data u_M, v_M can be constructed with the help of Theorem 10; see Section 4.9. One may ask whether also singular initial data in $\mathcal{G}_0 \setminus L^\infty(\Omega)$ occur at arbitrarily high energy, since these can not be found by a direct comparison with solutions of (2.1). They can, however, be found by adding to u_M a small perturbation in $H_0^1(\Omega) \setminus L^\infty(\Omega)$ and using the fact that $\mathcal{N}_+$, $\mathcal{B}$, and $\mathcal{G}_0$ are all *open* in $H_0^1(\Omega)$.

Next, we establish some general criteria to decide whether a given initial datum at possibly high energy level gives rise to vanishing or blow-up. Recall the definition of λ_∞ in Section 2.

Theorem 13. *If* $p \leq 1 + \frac{4}{n}$*, then the* L^2 *ball* $\{u \in H_0^1(\Omega) : \|u\|_2 < \lambda_\infty\}$ *is contained in* $\mathcal{G}_0$*, and it is positively invariant under the semiflow* $S(t)$*. Moreover,* λ_∞ *is the largest radius with this property.*

In [44] it was proved that the zero solution is stable in $L^r(\Omega)$ if and only if $p \le 1+\frac{2r}{n}$. Theorem 13 provides optimal information for the case $r=2$ in terms of the variationally characterized value λ_∞. Next we give a criterion for blow-up.

Theorem 14. *Assume that $u_0 \in H^1_0(\Omega)$ satisfies*

$$\|u_0\|_2^{p+1} \;\ge\; \frac{2(p+1)}{p-1}\,|\Omega|^{(p-1)/2}\,J(u_0)\ . \tag{3.3}$$

Then, $u_0 \in \mathcal{N}_- \cap \mathcal{B}$.

With the help of this criterion we can exhibit a class of initial data in $\mathcal{N}_-$ with arbitrarily high energy which gives rise to blow-up.

Theorem 15. *For any $M>0$ there exists $u_M \in \mathcal{N}_-$ such that $J(u_M) \ge M$ and $u_M \in \mathcal{B}$.*

We deduce Theorems 13 and 14 from more general criteria of variational type; see Lemma 8 below.

3.5. **Some open problems.** By Theorem 9, low energy data u_0 lead to vanishing if $u_0 \in \mathcal{N}_+$ and to blow-up if $u_0 \in \mathcal{N}_-$. On the other hand, Theorem 12 shows that initial data in $\mathcal{N}_+$ with high energy in general can lead to both vanishing and blow-up. Moreover, by Theorem 15 we exhibit a class of initial data $u_0 \in \mathcal{N}_-$ which gives rise to blow-up. This naturally leads to the following question: do we have $\mathcal{G}_0 \subset \mathcal{N}_+$, or $\mathcal{N}_- \subset \mathcal{B}$? A simpler question related to Theorem 11 is the following: for $u_0 \in \mathbb{K}\setminus\{0\}$ do we have $\alpha^* u_0 \in \mathcal{N}_+ \cup \mathcal{N}$?

For all $u_0 \in H^1_0(\Omega)$ with $\|u_0\|=1$ let

$$\underline{\alpha}(u_0)=\inf\{\alpha>0:\alpha u_0\in\mathcal{B}\}\qquad\text{and}\qquad\overline{\alpha}(u_0)=\sup\{\alpha>0:\alpha u_0\in\mathcal{G}\};$$

by Theorem 9 we have $\sqrt{2d} < \underline{\alpha}(u_0) \le \overline{\alpha}(u_0) < \infty$, where $d>0$ is the mountain pass level. If $u_0 \in \pm\mathbb{K}$, Theorem 11 ensures that $\underline{\alpha}(u_0)=\overline{\alpha}(u_0)=\alpha^*(u_0)$. Do we have $\underline{\alpha}(u_0)=\overline{\alpha}(u_0)$ also for *nodal* u_0? If affirmative, then $\partial\mathcal{G}$ would simply be a surface of codimension 1 such that any half line starting from the origin $0 \in H^1_0(\Omega)$ intersects $\partial\mathcal{G}$ exactly once. Note that if u_0 is nodal, then $u_0^+ > u_0 > u_0^-$ so that, according to Theorems 3, 4, and the comparison principle, we have $\underline{\alpha}(u_0) \ge \min\{\alpha^*(u_0^+),\alpha^*(u_0^-)\}$. One may also weaken the just-raised question and ask whether the equality $\partial\mathcal{G}=\partial\mathcal{G}_0=\partial\mathcal{B}$ holds. If this is true, then in particular every nontrivial stationary solution of (2.1) lies on $\partial\mathcal{G}_0$. Moreover, we then would expect that every nondegenerate nontrivial stationary solution can be connected to zero via a heteroclinic orbit.

A further challenging problem seems to be the localization of heteroclinics in $\partial\mathcal{G}$. Take two (possibly nodal) solutions u_1 and u_2 of (2.1) such that $J(u_2) < J(u_1)$. Under which additional assumptions can we find a *global* solution $u : \mathbb{R} \to H_0^1(\Omega)$ of (1.1) such that $u(t) \to u_1$ as $t \to -\infty$ and $u(t) \to u_2$ as $t \to \infty$? In view of Theorem 8, these orbits lie entirely over $\partial\mathcal{G}$. Some promising results on heteroclinics in the multidimensional case can be found in [2] for a somewhat different class of equations.

We think that progress on these open problems can lead to a much better understanding of the evolution of (1.1).

4. Proof of the main results

4.1. **Proof of Theorem 1.** Let u be a nodal solution of (2.1) and note first that

$$|u(x)|^{p-1}u(x)u^+(x) = |u^+(x)|^{p+1} \qquad \text{for a.e. } x \in \Omega \, . \tag{4.1}$$

Clearly, u^+ is not a minimizer for the Sobolev ratio (2.3); otherwise, a multiple of u^+ would be a (smooth) solution of (2.1) so that by the maximum principle we would have $u^+ > 0$ in Ω, contradicting $u \notin \mathbb{K}$. Hence, by the (strict) Sobolev inequality and an integration by parts, we obtain

$$S_{p+1}\|u^+\|_{p+1}^2 < \|u^+\|^2 = \int_\Omega \nabla u \nabla u^+ = -\int_\Omega \Delta u \, u^+ \; ;$$

therefore, using (4.1) and the fact that u solves (2.1), we infer

$$S_{p+1}\|u^+\|_{p+1}^2 < \int_\Omega |u|^{p-1}u \, u^+ = \int_\Omega |u^+|^{p+1} \; .$$

We may so conclude that

$$\|u^+\|_{p+1} > S_{p+1}^{1/(p-1)} \; . \tag{4.2}$$

Arguing similarly for the negative part u^-, we also get

$$\|u^-\|_{p+1} > S_{p+1}^{1/(p-1)} \; . \tag{4.3}$$

By (4.2)-(4.3) and recalling that $u \in \mathcal{N}$ (so that $K(u) = 0$) we have

$$\begin{aligned} J(u) &= \frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1} = \frac{p-1}{2(p+1)}\Big(\|u^+\|_{p+1}^{p+1} + \|u^-\|_{p+1}^{p+1}\Big) \\ &> \frac{p-1}{p+1}S_{p+1}^{(p+1)/(p-1)} = 2d \end{aligned}$$

the last equality being a consequence of (2.4).

