EXISTENCE AND MULTIPLICITY RESULTS FOR QUASILINEAR ELLIPTIC DIFFERENTIAL SYSTEMS

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ABSTRACT. We use a nonsmooth critical point theory to prove existence results for a variational system of quasilinear elliptic equations in both the sublinear and superlinear cases. We extend a technique of Bartsch to obtain multiplicity results when the system is invariant under the action of a compact Lie group. The problem is rather different from its scalar version, because a suitable condition on the coefficients of the system seems to be necessary in order to prove the convergence of the Palais-Smale sequences. Such condition is in some sense a restriction to the "distance" between the quasilinear operator and a semilinear one.

1. INTRODUCTION

We consider the following kind of systems of $m \geq 1$ quasilinear elliptic equations in an open set (not necessarily bounded) $\Omega \subset \mathbb{R}^n$, $n \geq 3$:

$$-\sum_{ij} D_j(a_{ij}(x,u)D_i u_k) + \frac{1}{2}\sum_{ijl} \frac{\partial a_{ij}}{\partial u_k}(x,u)D_i u_l D_j u_l = \frac{\partial F}{\partial u_k}(x,u)$$

where $u : \Omega \to \mathbb{R}^m$, the indices i, j run from 1 to n, the indices k, l run from 1 to m and the assumptions on a_{ij} and F are given in next section. In the sequel the sum over repeated indices is understood and we assume a more concise vectorial notation by setting $\nabla = \left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m}\right\}$ and denoting the scalar product in $\mathbb{R}^m$ by $\langle\cdot,\cdot\rangle$; the system takes the form

$$-D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\nabla a_{ij}(x,u)\langle D_i u, D_j u\rangle = \nabla F(x,u). \tag{1.1}$$

In order to prove the existence of weak solutions of (1.1) in a suitable functional space E, we look for critical points of the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \frac{1}{2}a_{ij}(x,u)\langle D_i u, D_j u\rangle - F(x,u); \tag{1.2}$$

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in fact this functional is not locally Lipschitz continuous if the coefficients a_{ij} depend on u; however, as pointed out in [1, 8], the Gâteaux-derivative of J exists in the smooth directions, i.e. it is possible to evaluate

$$J'(u)[\phi] = \int_\Omega a_{ij}(x,u)\langle D_i u, D_j\phi\rangle + \frac{1}{2}\langle\nabla a_{ij}(x,u),\phi\rangle\langle D_i u, D_j u\rangle - \langle\nabla F(x,u),\phi\rangle$$

for all $u \in E$ and $\phi \in C_c^\infty(\Omega,\mathbb{R}^m)$. By using the nonsmooth critical point theory developed in [12, 14] it is possible to define critical points in a generalized sense: according to this theory, a critical point u of J satisfies $J'(u)[\phi] = 0$ for all smooth ϕ with compact support, hence it solves (1.1) in distributional sense and, a posteriori, it is a weak solution.

The study of this system is motivated by its applications to some problems in differential geometry. The following example has been pointed out in [20]: let M be a Riemannian manifold, let $s : M \to \mathbb{R}^m$ and consider a Lagrangian integral

$$E(s) = \int_M L(x, s(x), ds(x))d\mu_M$$

where

$$L(x, s(x), ds(x)) = g(x)G_{kl}^{ij}(x, s(x))\frac{\partial s^k(x)}{\partial x^i}\frac{\partial s^l(x)}{\partial x^j};$$

then the functional J looks like $E(s)$ in local coordinates. More generally, the functional J may represent some Lagrangian integral for functions defined on a manifold whose metric tensor depends on the function itself. See also [16, 17] for more examples and details.

The nonsmooth critical point theory has been widely used for the study of scalar equations of the kind of (1.1), see e.g. [2, 8, 9, 11]; this class of equations has also been studied in [1, 18, 21] with different approaches. To our knowledge, systems have only been considered in [19], where a nonlinear eigenvalue problem in a bounded set is treated. The study of a system is more delicate than the study of a single equation; in particular the verification of the generalized Palais-Smale condition cannot be carried on by means of the same arguments used for an equation. Since we assume the Nemitsky operator to be compact, the problem does not arise from a lack of some compact embedding, but from a strong coupling of the equations in (1.1) due to the presence of a quasilinear operator. For this reason, (1.1) is also more difficult to study than a semilinear system; more precisely, in the semilinear version of this problem the coupling of the equations is only given by the Nemitsky operator, while in system (1.1) the terms a_{ij} as well depend on u and therefore the coupling is stronger. Such a coupling is controlled by the gradient ∇a_{ij} which is precisely the term responsible of the loss of smoothness of the functional. Moreover, this vector measures the "change of coercivity" of the quasilinear differential operator as well, hence one does not expect the behavior of such an operator to be too different from a semilinear one if ∇a_{ij} is small in some sense. Therefore, one can try to recover existence and multiplicity results similar to those well known for the semilinear operators by assuming that ∇a_{ij} is small, but even in this case standard methods do not apply directly and to prove the Palais-Smale condition we introduce a new kind of test functions which are strictly related to the growth of ellipticity of the quasilinear operator. On the other hand, when the functional J admits a mountain pass geometry, the Palais-Smale condition may also be proved by a variant of the method introduced in [19] which we extend to unbounded domains: this method too requires ∇a_{ij} to be small, but in a quite different fashion.

Multiplicity results are obtained when the system is invariant under the action of a compact Lie group: the smooth case is widely treated in the book of Bartsch [3]. We extend some results to our nonsmooth framework and we apply them to the system (1.1) in both the sublinear and superlinear case; more precisely, either we give a lower bound to the number of solutions or we prove the existence of infinitely many solutions, depending on the assumptions we take on the system. The standard multiplicity result of Bartsch (see Theorem 2.25 in [3]) comes as a corollary.

This paper is organized as follows: in Section 2 we state our main existence results in the cases where the term $\nabla F(x, \cdot)$ in (1.1) is either sublinear or superlinear. In Section 3 we state multiplicity results for the system under the additional assumption that it is invariant with respect to the action of a compact Lie group. In Section 4 we briefly recall some definitions of nonsmooth critical point theory and state the abstract version of the theorems we use; in particular we give the nonsmooth version of a multiplicity theorem of Bartsch. In Sections 5 and 6 we prove the boundedness and convergence of Palais-Smale sequences. In Sections 7 and 8 we give the proofs of our results. Finally in the Appendix we give some examples and remarks.

2. Existence of a solution

Denote by $\mathcal{D} = \mathcal{D}^{1,2}(\Omega, \mathbb{R}^m)$ the closure of $C_c^\infty(\Omega)$ (the space of smooth vector functions with compact support in Ω) with respect to the norm induced by the scalar product $(\psi, \phi) = \int_\Omega \langle D_i\psi, D_i\phi\rangle$. If Ω is bounded, then $\mathcal{D} = H_0^1(\Omega, \mathbb{R}^m)$.

The following condition (A1) is standard in this kind of problems and it is assumed throughout the paper: the matrix $[a_{ij}(x, s)]$ satisfies an ellipticity property and the matrix $[\langle s, \nabla a_{ij}(x, s)\rangle]$ is semipositive definite. More precisely:

(A1) The matrix $[a_{ij}(x, s)]$ satisfies assumption (A1) if

$$\begin{cases} a_{ij} \equiv a_{ji} \\ a_{ij}(x, s) \in L^\infty(\Omega \times \mathbb{R}^m, \mathbb{R}) \\ \nabla a_{ij}(x, s) \in L^\infty(\Omega \times \mathbb{R}^m, \mathbb{R}^m) \\ a_{ij}(x, \cdot) \in C^1(\mathbb{R}^m) \text{ for a.e. } x \in \Omega \\ \lim\limits_{|s|\to\infty} a_{ij}(x, s) = A_{ij}(x) \text{ for a.e. } x \in \Omega \end{cases} \tag{2.1}$$

and there exists $\nu > 0$ such that for a.e. $x \in \Omega$, all $s \in \mathbb{R}^m$ and all $\xi \in \mathbb{R}^n$

$$a_{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2 \tag{2.2}$$

and

$$\langle s, \nabla a_{ij}(x,s)\rangle \xi_i\xi_j \geq 0. \tag{2.3}$$

The following assumption (A2) is the first kind of control required on the matrix $[a_{ij}(x,s)]$ and it can be used for both the superlinear and sublinear cases. In Theorem 2.3 below we replace it with another assumption, which is simpler (but not weaker) and only works for the superlinear case.

