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EXISTENCE OF GROUND STATES AND FREE BOUNDARY PROBLEMS FOR QUASILINEAR ELLIPTIC OPERATORS

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Abstract. We prove the existence of non-negative non-trivial solutions of the quasilinear equation $\Delta_m u + f(u) = 0$ in $\mathbb{R}^n$ and of its associated free boundary problem, where Δ_m denotes the m-Laplace operator. The nonlinearity $f(u)$, defined for $u > 0$, is required to be Lipschitz continuous on $(0, \infty)$, and in L^1 on $(0, 1)$ with $\int_0^u f(s)\, ds < 0$ for small $u > 0$; the usual condition $f(0) = 0$ is thus completely removed. When $n > m$, existence is established essentially for all subcritical behavior of f as $u \to \infty$, and, with some further restrictions, even for critical and supercritical behavior. When $n = m$ we treat various exponential growth conditions for f as $u \to \infty$, while when $n < m$ no growth conditions of any kind are required for f. The proof of the main results moreover yield as a byproduct an a priori estimate for the supremum of a ground state in terms of n, m and elementary parameters of the nonlinearity. Our results are thus new and unexpected even for the semilinear equation $\Delta u + f(u) = 0$.

The proofs use only straightforward and simple techniques from the theory of ordinary differential equations; unlike well known earlier demonstrations of the existence of ground states for the semilinear case, we rely neither on critical point theory [6] nor on the Emden-Fowler inversion technique [2, 3].

1. Introduction. Let $\Delta_m u = \operatorname{div}(|Du|^{m-2} Du)$, $m > 1$, denote the degenerate m-Laplace operator. We study the existence of radial ground states of the quasilinear elliptic equation

$$\Delta_m u + f(u) = 0 \qquad \text{in } \mathbb{R}^n,\ n > 1, \tag{1.1}$$

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and the existence of positive radial solutions of the homogeneous Dirichlet-Neumann free boundary problem

$$\begin{aligned} &\Delta_m u + f(u) = 0, \quad u > 0 \quad \text{in } B_R, \\ &u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B_R, \end{aligned} \tag{1.2}$$

where B_R is an open ball in $\mathbb{R}^n$ with radius $R > 0$. Here, a *ground state* is a non-negative, non-trivial continuously differentiable distribution solution $u = u(x)$ of (1.1) which tends to zero as $|x|$ approaches infinity.

We shall assume throughout the paper the following general condition

(H1) f is locally Lipschitz continuous on $(0, \infty)$, and $\int_0 |f(s)|ds < \infty$.

By (H1) it is clear that $F(s) = \int_0^s f(t)dt$ exists and is continuous on $[0, \infty)$, and $F(0) = 0$. We shall then assume further

(H2) There exists $\beta > 0$ such that $F(s) < 0$ for $0 < s < \beta$, $F(\beta) = 0$ and $f(\beta) > 0$.

The behavior of f near zero is of crucial importance to our results. In the following considerations we shall identify two mutually exclusive situations:

1. *Regular case.* f is continuous on $[0, \infty)$; clearly (H2) implies that $f(0) \leq 0$

2. *Singular case.* f cannot be extended as a continuous function to $[0, \infty)$; we leave the value of f at 0 undefined.

When $m = 2$ we recall that (1.1) reduces to the classical nonlinear Euclidean scalar field equation

$$\Delta u + f(u) = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.3}$$

Under the assumption that f is regular and $f(0) = 0$, together with conditions on the behavior of f for large u, there are in the literature a number of well-known existence theorems for radial ground states of (1.3); see in particular [11, 6, 2, 3, 14].

Much less is known about ground states for the degenerate equation (1.1). Citti [10] has proved existence when $1 < m < n$, $f(0) = 0$, and f is bounded in $[0, \infty)$, while Franchi, Lanconelli and Serrin [16] have considered the case when $f(s)$ is "sublinear" for large s, in the sense that either $f(\gamma) = 0$ for some finite $\gamma > \beta$ or

$$\liminf_{s \to \infty} \frac{|F(s)|^{1/m}}{s} < \infty$$

(for example, $f(s) \le \text{const.}\, s^{m-1}$ for large s).

On the other hand, *if* $f(0) < 0$, *then ground states for* (1.1) *and* (1.3) *cannot exist*, see Theorem 1 below. At the same time, it was shown by Kaper and Kwong [21] that the *free boundary* problem (1.2) has solutions for the case $m = 2$ when $f(0) < 0$ and $f(s)/(s-\beta)$ is non-increasing for $s > \beta$. Moreover, it was conjectured by Pucci and Serrin (see [34, Section 6.1]) that if $f(0) < 0$, or even if f is singular, then solutions of the free boundary problem *can* be found for all $m > 1$. We shall address this question in detail in Theorem 1.

The purpose of this paper is to extend and unify the above results, by considering throughout the general conditions (H1)-(H2) on $f(u)$. *Thus we allow f to be different from zero, or even singular, at* 0, *and additionally we generalize earlier growth restrictions on $f(u)$ as $u \to \infty$.* Our results are thus new even for the classical field equation (1.3).

In what follows, the two conditions

$$|F(s)| \le \Phi(s) \quad \text{for} \quad 0 < s \le \eta \qquad \text{and} \qquad \int_0 |\Phi(s)|^{-1/m}\, ds = \infty, \tag{1.4}$$

where $\eta > 0$ and $\Phi : [0,\eta) \to \mathbb{R}$ is a non-decreasing function with $\Phi(0) = 0$, and

$$\int_0 |F(s)|^{-1/m}\, ds < \infty \tag{1.5}$$

will be important. Clearly, (1.4) is valid if

$$f(s) \le 0 \quad \text{for} \quad 0 < s \le \eta \qquad \text{and} \qquad \int_0 |F(s)|^{-1/m}\, ds = \infty.$$

We define the constant γ by $\gamma = \min\{s > \beta;\ f(s) = 0\}$ and put $\gamma = \infty$ if $f(s) > 0$ for all $s \ge \beta$. Finally we introduce a function related to the Pohozaev identity, namely $Q(s) = nmF(s) - (n-m)sf(s)$. Our main existence result is then

Theorem 1. *Let hypotheses* (H1) *and* (H2) *hold and suppose that one of the following three conditions is satisfied:*
(C1) $\gamma < \infty$. *There exists $k_0 > 0$ such that*

$$\limsup_{s \uparrow \gamma} \frac{f(s)}{(\gamma - s)^{m-1}} < k_0. \tag{1.6}$$

(C2) $\gamma = \infty$ *and* $n < m$.
(C3) $\gamma = \infty$ *and* $n \geq m$. $Q(s)$ *is locally bounded below near* $s = 0$, *and there exists* $b > \beta$ *and* $k \in (0, 1)$ *such that* $Q(s) \geq 0$ *for all* $s \geq b$ *and*

$$\limsup_{s\to\infty} Q(s_2)\Big(\frac{s^{m-1}}{f(s_1)}\Big)^{n/m} = \infty, \tag{1.7}$$

where s_1, s_2 *is an arbitrary pair of numbers in* $[ks, s]$.
Then we have the following results:

(i) *When* f *is regular and* $f(0) = 0$ *there exists a radial ground state* u *for equation* (1.1). *If* (1.4) *is satisfied, then* u *is positive and the free boundary problem* (1.2) *has no radial solution; if* (1.5) *is satisfied, then* u *has compact support and accordingly is also a radial solution of* (1.2) *for some* $R > 0$.

(ii) *When* $\limsup_{s\downarrow 0} f(s) < 0$, *there are no ground states of* (1.1), *and* (1.2) *has a radial solution for some* $R > 0$.

(iii) *When* f *is singular and* $\limsup_{s\downarrow 0} f(s) \geq 0$, *then either* (1.1) *has a positive radial ground state or* (1.2) *has a radial solution for some* $R > 0$. *If* (1.4) *is satisfied, then the first case occurs; if* (1.5) *is satisfied, the second case occurs.*

Finally, if $r = |x|$, *the function* $u = u(r)$ *obeys* $u'(r) < 0$ *for all* $r > 0$ *such that* $u(r) > 0$.

Note that when $m \leq 2$, (1.6) is automatically satisfied in view of (H1). In fact, (1.6) could be omitted even for $m > 2$ by arguing as in Section 2.2 of [16]; we shall not pursue this here since our main interest is in conditions (C2) and (C3). We underline the fact that in case (C2) there are *no* conditions on f apart from (H1)-(H2). The local boundedness assumption on Q in condition (C3) is automatic if $n = m$ or f is regular, or even if f is singular and bounded above by Const. s^{-1}.