4.2. **Proof of Theorem 2.** (i) Take $u \in C_c^\infty(\Omega) \cap \mathcal{N}$. This is possible by taking eventually $v = \gamma u$ with

$$\gamma = \Big(\frac{\|u\|^2}{\|u\|_{p+1}^{p+1}}\Big)^{1/(p-1)} .$$

With no loss of generality, we may assume that $O \in \Omega' :=$supp(u). For any $k \in \mathbf{N}$ put

$$u_k(x) = \begin{cases} k^{2/(p-1)} u(kx) & \text{if } x \in \frac{\Omega'}{k} \\ 0 & \text{if } x \in \Omega \setminus \frac{\Omega'}{k} \end{cases} . \tag{4.4}$$

Then, since $\frac{4}{p-1} < n$, we have

$$\|u_k\|_2^2 = k^{4/(p-1)} \int_{\Omega'/k} u^2(kx)dx = k^{4/(p-1)-n} \int_{\Omega'} u^2(y)dy \to 0 \quad \text{as } k \to \infty . \tag{4.5}$$

Moreover, similar calculations lead to

$$\|u_k\|_{p+1}^{p+1} = k^{2(p+1)/(p-1)-n} \|u\|_{p+1}^{p+1} = k^{2(p+1)/(p-1)-n} \|u\|^2 = \|u_k\|^2$$

which shows that $\{u_k\} \subset \mathcal{N}$. Together with (4.5), this proves (i).

(ii) By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{p+1}^{p+1} \le C\|u\|^{n(p-1)/2} \cdot \|u\|_2^\alpha \qquad \text{for all } u \in H_0^1(\Omega) , \tag{4.6}$$

where $\alpha = p+1-\frac{n(p-1)}{2} > 0$ since $p < \frac{n+2}{n-2}$. If $u \in \mathcal{N}$, (4.6) becomes

$$\|u\|^{2-n(p-1)/2} \le C\|u\|_2^\alpha \qquad \text{for all } u \in \mathcal{N} . \tag{4.7}$$

Recalling (2.9) and $p \le 1 + \frac{4}{n}$ (i.e. $2 - n(p-1)/2 \ge 0$), (4.7) proves (ii).

(iii) By (2.9) and the assumption that $u \in \mathcal{N}_a$, the left-hand side of (4.7) remains bounded away from 0 no matter what the sign of $2 - n(p-1)/2$ is. This proves $\lambda_a > 0$. Moreover, $\Lambda_a < \infty$ just follows from the Poincaré inequality.

4.3. **Proof of Theorem 6.** Let $u_0 \in \mathcal{G}$ and let $u(t) := S(t)u_0$. If we differentiate the map $t \mapsto J(u(t))$ with respect to t and we use (1.1), we get

$$\frac{d}{dt}J(u(t)) = -\int_\Omega u_t^2(t) \qquad \text{for all } t \in (0, T^*) . \tag{4.8}$$

Thus, $t \mapsto J(u(t))$ is decreasing; since it is also bounded from below by Theorem 4, we know that

$$\lim_{t\to\infty} J(u(t)) = L \qquad \text{for some } L \in (-\infty, J(u_0)) . \tag{4.9}$$

By the Fubini theorem, Hölder's inequality and (4.8), we obtain

$$\int_\Omega |u(x,t+k)-u(x,t)|dx = \int_\Omega \Big| \int_t^{t+k} u_s(x,s)\,ds\Big| dx \le \int_t^{t+k}\int_\Omega |u_s(x,s)|\,dx\,ds$$

$$\le \sqrt{k|\Omega|}\Big\{\int_t^{t+k}\int_\Omega u_s^2(x,s)\,dx\,ds\Big\}^{1/2} = \sqrt{k|\Omega|}\ \{J(u(t)) - J(u(t+k))\}^{1/2}\ . \tag{4.10}$$

If we let $t \to \infty$, then the right-hand side of (4.10) tends to 0 (recall (4.9)). This method to obtain $\|u(t+k)-u(t)\|_1 \to 0$ is picked from [9]. To see that $\|u(t+k)-u(t)\|_{C^1} \to 0$, assume by contradiction that there exist $k > 0$ and a sequence $t_n \to \infty$ such that

$$\|u(t_n) - u(t_n+k)\|_{C^1} \ge \varepsilon > 0 \qquad \text{for all} \quad n.$$

By Theorem 4 (*ii*), we may pass to a subsequence such that $u(t_n) \to u_1 \in C_0^1(\Omega)$ and $u(t_n+k) \to u_2 \in C_0^1(\Omega)$. Hence

$$\|u_1 - u_2\|_{C^1} = \lim_{n\to\infty} \|u(t_n) - u(t_n+k)\|_{C^1} \ge \varepsilon.$$

On the other hand, in view of (4.10),

$$\|u_1 - u_2\|_1 = \lim_{n\to\infty} \|u(t_n) - u(t_n+k)\|_1 = 0\ .$$

This is a contradiction, and Theorem 6 is proved.

4.4. **Proof of Theorem 7.** Note first that (4.8) gives

$$J'(u_0) \ne 0 \implies J(u(t)) < J(u_0) \quad \text{for all } t \in (0,T^*)\ . \tag{4.11}$$

Then,

$$J(u_0) > J(u(t)) = \frac{1}{2}K(u(t)) + \frac{p-1}{2(p+1)}\|u(t)\|_{p+1}^{p+1} \qquad \text{for all } t \in (0,T^*)\ . \tag{4.12}$$

Next, multiply (1.1) by $u(t)$, integrate by parts and use Hölder's inequality to obtain

$$-K(u(t)) = \int_\Omega u(t)u_t(t) \le \|u(t)\|_2\|u_t(t)\|_2 \qquad \text{for all } t \in [0,T^*)\ . \tag{4.13}$$

By (4.13) and Young's inequality, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$-K(u(t)) \le \varepsilon\|u(t)\|_{p+1}^{p+1} + C_\varepsilon\|u_t(t)\|_2^{(p+1)/p} \qquad \text{for all } t \in [0,T^*)\ , \tag{4.14}$$

where we also used Hölder's inequality. By Theorem 3, we know that $\|u(t)\|_{p+1} \to \infty$ as $t \to T^*$; therefore, by taking ε sufficiently small and by combining (4.12) and (4.14), we get

$$c\|u(t)\|_{p+1}^{p+1} \leq \|u_t(t)\|_2^{(p+1)/p} \quad \text{ for } t \to T^* \ .$$

By taking the liminf of the ratio as $t \to T^*$, we obtain the first statement.