(A2) Assumption (A2) holds if there exist $K > 0$ and a function $\psi : [0,+\infty) \to [0,+\infty)$ continuously differentiable almost everywhere and satisfying

$$\begin{cases} \text{(i) } \psi(0) = 0 \text{ and } \lim\limits_{t\to+\infty} \psi(t) = K \\ \text{(ii) } \psi'(t) \geq 0 \text{ for a.e. } t \in [0,+\infty) \\ \text{(iii) } \psi' \text{ is bounded and non-increasing} \\ \text{(iv) } \sum\limits_{k=1}^{m} \left|\frac{\partial}{\partial s_k} a_{ij}(x,s)\xi_i\xi_j\right| \leq 2e^{-4K}\psi'(|s|)a_{ij}(x,s)\xi_i\xi_j \text{ for all } s \in \mathbb{R}^m \\ \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and for a.e. } x \in \Omega. \end{cases} \tag{2.4}$$

In some sense, ψ is a measure of the growth of ellipticity of the differential operator; we assume that such growth is "not too large" and we refer to the appendix for some examples.

(F0) The function $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies assumption (F0) if it is measurable with respect to the first n variables and C^1 with respect to the other ones; furthermore

$$F(x,0) = 0 \text{ for a.e. } x \in \Omega.$$

We first consider the sublinear case.

(F1) The function F satisfies (F1) if $F(x,s) = b(x)|s|^2 + F_1(x,s)$ where

$$b \in L^{\frac{n}{2}}(\Omega,\mathbb{R}) \tag{2.5}$$

and F_1 satisfies the following: there exist $\alpha \in L^{\frac{2n}{n+2}}(\Omega,\mathbb{R})$ and $\beta \in L^{\frac{n}{2}}(\Omega,\mathbb{R})$ such that

$$|\nabla F_1(x,s)| \leq \alpha(x) + \beta(x)|s| \text{ for all } s \in \mathbb{R}^m \text{ and for a.e. } x \in \Omega, \tag{2.6}$$

$$\lim_{|s|\to\infty} \frac{\nabla F_1(x,s)}{|s|} = 0 \text{ uniformly w.r.t. } x \in \Omega, \tag{2.7}$$

$$F_1(x,s) \to +\infty \text{ if } |s| \to \infty \text{ for a.e. } x \in \Omega, \tag{2.8}$$

$$2F_1(x,s) - \langle s, \nabla F_1(x,s)\rangle \to +\infty \text{ if } |s| \to \infty \text{ for a.e. } x \in \Omega, \tag{2.9}$$

$$2F_1(x,s) - \langle s, \nabla F_1(x,s)\rangle \geq 0 \text{ for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}^m, \tag{2.10}$$

$$F_1(x,s) \geq 0 \text{ for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}^m. \tag{2.11}$$

Under the above assumptions we prove:

Theorem 2.1. *Assume (A1), (A2), (F0) and (F1). Then (1.1) admits a weak solution $u \in \mathcal{D}$.*

Remark 2.1. If $\nabla F_1(x,0) = 0$ it is not possible to exclude that the solution obtained with the previous theorem is trivial, but a nontrivial solution could be obtained by using the Morse theory as in [11].

Next we consider the case where the functional has a mountain pass geometry.

(F2) We say that the function F satisfies (F2) if $F(x,s) = F_2(x,s)$ and there exists $p \in (2, 2^*)$ such that

$$0 \leq pF_2(x,s) \leq \langle s, \nabla F_2(x,s)\rangle \qquad \text{for all } s \in \mathbb{R}^m \text{ and for a.e. } x \in \Omega, \tag{2.12}$$

$F_2(x,s)\not\equiv 0$ and there exist $q \in (2,2^*)$, $\delta \in (2,q)$, $\alpha \in L^{\frac{2n}{2n+(2-n)\delta}}(\Omega)$ and $\beta \in L^{\frac{2n}{2n+(2-n)q}}(\Omega)$ such that

$$|\nabla F_2(x,s)| \leq \alpha(x)|s|^{\delta-1} + \beta(x)|s|^{q-1} \quad \text{for all } s \in \mathbb{R}^m \text{ and for a.e. } x \in \Omega. \tag{2.13}$$

The following assumption is standard for superlinear problems: it relates the properties of the matrix $[a_{ij}]$ with the function F_2 and it will be used to prove that the Palais-Smale sequences are bounded.

(AF) We say that assumption (AF) is satisfied if there exists $\gamma \in (0, p-2)$ such that (p as in (F2))

$$\langle s, \nabla a_{ij}(x,s)\rangle \xi_i\xi_j \leq \gamma a_{ij}(x,s)\xi_i\xi_j \quad \text{for a.e. } x \in \Omega, \text{ for all } s \in \mathbb{R}^m \text{ and } \xi \in \mathbb{R}^n\ . \tag{2.14}$$

Theorem 2.2. *Assume (A1), (A2), (AF), (F0) and (F2). Then (1.1) admits a nontrivial weak solution $u \in \mathcal{D}$.*

We also prove a similar existence result under different assumptions; we first replace (F2) with

(F3) We say that the function F satisfies (F3) if $F(x,s) = F_2(x,s)$, F_2 is not trivial, there exists $p \in (2, 2^*)$ such that (2.12) holds and there exist $q \in (2, 2^*)$ and $\beta \in L^{\frac{2n}{2n+(2-n)q}}(\Omega) \cap L^\infty(\Omega)$ such that

$$|\nabla F_2(x,s)| \le \beta(x)|s|^{q-1} \quad \text{for all } s \in \mathbb{R}^m \text{ and for a.e. } x \in \Omega. \tag{2.15}$$

Then it is possible to replace (A2) with a different assumption on $[\nabla a_{ij}]$ (see [19]):

($\mathbf{A}_L$) The matrix $[\nabla a_{ij}]$ satisfies (A_L) if $|\nabla a_{ij}(x,s)| \le L$ for all $s \in \mathbb{R}^m$ and a.e. $x \in \Omega$.

Our last existence result reads:

Theorem 2.3. *Assume (A1), (AF), (F0), (F3). There exists a constant* $L > 0$ *such that if* $[\nabla a_{ij}]$ *satisfies* (A_L)*, then (1.1) admits a nontrivial weak solution* $u \in \mathcal{D} \cap L^\infty(\Omega)$.

3. Multiplicity results

In this section we consider the case when equation (1.1) is invariant with respect to an action of some compact Lie group. In order to proceed we recall some definitions and known results from representation theory. More information can be found in [3, 6].

Definition 3.1. *Let G be a compact Lie group; G is said to be* solvable *if there exists a sequence $G_0 \subset G_1 \subset \ldots \subset G_r = G$ of subgroups of G such that G_0 is a maximal torus of G, G_{i-1} is a normal subgroup of G_i and $G_i/G_{i-1} \cong \mathbf{Z}/p_i$, $1 \le i \le r$. Here the p_i's are prime numbers.*

Remark 3.1. If G is abelian, then G is isomorphic to the product of a torus with a finite abelian group [6, Corollary I.3.7]. In particular, all abelian compact Lie groups are solvable.

Definition 3.2. *Fix a compact Lie group G. A Hilbert space $(E, \langle\cdot,\cdot\rangle)$ is called a $G-$Hilbert space if there exists a continuous action*

$$G \times E \to E, (g,x) \mapsto gx$$

preserving the scalar product, i.e. $\langle gx, gy\rangle = \langle x, y\rangle$ for all $x, y \in E$ and all $g \in G$.

Definition 3.3. *Let $E, \tilde{E}$ be two $G-$Hilbert spaces. A subset B of E is said to be* invariant *if $gB \subset B$ for all $g \in G$. A functional $I : E \to \mathbb{R}$ is* invariant *if $I(gx) = I(x)$ for all g, x, and a function $f : E \to \tilde{E}$ is* equivariant *if $f(gx) = gf(x)$ for all g, x.*

Definition 3.4. *The space* $E^G := \{x \in E : gx = x \text{ for all } g \in G\}$ *is called the* fixed point space *of (the representation of)* G*, and the* orbit *of* x *is defined by* $\mathcal{O}_G(x) := \{gx : g \in G\}$*. We say that* $x, y \in E$ *are geometrically distinct if* $y \notin \mathcal{O}_G(x)$*.*

Definition 3.5. *Let* V *be a finite-dimensional representation space of* G*.* V *is called* admissible *if for each open, bounded and invariant neighborhood* $\mathcal{U}$ *of 0 in* V^k $(k \geq 1)$ *and each equivariant map* $f : \overline{\mathcal{U}} \to V^{k-1}$*,* $f^{-1}(0) \cap \partial\mathcal{U} \neq \emptyset$*. The corresponding representation* ρ *will also be called* admissible*.*

It is known (see [3, Theorem 3.7]) that V is admissible if and only if there exist subgroups $K \subset H$ of G such that K is normal in H, H/K is solvable, $V^K \neq 0$ and $V^H = 0$; moreover, if G is solvable, then any finite-dimensional representation space V with $V^G = 0$ is admissible.