Condition (1.7) is similar to one first introduced by Castro and Kurepa [7] for the classical Laplace operator in a ball and later used in [17, 18] for quasilinear operators. Note finally that, in the important case when Q and f are monotone increasing for large u, condition (1.7) can be written more simply

$$\limsup_{s\to\infty} Q(ks)\Big(\frac{s^{m-1}}{f(s)}\Big)^{n/m} = \infty. \tag{1.7$'$}$$

Note in particular the assertion of part (ii) of the theorem, that under the condition $\limsup_{s\downarrow 0} f(s) < 0$ there are no ground states of (1.1), *even for the nonradial case.* On the other hand, if f is redefined to be 0 when $u = 0$ (thus making it certainly discontinuous), a radial distributional compact support ground state on $\mathbb{R}^n$ *is* obtained. See the remark preceding Section 3.1.

In part (iii) of Theorem 1, the two cases are distinguished exactly by the convergence or divergence of the integrals in (1.4)–(1.5) *provided that* $|F(s)| \le \Phi(s)$, or in particular if $f(s) \le 0$ for $0 < s \le \eta$. When *neither* (1.4) *nor* (1.5) holds, then we cannot tell which case occurs. It is interesting to note however that there can exist positive ground states of (1.1) even if f is singular near zero.

Our proofs use only standard techniques from the theory of ordinary differential equations, the shooting method and variational identities. In particular, unlike earlier well-known demonstrations of the existence of ground states, we rely neither on critical point theory [6] nor on the Emden-Fowler inversion technique [2, 3]. It is additionally worth emphasizing that the proof of Theorem 1 yields as a byproduct an *a priori estimate* for the supremum norm of ground states in terms of n, m and the nonlinearity f; see Theorem 2 in Section 4.

A case of particular interest is the polynomial function

$$f(s) = -s^p + s^q, \qquad p < q. \tag{1.8}$$

Clearly (1.8) satisfies (H1)-(H2) when $p > -1$, while $\gamma = \infty$. First, for the case when $n > m$ the principal condition $(1.7) - (1.7')$ is satisfied when $q < \sigma$, where σ is the critical Sobolev exponent

$$\sigma = \frac{(m-1)n + m}{n - m}.$$

This shows that (1.7) is essentially a *subcritical assumption* on f for large u. It is also worth remarking that when $q \ge \sigma$, there are no ground states of (1.1) for the function (1.8), see [29, Theorem 3.2 and pages 180-181]. In the case $n \le m$, by (C2), (C3) and $(1.7')$, we see that any powers $-1 < p < q$ are allowed in (1.8); therefore, from Theorem 1 we get the following

Corollary 1. *Let f be as in* (1.8). *Then*

(i) *there exists a radial ground state u of* (1.1) *provided either*

$$n \le m, \quad p > 0 \qquad \text{or} \qquad n > m, \quad 0 < p < q < \sigma;$$

moreover, u is positive for all $r > 0$ if and only if $p \ge m - 1$;

(ii) *there exists a positive radial solution of* (1.2) *for some* $R > 0$ *provided either*

$$n \leq m, \quad -1 < p < m-1 \qquad or \qquad n > m, \quad -1 < p < m-1, \quad q < \sigma.$$

The results of Corollary 1 can be applied to obtain existence of solutions for several problems in physics. For example, when $p = 0$ and $q = 1/2$ in $\mathbb{R}^2$, (1.3) has been suggested as a model in plasma physics for Tokamak equilibria with magnetic islands, see [28] and [21]. Also, as pointed out in [9], when $-1 < p < 0$ and $q = 1$, equation (1.3) is related to the blow-up of self-similar solutions of a singular nonlinear parabolic problem, a model proposed in [26] for studying force-free magnetic fields in a passive medium.

Theorem 1 shows a striking difference between the cases $n \geq m$ and $n < m$. There is also an important difference between the cases $n = m$ and $n > m$. When $n > m$ we have already noted the subcritical requirement $q < \sigma$ for the nonlinearity (1.8).. When $n = m$, however, there is no critical Sobolev exponent σ, and no optimal embedding $W^{1,m}(\mathbb{R}^n) \to L^{\sigma+1}(\mathbb{R}^n)$. Instead, the appropriate embedding is into an Orlicz space [38], and critical growth means exponential growth. For the classical field equation (1.3) such exponential behavior for f was treated, for example, in [5] and [3].

On the other hand, Theorem 1 is not directly satisfactory for exponential growth since condition (1.7) then fails. To include such behavior, especially for the general equation (1.1) when $n = m$, we need a refinement of condition (1.7).

Theorem 1′. *Let hypotheses* (H1) *and* (H2) *hold, and suppose that* $\gamma = \infty$ *and* $n = m$. *Assume moreover that there exist constants* $b > \beta$, $\rho > 0$ *and* $\alpha > b + \rho$ *such that*

$$\frac{\rho^{n-1} F(\alpha - \rho)}{f(\overline{\alpha})} > \Gamma, \tag{1.9}$$

where $\overline{\alpha}$ *is an arbitrary number in* $[\alpha - \rho, \alpha]$ *and* Γ *is a constant depending only on* n *and on the behavior of* f *on the bounded interval* $[0, b]$, *see* (5.3). *Then the conclusions of Theorem* 1 *hold.*

The constant Γ has an extremely simple form in the natural case when $f(s)$ has only a single positive zero, say at $s = a$. Then clearly $a < \beta$ and $\overline{F} = -F(a)$, so we can take

$$\Gamma = \frac{2}{n}\big[(n-1)b\big(2 - \frac{F(a)}{F(b)}\big)\big]^n,$$

or in the important subcase $n = m = 2$ (see footnote 1)

$$\Gamma = b^2\Big(\frac{3}{2} - \frac{F(a)}{F(b)}\Big)^2.$$

In Section 5, as a second major goal of the paper, we describe a class of exponentially growing functions f satisfying (1.9). In particular, behaviors at infinity like $f(s) \sim s^p \exp(\lambda s^q)$ are allowed for $-\infty < p < \infty$, $0 \le q < 1$ and all $\lambda > 0$, and for $q \ge 1$ provided λ lies in some appropriate range. When $m = n = 2$ Atkinson and Peletier [3] and Berestycki, Gallouët and Kavian [5] have obtained existence results for all $\lambda > 0$ and for the range $1 \le q < 2$, and also for various further cases when $q \ge 2$. In Example 3 of Section 5 we comment further on the relation between our work and theirs.

The paper is organized as follows: in the next section we present some preliminary results on the behavior of radial solutions; in Section 3 we prove Theorem 1 and in Section 4 we obtain an important a priori estimate for the supremum norm of ground states. We prove Theorem $1'$ in Section 5, and finally discuss the symmetry and uniqueness of ground states of (1.1) and (1.3) in Section 6.

2. Preliminary results. We maintain the hypotheses (H1)-(H2) without further comment. Observe that a nonnegative, nontrivial radial solution $u = u(r)$ of (1.1) is in fact a solution of the ordinary differential initial value problem

$$\begin{aligned} &(|u'|^{m-2}u')' + \frac{n-1}{r}|u'|^{m-2}u' + f(u) = 0,\\ &u(0) = \alpha > 0, \quad u'(0) = 0 \end{aligned} \tag{2.1}$$

for some initial value α, where for our purposes the dimension n may be considered as any real number greater than 1. The equation (2.1) can be rewritten as

$$(r^{n-1}|u'|^{m-2}u')' + r^{n-1}f(u) = 0, \tag{2.2}$$

or equivalently, with $w = w(r) = |u'(r)|^{m-2}u'(r)$,

$$(r^{n-1}w)' = -r^{n-1}f(u) \tag{2.3}$$

where of course $w(0) = 0$.

Lemma 1.1.1 and Corollary 1.2.5 in [16] show that any non-trivial radial solution of (1.1) or (1.2) has initial value $\alpha > \beta$; therefore we take

$$u(0) = \alpha \in [\beta, \gamma), \tag{2.4}$$

where the case $u(0) = \beta$ is included for convenience in later work.

For definiteness in what follows, we understand that a (classical) solution of (2.1) is a function u which, together with $w = |u'|^{m-2}u'$, is of class C^1 on its domain of definition and satisfies (2.1) there.

Lemma 2.1. *Let* (2.4) *be valid. Then* (2.1) *has a unique (classical) solution* u *in a neighborhood of the origin, which satisfies* $u'(r) < 0$ *for small* $r > 0$.

Proof. Local existence and uniqueness of solutions of the Cauchy problem (2.1) is well-known, see for example [29], [30] and Propositions A1, A4 of [16]. As proved in Lemma 1.1.1 of [16], also

$$w'(0) = -\frac{1}{n}f(u(0)) = -\frac{1}{n}f(\alpha).$$

By (2.4) and assumption (H2) we have $f(\alpha) > 0$. Hence $w'(0) < 0$ so that $u'(r) < 0$ for small $r > 0$. □

Continuation of the solution given by Lemma 2.1 is standard. We denote by $J = (0, R)$, $R \leq \infty$, the *maximal open interval of continuation under the restriction*

$$u > 0, \quad -\infty < u' < 0 \quad \textit{in } J.$$

Since clearly $0 < u < \alpha$ in J, it is also standard that the continuation and the corresponding interval J is uniquely determined.