In order to obtain the second statement, it suffices to note that (4.12) entails $K(u(t)) \to -\infty$ (namely, $\|u(t)\|^2 \ll \|u(t)\|_{p+1}^{p+1}$) as $t \to T^*$.

4.5. **Proof of Theorem 8.** We first prove the "topological" part of Theorem 8, namely

Lemma 1. *The set $\mathcal{G}$ is closed in $H_0^1(\Omega)$. The set $\mathcal{B}$ is open in $H_0^1(\Omega)$. The set $\mathcal{G}_0$ is open in $H_0^1(\Omega)$. The sets $\mathcal{G}$, $\mathcal{B}$, $\mathcal{G}_0$, $\partial\mathcal{G}$, and $\partial\mathcal{G}_0$ are invariant under the semiflow of* (1.1).

Proof. Since $\mathcal{B} := \{u_0 \in H_0^1(\Omega) : T^*(u_0) < \infty\}$, we deduce from Theorem 5 that $\mathcal{B}$ is open in $H_0^1(\Omega)$. Hence $\mathcal{G} \subset H_0^1(\Omega)$ is closed. Next we note that the origin of $H_0^1(\Omega)$ is a nondegenerate local minimum for the functional J. Hence, the continuous dependence (Theorem 5) implies that $\mathcal{G}_0$ is open.

It is clear that $\mathcal{G}$, $\mathcal{G}_0$, and $\mathcal{B}$ are invariant under the semiflow of (1.1). We now prove that also $\partial\mathcal{G}$ is invariant. By contradiction, assume that there exists $u_0 \in \partial\mathcal{G}$ and $\gamma > 0$ such that $S(\gamma)u_0 \in \mathrm{int}(\mathcal{G})$. By Theorem 5, there is an $H_0^1(\Omega)$ neigborhood U of u_0 which by $S(\gamma)$ is mapped into $\mathrm{int}(\mathcal{G})$. Hence $U \subset \mathcal{G}$, which contradicts the fact that $u_0 \in \partial\mathcal{G}$. The invariance of $\partial\mathcal{G}_0$ follows from a similar argument. □

The rest of the section is occupied with the proof of the identity $\mathcal{G}_0 = \mathrm{int}(\mathcal{G})$. A rather implicit proof of this identity can be obtained by suitably combining different results of P.L. Lions (see [28] and [27]) and reinterpreting them in the light of more recent a priori estimates; see Theorem 4. A comprehensive abstract approach to this kind of problems has already been developed by Hirsch [22]. However, since in our special context the arguments simplify considerably, we give a self-contained proof here. We need three lemmas.

Lemma 2. *If u is a nontrivial solution of* (2.1)*, then $J''(u)(u,u) < 0$, and the first eigenvalue of the eigenvalue problem*

$$-\Delta\psi - p|u|^{p-1}\psi = \lambda\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \tag{4.15}$$

is negative.

Proof. A nontrivial solution u of (2.1) satisfies $\int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}$, and hence $J''(u)(u,u) = \int_\Omega |\nabla u|^2 - p\int_\Omega |u|^{p+1} < 0$. Consequently, the first eigenvalue of (4.15) is negative. $\square$

Lemma 3. *Assume that $v_1, v_2 \in H_0^1(\Omega)\setminus\{0\}$ solve (2.1) with $v_1 \le v_2$. Then, either $v_1 < 0 < v_2$ or $v_1 \equiv v_2$.*

Proof. Suppose that $v_1 \not\equiv v_2$. Then, by comparison, $v_1 < v_2$ in Ω and $\frac{\partial v_1}{\partial \nu} > \frac{\partial v_2}{\partial \nu}$ on $\partial\Omega$. By Lemma 2, the first eigenvalues λ_{v_1} and λ_{v_2} of the Dirichlet eigenvalue problems

$$-\Delta\psi - p|v_i|^{p-1}\psi = \lambda\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \qquad i = 1,2,$$

are negative. Denote by e_1 (respectively e_2) corresponding positive first eigenfunctions; then

$$J''(v_1)(e_1,e_1) < 0 \qquad \text{and} \qquad J''(v_2)(e_2,e_2) < 0$$

Since J'' is continuous, we have

$$J(v_1 + \delta e_1) = J(v_1) + \frac{\delta^2}{2} J''(v_1)(e_1,e_1) + o(\delta^2) < J(v_1) \tag{4.16}$$

and

$$J(v_2 - \delta e_2) = J(v_2) + \frac{\delta^2}{2} J''(v_2)(e_2,e_2) + o(\delta^2) < J(v_2) \tag{4.17}$$

for sufficiently small $\delta > 0$. Consider now the closed set

$$Q = \{w \in H_0^1(\Omega) \,:\, v_1 \le w \le v_2 \text{ a.e. in } \Omega\} \subset H_0^1(\Omega),$$

and put $m := \inf_{u\in Q} J(u)$. Since $v_1 < v_1 + \delta e_1 < v_2 - \delta e_2 < v_2$ for small $\delta > 0$; (4.16) and (4.17) imply that

$$m < \min\{J(v_1), J(v_2)\}. \tag{4.18}$$

We claim that m is achieved by a function $w \in Q$. Indeed, let $\{w_n\}_n \subset Q$ be a minimizing sequence for $J|_Q$. Then

$$\begin{aligned} \|w_n\|^2 &= 2J(w_n) + \frac{2}{p+1}\int_\Omega |w_n|^{p+1} \\ &\le 2J(w_n) + \frac{2}{p+1}\int_\Omega (|v_1|^{p+1} + |v_2|^{p+1}) \le C, \end{aligned}$$

where $C > 0$ is a constant independent of n. Passing to a subsequence, we have $w_n \rightharpoonup w \in H_0^1(\Omega)$ and

$$w_n \to w \qquad \text{a.e. in } \Omega, \qquad \int_\Omega |w_n|^{p+1} \to \int_\Omega |w|^{p+1}.$$

We conclude that $w \in Q$ and that

$$J(w) = \frac{1}{2}\|w\|^2 - \frac{1}{p+1}\|w\|_{p+1}^{p+1} \le \frac{1}{2}\liminf_{n\to\infty}\|w_n\|^2 - \frac{1}{p+1}\lim_{n\to\infty}\|w_n\|_{p+1}^{p+1}$$
$$= \liminf_{n\to\infty} J(w_n) = m.$$

This forces $J(w) = m$ so that w is a minimizer for $J|_Q$. By (4.18) we have $w \neq v_1$, $w \neq v_2$. Moreover, the comparison principle implies that $S(t)w \in Q$ and therefore $J(S(t)w) \ge m$ for all $t \ge 0$. On the other hand, by (4.8) we know that $t \mapsto J(S(t)w)$ is strictly decreasing along nonconstant trajectories. These two facts enable us to conclude that $S(t)w = w$ for every $t \ge 0$. Consequently, w is a solution of (2.1) and by comparison we have $v_1 < w < v_2$ in Ω and $\frac{\partial v_1}{\partial \nu} > \frac{\partial w}{\partial \nu} > \frac{\partial v_2}{\partial \nu}$ on $\partial\Omega$. Hence, for $|t|$ sufficiently small we have $(1+t)w \in Q$ so that the minimization property of w yields

$$J''(w)(w,w) = 2\lim_{t\to 0}\frac{J((1+t)w) - J(w)}{t^2} \ge 0.$$

By Lemma 2, this implies $w \equiv 0$ and completes the proof. $\square$

Remark 4. The result and the proof of Lemma 3 carry over to the case where v_1 is a subsolution and v_2 is a supersolution of (2.1).