Consider now a representation of G in $\mathbb{R}^m$: then a natural representation of G in $\mathcal{D}$ or H is given by $g(u)(x) := g(u(x))$.

Definition 3.6. *Let* G *be a compact Lie group,* X *a* $G-$*Hilbert space,* $V \equiv \mathbb{R}^m$ *an admissible representation space for* G *and* $\Sigma = \{A \subset X : A \text{ is closed and invariant}\}$*. We define an index* $\gamma : \Sigma \to \mathbb{N} \cup \{+\infty\}$ *in the following way:* $\gamma(A)$ *is the smallest integer* k *such that there exists an equivariant map* $\Psi : A \to \{x \in (\mathbb{R}^m)^k : |x| = 1\}$*. More precisely, the map* Ψ *satisfies* $\Psi(gx) = (\Psi_1(gx), \ldots, \Psi_k(gx)) = (g\Psi_1(x), \ldots, g\Psi_k(x))$*.*

Remark 3.2. The index γ corresponds to the $\mathcal{A}$-*genus* defined in [3] with $\mathcal{A} = \{x \in V : |x| = 1\}$, and therefore it satisfies the usual properties of indices.

In order to establish the geometrical properties of the functional J in the sublinear case, we consider the linear self-adjoint operator $L^\infty : \mathcal{D} \to \mathcal{D}$ implicitly defined by

$$(L^\infty u, v) = \int_\Omega A_{ij}(x)\langle D_i u, D_j v\rangle - b(x)\langle u, v\rangle. \tag{3.1}$$

It is well known (see [15] for an extensive treatment of the topic) that, under the assumptions we take on A_{ij} and b, the whole spectrum $\sigma(L^\infty)$ but a finite set of eigenvalues with finite multiplicity is contained in some interval $[\mu_{\min}, \mu_{\max}]$ with $0 < \mu_{\min} \leq \mu_{\max} < +\infty$. As L^∞ is self-adjoint, there exist orthogonal subspaces $\mathcal{D}^+$, $\mathcal{D}^0$ and $\mathcal{D}^-$ of $\mathcal{D}$ such that $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^0 \oplus \mathcal{D}^-$ and L^∞ is positive definite on $\mathcal{D}^+$, negative definite on $\mathcal{D}^-$ and $\mathcal{D}^0 = \ker L^\infty$; by a symmetry argument the (finite) dimensions of $\mathcal{D}^-$ and $\mathcal{D}^0$ are both multiples of m: we set $q = (\dim \mathcal{D}^0 + \dim \mathcal{D}^-)/m$.

In order to exploit the symmetry of the equation, we need to consider its linearization in 0. Let $\alpha \equiv 0$ in (2.6), we have $F_1(x, 0) = 0$ and $\nabla F_1(x, 0) = 0$, set

$$f_0(x) = \limsup_{s \to 0} \frac{2F_1(x, s)}{|s|^2}$$

and define the linear self-adjoint operator $L^0 : \mathcal{D} \to \mathcal{D}$ by

$$(L^0 u, v) = \int_\Omega a_{ij}(x,0)\langle D_i u, D_j v\rangle - (b(x) + f_0(x))\langle u, v\rangle.$$

The operator L^0 has the same properties of L^∞: in particular the codimension of its positive subspace is a multiple of m, which we denote by pm. We prove the following:

Theorem 3.1. *Assume (A1), (A2), (F0), (F1) (with $\alpha \equiv 0$) and assume that equation (1.1) is invariant under some admissible representation of a compact Lie group G. Let p and q be defined as above. If $q > p$, then (1.1) admits at least $q - p$ geometrically distinct weak solutions in $\mathcal{D}$.*

The next theorems handle the case where the functional has a mountain pass geometry.

Theorem 3.2. *Assume (A1), (A2), (AF), (F0), (F2) and assume that (1.1) is invariant under some admissible representation of a compact Lie group G. Then (1.1) admits infinitely many geometrically distinct weak solutions in $\mathcal{D}$.*

Theorem 3.3. *Assume (A1), (AF), (F0), (F3) and assume that (1.1) is invariant under some admissible representation of a compact Lie group G. There exists a sequence $\{L_k\} \subset (0, +\infty)$ such that if $[\nabla a_{ij}]$ satisfies (A_{L_k}), then (1.1) admits at least k geometrically distinct weak solutions in $\mathcal{D} \cap L^\infty(\Omega)$.*

Remark 3.3. In general the sequence $\{L_k\}$ may vanish, therefore in order to obtain more solutions we need (1.1) to be closer to a semilinear system. On the other hand both Theorems 3.2 and 3.3 imply that (1.1) admits infinitely many geometrically distinct solutions if $\nabla a_{ij}(x,s) = 0$, that is the standard multiplicity result in the semilinear case.

4. Variational setting

We briefly recall some basic definitions of the nonsmooth critical point theory introduced in [12, 14].

Definition 4.1. *Let (X,d) be a metric space, $I \in C(X,\mathbb{R})$ and let $x \in X$. We denote by $|dI|(x)$ the supremum of the $\sigma \in [0,+\infty)$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : B(x,\delta) \times [0,\delta] \longrightarrow B(x, 2\delta)$$

such that for all $y \in B(x,\delta)$ and for all $t \in [0,\delta]$ we have

$$d(\mathcal{H}(y,t), y) \le t \text{ and } I(\mathcal{H}(y,t)) \le I(y) - \sigma t$$

where $B(x,r) := \{y \in X,\ d(x,y) < r\}$; $|dI|(x)$ is called the weak slope of I at x.

Definition 4.2. *Let (X,d) be a metric space and $I \in C(X,\mathbb{R})$; a point $x \in X$ is said to be critical for I if $|dI|(x) = 0$. A real number c is said to be a critical value for I if there exists $x \in X$ such that $I(x) = c$ and $|dI|(x) = 0$.*

The compactness condition of Palais-Smale (PS) has been defined in this context (see [14]). We prove that it holds in the superlinear case, but we do not know if it is satisfied in the sublinear case. In the latter case we prove instead that the functional J satisfies a weaker condition which is due to Cerami [10] in the smooth context; in our framework Palais-Smale-Cerami (PSC) sequences, PSC condition and the Y-differentiability have been defined in [2] as follows:

Definition 4.3. *Let X be a Banach space and let $I \in C(X, \mathbb{R})$. A sequence $\{x_m\} \subset X$ is called a PSC sequence if there exists $K > 0$ such that $|I(x_m)| \leq K$ and $(1 + \|x_m\|)|dI|(x_m) \to 0$.*

Definition 4.4. *Let X be a Banach space and let $I \in C(X, \mathbb{R})$. I satisfies the PSC condition if all its PSC sequences are precompact.*

Definition 4.5. *Let X be a Banach space, let $I \in C(X, \mathbb{R})$ and let Y be a dense subspace of X. If the directional derivative of I exists for all x in X in all the directions $y \in Y$ we say that I is weakly Y-differentiable and we call weak Y-slope in x the extended real number*

$$\|I'_Y(x)\|_* := \sup\{I'(x)[\phi] : \ \phi \in Y, \ \|\phi\|_X = 1\}.$$

We can now state the version of the saddle point theorem which we use:

Theorem 4.1. *Let $\mathcal{D} = V \oplus W$, where $V \neq \{0\}$ is finite dimensional; let J be defined as in (1.2) and assume that*

(i) *J satisfies the PSC condition*
(ii) *there exists $\beta \in \mathbb{R}$ such that $J(x) \geq \beta$ for all $x \in W$*
(iii) *there exist $\alpha < \beta$ and $R > 0$ such that $J(x) \leq \alpha$ for all $x \in \partial B_R \bigcap V$*

Then (1.1) has a weak solution $u \in \mathcal{D}$.