In the sequel we understand that every solution u *of* (2.1) *is continued exactly to the corresponding maximal domain* J.

Since u is decreasing and positive on J it is obvious that $\lim_{r\uparrow R} u(r)$ exists and is non-negative. We denote this important limit by l.

Now define the energy function

$$E(r) = \frac{m-1}{m}|u'(r)|^m + F(u(r)), \quad r \in J. \tag{2.5}$$

By a straightforward calculation one finds that E is continuously differentiable on J and

$$\frac{dE(r)}{dr} = -\frac{n-1}{r}|u'(r)|^m. \tag{2.6}$$

Hence $E(r)$ is decreasing and moreover bounded below since $F(u)$ is bounded below, see (H2). This shows in particular that $|u'(r)|$ is bounded on J.

More detailed characterizations of solutions are given in the following three lemmas. We assume throughout that (2.4) holds.

Lemma 2.2. *If $R = \infty$ then $\lim_{r\to\infty} u'(r) = 0$.*

Proof. Since $E(r)$ is decreasing and bounded below, it is convergent as $r \to \infty$. Then, by (2.5) and the fact that $F(u(r)) \to F(l)$, we see that $u'(r)$ also approaches a limit, necessarily zero, as $r \to \infty$. □

Lemma 2.3. *If R is finite then either $u(r) \to 0$ or $u'(r) \to 0$ as $r \to R$. In the first case, $u'(R) = \lim_{r\uparrow R} u'(r)$ exists and $u'(R) \leq 0$.*

Proof. Since u' is negative and bounded on J, and R is assumed finite, the only obstacle to continuation on J is for either u or u' to approach zero as $r \uparrow R$. This proves the first part of the lemma.

Next, since E approaches a limit as r tends to R it is clear that also u' approaches a limit. But $u' < 0$ in J and the conclusion follows. □

Lemma 2.4. *$l \in [0, \beta)$.*

Proof. Suppose for contradiction that $l \geq \beta$. Then obviously $u > \beta$ in J, so by (2.3) and (H2)

$$(r^{n-1}w(r))' < 0,$$

that is, $r^{n-1}w(r)$ is decreasing on J.

Now, if R is finite, then by the first part of Lemma 2.3 we have $u'(r) \to 0$ as $r \uparrow R$. In turn, $r^{n-1}w(r)$ approaches 0 as $r \uparrow R$, while also $r^{n-1}w(r)$ takes the value 0 at $r = 0$. But this is absurd since $r^{n-1}w(r)$ is decreasing on J.

If $R = \infty$, then by (2.1) and Lemma 2.2 we get $\lim_{r\to\infty} w'(r) = -f(l)$, where the right hand side is negative by the assumption $l \geq \beta$. This is of course impossible since $w(r) \to 0$ as $r \to \infty$. The proof is complete. □

In the proof of Theorem 1 we will also need the following results, the first of which is an obvious consequence of Lemma 2.4.

Lemma 2.5. *Suppose $b \in (\beta, \gamma)$ and $\alpha \in (b, \gamma)$. Then there exists a unique value $\overline{R} = \overline{R}(\alpha) \in J$ such that $u(\overline{R}) = b$.*

Proposition 2.6. (Continuous dependence on initial data). *Let u be a solution of (2.1) with maximal domain J. Then for any $r_0 \in J$ and $\epsilon > 0$, there exists $\delta > 0$ such that if v is a solution of (2.1) with $|u(0) - v(0)| < \delta$, then $v(r)$ is defined on $[0, r_0]$ and*

$$\sup_{r\in[0,r_0]} (|u(r) - v(r)| + |u'(r) - v'(r)|) < \epsilon.$$

This can be obtained by combining the ideas of Propositions A3 and A4 in [16].

To conclude the section, we state a Pohozaev-type identity due to Ni, Pucci and Serrin, see [29], [30], [33], and also [15].

Proposition 2.7. (Ni-Pucci-Serrin). *Let u be a solution of* (2.1) *with domain J. Let $Q(s) = nmF(s) - (n-m)sf(s)$ and*

$$\begin{aligned} P(r) &= (n-m)r^{n-1}u(r)u'(r)|u'(r)|^{m-2} + (m-1)r^n|u'(r)|^m + mr^nF(u(r)) \\ &= (n-m)r^{n-1}u(r)u'(r)|u'(r)|^{m-2} + mr^nE(r). \end{aligned}$$

Then

$$P(r) = \int_0^r Q(u(t))t^{n-1}dt \qquad \forall r \in J.$$

3. Proof of Theorem 1. Let $\beta \le \alpha < \gamma$ and let u_α be a corresponding solution of (2.1) with maximal domain $J_\alpha = (0, R_\alpha)$, $R_\alpha \le \infty$. Also set $l_\alpha = \lim_{r\uparrow R_\alpha} u_\alpha(r)$; of course $l_\alpha \in [0, \beta)$ by Lemma 2.4. Define the pair of sets, $I^- = \{\alpha \in [\beta,\gamma) : R_\alpha < \infty,\ l_\alpha = 0,\ u'_\alpha(R_\alpha) < 0\}$, $I^+ = \{\alpha \in [\beta,\gamma) : l_\alpha > 0\}$. Clearly I^+ and I^- are disjoint; we shall show

Claim 1. $\beta \in I^+$,

Claim 2. I^+ is open in $[\beta,\gamma)$,

Claim 3. I^- is non-empty,

Claim 4. I^- is open.

Deferring the proof of these claims until subsections 3.1 - 3.3 below, we turn to the demonstration of Theorem 1. First, in view of Claims 1-4 there must be some $\alpha^* \in (\beta,\gamma)$ which is neither in I^+ nor in I^-. We denote the corresponding solution by u^*, with domain $J^* = (0, R^*)$. Then $l^* = 0$ (notation obvious) since α^* is not in I^+. Moreover, since α^* is also not in I^-, either $R^* = \infty$ or R^* *is finite and* ${u^*}'(R^*) = 0$, see Lemma 2.3. In the first case u^* is a positive ground state of (1.1), in the second a solution of (1.2) with $R = R^*$.

When f is regular and $f(0) = 0$, the solution in the second case, when it is extended to all $r > R^*$ by the value 0, becomes a compactly supported ground state of (1.1). The first statement of (i) is proved.

To obtain the remaining part of (i), observe that if (1.4) holds then Proposition 1.3.2 of [16] applies and the ground state u is necessarily positive, while if (1.5) is satisfied then u has compact support in view of Proposition 1.3.1 of [16].

In case (ii), there are no ground states of (1.1), radial or not. This is an immediate consequence of Theorem 2 of [36] together with Remark 2 at the end of the proof of that theorem. Indeed from this result any possible ground state in case (ii) would have compact support, which would violate the differential equation for large values of $|x|$. [If f is *redefined* to be 0 when $u = 0$ (thus making it certainly discontinuous), a distributional compact support ground state on $\mathbb{R}^n$ *is* however obtained. See the following remark.]

Case (iii) is obtained in almost exactly the same way as case (i).

Remark. Under the assumptions of Theorem 1, if (1.1) has no ground states, then (1.2) has a radial solution for some $R > 0$. In this case, let u be such a solution and extend it to $\mathbb{R}^n$ by $u(r) \equiv 0$ for $r \geq R$. The resulting extension (still called u) is obviously in $C^1(\mathbb{R}^n)$. Moreover, if we redefine f to have the value 0 when $u = 0$, then the extension becomes a distributional compactly support ground state of (1.1) provided $\limsup_{u\downarrow 0} f(u) < \infty$.

To see this, let $\phi \in C_0^\infty(\mathbb{R}^n)$ and let B_r be a ball in $\mathbb{R}^n$ with radius $r < R$. Multiplying (1.2) by ϕ, using Lemma 2.1 and integrating by parts, we get

$$\int_{B_r} |Du|^{m-2} Du \cdot D\phi = \int_{B_r} f(u)\phi + \oint_{\partial B_r} |Du|^{m-2} \frac{\partial u}{\partial n} \phi.$$

Let $r \uparrow R$; then the left hand side approaches a limit while the second term on the right approaches 0. Hence the first term on the right side also approaches a limit. Writing $f(u) = [f(u) - M] + M$, where $M = \sup_{0<u<\alpha} f(u)$ and $M < \infty$ since $\limsup_{u\downarrow 0} f(u) < \infty$, it follows easily that $f(u(r)) - M$, a nonpositive function, is in $L^1(0, R)$. Therefore, $f(u(r)) \in L^1(0, R)$ and we get

$$\int_{B_R} |Du|^{m-2} Du \cdot D\phi = \int_{B_R} f(u)\phi.$$

But then, since $u(r) \equiv 0$ outside B_R and $f(0) = 0$, we find finally that

$$\int_{\mathbb{R}^n} |Du|^{m-2} Du \cdot D\phi = \int_{\mathbb{R}^n} f(u)\phi$$

as required.