Lemma 4. *Let $u_0 \in \mathcal{G} \setminus \mathcal{G}_0$. Then*

(i) if $\omega(u_0) \subset \mathcal{S}_+ \cup \mathcal{S}_n$ (cf. Corollary 1), then $v_0 \in \mathcal{B}$ for every $v_0 \ge u_0$, $v_0 \neq u_0$.

(ii) if $\omega(u_0) \subset \mathcal{S}_- \cup \mathcal{S}_n$, then $v_0 \in \mathcal{B}$ for every $v_0 \le u_0$, $v_0 \neq u_0$.

Proof. We only prove (i). Let $u_0 \in \mathcal{G}$ and $v_0 \ge u_0$, $v_0 \neq u_0$. Then $v_0 \notin \mathcal{G}_0$ by comparison and the assumption on $\omega(u_0)$. We denote $u(t) := S(t)u_0$ and $v(t) := S(t)v_0$; we assume by contradiction that $v_0 \in \mathcal{G} \setminus \mathcal{G}_0$ and we distinguish the following cases.

Case 1: There is $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that $\|v(t_n) - u(t_n)\|_{C^1} \ge \varepsilon$ for all n.

Case 2: $\|v(t) - u(t)\|_{C^1} \to 0$ as $t \to \infty$.

If Case 1 occurs, by compactness of $\omega(u_0)$ and $\omega(v_0)$ (see Theorem 4 (iii)), we may pass to a subsequence such that $u(t_n) \to \hat{u}$ and $v(t_n) \to \hat{v}$ in $C_0^1(\Omega)$, where $\hat{u}, \hat{v}$ are nontrivial solutions of (2.1). By comparison we have $\hat{u} \le \hat{v}$, whereas $\hat{u}$ is not negative by assumption. Hence, Lemma 3 yields $\hat{u} = \hat{v}$. But this is impossible, since

$$\|\hat{v} - \hat{u}\|_{C^1} = \lim_{n\to\infty}\|v(t_n) - u(t_n)\|_{C^1} \ge \varepsilon.$$

We now suppose that Case 2 occurs. For every $\phi \in \omega(u_0)$, let λ_ϕ be the first eigenvalue of the Dirichlet eigenvalue problem

$$-\Delta\psi - p|\phi|^{p-1}\psi = \lambda\psi \quad \text{in} \quad \Omega, \qquad \psi = 0 \quad \text{on} \quad \partial\Omega,$$

and let e_ϕ denote the unique positive L^∞ normalized eigenfunction corresponding to λ_ϕ. Then

$$\lambda_0 := \sup_{\phi\in\omega(u_0)} \lambda_\phi < 0$$

by Lemma 2 and the compactness of $\omega(u_0)$ in $C^1_0(\Omega)$. Moreover, let $\theta \in C(\overline{\Omega})$ denote the distance function to the boundary $\partial\Omega$; that is, $\theta(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$. Then, again by compactness, there are $C_1, C_2 > 0$ such that

$$C_1\theta(x) \le e_\phi(x) \le C_2\theta(x) \qquad \text{for all } \ \phi \in \omega(u_0),\ x \in \Omega. \tag{4.19}$$

Let $w(t) = v(t) - u(t)$; then $w(x,t) > 0$ for $x \in \Omega, t > 0$, and w solves the equation

$$w_t = \Delta w + V(t)w$$

where $V(t) := V(\cdot, t) \in L^\infty(\Omega)$ is given by

$$V(x,t) = p\int_0^1 |u(x,t) + sw(x,t)|^{p-1}\, ds \qquad \text{for } x \in \Omega,\ t \ge 0.$$

Now fix $\tau > 0$ such that $C_2 \le C_1 e^{\frac{|\lambda_0|}{2}\tau}$. We claim that

$$\inf_{\phi\in\omega(u_0)} \sup_{t\le s\le t+\tau} \|V(s) - p|\phi|^{p-1}\|_\infty \ \to\ 0 \qquad \text{as} \quad t \to \infty. \tag{4.20}$$

Indeed, suppose by contradiction that for a sequence $t_n \to \infty$ and some $\varepsilon > 0$ we have

$$\inf_{\phi\in\omega(u_0)} \sup_{t_n\le s\le t_n+\tau} \|V(s) - p|\phi|^{p-1}\|_\infty > \varepsilon \qquad \text{for all} \quad n. \tag{4.21}$$

By Theorems 4 and 6 there exist $\phi \in \omega(u_0)$ and a subsequence – still denoted by t_n – such that

$$\sup_{t_n\le s\le t_n+\tau} \|u(s) - \phi\|_\infty \to 0 \qquad \text{as} \quad n \to \infty.$$

Moreover, recalling that $\|w(t)\|_{C^1} \to 0$ as $t \to \infty$, we obtain

$$\sup_{t_n\le s\le t_n+\tau} \|V(s) - p|\phi|^{p-1}\|_\infty \to 0 \qquad \text{as} \quad n \to \infty.$$

This contradicts (4.21) and proves (4.20). We may therefore take $T_0 > 0$ such that

$$\inf_{\phi\in\omega(u_0)} \sup_{t\le s\le t+\tau} \|V(s) - p|\phi|^{p-1}\|_\infty \le \frac{|\lambda_0|}{2} \qquad \text{for} \quad t \ge T_0. \tag{4.22}$$

Next, we claim that

$$\int_\Omega w(t+\tau)\theta \geq \int_\Omega w(t)\theta \qquad \text{for} \quad t \geq T_0. \tag{4.23}$$