Proof. The functional J is of the type

$$J(u) = \int_\Omega L(x, u, \nabla u) dx,$$

where $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to \mathbb{R}$ satisfies the following assumptions:

$$\begin{array}{l} L(x, s, \xi) \text{ is measurable with respect to } x \text{ for all } (s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn} \\ L(x, s, \xi) \text{ is of class } C^1 \text{ with respect to } (s, \xi) \text{ for a.e. } x \in \Omega \end{array}$$

and there exist $h_1 \in L^1(\Omega, \mathbb{R})$, $h_2 \in L^1_{loc}(\Omega, \mathbb{R})$, $h_3 \in L^\infty_{loc}(\Omega, \mathbb{R})$ and $c > 0$ such that for all $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$ and a.e. $x \in \Omega$ the following inequalities hold:

$$\begin{array}{c} |L(x, s, \xi)| \leq h_1(x) + c(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\frac{\partial L}{\partial s}(x, s, \xi)\right| \leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2) \\ \left|\frac{\partial L}{\partial \xi}(x, s, \xi)\right| \leq h_2(x) + h_3(x)(|s|^{\frac{2n}{n-2}} + |\xi|^2). \end{array}$$

With the above growth conditions and by adapting Theorem 1.5 in [8] to our case, we infer that J is continuous, weakly $C^\infty_c(\Omega)-$differentiable and that the weak slope gives an upper estimate of the weak $C_c^\infty(\Omega)-$slope, i.e.

$$|dJ|(u) \geq \|J'_{C_c^\infty}(u)\|_*. \tag{4.1}$$

In particular, if u is a critical point of J, then equation (1.1) is satisfied in distributional sense. Therefore, as $D_j(a_{ij}(x,u)D_iu) + \nabla F(x,u) \in \mathcal{D}^*$(the dual space of $\mathcal{D}$) we also have $\nabla a_{ij}(x,u)\langle D_iu, D_ju\rangle \in \mathcal{D}^*$and the system is solved in a weak sense. To complete the proof it suffices to reason as for Theorems 3 and 5 in [2]. □

Analogously one can prove the following version of the mountain pass theorem:

Theorem 4.2. *Let J be defined as in (1.2) and assume that*

(i) *J satisfies the PS condition*
(ii) *there exist $\alpha, r \in (0, +\infty)$ such that $J(x) \geq \alpha$ for all $x \in \partial B_r$*
(iii) *there exist $R > r$ and $y \in \partial B_R$ such that $J(y) \leq 0$.*

Then equation (1.1) has a nontrivial weak solution $u \in \mathcal{D}$.

The following Lemma extends a result in [5]:

Lemma 4.1. *Let Ω be (any) open set in $\mathbb{R}^n$ let $T \in [\mathcal{D}^{1,2}(\Omega,\mathbb{R}^m)]^* \cap L^1_{loc}(\Omega,\mathbb{R}^m)$ and $u \in \mathcal{D}^{1,2}(\Omega,\mathbb{R}^m)$ satisfying $\langle T, u\rangle \geq f$ in Ω for some function $f \in L^1(\Omega,\mathbb{R})$. Then $\langle T, u\rangle \in L^1(\Omega,\mathbb{R})$ and the duality product (T, u) equals $\int_\Omega \langle T, u\rangle$.*

Proof. The proof follows by inspection of the proof in [5]. □

Remark 4.1. The previous Lemma and conditions (2.2)-(2.6) imply that

$$\langle u, \nabla a_{ij}(x,u)\rangle\langle D_iu, D_ju\rangle \in L^1(\Omega,\mathbb{R})$$

for all $u \in \mathcal{D}$ and therefore $J'(u)[u]$ is well defined, see [8, 9] for details.

In order to deal with a symmetric functional, we adapt some results of Bartsch [3] to our case.

Theorem 4.3. *Let G be a compact Lie group and let X be a $G-$Hilbert space. Consider an admissible representation of G on $\mathbb{R}^m$, and assume that $X = \bigoplus_i X_i$ where $X_i \simeq \mathbb{R}^m$ and each X_i is isomorphic to $\mathbb{R}^m$ as a representation of G. Let $I : X \to \mathbb{R}$ be a continuous $G-$invariant functional satisfying the PSC condition and $I(0) = 0$; assume moreover that there exist two integers p and q ($q > p$), such that*

(i) *There exist $\rho, \beta > 0$ such that $I(x) \geq \beta$ for all $x \in \partial B_\rho \bigcap \bigoplus_{i=p+1}^{\infty} X_i$.*
(ii) *There exists $R > \rho$ such that $I(x) \leq 0$ for all $x \in \partial B_R \bigcap \bigoplus_{i=1}^{q} X_i$.*

Then I admits at least $q - p$ critical orbits.

Proof. The smooth version of this theorem is Theorem 2.25 in [3]. The proof follows the same lines, with two distinctions: 1. The classical deformation lemma does not hold in the framework of nonsmooth critical point theory, but an equivariant deformation is provided by Theorem 1.2.5 in [9] for functionals satisfying PS and can be easily extended to functionals only satisfying PSC. 2. In the mentioned theorem it is assumed that condition (ii) holds for all integers q (mountain pass case), but the saddle point case is an easy variant. □

Another variant of the same theorem is the following (see also Theorem 1.5 in [8] where a $\mathbb{Z}_2$ symmetry is considered).

Theorem 4.4. *Let G be a compact Lie group and let X be a $G-$Hilbert space. Consider an admissible representation of G on $\mathbb{R}^m$, and assume that $X = \bigoplus_i X_i$ where $X_i \simeq \mathbb{R}^m$ and each X_i is isomorphic to $\mathbb{R}^m$ as a representation of G. Let $I : X \to \mathbb{R}$ be a continuous $G-$invariant functional, $I(0) = 0$; assume moreover that there exists an integer p such that*

(i) *There exist $\rho, \beta > 0$ such that $I(x) \geq \beta$ for all $x \in \partial B_\rho \bigcap \bigoplus_{i=p+1}^{\infty} X_i$.*

(ii) *For every integer k there exists $R > \rho$ such that $I(x) \leq 0$ for all $x \in \partial B_R \bigcap \bigoplus_{i=1}^{k} X_i$.*

Then there exists a diverging sequence $\{c_k\} \subset [\beta, +\infty)$ such that for every integer k there exists a PS sequence $\{u_k^h\}$ at level c_k.

5. Boundedness of PS sequences

5.1. **The sublinear case.** For all $\omega \subset \Omega$ and $p \geq 1$ we set $\omega^c = \Omega \setminus \omega$, $\|u\|_{L^p(\omega)} = \left(\int_\omega |u|^p\right)^{1/p}$ and $\|u\|_p = \left(\int_\Omega |u|^p\right)^{1/p}$.

Lemma 5.1. *Assume (F0) and (F1). If $\{u^h\} \subset \mathcal{D}$ is a sequence such that $\|u^h\| \to \infty$, then*

$$\frac{\int_\Omega F_1(x, u^h)}{\|u^h\|^2} \to 0 \text{ as } h \to \infty.$$

Proof. Let $\{u^h\} \subset \mathcal{D}$ be such that $\|u^h\| \to \infty$; we claim that there exists a sequence $\{\varepsilon^h\} \subset \mathbb{R}^+$ such that $\varepsilon^h \to 0$ and, for a.e. $x \in \Omega$

$$\left|F_1(x, u^h(x))\right| \leq \alpha(x)\|u^h\|^{1/2} + \frac{\beta(x)}{2}\|u^h\| + \varepsilon^h |u^h(x)|^2 \,. \tag{5.1}$$

Consider $h \in \mathbb{N}$ and $x \in \Omega$, let γ be the segment connecting 0 to $u^h(x)$ and let $v^h(x) = \frac{u^h(x)}{|u^h(x)|}$: we have

$$\begin{aligned} \left|F_1(x, u^h(x))\right| &= \left|\int_\gamma \langle \nabla F_1(x, s), ds\rangle\right| \leq \int_0^1 |\langle \nabla F_1(x, tu^h(x)), u^h(x)\rangle| dt \\ &\leq \int_0^{|u^h(x)|} \left|\nabla F_1\left(x, tv^h(x)\right)\right| dt. \end{aligned}$$

If $|u^h(x)| \le \|u^h\|^{1/2}$, by (2.6) we have

$$\int_0^{|u^h(x)|} |\nabla F_1(x, tv^h)| dt \le \int_0^{\|u^h\|^{1/2}} (\alpha(x) + \beta(x)t) dt \le \alpha(x)\|u^h\|^{1/2} + \frac{\beta(x)}{2}\|u^h\|,$$

and if $|u^h(x)| > \|u^h\|^{1/2}$, by Hölder inequality we get

$$\begin{aligned} \int_{\|u^h\|^{1/2}}^{|u^h(x)|} \frac{|\nabla F_1(x, tv^h)|}{t} t dt &\le \left[\int_{\|u^h\|^{1/2}}^{|u^h(x)|} t^2 dt\right]^{1/2} \cdot \left[\int_{\|u^h\|^{1/2}}^{|u^h(x)|} \left|\frac{\nabla F_1(x, tv^h)}{t}\right|^2 dt\right]^{1/2} \\ &\le |u^h(x)|^{3/2} \cdot \varepsilon^h |u^h(x)|^{1/2}, \end{aligned}$$

where ε^h depends on $\|u^h\|$ and by (2.7) $\varepsilon^h \to 0$ as $h \to \infty$; combining this with the previous inequality we obtain (5.1). If Ω is bounded, then we are done by integrating (5.1) on Ω; otherwise choose $\varepsilon > 0$ and let $\omega \subset \Omega$ be a bounded open set such that $\|\beta\|_{L^{\frac{n}{2}}(\omega^c)} < \varepsilon$, where $\omega^c = \Omega \setminus \omega$. By (2.6), Hölder inequality and the continuous embedding $\mathcal{D} \subset L^{2^*}(\Omega)$ we have