3.1. Proof of Claims 1 and 2.

Proof of Claim 1. Let u be a solution of (2.1) with $u(0) = \beta$, defined on the corresponding maximal domain $J = (0, R)$. By (2.5) and condition (H2) we have $E(0) = 0$ and, since E is decreasing, $E(r) < 0$ on J. Pick

$r_0 \in J$; then $E(r_0) < 0$ and $E(r) \le E(r_0)$ for all $r \in (r_0, R)$. It follows that $F(u(r)) \le E(r_0) < 0$ on this interval and thus $l > 0$. Hence $\beta \in I^+$. □

Proof of Claim 2. Let $\overline{\alpha} \in I^+$. We denote the corresponding solution by $\overline{u}$ and its domain by $\overline{J} = (0, \overline{R})$. Of course $l = \bar{l} \in (0, \beta)$.

From Lemmas 2.2 and 2.3 we have $\overline{u}'(r) \to 0$ as $r \uparrow \overline{R}$. Hence

$$\lim_{r\uparrow\overline{R}} \overline{E}(r) = F(\bar{l}) < 0.$$

Choose r_0 in $\overline{J}$ so that $\overline{E}(r_0) < 0$. If α is taken sufficiently close to $\overline{\alpha}$ and u denotes the corresponding solution of (2.1) with $u(0) = \alpha$, then applying Proposition 2.6 we can arrange that $(0, r_0] \subset J$, where J is the domain of u, while also $E(r_0) \le \frac{1}{2}\overline{E}(r_0) < 0$. As in the proof of Claim 1, this implies that $\alpha \in I^+$. □

3.2. Proof of Claim 3. Since $F(0) = 0$ and $F(s) > 0$ for $\beta < s < \gamma$, we can define

$$\overline{F} = -\min_{0<s<\beta} F(s) > 0;$$

the minimum of course is negative by condition (H2). To prove Claim 3 we shall use the following crucial result, in the spirit of Lemma 2.1.1 in [16].

Lemma 3.1. *Let $b \in (\beta, \gamma)$ be fixed and, for any $\alpha \in (b, \gamma)$, let $\overline{R} = \overline{R}(\alpha)$ denote the unique value of r when the solution u of* (2.1) *reaches b, see Lemma* 2.5. *Then $\alpha \in I^-$ provided*

$$\overline{R} \ge C(b) \equiv (n-1)\Big(\frac{m}{m-1}\Big)^{\frac{m-1}{m}} \frac{b}{F(b)} \cdot \big(\overline{F} + F(b)\big)^{\frac{m-1}{m}}. \tag{3.1}$$

Lemma 3.1 states that any solution of the initial value problem (2.1) which waits *sufficiently long* before crossing the line $u = b$ will eventually reach the axis $u = 0$ with non-zero slope. Thus in fact it is only solutions which cross the line $u = b$ *before* r reaches the value $C(b)$ which can be candidates for being ground states.

It is exactly this paradoxical situation which makes the existence problem for ground states such a delicate matter.

Proof of Lemma 3.1. Let $\alpha \in (b, \gamma)$, and let u be the solution of (2.1) with $u(0) = \alpha$, defined on the maximal domain $J = (0, R)$, $R \le \infty$. Suppose for contradiction that $\alpha \notin I^-$; then one sees easily from Lemmas 2.2–2.4

that $l \in [0, \beta)$ and $u'(r) \to 0$ as $r \uparrow R$. Let $M = \sup_{[\overline{R},R)} |u'(r)| = |u'(R_1)|$, where $R_1 \in [\overline{R}, R)$; therefore formula (2.5) at $r = \overline{R}$ together with (2.6) integrated on $(\overline{R}, R)$ gives

$$\begin{aligned} F(b) < E(\overline{R}) &= E(R) + (n-1)\int_{\overline{R}}^{R} \frac{|u'(r)|^m}{r} dr \\ &\le (n-1)\frac{M^{m-1}}{\overline{R}} \int_{\overline{R}}^{R} |u'(r)| dr \\ &= (n-1)\frac{M^{m-1}}{\overline{R}} \big(u(\overline{R}) - l\big) \le (n-1)\frac{M^{m-1}}{\overline{R}} b, \end{aligned} \tag{3.2}$$

where $E(R) = \lim_{r\uparrow R} E(r) = F(l) \le 0$ by (H2).

Similarly, (2.5) at $r = R_1$ together with (2.6) integrated on (R_1, R) yields

$$\begin{aligned} \frac{m-1}{m} M^m &= E(R_1) - F(u(R_1)) \\ &\le (n-1)\int_{R_1}^{R} \frac{|u'(r)|^m}{r} dr - F(u(R_1)) \le (n-1)\frac{M^{m-1}}{\overline{R}} b + \overline{F}. \end{aligned} \tag{3.2$'$}$$

By (3.1) and (3.2) we get

$$M^{m-1} > \frac{\overline{R}}{n-1}\frac{F(b)}{b} \ge \Big[\frac{n-1}{\overline{R}}\frac{b}{F(b)}\frac{m}{m-1}(\overline{F} + F(b))\Big]^{m-1},$$

so also

$$\frac{m-1}{m} M > \frac{n-1}{\overline{R}}\frac{b}{F(b)}(\overline{F} + F(b)).$$

Now from (3.2$'$) and these estimates for M, we find that

$$\begin{aligned} \overline{F} &\ge M^{m-1}\Big(\frac{m-1}{m} M - (n-1)\frac{b}{\overline{R}}\Big) \\ &> \frac{\overline{R}}{n-1}\frac{F(b)}{b} \cdot \frac{(n-1)b}{\overline{R}}\Big(\frac{\overline{F} + F(b)}{F(b)} - 1\Big) = \overline{F}, \end{aligned}$$

a contradiction. □

We shall apply this lemma to prove Claim 3 under the three different conditions (C1), (C2) and (C3).

Condition (C1). *Assume* $\gamma < \infty$ *and* (1.6).

Pick an $\epsilon \in (0, \gamma - \beta)$ such that if $\gamma - \epsilon \le s < \gamma$, then

$$\frac{f(s)}{(\gamma - s)^{m-1}} < k_0 + 1,$$

and let $b = \gamma - \epsilon$. For any $\alpha \in (b, \gamma)$ let u be a solution of (2.1) and let $\overline{R} = \overline{R}(\alpha)$ be the unique value of r where $u(r) = b$. Then

$$b \le u(r) < \gamma, \quad u'(r) < 0 \quad \text{and} \quad f(u(r)) > 0 \quad \text{for all} \quad r \in (0, \overline{R}].$$

Using these inequalities and equation (2.3) there holds, for $r \in [0, \overline{R}]$,

$$\begin{aligned} |u'(r)|^{m-1} = |w(r)| &= \frac{1}{r^{n-1}} \int_0^r t^{n-1} f(u(t)) dt \\ &\le \frac{r}{n} \sup_{[0,r]} f(u(t)) \le \frac{r}{n} (k_0 + 1)(\gamma - u(r))^{m-1}. \end{aligned}$$

In turn,

$$|u'(r)| \le c r^{1/(m-1)} (\gamma - u(r)) \le c \overline{R}^{1/(m-1)} \Big(\gamma - \alpha + \int_0^r |u'(t)| dt \Big) \quad \forall r \in [0, \overline{R}]$$

where $c = ((k_0 + 1)/n)^{1/(m-1)}$. Applying Gronwall's inequality, we obtain

$$|u'(r)| \le c \overline{R}^{1/(m-1)} (\gamma - \alpha) \exp(c \overline{R}^{m/(m-1)}) \qquad \forall r \in [0, \overline{R}]$$

which shows that $\overline{R}(\alpha) \to \infty$ as $\alpha \uparrow \gamma$. Indeed, if $\overline{R}(\alpha)$ remains bounded, then the previous inequality implies that $\max_{r \in [0, \overline{R}]} |u'(r)| \to 0$, contradicting

$$\alpha - b = u(0) - u(\overline{R}) \le \overline{R} \max_{r \in [0, \overline{R}]} |u'(r)|.$$

Thus (3.1) is satisfied if α is sufficiently close to γ, and so $I^- \neq \emptyset$.

Condition (C2). *Assume* $\gamma = \infty$ *and* $n < m$.