Indeed, by (4.22) and compactness, for any $t \geq T_0$ we may find $\phi \in \omega(u_0)$ such that $\|V(s) - p|\phi|^{p-1}\|_\infty \leq \frac{|\lambda_0|}{2}$ for all $s \in [t, t+\tau]$. Then

$$\begin{aligned}\frac{d}{ds}\int_\Omega w(s)e_\phi &= \int_\Omega (\Delta w(s) + V(s)w(s))e_\phi = \int_\Omega w(s)(\Delta e_\phi + V(s)e_\phi)\\ &= \int_\Omega [V(s) - p|\phi|^{p-1} - \lambda_\phi]w(s)e_\phi \geq \frac{|\lambda_0|}{2}\int_\Omega w(s)e_\phi\end{aligned}$$

for $s \in [t, t+\tau]$. Combining this with (4.19), we get

$$C_2\int_\Omega w(t+\tau)\theta \geq \int_\Omega w(t+\tau)e_\phi \geq e^{\frac{|\lambda_0|}{2}\tau}\int_\Omega w(t)e_\phi \geq C_1 e^{\frac{|\lambda_0|}{2}\tau}\int_\Omega w(t)\theta.$$

By our choice of τ this amounts to (4.23). An iterated use of (4.23) gives

$$\int_\Omega w(T_0 + l\tau)\theta \geq \int_\Omega w(T_0)\theta > 0$$

for every $l \in \mathbb{N}$, which contradicts the assumption that $\|w(t)\|_{C^1} \to 0$ as $t \to \infty$. The proof is finished. $\square$

We may now conclude the proof of Theorem 8. From Lemma 4 it follows that $\mathcal{G} \setminus \mathcal{G}_0 \subset \partial\mathcal{G}$. Since by Lemma 1 the set $\mathcal{G}_0$ is open, we infer that $\mathrm{int}(\mathcal{G}) = \mathcal{G}_0$.

4.6. **Proof of Theorem 9.** We first recall the following:

Lemma 5. *We have $J(u) > 0$ for any $u \in \mathcal{N}_+$. Moreover, for all $u \in \mathcal{N}$, we have $J(u) = \max_{s\geq 0} J(su)$. Finally, for any $k > 0$ the set $J^k \cap \mathcal{N}_+$ is bounded in $H_0^1(\Omega)$.*

Proof. For the first two statements it suffices to study the monotonicity of the map $s \mapsto J(su)$ for all $u \in H_0^1(\Omega)$ such that $\|u\| = 1$. For the third statement, note that the two conditions $J(u) < k$ and $K(u) > 0$ yield $\|u\|^2 < k\frac{2(p+1)}{p-1}$. $\square$

Let $u(t) := S(t)u_0$. Since $J(u_0) < d$, by (4.8) we infer that $J(u(t)) < d$ for all $t \in [0, T^*(u_0))$. Therefore, (2.7) tells us that $u(t) \notin \mathcal{N}$ for all $t \in [0, T^*(u_0))$ and that $\{u(t)\}$ cannot approach $\mathcal{N}$ as $t \to T^*$. Hence, if $u_0 \in \mathcal{N}_-$ then $u(t) \in J^d \cap \mathcal{N}_-$ for all t; since there are no solutions of (2.1) in $\mathcal{N}_-$, $\{u(t)\}$ blows up in finite time by Theorems 3-4. On the other hand, if $u_0 \in \mathcal{N}_+$ then $u(t) \in J^d \cap \mathcal{N}_+$ for all t and $\{u(t)\}$ remains bounded in view of Lemma 5; by Theorem 3, this shows that $u_0 \in \mathcal{G}$. Since $u \equiv 0$ is the only solution of (2.1) in J^d, we conclude that $u_0 \in \mathcal{G}_0$.

4.7. **Proof of Theorem 10.** This is essentially a consequence of Lemma 4. Let v be a nontrivial solution of (2.1), so that $v \in \mathcal{G} \setminus \mathcal{G}_0$. If $v^+ \neq 0$, then Lemma 4 (i) yields $u_0 \in \mathcal{B}$ for every $u_0 \geq v$, $u_0 \neq v$. Analogously, if $v^- \neq 0$, then Lemma 4 (ii) yields $u_0 \in \mathcal{B}$ for every $u_0 \leq v$, $u_0 \neq v$. Finally, suppose that $v > 0$ and $-v \leq u_0 \leq v$; then $u_0 \in \mathcal{G}$ by comparison. Moreover, $-v \neq u_0 \neq v$ implies that $-v < S(1)u_0 < v$ in Ω and $-\frac{\partial v}{\partial \nu} > \frac{\partial}{\partial \nu}[S(1)u_0] > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$. Hence $S(1)u_0$ belongs to the interior of $\mathcal{G} \cap C^1_0(\Omega)$ with respect to the C^1 topology. By Theorems 5 and 8 we conclude that $u_0 \in \mathrm{int}(\mathcal{G}) = \mathcal{G}_0$.

4.8. **Proof of Theorem 11.** Throughout this section we fix $u_0 \in \mathbb{K} \setminus \{0\}$ and for all $\alpha \geq 0$ we put $u^\alpha(t) := S(t)[\alpha u_0]$ (the unique local solution of (1.1) with initial datum $u^\alpha(0) = \alpha u_0$) and $T_\alpha := T^*(\alpha u_0)$.

By Theorem 3 (ii), $u^\alpha(t) \in \mathbb{K}$ for all $t \in [0, T_\alpha)$. Therefore, u^α also solves the equation $u_t - \Delta u = |u|^p$ where the nonlinearity is now *convex*. Define

$$\alpha^* := \inf\{\alpha \geq 0 : \alpha u_0 \in \mathcal{B}\} \ .$$

By [27, Theorem 2.1] (see also [8, Remarque 13]), we know that statements $(i) - (iii)$ hold. By comparison, we have that $\alpha \mapsto T_\alpha$ is decreasing over (α^*, ∞). Hence, to conclude the proof of Theorem 11, we only have to prove (3.1) and (3.2). To this end, we introduce the following classes

$$\begin{aligned} \mathbb{L} : = \Big\{ & \psi \in \mathrm{Lip}[0,\infty) : \psi(0) = 1,\, \psi \geq 0,\, \psi(t) = 0 \text{ for all } t \geq 1, \\ & \frac{|\psi'|^{p/(p-1)}}{\psi^{1/(p-1)}} \in L^1(0,1) \Big\} \\ \mathbb{T} : = \Big\{ & w \in W^{2,1}(\Omega) : w \geq 0 \text{ a.e. in } \Omega,\, \frac{|\Delta w|^{p/(p-1)}}{w^{1/(p-1)}} \in L^1(\Omega) \Big\}. \end{aligned}$$

Note that $\mathbb{L}, \mathbb{T} \neq \varnothing$; in particular, $\mathbb{T}$ contains any positive solution of (2.1), the first positive eigenfunction of $-\Delta$, and suitable functions $w \in C^2_c(\Omega)$; see the proof of [30, Theorem 2.5].