$$\begin{aligned} \left|\int_{\omega^c} F_1(x, u^h)\right| &\le \|\alpha\|_{L^{\frac{2n}{n+2}}(\omega^c)} \|u^h\|_{L^{2^*}(\omega^c)} + \frac{1}{2}\|\beta\|_{L^{\frac{n}{2}}(\omega^c)} \|u^h\|^2_{L^{2^*}(\omega^c)} \\ &\le c\|u^h\| + \varepsilon c\|u^h\|^2; \end{aligned}$$

furthermore, by integrating (5.1) on ω we have

$$\left|\int_\omega F_1(x, u^h)\right| \le c\|u^h\| + \varepsilon^h \|u^h\|^2_{L^2(\omega)},$$

and these two inequalities yield the result by the arbitrariness of ε. □

Lemma 5.2. *Assume (A1), (F0) and (F1). There exist a bounded set $\omega \subset \Omega$ and $\eta > 0$ such that for all sequences $\{u^h\} \subset \mathcal{D}$ satisfying $\sup J(u^h) < \infty$ and $\|u^h\| \to \infty$ the following inequality holds:*

$$\|u^h\| \le \eta \|u^h\|_{L^2(\omega)}$$

(if Ω itself is bounded, then the statement holds for $\omega = \Omega$).

Proof. Fix $\varepsilon > 0$ and choose ω_ε so that $\|b\|_{L^{\frac{n}{2}}(\omega_\varepsilon^c)} < \frac{\varepsilon}{2}$; the restriction of b to ω_ε is in $L^{\frac{n}{2}}(\omega_\varepsilon)$, therefore there exist two functions $\tilde{b}_2$ and b_3 such that $\tilde{b}_2 \in L^\infty(\omega_\varepsilon)$, $b_3 \in L^{\frac{n}{2}}(\omega_\varepsilon)$, $\|b_3\|_{L^{\frac{n}{2}}(\omega_\varepsilon)} < \frac{\varepsilon}{2}$ and $b(x) = \tilde{b}_2(x) + b_3(x)$ for a.e. $x \in \omega_\varepsilon$. Now let

$$b_1(x) = \begin{cases} b_3(x) & \text{if } x \in \omega_\varepsilon \\ b(x) & \text{if } x \notin \omega_\varepsilon \end{cases} \qquad \text{and} \qquad b_2(x) = \begin{cases} \tilde{b}_2(x) & \text{if } x \in \omega_\varepsilon \\ 0 & \text{if } x \notin \omega_\varepsilon. \end{cases}$$

So far we have proved that for all $\varepsilon > 0$ there exist an open bounded set ω_ε and two functions $b_1 \in L^{\frac{n}{2}}(\Omega)$ and $b_2 \in L^\infty(\Omega)$ such that $b = b_1 + b_2$, $\|b_1\|_{\frac{n}{2}} < \varepsilon$ and

$\mathrm{supp} b_2 \subset \omega_\varepsilon$; hence for all $u \in \mathcal{D}$ we have

$$\left| \int_\Omega b(x)|u|^2 \right| \le \int_\Omega |b_1(x)||u|^2 + \int_{\omega_\varepsilon} |b_2(x)||u|^2$$
$$\le \|b_1\|_{\frac{n}{2}} \|u\|^2_{\frac{2n}{n-2}} + \|b_2\|_\infty \|u\|^2_{L^2(\omega_\varepsilon)} \le c\varepsilon\|u\|^2 + \|b_2\|_\infty \|u\|^2_{L^2(\omega_\varepsilon)}.$$

Then, by (2.2) and Lemma 5.1 we obtain

$$J(u^h) \ge c\|u^h\|^2 - c\varepsilon\|u^h\|^2 - \|b_2\|_\infty \|u^h\|^2_{L^2(\omega_\varepsilon)} + o(\|u^h\|^2) \tag{5.2}$$

and the result follows by choosing ε small enough and setting $\omega = \omega_\varepsilon$. □

Lemma 5.3. *Assume (A1), (F0) and (F1). Then all the PSC sequences for J are bounded in $\mathcal{D}$.*

Proof. By contradiction, let $\{u^h\}$ be a diverging PSC sequence; by Remark 4.1 for h large we can evaluate $J'(u^h)[u^h] - 2J(u^h)$ and taking into account (2.3) and (4.1) we have

$$O(1) \ge \int_\Omega 2F_1(x, u^h) - \langle \nabla F_1(x, u^h), u^h \rangle. \tag{5.3}$$

Let $v^h(x) = \frac{u^h(x)}{\|u^h\|}$, then there exists $v \in \mathcal{D}$ such that, up to a subsequence, $v^h \rightharpoonup v$ and therefore $v^h \to v$ in L^2_{loc} and $v^h(x) \to v(x)$ for a.e. $x \in \Omega$; Lemma 5.2 implies that $v \ne 0$. By (2.9) we infer that $2F_1(x, u^h) - \nabla F_1(x, u^h)u^h \to +\infty$ on a subset of Ω with positive measure, hence by (2.10) and Fatou Lemma we infer

$$\int_\Omega [2F_1(x, u^h) - \langle \nabla F_1(x, u^h), u^h \rangle] \to +\infty,$$

which contradicts (5.3). □

5.2. **The superlinear case.** Let A_{ij} be as in (2.1) and define the (smooth) functional

$$J_\infty(u) = \frac{1}{2} \int_\Omega A_{ij}(x) \langle D_i u, D_j u \rangle - \int_\Omega F_2(x, u).$$

By (2.12) the function $F_2(x, \cdot)$ is superquadratic at $+\infty$ and we can choose $\bar{v} \in \mathcal{D}$ satisfying $J_\infty(\bar{v}) < 0$; now define

$$\Gamma := \{\gamma \in C([0,1];\ \mathcal{D}),\ \gamma(0) = 0,\ \gamma(1) = \bar{v}\}$$

and

$$M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\infty(\gamma(t)).$$

We prove that J admits a bounded PS sequence and give an estimate of the norm:

Lemma 5.4. *Assume (A1), (AF), (F0) and (F2). There exists a PS sequence* $\{u^h\} \subset \mathcal{D}$ *satisfying*

$$\lim_{h\to+\infty} \|u^h\|^2 \le \frac{2Mp}{\nu(p-2-\gamma)} \ .$$

Remark 5.1. Recall that (F3) implies (F2).

Proof. First note that $J(0) = 0$. Next, observe that there exist $\rho, R > 0$ such that $J(u) \ge R$ for all $\|u\| = \rho$: indeed, by (2.2), (2.13), Hölder inequality and Sobolev embedding Theorem we infer

$$J(u) \ge C_1\|u\|^2 - C_2\|u\|^\delta - C_3\|u\|^q \qquad \text{for all } u \in \mathcal{D}$$

(note that if (F3) is assumed, then $C_2 = 0$), which yields ρ and R. Let

$$\alpha := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

then we have $\alpha \le M$ because $J \le J_\infty$. We obtain a PS sequence for J at level α by applying the mountain pass Lemma in the nonsmooth version [14]: we claim that such a sequence satisfies the above estimate. Since $\{u^h\} \subset \mathcal{D}$ satisfies $|J(u^h)| \le M + o(1)$, by (2.12) we get

$$I^h := \frac{1}{2}\int_\Omega a_{ij}(x,u^h)\langle D_i u^h, D_j u^h\rangle - \frac{1}{p}\int_\Omega \langle \nabla F_2(x,u^h), u^h\rangle \le M + o(1) \ ;$$

by (2.14) we can evaluate $J'(u^h)[u^h]$ and as $\{u^h\}$ is a PS sequence we have

$$|J'(u^h)[u^h]| \le o(\|u^h\|) \ .$$

Therefore, by (2.14) and computing $I^h - \frac{1}{p}J'(u^h)[u^h]$ we get

$$\frac{p-2-\gamma}{2p}\int_\Omega a_{ij}(x,u^h)\langle D_i u^h, D_j u^h\rangle \le o(\|u^h\|) + M + o(1) \ ;$$

by (2.2) this proves that $\{u^h\}$ is bounded and that

$$\frac{(p-2-\gamma)\nu}{2p}\|u^h\|^2 \le \frac{p-2-\gamma}{2p}\int_\Omega a_{ij}(x,u^h)\langle D_i u^h, D_j u^h\rangle \le M + o(1) \ ,$$

that is, the required estimate. ∎

6. Compactness of PS sequences

6.1. The case of assumption (A2).

Lemma 6.1. *Assume (A1) and (A2), let* $\{u^h\} \subset \mathcal{D}$ *be a bounded sequence and set*

$$w^h = -D_j(a_{ij}(x,u^h)D_i u^h) + \frac{1}{2}\nabla a_{ij}(x,u^h)\langle D_i u^h, D_j u^h\rangle.$$