Fix $b > \beta$, and put $\overline{C} = C(b)$, the constant defined in (3.1). Suppose for contradiction that I^- is empty. Then by Lemma 3.1, for any $\alpha > b$ there holds

$$\overline{R} = \overline{R}(\alpha) < \overline{C}, \tag{3.3}$$

where we recall that $\overline{C}$ is independent of α. Define

$$v(r) = r^{\frac{n-1}{m-1}}|u'(r)|, \qquad r \in J;$$

then by equation (2.2) one has $(v^{m-1})' = r^{n-1}f(u)$, and therefore v is increasing on $[0, \overline{R}]$. Let $V = v(\overline{R})$; then $v(r) \le V$, or equivalently,

$$|u'(r)| \le V r^{-\frac{n-1}{m-1}} \qquad \text{on } [0, \overline{R}].$$

Integrating this inequality over $[0, \overline{R}]$ leads to

$$\alpha - b \le \frac{m-1}{m-n}\overline{R}^{\frac{m-n}{m-1}}V.$$

Combining this with (3.3), there results

$$V \ge (\alpha - b)\frac{m-n}{m-1}\overline{C}^{\frac{n-m}{m-1}}. \tag{3.4}$$

Now we introduce the function

$$D(r) = r^{\widetilde{m}}E(r) = \frac{m-1}{m}(v(r))^m + r^{\widetilde{m}}F(u(r)), \tag{3.5}$$

where $\widetilde{m} = m(n-1)/(m-1)$. Using (2.6), it follows that

$$D'(r) = \widetilde{m}r^{\widetilde{m}-1}F(u(r)). \tag{3.6}$$

Let $r \in (\overline{R}, R)$ and integrate (3.6) on $[\overline{R}, r]$ to obtain

$$\frac{m-1}{m}(v(r))^m = \frac{m-1}{m}V^m + \overline{R}^{\widetilde{m}}F(b) - r^{\widetilde{m}}F(u(r)) + \widetilde{m}\int_{\overline{R}}^{r} t^{\widetilde{m}-1}F(u(t))dt.$$

Define for any constant $a > 0$ $R_a = \min\{\overline{C} + a, R\}$. We assert that $R_a > \overline{R}$. This is obvious if $R_a = R$; otherwise, if $R_a = \overline{C} + a < R$ then $R_a > \overline{R} + a\ (> \overline{R})$ by (3.3). Clearly $0 < u(r) < b$ for $\overline{R} < r < R_a$ and we find with the help of (3.3) that

$$(v(r))^m \ge V^m - \frac{m}{m-1}(\overline{C} + a)^{\widetilde{m}}(F(b) + \overline{F}), \quad \overline{R} < r < R_a.$$

It now follows from (3.4) that if $\overline{R} < r < R_a$, and α is sufficiently large, then

$$(v(r))^m > \left(\overline{C} + a\right)^{\widetilde{m}} \left(\frac{b}{a}\right)^m; \tag{3.7}$$

in turn, $u'(r) < -b/a$. Now, if $R_a = R$, then $u'(R) = 0$ since $\alpha \notin I^-$, contradicting $u'(R) = \lim_{r\uparrow R} u'(r) \le -b/a$. Thus $R_a = \overline{C} + a$, leading to $u'(r) < -b/a$ for $\overline{R} < r < \overline{C} + a$. Integrating from $\overline{R}$ to $R_a = \overline{C} + a$ gives

$$u(R_a) < b - \frac{b}{a}(R_a - \overline{R}) = \frac{b}{a}(\overline{R} - \overline{C}).$$

Since $\overline{R} < \overline{C}$ by the main condition (3.3), we obtain $u(R_a) < 0$, contradicting the fact that $u(r) > 0$ for $0 \le r < R$. This shows that I^- is not empty.

Condition (C3). *Assume* $\gamma = \infty$, $n \ge m$ *and* (1.7).

We may find $b > \beta$ such that $Q(s) \ge 0$ if $s \ge b$. Choose $\alpha > b/k$, where k is the constant specified in condition (C3), and let u be a corresponding solution of (2.1), with domain $J = (0, R)$. Define R_k to be the first (unique by Lemma 2.5) point where u reaches $k\alpha$: clearly $R_k < \overline{R}$. Also define $\overline{\alpha} = \overline{\alpha}(k, \alpha)$ by $f(\overline{\alpha}) = \max_{s\in[k\alpha,\alpha]} f(s)$. In fact, we can take $\overline{\alpha} = k_1\alpha$ for some $k_1 \in [k, 1]$. For any $r \in (0, R_k)$, integration of the identity (2.3) over $[0, r]$ gives

$$r^{n-1}|u'(r)|^{m-2}u'(r) = -\int_0^r t^{n-1} f(u(t))dt \ge -\frac{f(\overline{\alpha})}{n} r^n;$$

thus

$$|u'(r)|^{m-2}u'(r) \ge -\frac{f(\overline{\alpha})}{n} r,$$

or equivalently,

$$u'(r) \ge -\Big(\frac{f(\overline{\alpha})}{n}\Big)^{1/(m-1)} r^{1/(m-1)}.$$

Integrating this over $[0, R_k]$ leads to

$$\alpha(1-k) \le \frac{m-1}{m}\Big(\frac{f(\overline{\alpha})}{n}\Big)^{1/(m-1)} R_k^{m/(m-1)},$$

and therefore

$$R_k \ge \Big(\frac{dn\alpha^{m-1}}{f(\overline{\alpha})}\Big)^{1/m}, \quad \text{where} \quad d = \Big[(1-k)\frac{m}{m-1}\Big]^{m-1}. \tag{3.8}$$

Since $Q(s) \geq 0$ for $s \geq b$, and $Q(s)$ is locally bounded below near $s = 0$ by hypothesis (C3), we can define $\overline{Q} = -\inf_{s>0} Q(s) < \infty$. Also $Q(\beta) = -(n-m)\beta f(\beta) < 0$ by (H2), so in fact $\overline{Q} > 0$.

Now assume for contradiction that I^- is empty. Applying Proposition 2.7, we have for $\overline{R} < r < R$,

$$\begin{aligned} mr^n E(r) &\geq \int_0^r Q(u(t))t^{n-1}dt = \Big(\int_0^{R_k} + \int_{R_k}^{\overline{R}} + \int_{\overline{R}}^r\Big)Q(u(t))t^{n-1}dt \\ &\geq \Big(\int_0^{R_k} + \int_{\overline{R}}^r\Big)Q(u(t))t^{n-1}dt \quad \text{since } Q(u(t)) \geq 0 \text{ for } 0 < t < \overline{R} \\ &\geq \int_0^{R_k} Q(u(t))t^{n-1}dt - \overline{Q}\int_{\overline{R}}^r t^{n-1}dt \\ &\geq Q(k_2\alpha)\frac{R_k^n}{n} - \overline{Q}\frac{r^n}{n}, \text{ where } \ Q(k_2\alpha) = \min_{s\in[k\alpha,\alpha]} Q(s),\ k_2 \in [k,1] \\ &\geq \frac{Q(k_2\alpha)}{n}\Big(\frac{dn\alpha^{m-1}}{f(k_1\alpha)}\Big)^{n/m} - \frac{\overline{Q}}{n}r^n \qquad \text{by (3.8).} \end{aligned}$$

Let $R_a = \min\{\overline{C}+a, R\}$ be as in the previous case. Recall that $R_a \in (\overline{R}, R)$. Now using (1.7) and noting that $R_a \leq \overline{C} + a$, we may fix $\alpha > b/k$ so large that

$$E(r) \geq F(b) + \frac{m-1}{m}\left(\frac{b}{a}\right)^m, \qquad \overline{R} < r < R_a. \tag{3.9}$$

It follows that

$$R_a = \overline{C} + a < R \tag{3.10}$$

for this α; indeed otherwise $R_a = R < \infty$ and $u'(R_a) = 0$, yielding (see (2.5))

$$E(R_a) = F(u(R_a)). \tag{3.11}$$

But $F(u(R_a)) < F(b)$ since $u(R_a) < b$, while $E(R_a) > F(b)$ from (3.9). Hence (3.11) gives a contradiction, and (3.10) is proved.

Finally, by (2.5), (3.9), since $F(s) < F(b)$ when $s < b$, we obtain

$$|u'(r)| > b/a, \quad \overline{R} < r \leq R_a,$$

which, as in the previous case, also leads to $u(R_a) < 0$, contradicting the fact that $u(r) > 0$ for $0 \leq r < R$. Hence I^- must be non-empty, and must in fact contain the value $\alpha > b/k$ for which (3.9) holds, completing the proof of Claim 3. $\square$