We prove the following general result:

Lemma 6. *Assume that $T_\alpha < \infty$. Then for all $\psi \in \mathbb{L}$, all $w \in \mathbb{T}$, all $\gamma \geq T_\alpha^{-1}$, and all $\delta \in (0, p^{1/p})$, we have*

$$\alpha \int_\Omega u_0 w \leq \frac{p-1}{p} \Big[\frac{\gamma^{-1}}{\delta^{p/(p-1)}} \int_0^1 \psi(t)dt \int_\Omega \frac{|\Delta w|^{p/(p-1)}}{w^{1/(p-1)}}$$

$$+\frac{\|w\|_1\gamma^{1/(p-1)}}{(p-\delta^p)^{1/(p-1)}}\int_0^1\frac{|\psi'(t)|^{p/(p-1)}}{\psi^{1/(p-1)}(t)}dt\Big].$$

Proof. For all $\gamma \geq T_\alpha^{-1}$ let $\phi^\gamma(x,t) = \psi(\gamma t)w(x)$ so that $\phi^\gamma(x,t) \equiv 0$ for $t \geq 1/\gamma$; multiply (1.1) by ϕ^γ and integrate by parts over $\Omega \times [0, 1/\gamma]$ to obtain

$$\alpha\int_\Omega u_0(x)w(x)\,dx + \int_0^{1/\gamma}\int_\Omega (u^\alpha)^p(x,t)\psi(\gamma t)w(x)\,dx\,dt \tag{4.24}$$
$$= -\int_0^{1/\gamma}\int_\Omega u^\alpha(x,t)\psi(\gamma t)\Delta w(x)\,dx\,dt - \gamma\int_0^{1/\gamma}\int_\Omega u^\alpha(x,t)\psi'(\gamma t)w(x)\,dx\,dt$$
$$\leq \int_0^{1/\gamma}\int_\Omega u^\alpha(x,t)\psi(\gamma t)|\Delta w|(x)dx\,dt + \gamma\int_0^{1/\gamma}\int_\Omega u^\alpha(x,t)|\psi'(\gamma t)|w(x)dx\,dt.$$

In order to estimate further (4.24), we make use of Young's inequality in the following form:

$$ab \leq \frac{\delta^p}{p}a^p + \frac{p-1}{p\delta^{p/(p-1)}}b^{p/(p-1)} \qquad \text{for all } a, b, \delta > 0.$$

More precisely, for all $\delta > 0$ we have

$$u^\alpha\psi|\Delta w| = u^\alpha[\psi w]^{1/p}\cdot\frac{\psi|\Delta w|}{[\psi w]^{1/p}} \leq \frac{\delta^p}{p}(u^\alpha)^p\psi w + \frac{p-1}{p\delta^{p/(p-1)}}\psi\frac{|\Delta w|^{p/(p-1)}}{w^{1/(p-1)}}\ ;$$

similarly, for all $\eta > 0$, we have

$$\gamma u^\alpha|\psi'|w = u^\alpha[\psi w]^{1/p}\frac{\gamma|\psi'|w}{[\psi w]^{1/p}} \leq \frac{\eta^p}{p}(u^\alpha)^p\psi w + \frac{p-1}{p\eta^{p/(p-1)}}\gamma^{p/(p-1)}\frac{|\psi'|^{p/(p-1)}}{\psi^{1/(p-1)}}w.$$

Taking these into account, and assuming that $\delta^p + \eta^p = p$, the estimate in (4.24) yields the result after the change of variables $\gamma t \mapsto t$. □

We now make particular choices of the functions ψ and w in Lemma 6. We first take

$$\psi_q(t) := \begin{cases} (1-t)^q & \text{if } t \in [0,1] \\ 0 & \text{if } t \geq 1\ ; \end{cases}$$

note that if $q > \frac{1}{p-1}$, then $\psi_q \in \mathbb{L}$ and

$$\int_0^1 \psi_q(t)dt = \frac{1}{q+1}\ , \qquad \int_0^1\frac{|\psi_q'(t)|^{p/(p-1)}}{\psi_q^{1/(p-1)}(t)}dt = \frac{q^{p/(p-1)}}{q-\frac{1}{p-1}}\ .$$

In order to prove (3.1), let v be a positive mountain pass solution of (2.1); then

$$\int_\Omega \frac{|\Delta v|^{p/(p-1)}}{v^{1/(p-1)}} = \int_\Omega v^{p+1} = S_{p+1}^{(p+1)/(p-1)} .$$

Let $\alpha > \alpha^*$ (so that $T_\alpha < \infty$) and take $\delta^p = \frac{p}{2}$, $w = v$, and $\psi = \psi_q$ with $q = \frac{2}{p-1}$; with these choices, Lemma 6 entails

$$\alpha \int_\Omega u_0 v \leq \frac{(p-1)^2\, 2^{1/(p-1)}}{p^{p/(p-1)}} \Big[\frac{S_{p+1}^{(p+1)/(p-1)}}{p+1} \frac{1}{\gamma} + \Big(\frac{2}{p-1}\Big)^{p/(p-1)} \|v\|_1 \gamma^{1/(p-1)} \Big] \tag{4.25}$$

for all $\gamma \geq T_\alpha^{-1}$. Since $T_{\alpha^*} = +\infty$, if we let $\alpha \downarrow \alpha^*$ in (4.25), then (4.25) holds true for any $\gamma > 0$ whenever $\alpha = \alpha^*$; then, by minimizing the right-hand side of (4.25) with respect to γ we get (3.1).

In order to prove (3.2), take a sequence $\{u_m\} \subset C_c^\infty(\Omega)$ such that $u_m > 0$ in the interior of its support and $u_m \to u_0$ in $L^2(\Omega)$ as $m \to \infty$; as noticed above, we have $\{u_m\} \subset \mathbb{T}$. Assume that α is sufficiently large to ensure $T_\alpha < \infty$; then, by applying Lemma 6 with $\psi = \psi_q$ and $w = u_m$ we obtain

$$\begin{aligned} &\alpha \int_\Omega u_0 u_m \\ &\leq \frac{p-1}{p} \Big[\frac{\gamma^{-1}}{(q+1)\delta^{p/(p-1)}} \int_\Omega \frac{|\Delta u_m|^{p/(p-1)}}{u_m^{1/(p-1)}} + \frac{\|u_m\|_1 \gamma^{1/(p-1)}}{(p-\delta^p)^{1/(p-1)}} \frac{q^{p/(p-1)}}{q - \frac{1}{p-1}} \Big] \end{aligned} \tag{4.26}$$

for all $q > \frac{1}{p-1}$, all m, all $\gamma \geq T_\alpha^{-1}$, and all $\delta \in (0, p^{1/p})$. For any sufficiently large $\alpha = \alpha(m, \delta) > 0$ let $\gamma_\alpha > 1$ be the unique value of γ for which equality holds in (4.26). Then, (4.26) implies that $T_\alpha \leq \gamma_\alpha^{-1}$; hence, letting $\alpha \to \infty$ (i.e. $\gamma_\alpha \to \infty$) in (4.26) yields