If $\{w^h\} \subset \mathcal{D}^*$ *and it is strongly convergent to some* w*, then* $\{u^h\}$ *is precompact.*

Proof. Since $\{u^h\}$ is bounded, then $u^h \rightharpoonup u$ for some u up to a subsequence. Each component u_l^h satisfies (2.5) in [7] and since Theorem 2.1 in the same paper can be extended to unbounded domains, we infer that $D_i u_l^h \to D_i u_l$ a.e. in Ω for all $l = 1, \dots, m$ (see also [13]). We first prove that

$$\int_\Omega a_{ij}(x,u)\langle D_i u, D_j u\rangle + \frac{1}{2}\int_\Omega \langle \nabla a_{ij}(x,u), u\rangle\langle D_i u, D_j u\rangle = w[u] \tag{6.1}$$

where $w[u]$ represents the duality product between $w \in \mathcal{D}^*$ and $u \in \mathcal{D}$. With some abuse of notation we denote by ψ the odd extension to the whole real line of the function defined in (2.4) and let $v^h = \varphi \exp[\psi(u) - \psi(u^h)]$, where $\varphi \in \mathcal{D} \cap L^\infty$, $\varphi \geq 0$ and

$$\exp[\psi(u) - \psi(u^h)] = (\exp[\psi(u_1) - \psi(u_1^h)], \dots, \exp[\psi(u_m) - \psi(u_m^h)]).$$

For all h we have $\left(\partial_k = \frac{\partial}{\partial s_k}\right)$

$$\begin{aligned}\int_\Omega a_{ij}(x,u^h) D_i u_l^h (D_j\varphi + \varphi\psi'(u_l) D_j u_l) \exp[\psi(u_l) - \psi(u_l^h)] \\ + \frac{1}{2}\int_\Omega \partial_k a_{ij}(x,u^h)\exp[\psi(u_k) - \psi(u_k^h)] D_i u_l^h D_j u_l^h \varphi \\ - \int_\Omega a_{ij}(x,u^h) D_i u_l^h D_j u_l^h \psi'(u_l^h)\exp[\psi(u_l) - \psi(u_l^h)]\varphi = w^h[v^h];\end{aligned}$$

let us study the behavior of each term of the previous equality as $h \to \infty$. First of all, as $v^h \rightharpoonup v = (\varphi, \dots, \varphi)$, then

$$w^h[v^h] \to w[v]. \tag{6.2}$$

Next, since $u^h \rightharpoonup u$, by Lebesgue Theorem we obtain

$$\begin{aligned}\int_\Omega a_{ij}(x,u^h) D_i u_l^h (D_j\varphi + \varphi\psi'(u_l) D_j u_l)\exp[\psi(u_l) - \psi(u_l^h)] \to \\ \int_\Omega a_{ij}(x,u) D_i u_l (D_j\varphi + \varphi\psi'(u_l) D_j u_l).\end{aligned} \tag{6.3}$$

Finally, note that by (A2), for all $\xi \in \mathbb{R}^n$ we get

$$\left|\partial_k a_{ij}(x,u^h)\exp[\psi(u_k) - \psi(u_k^h)]\xi_i\xi_j\right| \leq \sum_k \left|\partial_k a_{ij}(x,u^h)\xi_i\xi_j\right| e^{2K};$$

next, by using (iv) of (2.4), we infer

$$\sum_k \left|\partial_k a_{ij}(x,u^h)\xi_i\xi_j\right| e^{2K} \leq 2e^{-2K}\psi'(|u^h|) a_{ij}(x,u^h)\xi_i\xi_j;$$

moreover we have

$$2e^{-2K}\psi'(|u^h|) a_{ij}(x,u^h)\xi_i\xi_j \leq 2\psi'(u_l^h)\exp[\psi(u_l) - \psi(u_l^h)] a_{ij}(x,u^h)\xi_i\xi_j.$$

The last three inequalities yield

$$\begin{aligned}&\frac{1}{2}\partial_k a_{ij}(x,u^h)\exp[\psi(u_k)-\psi(u_k^h)]D_iu_l^hD_ju_l^h\\ \le\ & a_{ij}(x,u^h)D_iu_l^hD_ju_l^h\psi'(u_l^h)\exp[\psi(u_l)-\psi(u_l^h)].\end{aligned}$$

Hence, we can apply Fatou Lemma to obtain

$$\limsup_{h\to\infty}\left\{\frac{1}{2}\int_\Omega \partial_k a_{ij}(x,u^h)\exp[\psi(u_k)-\psi(u_k^h)]D_iu_l^hD_ju_l^h\varphi\right.$$
$$\left.-\int_\Omega a_{ij}(x,u^h)D_iu_l^hD_ju_l^h\psi'(u_l^h)\exp[\psi(u_l)-\psi(u_l^h)]\varphi\right\}\le$$
$$\frac{1}{2}\int_\Omega\sum_k \partial_k a_{ij}(x,u)D_iu_lD_ju_l\varphi-\int_\Omega a_{ij}(x,u)D_iu_lD_ju_l\psi'(u_l)\varphi,$$

which, together with (6.2) and (6.3), yields

$$\int_\Omega a_{ij}(x,u)\langle D_iu,D_jv\rangle+\frac{1}{2}\int_\Omega\langle\nabla a_{ij}(x,u),v\rangle\langle D_iu,D_ju\rangle\ge w[v]$$

for all test functions $v=(\varphi,\dots,\varphi)$ with $\varphi\in\mathcal{D}\cap L^\infty$, $\varphi\ge 0$. We obtain the opposite inequality by using the test functions $v^h=\varphi\exp[\psi(u^h)-\psi(u)]$ (with $\varphi\ge 0$) and therefore we have

$$\int_\Omega a_{ij}(x,u)\langle D_iu,D_jv\rangle+\frac{1}{2}\int_\Omega\langle\nabla a_{ij}(x,u),v\rangle\langle D_iu,D_ju\rangle= w[v]$$

for all $v=(\varphi,\dots,\varphi)$ with $\varphi\in\mathcal{D}\cap L^\infty$ (and not necessarily $\varphi\ge 0$). Consider now the test functions

$$v^h=\varphi(\sigma_1\exp[\sigma_1(\psi(u_1)-\psi(u_1^h))],\dots,\sigma_m\exp[\sigma_m(\psi(u_m)-\psi(u_m^h))]),$$

where $\varphi\in\mathcal{D}\cap L^\infty$, $\varphi\ge 0$ and $\sigma_l=\pm 1$ for all l. By taking all possible choices of σ_i and by reasoning as in the previous steps we infer

$$\int_\Omega a_{ij}(x,u)\langle D_iu,D_jv\rangle+\frac{1}{2}\int_\Omega\langle\nabla a_{ij}(x,u),v\rangle\langle D_iu,D_ju\rangle= w[v]$$

for all test functions $v=(\sigma_1\varphi,\dots,\sigma_m\varphi)$ (not necessarily $\varphi\ge 0$), and since every function $v\in\mathcal{D}\cap L^\infty$ can be written as a linear combination of such functions, by a density argument we infer (6.1). Now, by taking the same steps as in the proof of inequality (2.3.10) in [8] we get

$$\limsup_{h\to\infty}\int_\Omega a_{ij}(x,u^h)\langle D_iu^h,D_ju^h\rangle\le\int_\Omega a_{ij}(x,u)\langle D_iu,D_ju\rangle. \tag{6.4}$$

Finally, by (2.2) we have

$$\nu\|u^h-u\|^2\le\int_\Omega a_{ij}(x,u^h)\left(\langle D_iu^h,D_ju^h\rangle-2\langle D_iu^h,D_ju\rangle+\langle D_iu,D_ju\rangle\right):$$

hence, by Lebesgue Theorem and (6.4) we obtain

$$\limsup_{h\to\infty} \|u^h - u\|^2 \leq 0$$

which proves that $u^h \to u$ in $\mathcal{D}$. □

With the result of the previous lemma we can prove the compactness of PSC sequences:

Lemma 6.2. *Assume (A1), (A2), (F0) and (F1); let $\{u^h\} \subset \mathcal{D}$ be a PSC sequence for the functional J. Then $\{u^h\}$ is precompact.*

Proof. Let $\{u^h\}$ be a PSC sequence, by Lemma 5.3 we know that $\{u^h\}$ is bounded and that $u^h \rightharpoonup u$ for some u up to a subsequence. By a standard procedure (see e.g. Theorem 2.2.7 in [9]) up to a further subsequence we have $b(x)u^h \to b(x)u$ and $\nabla F_2(x, u^h) \to \nabla F_2(x, u)$ in $L^{\frac{2n}{n+2}}(\Omega; \mathbb{R}^m)$; then $\{u^h\}$ satisfies the assumptions of Lemma 6.1 and therefore it has a converging subsequence. □

Similarly, we can prove

Lemma 6.3. *Assume (A1), (A2), (AF), (F0) and (F2) and let $\{u^h\} \subset \mathcal{D}$ be a PS sequence for the functional J. Then $\{u^h\}$ is precompact.*

6.2. The case of assumption ($\mathbf{A}_L$).

Lemma 6.4. *Assume (A1), (F0) and (F3). Let $\{u^h\}$ be a PS sequence satisfying $\|u^h\| \leq K$ for some $K > 0$; then, there exists $u \in \mathcal{D} \cap L^\infty$ such that $u^h \rightharpoonup u$ (up to a subsequence) and*

$$\|u\|_\infty \leq c(K)$$

for some function c which is non-decreasing with respect to K.