3.3. Proof of Claim 4. Let $\alpha \in I^-$ and $\{\alpha_i\}$ be a sequence approaching α as $i \to \infty$. Let u be the solution of (2.1) corresponding to $u(0) = \alpha$ with maximal domain $J = (0, R)$, $R < \infty$, and u_i the solution corresponding to $u_i(0) = \alpha_i$ with maximal domain $J_i = (0, R_i)$, $R_i \le \infty$. Let $E(r)$, defined in (2.5), be the energy function of u; denote by $E_i(r)$ the corresponding energy associated with u_i. Write $d = E(R)/2$; clearly $d > 0$ since $\alpha \in I^-$. By Proposition 2.6 we can choose $r_0 \in (R/2, R)$ such that $2d < E(r_0) < 3d$ and

$$R_i > r_0, \quad d < E_i(r_0) < 4d, \qquad u_i(r_0) < 2u(r_0) < \beta, \tag{3.13}$$

for i sufficiently large. Integrating (2.6) over $[r_0, R_i]$ yields

$$\begin{aligned}
\big|E_i(R_i) - E_i(r_0)\big| &= \Big| \int_{r_0}^{R_i} \frac{n-1}{r} |u_i{}'(r)|^m dr \Big| \\
&= \Big| \int_{u_i(r_0)}^{u_i(R_i)} \frac{n-1}{r} |u_i{}'|^{m-1} du \Big| \qquad (\text{if } R_i = \infty, \text{ then } u_i(R_i) = \lim_{r\to\infty} u_i(r)) \\
&\le \frac{n-1}{r_0} \sup_{r_0 \le r < R_i} |u_i{}'|^{m-1} |u_i(R_i) - u_i(r_0)| \\
&\le \frac{n-1}{r_0} u_i(r_0) \sup_{r_0 \le r < R_i} |u_i{}'|^{m-1} \le \frac{4(n-1)}{R} u(r_0) \sup_{r_0 \le r < R_i} |u_i{}'|^{m-1}.
\end{aligned} \tag{3.14}$$

Moreover, using (2.5) we get, for any $r \in [r_0, R_i)$,

$$\begin{aligned}
\frac{m-1}{m} |u_i{}'(r)|^m &= E_i(r) - F(u_i(r)) \\
&\le E_i(r_0) - F(u_i(r)) \qquad \text{since } E_i \text{ is decreasing} \\
&\le 4d + \overline{F},
\end{aligned}$$

where $\overline{F}$ is given at the beginning of Section 3.2. Therefore,

$$\sup_{r_0 \le r < R_i} |u_i{}'|^{m-1} \le \big(\frac{m}{m-1} (4d + \overline{F}) \big)^{(m-1)/m} \equiv \overline{d},$$

and (3.14) now gives

$$\big|E_i(R_i) - E_i(r_0)\big| \le \frac{4(n-1)}{R} \overline{d} u(r_0).$$

Note that this remains valid if we replace r_0 by any $r \in (r_0, R)$; in particular if $r \uparrow R$ we get $u(r) \to 0$ and thus $E_i(R_i) \geq d$ because of (3.13), which shows that $\alpha_i \in I^-$ for sufficiently large i. Thus I^- is open. $\square$

4. A priori estimates. The proof of Claim 3 in Section 3 gives, as a by product, an a priori estimate for the supremum of a ground state u in terms of n, m and the nonlinearity f. First, in the simple case (C1) it is clear that $u(r) < \gamma$ for all $r \geq 0$.

To treat case (C2), let $b > \beta$ and $\overline{C} = C(b)$ be given by (3.1). Suppose $\alpha > b$ is not in I^-; then (3.3) and (3.4) hold and we can proceed as in the proof of Claim 3 to obtain (3.7), provided that

$$\Big(\frac{m-n}{m-1}\Big)^m \overline{C}^{\frac{m(n-m)}{m-1}}(\alpha - b)^m > \Big[\frac{m}{m-1}(\overline{F} + F(b)) + \Big(\frac{b}{a}\Big)^m\Big](\overline{C} + a)^{\tilde{m}},$$

or equivalently

$$\alpha > \widehat{\alpha} \equiv b + \frac{m-1}{m-n}\, \overline{C}^{\frac{m-n}{m-1}} \Big[\frac{m}{m-1}(\overline{F} + F(b)) + \Big(\frac{b}{a}\Big)^m\Big]^{1/m} (\overline{C} + a)^{\frac{n-1}{m-1}}, \tag{4.1}$$

where $a > 0$ is an arbitrary number. By the final argument in the proof of Claim 3 one then obtains a contradiction. Consequently, when (4.1) is valid, then $\alpha \in I^-$ and in turn a radial ground state u of (1.1) (or a radial solution of (1.2)) has the upper bound $\widehat{\alpha}$.

Next, we consider case (C3). Let b be such that $Q(s) \geq 0$ for $s \geq b$, and $k \in (0, 1]$. Suppose $\alpha > 0$ is not in I^-. Then we can proceed as in the proof of Claim 3 to obtain (3.9), provided that

$$\begin{gathered} Q(k_2\alpha)\Big(\frac{dn\alpha^{m-1}}{f(k_1\alpha)}\Big)^{n/m} \geq \Big[\overline{Q} + nmF(b) + n(m-1)\Big(\frac{b}{a}\Big)^m\Big](\overline{C} + a)^n, \\ \alpha > b/k, \qquad k_1,\ k_2 \in [k, 1], \qquad d = [(1-k)m/(m-1)]^{m-1}, \end{gathered} \tag{4.2}$$

where we recall that $\overline{C} = C(b)$ is given by (3.1) and $a > 0$ is an arbitrary number. Then by the remaining argument of Claim 3 one obtains a contradiction. Consequently, when (4.2) is valid, then $\alpha \in I^-$ and in turn the corresponding solution of (2.1) cannot be a ground state or a solution of (1.2). This proves the following a priori estimates.

Theorem 2. *Let hypotheses* (H1) *and* (H2) *hold. Let u be a radial ground state of* (1.1) *or a radial solution of* (1.2).

(i) *Let $n < m$ and $\gamma = \infty$. Then*

$$u(r) < \widehat{\alpha} \quad \textit{for all} \quad r \geq 0, \tag{4.3}$$

where $\widehat{\alpha}$ is the constant given in (4.1)*, and b is any number larger than β.*

(ii) *Let $n \geq m$, $\gamma = \infty$. Suppose $Q(s)$ is locally bounded below near $s = 0$ and $Q(s) \geq 0$ for $s \geq b$ $(> \beta)$. If* (4.2) *holds for all $\alpha \geq \widehat{\alpha}$, then*

$$u(r) < \widehat{\alpha} \quad \textit{for all} \quad r \geq 0. \tag{4.4}$$

In connection with case (i), consider the example

$$f(u) = -u + u^3 \quad m = 4, \quad n = 3. \tag{4.5}$$

Then $F(u) = -\frac{1}{2}u^2 + \frac{1}{4}u^4, \quad \beta = \sqrt{2}, \overline{F} = \frac{1}{4}$. Choose $b = 2$; then $F(b) = 2$, $\overline{C} = 2 \cdot 3^{3/4}$, see (3.1). Moreover, $\widehat{\alpha} = 2 + 3 \cdot \big(3 + 16/a^4\big)^{1/4}\, \overline{C}^{1/3} (\overline{C} + a)^{2/3}$; taking $a = 2$, which approximately minimizes the right hand side, we get $\widehat{\alpha} \asymp 26.6501$. Thus for example (4.5) any radial ground state u for (1.1) satisfies

$$|u(r)| < 27 \qquad \text{for all } r \geq 0.$$

Note finally that for this example condition (1.5) is satisfied, so that from Theorem 1(i) any ground state necessarily has compact support.

It is more complicated to find explicit a priori estimates for the case (ii). From a practical point of view, even for simple cases, the estimate (4.4) can give an extremely large value for the supremum. Consider the example

$$f(u) = -u + u^3, \quad m = 2, \quad n = 3 \tag{4.6}$$

for which the corresponding ground state is known to be symmetric, positive and unique. Then $F(u)$, β, $\overline{F}$ are as above. We have also $Q(u) = -2u^2 + \frac{1}{2}u^4$, $\overline{Q} = 2$, $d = 2(1 - k)$. Choosing $b = 2$ then gives $F(b) = 2$ and $\overline{C} = 3\sqrt{2}$. Now, taking $k_2 = k$ and $k_1 = 1$, the left side of (4.2) is

$$\big(-2k^2\alpha^2 + \frac{1}{2}k^4\alpha^4\big)\big(\frac{6(1-k)}{-1+\alpha^2}\big)^{3/2},$$

and so inequality (4.2) becomes

$$3\sqrt{6}(1-k)^{3/2}k^4\alpha\cdot\frac{1-4/(k^2\alpha^2)}{\big(1-1/\alpha^2\big)^{3/2}}\geq\big(14+\frac{12}{a^2}\big)\big(3\sqrt{2}+a\big)^3. \tag{4.7}$$

Finally, choosing $a=4/3$, which approximately minimizes the right hand side of (4.7), and taking $k=8/11$, condition (4.7) is satisfied for all $\alpha\geq 12,500$. Thus an a priori estimate for the supremum of the ground state for (4.6) is

$$|u(r)|<12,500 \qquad \text{for all } r\geq 0.$$

The case $n=m=2$, $f(u)=-u+u^3$ is also instructive. Here β, $\overline{F}$, d are the same as for (4.6), while $Q=4F$ and $\overline{Q}=1$. Again let $b=2$, then $F(b)=2$, $\overline{C}=3\sqrt{2}/2$, and (4.2) becomes

$$4(1-k)k^4\alpha^2\cdot\frac{1-2/(k^2\alpha^2)}{1-1/\alpha^2}\big(\frac{4(1-k)}{-1+\alpha^2}\big)\geq\big(9+\frac{8}{a^2}\big)\big(\frac{3\sqrt{2}}{2}+a\big)^2. \tag{4.8}$$

Choosing $a=5/4$ and $k=4/5$, condition (4.8) is satisfied for all $\alpha>22.5$, a much more practical value. Thus an a priori estimate for the supremum of the ground state in this case is

$$|u(r)|<22.5 \qquad \text{for all } r\geq 0.$$

5. The case $n=m$: Exponential and supercritical growth. First we give the proof of Theorem $1'$. We need to show Claims 1-4 of Section 3; clearly only Claim 3 needs further argument.