$$\limsup_{\alpha \to \infty} \; \alpha T_\alpha^{1/(p-1)} \leq \frac{p-1}{p} \frac{\|u_m\|_1}{(p-\delta^p)^{1/(p-1)}} \frac{q^{p/(p-1)}}{q - \frac{1}{p-1}} \Big(\int_\Omega u_0 u_m \Big)^{-1} \tag{4.27}$$

for all $q > \frac{1}{p-1}$, all m and all $\delta \in (0, p^{1/p})$. The number $q > \frac{1}{p-1}$ which minimizes the right-hand side of (4.27) is $q = \frac{p}{p-1}$; with this choice of q and taking the infimum with respect to $\delta \in (0, p^{1/p})$, (4.27) becomes

$$\limsup_{\alpha \to \infty} \; \alpha^{p-1} T_\alpha \;\leq\; \frac{1}{p-1} \Big(\frac{\|u_m\|_1}{\int_\Omega u_0 u_m} \Big)^{p-1} \tag{4.28}$$

for all m. Finally, letting $m \to \infty$ in (4.28) proves (3.2).

4.9. **Proof of Theorem 12.** Let $M > 0$ and let v denote a positive solution of (2.1). Let $c > 0$ and $\Omega' \subset \Omega$ be an open subset such that $v > c$ on Ω'. For $k > 0$, pick a positive function $\phi_k \in C_0^1(\Omega')$ such that

$$\|\phi_k\| \geq k \qquad \text{and} \qquad \|\phi_k\|_\infty \leq c.$$

Fix $k > 0$, and put $w_+ := v + \phi_k$, $w_- := v - \phi_k$. Then $w_\pm \in \mathbb{K}$, and

$$\|w_\pm\| \geq \|\phi_k\| - \|v\| \geq k - \|v\| \ ,$$

$$\|w_\pm\|_{p+1} \leq \|v\|_{p+1} + \|\phi_k\|_{p+1} \leq \|v\|_{p+1} + c|\Omega'|^{1/(p+1)} \ .$$

Hence, for k sufficiently large we have both $J(w_\pm) \geq M$ and $K(w_\pm) > 0$, hence $w_\pm \in \mathcal{N}_+$. For such a number k, take $u_M = w_-$ and $v_M = w_+$. Since $0 \leq u_M \leq v$ we have $u_M \in \mathcal{G}_0$ by Theorem 10 (*iii*), whereas $v_M \in \mathcal{B}$ by Theorem 10 (*i*).

4.10. **Proof of Theorem 13.** The weak dissipativity and antidissipativity of the flow of (1.1) are explained by the following easy observation.

Lemma 7. *Let $u_0 \in H_0^1(\Omega)$, and put $u(t) = S(t)u_0$ for $t \in [0, T^*(u_0))$. Then*

$$\frac{d}{dt}\|u(t)\|_2^2 = -2K(u(t)) \qquad \textit{for all } t \in (0, T^*(u_0)) \ . \tag{4.29}$$

Proof. This follows immediately by multiplying (1.1) by $u(t)$ and integrating by parts. □

We now give an abstract criterion for vanishing (respectively blow-up) in terms of the variational values λ_a and Λ_a defined in Section 2.

Lemma 8. *If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \leq \lambda_{J(u_0)}$, then $u_0 \in \mathcal{G}_0$. If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \geq \Lambda_{J(u_0)}$, then $u_0 \in \mathcal{B}$.*

Proof. Put $u(t) := S(t)u_0$ for $t \in [0, T^*(u_0))$. Assume first that $u_0 \in \mathcal{N}_+$ satisfies $\|u_0\|_2 \leq \lambda_{J(u_0)}$. We claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T^*(u_0))$. By contradiction, if there is $s > 0$ such that $u(t) \in \mathcal{N}_+$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, then (4.29) and (4.11) imply that

$$\|u(s)\|_2 < \|u_0\|_2 \leq \lambda_{J(u_0)} \ , \qquad J(u(s)) < J(u_0).$$

This contradicts the definition of $\lambda_{J(u_0)}$ and proves the claim. Hence, Lemma 5 shows that the orbit $\{u(t)\}$ remains bounded in $H_0^1(\Omega)$ for $t \in [0, T^*(u_0))$ so that $T^*(u_0) = \infty$. Now for every $w \in \omega(u_0)$, by (4.29) and (4.8) we have

$$\|w\|_2 < \lambda_{J(u_0)} \qquad \text{and} \qquad J(w) \leq J(u_0).$$

By definition of $\lambda_{J(u_0)}$ we conclude that $\omega(u_0) \cap \mathcal{N} = \varnothing$, hence $\omega(u_0) = \{0\}$. In other words, $u_0 \in \mathcal{G}_0$, as claimed.

Assume now that $u_0 \in \mathcal{N}_-$ satisfies $\|u_0\|_2 \geq \Lambda_{J(u_0)}$. A similar argument as above shows that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T^*(u_0))$. Now if, by contradiction, $T^*(u_0) = \infty$, then for every $w \in \omega(u_0)$ we would have by (4.29) and (4.8)

$$\|w\|_2 > \Lambda_{J(u_0)} \qquad \text{and} \qquad J(w) \leq J(u_0).$$

By definition of $\lambda_{J(u_0)}$ we then infer that $\omega(u_0) \cap \mathcal{N} = \varnothing$. However, since $\operatorname{dist}(0, \mathcal{N}_-) > 0$, we also have $0 \notin \omega(u_0)$. This gives $\omega(u_0) = \varnothing$, contrary to the assumption that $u(t)$ is a global solution. We conclude that $T^*(u_0) < \infty$, as claimed. $\square$

We may now complete the proof of Theorem 13. By Theorem 2, since $p \leq 1 + \frac{4}{n}$, we have

$$\gamma = \inf_{u \in \mathcal{N}_-} \|u\|_2 = \inf_{u \in \mathcal{N}} \|u\|_2 > 0.$$

Hence $\{u \in H_0^1(\Omega) : \|u\|_2 < \gamma\} \subset \mathcal{N}_+$, and by (4.29) (and continuous dependence) it is the maximal open L^2 ball centered at zero which is invariant under the semiflow $S(t)$. Moreover, $\{u \in H_0^1(\Omega) : \|u\|_2 < \gamma\} \subset \mathcal{G}_0$ by Lemma 8.

Remark 5. In view of the previous proof, it is clear that we may also allow $\|u_0\|_2 = \lambda_\infty$ provided that $u_0 \notin \mathcal{N}$; more precisely, $(\{u \in H_0^1(\Omega) : \|u\|_2 \leq \lambda_\infty\} \cap \mathcal{N}_+) \subset \mathcal{G}_0$.