Proof. It follows the same lines of Lemma 3.3 in [19]. For all $r \geq 0$, $\eta > 0$ define the test functions

$$\varphi^h_{r,\eta} = \frac{u^h}{|u^h|} \min\{|u^h|, \eta\}^{r+1}$$

which we only denote by φ^h for simplicity. The sequence $\{\varphi^h\}$ is uniformly bounded in $\mathcal{D}$ and $\{\varphi^h\} \subset \mathcal{D} \cap L^\infty$, hence by (4.1) we have $J'(u^h)[\varphi^h] \to 0$ and if we define

$$A^h_1 = \int_\Omega a_{ij}(x, u^h)\langle D_i u^h, D_j \varphi^h\rangle + \frac{1}{2}\int_\Omega \langle \nabla a_{ij}(x, u^h), \varphi^h\rangle \langle D_i u^h, D_j u^h\rangle$$

$$A^h_2 = \int_\Omega \langle \nabla F_2(x, u^h), \varphi^h\rangle$$

we have

$$A^h_1 - A^h_2 \to 0\ . \qquad (6.5)$$

Let $u^h_\eta = \min\{|u^h|, \eta\}$, then, by reasoning as in the proof of Lemma 3.3 in [19] (taking $\lambda^* = 0$) we obtain

$$A_1^h \geq 4\nu(r+2)^{-2}\|(u_\eta^h)^{(r+2)/2}\|^2 \geq c(r+2)^{-2}\|u_\eta^h\|^{r+2}_{n(r+2)/(n-2)}$$

and, by (2.15)

$$A_2^h \leq c\int_\Omega \beta(x)|u^h|^{q-1}(u_\eta^h)^{r+1}.$$

Note that $\beta(x)(u_\eta^h)^{r+1} \to \beta(x)(u_\eta)^{r+1}$ for a.e. $x \in \Omega$ and $\left|\beta(x)(u_\eta^h)^{r+1}\right| \leq \eta^{r+1}\beta(x) \in L^{\frac{2n}{2n+(2-n)q}}$ so by Lebesgue Theorem $\beta(x)(u_\eta^h)^{r+1} \to \beta(x)(u_\eta)^{r+1}$ in $L^{\frac{2n}{2n+(2-n)q}}$. If Ω is unbounded, let $\Omega_\varepsilon \subset \Omega$ be a bounded set satisfying $\int_{\Omega\setminus\Omega_\varepsilon}(\beta(x))^{\frac{2n}{2n+(2-n)q}} \leq \varepsilon$, otherwise choose $\Omega_\varepsilon = \Omega$. As $u^h \to u$ in $L^{q-1}(\Omega_\varepsilon)$ we have

$$\left|\int_\Omega \beta(x)|u^h|^{q-1}(u_\eta^h)^{r+1} - \beta(x)|u|^{q-1}(u_\eta)^{r+1}\right|$$

$$\leq \eta^r\int_{\Omega\setminus\Omega_\varepsilon}\beta(x)\left(|u^h|^q + |u|^q\right) + \left|\int_{\Omega_\varepsilon}\beta(x)|u^h|^{q-1}(u_\eta^h)^{r+1} - \beta(x)|u|^{q-1}(u_\eta)^{r+1}\right|$$

$$\leq c\varepsilon + o(1),$$

therefore $\lim\limits_{h\to\infty} A_2^h \leq c\int_\Omega \beta(x)|u|^{q-1}(u_\eta)^{r+1}$ by the arbitrariness of ε. The previous estimates and (6.5) yield

$$\|u_\eta\|^{r+2}_{n(r+2)/(n-2)} \leq c(r+2)^2\int_\Omega \beta(x)|u|^{q-1}(u_\eta)^{r+1}$$

and by letting $\eta \to \infty$

$$\|u\|^{r+2}_{n(r+2)/(n-2)} \leq c(r+2)^2\int_\Omega \beta(x)|u|^{q+r} \leq c(r+2)^2\|u\|^{q+r}_{q+r}.$$

Now choosing $r = r_0 = 2^* - q$ we have

$$||u||^{2^*-q+2}_{\frac{n(2^*-q+2)}{n-2}} \leq c||u||^{2^*}_{2^*};$$

next define by induction $r_{i+1} = \frac{n}{n-2}(r_i+2) - q$ and conclude the proof by following the same steps as in the proof of Lemma 3.3 in [19]. □

In particular, from Lemma 6.4, we infer that the PS sequence $\{u^h\}$ found in Lemma 5.4 admits a weak limit (up to a subsequence) $u \in \mathcal{D} \cap L^\infty$ such that

$$\|u\|_\infty \leq c\left(\sqrt{\frac{2Mp}{\nu(p-2-\gamma)}}\right) =: U\,. \tag{6.6}$$

To apply the mountain pass Lemma we need now to prove the convergence of such PS sequence: the proof of the next result follows closely the proof of Lemma 3.4 in [19], but our setting is in a (possibly) unbounded domain and with different assumptions on F, therefore we give some details.

Lemma 6.5. *Assume (A1), (F0) and (F3); let $\{u^h\}$ be the PS sequence obtained by Lemma 5.4. There exists a constant $L > 0$ such that if (A_L) holds, then $\{u^h\}$ is precompact.*

Proof. Let u be the weak limit of $\{u^h\}$; by Lemma 5.4 we know that $u \in \mathcal{D} \cap L^\infty$ and therefore it is an admissible test function, we can evaluate $J'(u^h)[u^h - u]$ and $J'(u^h)[u^h - u] \to 0$. By (2.13) we have

$$\lim_{h \to \infty} \int_\Omega \langle \nabla F_2(x, u^h), u^h - u \rangle = 0 \ . \tag{6.7}$$

In what follows $\omega(\delta)$ denotes a generic function vanishing as $\delta \to 0$. For all $\delta > 0$ let $\Omega^\delta \subset \Omega$ be a bounded set (if Ω itself is bounded, then let $\Omega^\delta = \Omega$) such that $\int_{\Omega \setminus \Omega^\delta} |\nabla u|^2 \leq \delta$, then

$$\int_{\Omega \setminus \Omega^\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i u^h, D_j u^h \rangle =$$
$$\int_{\Omega \setminus \Omega^\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i(u^h - u), D_j(u^h - u) \rangle + \omega(\delta);$$

now let $\Omega_\delta \subset \Omega^\delta$ satisfy $|\Omega_\delta| \leq \delta$ and $u^h \to u$ uniformly on $\Omega^\delta \setminus \Omega_\delta$ (such set exists by Egorov Theorem), then

$$\int_{\Omega^\delta \setminus \Omega_\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i u^h, D_j u^h \rangle =$$
$$\int_{\Omega^\delta \setminus \Omega_\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i(u^h - u), D_j(u^h - u) \rangle + o(1);$$

finally we have

$$\int_{\Omega_\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i u^h, D_j u^h \rangle =$$
$$\int_{\Omega_\delta} \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i(u^h - u), D_j(u^h - u) \rangle + \omega(\delta).$$

These three equalities yield

$$\int_\Omega \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i u^h, D_j u^h \rangle = \tag{6.8}$$
$$\int_\Omega \langle \nabla a_{ij}(x, u^h), u^h - u \rangle \langle D_i(u^h - u), D_j(u^h - u) \rangle + \omega(\delta) + o(1).$$