Proof of Theorem $1'$. As in Case (C3) of Section 4, in order to show that a value α is in I^- it is enough to verify (4.2). In the present case, we have $m=n$, $Q=n^2F$; then setting $(1-k)\alpha=\rho$ (note that the condition $\alpha>b/k$ is thus equivalent to $\alpha>b+\rho$) it is easy to check that (4.2) reduces to

$$\rho^{n-1}\frac{F(\alpha-\rho)}{f(\overline{\alpha})}>\frac{(n-1)^{n-1}}{n^n}\big[\overline{F}+F(b)+\frac{n-1}{n}\big(\frac{b}{a}\big)^n\big](\overline{C}+a)^n, \tag{5.1}$$

where $a>0$ is an arbitrary number and (see (3.1))

$$\overline{C}=C(b)=(n-1)\big(\frac{n}{n-1}\big)^{\frac{n-1}{n}}\cdot\frac{b}{F(b)}\big(\overline{F}+F(b)\big)^{\frac{n-1}{n}}.$$

We take $a^{n+1} = \frac{n-1}{n}\frac{\overline{C}}{\overline{F}+F(b)}b^n$ in (5.1), which minimizes the right hand side; after a fairly long calculation, (5.1) then becomes

$$\rho^{n-1}\frac{F(\alpha-\rho)}{f(\overline{\alpha})} > \frac{(n-1)^n b^n}{n^{n+1}}\big(1+\big[\frac{n\overline{F}}{F(b)}+n\big]^{\frac{n}{n+1}}\big)^{n+1}. \tag{5.2}$$

Applying the elementary inequality

$$(x+y)^p \le 2^{p-1}(x^p+y^p), \quad x,\ y,\ p \ge 0,$$

we get

$$\big(1+\big[\frac{n\overline{F}}{F(b)}+n\big]^{\frac{n}{n+1}}\big)^{\frac{n+1}{n}} \le 2^{\frac{1}{n}}\big(1+n+\frac{n\overline{F}}{F(b)}\big).$$

Thus (5.2), and consequently (4.2), follows if

$$\rho^{n-1}\frac{F(\alpha-\rho)}{f(\overline{\alpha})} > \frac{2}{n}\big[(n-1)b\big(\frac{\overline{F}}{F(b)}+1+\frac{1}{n}\big)\big]^n,$$

which is just (1.9) if we define

$$\Gamma = \frac{2}{n}\big[(n-1)b\big(\frac{\overline{F}}{F(b)}+2\big)\big]^n, \quad \overline{F} = -\min_{0<s<\beta} F(s). \tag{5.3}$$

Theorem 1′ is proved.[1] □

It is worth remarking that the left side of (1.9) depends only on the behavior of f for large α, and the right side only on the behavior of f on $(0, b]$.

We shall now apply Theorem 1′ to the case of nonlinearities f with exponential growth as $u \to \infty$. Suppose that $\lambda \ge 0$, $q \ge 0$, and

$$f(s) = \omega(s)\exp(\lambda s^q) \quad \text{for} \quad s > \tau, \tag{5.4}$$

where $\omega(s)$ is a Lipschitz continuous function for $s > \tau$, with

$$\omega_1 s^{p_1} \le \omega(s) \le \omega_2 s^{p_2}, \quad \omega_1 > 0, \quad \omega_2 > 0, \quad p_1 \le p_2. \tag{5.5}$$

Then the following result holds.

[1] The additive constant 2 in (5.3) can be replaced by $1 + 1/n$, see the previous display line. We use 2 only for greater simplicity.

Theorem 3.1. *Assume that* (H1)-(H2) *and* (5.4)-(5.5) *are satisfied and that* $n = m > 1$, $\gamma = \infty$. *Then the conclusions of Theorem* 1 *hold if* $0 \le q < 1 - (p_2 - p_1)/n$.

Proof. Without loss of generality we may assume $\tau \ge \beta$. For any $s > \tau$ and $q > 0$, let $\rho = \rho(s) = s^{1-q}$. Then for any $s > \tau + \rho$ and $\overline{s} \in [s-\rho, s]$,

$$\rho^{n-1}(s)\frac{F(s-\rho)}{f(\overline{s})} \ge \frac{\omega_1 s^{(n-1)(1-q)}}{\omega_2 s^{p_2}\exp(\lambda s^q)}\int_\tau^{s-s^{1-q}} t^{p_1}\exp(\lambda t^q)dt.$$

By L'Hopital's rule,

$$\begin{aligned}
&\lim_{s\to\infty}\rho^{n-1}(s)\frac{F(s-\rho)}{f(\overline{s})} \\
&\ge \lim_{s\to\infty}\Big(\frac{\omega_1}{\omega_2}\frac{s^{(n-1)(1-q)}(s-s^{1-q})^{p_1}}{(p_2-(n-1)(1-q))s^{p_2-1}+\lambda q s^{p_2+q-1}} \\
&\times \exp\big[\lambda(s-s^{1-q})^q - \lambda s^q\big]\Big) \\
&= \frac{\omega_1\exp(-\lambda q)}{\lambda q\omega_2}\cdot\lim_{s\to\infty}\frac{(1-s^{-q})^{p_1}}{s^{nq+p_2-p_1-n}} = \infty,
\end{aligned}$$

since $nq + p_2 - p_1 - n < 0$. When $q = 0$, we take $\rho = s/2$ and repeat the above computation to get the same result. Now if α is chosen sufficiently large, then (1.9) is obviously satisfied and the proof is completed. □

If $q = 1 - (p_2 - p_1)/n$, the limit above is finite and the verification of (1.9) becomes more involved. We give one theorem and one example for this case; for clarity, we shall assume $p_1 = p_2$, though more general cases can be discussed similarly.

Theorem 3.2. *Assume that* (H1)-(H2) *and* (5.4)-(5.5) *are satisfied, and that* $n = m > 1$, $\gamma = \infty$. *If* $p_1 = p_2$ *and* $q = 1$, *then the conclusions of Theorem* 1 *hold if there exists* $b > \beta$ *such that*

$$\lambda b\Big(\frac{\overline{F}}{F(b)} + 2\Big) < \frac{1}{e}\Big(\frac{e}{2}\frac{n}{n-1}\frac{\omega_1}{\omega_2}\Big)^{1/n}. \tag{5.6}$$

Proof. Let $\rho = (n-1)/\lambda$. As in the proof of Theorem 3.1, we get

$$\lim_{s\to\infty}\rho^{n-1}\frac{F(s-\rho)}{f(\overline{s})} \ge \frac{\omega_1}{\omega_2}\Big(\frac{n-1}{e}\Big)^{n-1}\frac{1}{\lambda^n}.$$

Using (5.6) we see that this limit is larger than Γ, see (5.3), and thus (1.9) is satisfied for α sufficiently large. $\square$

Example 1. To illustrate Theorem 3.2 in a more specific way, consider the nonlinearity

$$f(s) = e^{\lambda s} - 1 - (\lambda + \mu)s, \tag{5.7}$$

where $\lambda > 0$ and $\mu > 0$. It follows that

$$F(s) = \frac{1}{\lambda}\big(e^{\lambda s} - 1 - \lambda s - \frac{\lambda^2}{2}s^2\big) - \frac{\mu}{2}s^2 \geq \frac{\lambda^2}{6}s^3 - \frac{\mu}{2}s^2.$$

The last quantity takes a minimum value at $s = 2\mu/\lambda^2$, and thus

$$\overline{F} \leq |F(2\mu/\lambda^2)| = \frac{2\mu^3}{3\lambda^4}.$$

Choose $b = 4\mu/\lambda^2$. We find $F(b) \geq \frac{8\mu^3}{3\lambda^4}$, which leads to $\overline{F}/F(b) \leq 1/4$, a constant independent of λ, μ. Thus

$$\lambda b\big(\frac{\overline{F}}{F(b)} + 2\big) \leq \frac{9\mu}{\lambda}.$$

To apply Theorem 3.2, we write $\omega(s) = 1 - [1 + (\lambda+\mu)s]/e^{\lambda s}$, and observe that correspondingly $p_1 = p_2$, $\omega_1 \leq \omega(s) \leq \omega_2 = 1$ and $\omega_1/\omega_2 \to 1$ as $s \to \infty$. Obviously, (5.6) follows if

$$\frac{\lambda}{\mu} > 9e\big(\frac{2}{e}\frac{n-1}{n}\big)^{1/n} = \Lambda_1(n). \tag{5.8}$$

For instance, if $\mu = 1$ then (5.8) is satisfied if $\lambda > \Lambda_1(n)$, or if $\lambda = 1$ then $\mu < \Lambda_1^{-1}(n)$. We mention that $\Lambda_1(2) = 9\sqrt{e}$ and $\Lambda_1(n) \to 9e$ as $n \to \infty$. It is easy to show that $\Lambda_1(n)$ is an increasing function of n, so that in fact we can use $\Lambda_1 = 9e$ for all $n > 1$.