4.11. **Proof of Theorem 14.** By using the strict Hölder inequality (strict because u_0 is not a constant) and (3.3), we get

$$|\Omega|^{(p-1)/2} \|u_0\|_{p+1}^{p+1} > \|u_0\|_2^{p+1} \geq \frac{2(p+1)}{p-1} \, |\Omega|^{(p-1)/2} \, J(u_0) \ .$$

Then, we readily infer $\|u_0\|_{p+1}^{p+1} > \|u_0\|^2$ and, in turn, $u_0 \in \mathcal{N}_-$. By the Hölder inequality, for any $u \in \mathcal{N}_{J(u_0)}$, we have

$$|\Omega|^{(1-p)/2} \|u\|_2^{p+1} \leq \|u\|_{p+1}^{p+1} = \|u\|^2 \leq \frac{2(p+1)}{p-1} \, J(u_0) \ .$$

Therefore, taking the supremum over $\mathcal{N}_{J(u_0)}$, we immediately get

$$\Lambda_{J(u_0)}^{p+1} \leq \frac{2(p+1)}{p-1} |\Omega|^{(p-1)/2} \, J(u_0) \ .$$

Hence, if u_0 satisfies (3.3), then $\|u_0\|_2 \geq \Lambda_{J(u_0)}$ and Lemma 8 shows that $u_0 \in \mathcal{B}$.

4.12. **Proof of Theorem 15.** Let $M > 0$, and let Ω_1, Ω_2 be two arbitrary disjoint open subdomains of Ω. Furthermore, let $v \in H_0^1(\Omega_1) \subset H_0^1(\Omega)$ be an arbitrary nonzero function. Then $\|\alpha v\|_2^{p+1} \geq \frac{2(p+1)}{p-1}|\Omega|^{(p-1)/2}M$ and $J(\alpha v) \leq 0$ for $\alpha > 0$ large. Fix such a number $\alpha > 0$ and pick a function $w \in H_0^1(\Omega_2)$ with $J(w) = M - J(\alpha v)$. Then $u_M := \alpha v + w$ satisfies $J(u_M) = J(\alpha v) + J(w) = M$ and

$$\|u_M\|_2^{p+1} \geq \|\alpha v\|_2^{p+1} \geq \frac{2(p+1)}{p-1}|\Omega|^{(p-1)/2}J(u_M);$$

hence, $u_M \in \mathcal{N}_- \cap \mathcal{B}$ by Theorem 14.

5. Appendix: the comparison principle

Throughout the paper we made extensive use of the following comparison principle for initial data $u_0 \in H_0^1(\Omega)$. This result is well established, but we could not find an exact reference. For the sake of completeness, we recall the proof here.

Proposition 1. *Let $u_0, v_0 \in H_0^1(\Omega)$ be such that $u_0 - v_0 \in \mathbb{K}$. Then, $[S(t)u_0 - S(t)v_0] \in \mathbb{K}$ for all $t \geq 0$. Moreover, if $u_0 \neq v_0$, then, for $t > 0$,*

$$S(t)u_0 - S(t)v_0 > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial}{\partial\nu}[S(t)u_0 - S(t)v_0] < 0 \quad \text{on } \partial\Omega. \tag{5.1}$$

Proof. Throughout this proof we put $u(t) := S(t)u_0$ and $v(t) := S(t)v_0$. We first prove the statement for $u_0, v_0 \in C_c^\infty(\Omega)$ so that $u, v \in C(\overline{\Omega} \times [0, T])$ for all $T < \overline{T} := \min\{T^*(u_0), T^*(v_0)\}$. By subtracting the two equations for u and v, we see that $w := u - v$ satisfies

$$\begin{cases} w_t - \Delta w = V(t)w & \text{in } \Omega \times (0, \overline{T}) \\ w(0) = u_0 - v_0 \geq 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \times (0, \overline{T}) \, , \end{cases} \tag{5.2}$$

Here $V(t) := V(\cdot, t) \in L^{\frac{p+1}{p-1}}(\Omega)$ is given by

$$V(x,t) = p\int_0^1 |u(x,t) + sw(x,t)|^{p-1}\, ds \qquad \text{for } x \in \Omega, t \geq 0.$$

Since u, v are continuous functions, for all $T \in (0, \overline{T})$ we have

$$M_T := \sup_{\Omega \times (0,T)} V(x,t) < \infty.$$

Taking this into account, if we multiply (5.2) by w^- and we integrate we get

$$\frac{1}{2}\frac{d}{dt}\|w^-(t)\|_2^2 = -\|w^-(t)\|^2 + \int_\Omega V(t)|w^-(t)|^2 \leq M_T\|w^-(t)\|_2^2$$

for all $t \in [0,T]$. By the Gronwall lemma and by arbitrariness of T, this proves that $w^-(t) \equiv 0$ and, in turn, the comparison principle for smooth initial data u_0 and v_0.

For general $u_0, v_0 \in H^1_0(\Omega)$, take two sequences $\{u_0^m\}, \{v_0^m\} \subset C_c^\infty(\Omega)$ such that $u_0^m \to u_0$ and $v_0^m \to v_0$ in $H^1_0(\Omega)$ as $m \to \infty$, $v_0^m \le v_0 \le u_0 \le u_0^m$ almost everywhere in Ω for all m. Recall that u and v (solutions corresponding to u_0 and v_0) are smooth functions for $t > 0$, see Theorem 3. If $v(X,T) > u(X,T)$ for some $(X,T) \in \Omega \times (0,\overline{T})$, then by Theorem 5 (see also [7, Theorem 1]) we would also have $v^m(X,T) > u^m(X,T)$ for sufficiently large m (here $v^m(t) = S(t)v_0^m$ and $u^m(t) = S(t)u_0^m$). This contradicts the just proved comparison principle for smooth initial data.

Finally, to see (5.1), suppose that $u_0 \ne v_0$ and fix $t > 0$. For $\delta \in (0,t)$ small enough, it is obvious that the C^1_0-functions $u_1 := S(\delta)u_0$ and $v_1 := S(\delta)v_0$ do not coincide (by backward uniqueness, this is true for all $\delta > 0$). Moreover, we already proved that $S(t)u_0 - S(t)v_0 \in \mathbb{K}$ for $t < \overline{T}$, in particular $u_1 - v_1 \in \mathbb{K}$. Moreover, $V(t) \ge 0$ for $t < \overline{T}$. Since $w(t) = S(t)u_0 - S(t)v_0$ satisfies the equation $w_t - \Delta w = V(t)w \ge 0$ on $[\delta, \overline{T})$ together with homogeneous Dirichlet boundary conditions, the strong parabolic maximum principle for initial data in $C^1_0(\Omega)$ implies that $w(t) > 0$ in Ω and $\frac{\partial}{\partial \nu} w(t) < 0$ on $\partial\Omega$ for $t \in (\delta, \overline{T})$. We conclude (5.1). $\square$

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