The following result avoids the restriction $q \leq 1 - (p_2 - p_1)/n$, but at the expense of more subtle considerations concerning the behavior of $f(s)$.

Theorem 3.3. *Assume that* (H1)-(H2) *and* (5.4)-(5.5) *are satisfied and that* $n = m > 1$, $\gamma = \infty$. *Then the statement of Theorem* 1 *holds if there is a* $b > \beta$ *such that*

$$b\big(\frac{\overline{F}}{F(b)} + 2\big) \leq \Theta, \tag{5.9}$$

where Θ *is a constant depending only on* n *and the parameters in* (5.4)-(5.5).

Proof. We choose $\alpha = \tau + 1$, $\rho = 1/2$ in (1.9). Then

$$\rho^{n-1}\frac{F(\alpha-\rho)}{f(\overline{\alpha})} \geq \frac{1}{2^{n-1}}\frac{\omega_1}{\omega_2}\frac{\int_\tau^{\tau+1/2} t^{p_1}\exp(\lambda t^q)dt}{(\tau+1)^{p_2}\exp(\lambda(\tau+1)^q)}$$
$$> \frac{1}{2^n}\frac{\omega_1}{\omega_2}\frac{\tau^{p_1}}{(\tau+1)^{p_2}}\exp(-\lambda[(\tau+1)^q - \tau^q]) = \overline{\Gamma}.$$

Thus (1.9) is satisfied if we choose

$$\Theta = \frac{1}{n-1}\big(\frac{n\overline{\Gamma}}{2}\big)^{1/n}. \qquad (5.10)$$

This completes the proof. □

It should be noted that the solutions in Theorem 1 which are obtained in this way have the a priori bound $|u(r)| < \tau + 1$.

Example 2. Consider the nonlinearity

$$f(s) = e^{\lambda s^q} - 1 - \mu s, \quad \lambda > 0, \quad \mu > 0, \quad q > 1. \qquad (5.11)$$

By rescaling we can first reduce to the case $\lambda = 1$. Then clearly $f(s) \geq s^q - \mu s$; thus

$$F(s) \geq -\frac{1}{2}\mu s^2\big(1 - \frac{2}{(q+1)\mu}s^{q-1}\big).$$

The last quantity takes a minimum value at $\overline{s} = \mu^{1/(q-1)}$, yielding

$$\overline{F} \leq \frac{q-1}{2(q+1)}\mu\overline{s}^2.$$

Choose $b = (q\mu)^{1/(q-1)} = q^{1/(q-1)}\overline{s}$. We find

$$F(b) \geq \frac{q-1}{2(q+1)}\mu b^2 = \frac{q-1}{2(q+1)}\mu\overline{s}^2 q^{2/(q-1)}.$$

Combining the last two inequalities gives

$$b\big(\frac{\overline{F}}{F(b)} + 2\big) \leq \big(2q^{\frac{1}{q-1}} + q^{-\frac{1}{q-1}}\big)\mu^{1/(q-1)}.$$

Using the previous elementary inequality, the right hand side can, for simplicity, be replaced by the larger quantity $4(\mu q)^{1/(q-1)}$.

To apply Theorem 3.3, we write (5.11) in the form of (5.4) with $\lambda = 1$, $p_1 = p_2$, and observe that $\frac{1}{2} \leq \omega(s) = 1 - (1+\mu s)/e^{s^q} \leq 1$, provided $\mu \leq 1$ and $s \geq 2$. Hence we can take $\omega_1 = 1/2$, $\omega_2 = 1$, $p_1 = p_2 = 0$ and $\tau = 2$ in (5.5). For $\rho = 1/2$ we then find (see the proof of Theorem 3.3)

$$\overline{\Gamma} = \frac{1}{2^{n+1}} \exp(2^q - 3^q).$$

In view of (5.10), condition (5.9) is now satisfied if

$$(\mu q)^{1/(q-1)} \leq \frac{1}{8(n-1)} \left(\frac{n}{4} \exp(2^q - 3^q)\right)^{1/n}.$$

Returning to the unscaled nonlinearity (5.11), we see that the conclusions of Theorem 1 now hold if

$$\frac{\mu}{\lambda^{1/q}} \leq \min\left\{1,\ \frac{1}{q}\Big(\frac{1}{8(n-1)}\Big)^{q-1}\Big(\frac{n}{4}\exp(2^q - 3^q)\Big)^{(q-1)/n}\right\}.$$

Since the function (5.11) is logarithmically convex for large s, this example is covered (when $n = m = 2$) by the results of [3]; also for $1 < q < 2$ they require no restrictions on λ and μ beyond λ, $\mu > 0$. See, however, the following Example 3.

Example 3. Using the procedures illustrated in Examples 1 and 2 we can easily treat other more complicated cases, e.g.,

$$f(s) = s^p(e^{\lambda s} - 1) - (\lambda + \mu)s^t \tag{5.12}$$

when $p + 1 = t > -1$, and

$$f(s) = s^p(e^{\lambda s^q} - 1) - \mu s^t \tag{5.13}$$

when $p + q > t > -1$.

The work of [5], when $n = m = 2$, applies to (5.12) only when $p = 0$, $t = 1$, and to (5.13) only when $t = 1$, $0 \leq q < 2$, $p > 1 - q$. The results of [3] do not apply to (5.12) because f is not logarithmically convex.

If the term $e^{\lambda s^q}$ in (5.13) be replaced by $e^{\lambda s^q} + \sin(e^{2\lambda s^q})$ when $s \geq (\frac{1}{\lambda}\log k\pi)^{1/q}$ and k is a large integer. Then, exactly as in Example 2, the conclusions of Theorem 1 continue to hold whenever $\mu/\lambda^{1/q}$ is suitably small.

On the other hand, the function $e^{\lambda s^q} + \sin(e^{2\lambda s^q})$ is not logarithmically convex for any q, so that the results of [3] no longer apply.

6. Comments. We add some comments here concerning whether ground states of (1.1) are necessarily radial, and whether radial ground states are themselves unique.

The classical paper of Gidas, Ni, Nirenberg [20] (see also [24] and [25]) showed that positive ground states of (1.3) are necessarily radially symmetric about some origin O, provided that f is of class $C^{1+\alpha}$ in $[0,\epsilon)$, with $f(0) = 0$, $f'(0) < 0$. Moreover, by virtue of the strong maximum principle the a priori condition of positivity in this result is in fact automatic.

Symmetry without recourse to the assumption $f'(0) < 0$ is more delicate. For simplicity, we shall assume that $f(0) = 0$ and that f is a decreasing function on some interval $(0,\,\epsilon)$. Then, if

$$\int_0 \frac{ds}{|F(s)|^{1/2}} = \infty, \tag{6.1}$$

ground states of (1.3) indeed remain positive and symmetric, see [37]. On the other hand, if (6.1) fails, then ground states necessarily have compact support, as shown in [36]. For this case, it was observed by [22] and [16] that two or more compact support ground states, with translated origins and disjoint supports, will still constitute a ground state, which is clearly not radial: thus, in such cases equation (1.3) (and even (1.1)) can have denumerable many non-radial ground states. To restore symmetry one can assume additionally that the open support of the ground state is *connected*, see [37].

Turning to equation (1.1), the results are more subtle. First, if (1.4) holds and $1 < m \le 2$, one can again assert that ground states are symmetric, see [13]. When $m > 2$, or if (1.4) fails, symmetry is known only under the fairly strong condition that the ground state in question has only a single critical point, see [37], [4].

For the study of uniqueness of radial ground states of (1.1) for the model nonlinearity (1.8) and also for other types of nonlinearities, the reader is referred to [16], [32], [34] and references therein. In the last reference, uniqueness is established both for problems (1.1) and (1.2) and whether or not $f(0) = 0$. The results here cover in particular the examples discussed at the end of Section 4. Finally, for the case $m = n > 1$ uniqueness holds for the model exponential nonlinearity (5.7), see [35].

Throughout the paper we have considered only ground states for the entire space $\mathbb{R}^n$, and solutions of the Dirichlet-Neumann problem (1.2). The related Dirichlet problem for (1.1) or (1.3) in a bounded domain is of course important in itself, both when $f(0) = 0$ and $f(0) \neq 0$. For these problems the papers [1, 7, 12, 8, 19, 23, 27] are particular relevant to the results here.

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