By (2.2) we have

$$
\begin{aligned}
\nu\|u^h-u\|^2 &\leq \int_\Omega a_{ij}(x,u^h)\langle D_i(u^h-u),D_j(u^h-u)\rangle \\
&= \int_\Omega a_{ij}(x,u^h)\langle D_iu^h,D_j(u^h-u)\rangle + o(1) \\
\text{by (6.7)} \quad &= J'(u^h)[u^h-u] - \frac{1}{2}\int_\Omega \langle \nabla a_{ij}(x,u^h),u^h-u\rangle\langle D_iu^h,D_ju^h\rangle + o(1) \\
\text{by (6.8)} \quad &= -\frac{1}{2}\int_\Omega \langle \nabla a_{ij}(x,u^h),u^h-u\rangle\langle D_i(u^h-u),D_j(u^h-u)\rangle + \omega(\delta) + o(1) \\
\text{by (2.14)} \quad &\leq \frac{1}{2}\int_\Omega \langle \nabla a_{ij}(x,u^h),u\rangle\langle D_i(u^h-u),D_j(u^h-u)\rangle + \omega(\delta) + o(1) \\
\text{by (6.6)} \quad &\leq U\|\nabla a_{ij}\|_\infty \int_\Omega |\nabla(u^h-u)|^2 + \omega(\delta) + o(1) \; ;
\end{aligned}
$$

and since δ may be chosen arbitrarily small, if $L < \frac{\nu}{U}$ we have $u^h \to u$ in the norm topology. □

7. Proofs of the sublinear theorems

Recall that in Section 3 we defined qm to be the number of nonpositive eigenvalues of L^∞ counted with their multiplicity. We first consider the case $q \geq 1$ and we prove that the geometrical requirements of the saddle point theorem hold.

Proposition 7.1. *Assume (A1), (F0) and (F1). Then*

(i) *there exists $\beta \in \mathbb{R}$ such that for all $u \in \mathcal{D}^+$ we have $J(u) \geq \beta$.*
(ii) *there exist $\alpha < \beta$ and $R > 0$ such that if $u \in \mathcal{D}^- \oplus \mathcal{D}^0$ and $\|u\| = R$, then $J(u) \leq \alpha$.*

Proof. Since $J(u^h)$ is bounded on bounded subsets of $\mathcal{D}$, then (i) holds if $J(u^h) \to +\infty$ for every sequence $\{u^h\} \subset \mathcal{D}^+$ such that $\|u^h\| \to \infty$. Consider a diverging sequence $\{u^h\} \subset \mathcal{D}^+$: by Lemma 5.1 $\int_\Omega F_1(x,u^h)/\|u^h\|^2 \to 0$, therefore it suffices to prove that for h large enough

$$
(7.1) \qquad \int_\Omega a_{ij}(x,u^h)\langle D_iv^h,D_jv^h\rangle - \int_\Omega b(x)|v^h|^2 \geq c > 0 \; ,
$$

where $v^h = \frac{u^h}{\|u^h\|}$. There exists $v \in \mathcal{D}$, $\|v\| \leq 1$, such that $v^h \rightharpoonup v$ and $\int_\Omega b|v^h|^2 \to \int_\Omega b|v|^2$ on a subsequence, since $b \in L^{\frac{n}{2}}$. To prove (7.1) we use the same device as in [11]. Let $l_h = \int_\Omega a_{ij}(x,u^h)\langle D_iv^h,D_jv^h\rangle$; as $\{l_h\}$ is bounded, on a subsequence $l_h \to l$ and two cases may occur:

1. $l > \int_\Omega A_{ij}(x)\langle D_iv,D_jv\rangle$. In this case inequality (7.1) follows because $v \in \mathcal{D}^+$.

2. $l \leq \int_\Omega A_{ij}(x)\langle D_i v, D_j v\rangle$. Then by (2.2) we have

$$\begin{aligned} \nu\|v^h - v\|^2 &\leq \int_\Omega a_{ij}(x,u^h)\langle D_i(v^h - v), D_j(v^h - v)\rangle \\ &= \int_\Omega a_{ij}(x,u^h)\left(\langle D_i v^h, D_j v^h\rangle - 2\langle D_i v^h, D_j v\rangle + \langle D_i v, D_j v\rangle\right), \end{aligned}$$

but $D_i v^h \rightharpoonup D_i v$ in L^2, and $a_{ij}(x,u^h)D_j v \to A_{ij}(x)D_j v$ in L^2 by Lebesgue dominated convergence theorem, therefore

$$\int_\Omega a_{ij}(x,u^h)\langle D_i v^h, D_j v\rangle \to \int_\Omega A_{ij}(x)\langle D_i v, D_j v\rangle,$$

$$\int_\Omega a_{ij}(x,u^h)\langle D_i v, D_j v\rangle \to \int_\Omega A_{ij}(x)\langle D_i v, D_j v\rangle\ ,$$

hence $v^h \to v$ in $\mathcal{D}$ and (7.1) follows.

To prove (ii) it suffices to prove that if $\{u^h\} \subset \mathcal{D}^- \oplus \mathcal{D}^0$ is a diverging sequence, then $J_0(u^h) \to -\infty$. Since $\dim\mathcal{D}^- + \dim\mathcal{D}^0 < +\infty$ and (2.8) holds, then $F_1(x,u^h) \to +\infty$ on a subset of Ω with positive measure; by (2.11) and Fatou Lemma we infer

$$\int_\Omega F_1(x,u^h) \to +\infty\ ;$$

the result follows by compactness taking into account (2.3) and the fact that if $u^h \in \mathcal{D}^- \oplus \mathcal{D}^0$, then the quadratic part of the functional is nonpositive. □

Proof of Theorem 2.1. By Lemma 6.2 and the above propositions, the assumptions of Theorem 4.1 are fulfilled and Theorem 2.1 is proved if $q \geq 1$.

If $q = 0$, then L^∞ is positive definite in $\mathcal{D}$, and by the same arguments as in the proofs of the previous propositions we infer that J is coercive; furthermore the functional satisfies the PSC condition, therefore it admits a minimum u. By a standard argument of nonsmooth critical point theory [14] we have $|dJ|(u) = 0$, hence u is a weak solution of (1.1) and the proof of Theorem 2.1 is complete.

Proof of Theorem 3.1. By the definition of the operator L^0 there exists an invariant subspace $\mathcal{D}_0^+ \subset \mathcal{D}$ of codimension pm such that $(L^0 u, u) \geq c||u||^2$ for some $c > 0$ and for all $u \in \mathcal{D}_0^+$. Furthermore, by (2.10) we have

$$\left\langle \nabla\left[\frac{F_1(x,s)}{|s|^2}\right], s\right\rangle = \frac{\langle\nabla F_1(x,s), s\rangle - 2F_1(x,s)}{|s|^2} \leq 0$$

which together with the semipositivity condition (2.3) yields

$$J(u) \geq \frac{1}{2}\int_\Omega a_{ij}(x,0)\langle D_i u, D_j u\rangle - (b(x) + f_0(x))|u|^2 = \frac{1}{2}(L^0 u, u)$$

for all $u \in \mathcal{D}$, which proves that

$$\liminf_{u\to 0,\ u\in\mathcal{D}_0^+} \frac{J(u)}{||u||^2} > 0,$$

therefore the hypotheses of Theorem 4.3 are fulfilled and the proof of Theorem 3.1 is complete.

8. Proofs of the superlinear theorems

Proof of Theorem 2.2. In the proof of Lemma 5.4 we proved that J has a mountain pass geometry; moreover, by Lemma 6.3 the functional satisfies the PS condition. The result then follows by applying Theorem 4.2.

Proof of Theorem 2.3. Also in this case J has a mountain pass geometry and a PS sequence may be constructed by the standard infmax procedure; if $\sup|\nabla a_{ij}(x,s)|$ is sufficiently small, by Lemma 6.5 we infer that the functional J satisfies the PS condition at the infmax level and the result follows again by applying Theorem 4.2.

Proof of Theorem 3.2. By Lemma 6.3, the functional J satisfies the PS condition; therefore Theorem 4.3 applies and yields the result.

Proof of Theorem 3.3. Observe that the assumptions of Theorem 4.4 are satisfied with $p = 0$, so we have a sequence $\{c_k\}$ such that for all k there exists a PS sequence $\{u_h^k\}$ for J at level c_k. Moreover, by definition of J_∞ and by Lemma 2.30 in [3] there exists a diverging sequence $\{c_k^\infty\}$ of critical levels of J_∞; by the infmax characterization of the levels $\{c_k\}$ and $\{c_k^\infty\}$ and by (2.3) we have $c_k \le c_k^\infty$ for all k. Hence, by the same argument of Lemma 5.4 we obtain a sequence $\{M_k\}$ such that $||u_h^k|| \le M_k$ for all k. Now fix an integer k: then Lemmas 6.4 and 6.5 yield a constant L_k such that if $|\nabla a_{ij}(x,s)| \le L_k$ for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^m$ then $\{u_h^r\}$ is precompact for all $r = 1, ..., k$; if all the c_r are different we obtain k distinct critical levels while if $c_r = c_{r+1}$ for some $r \le k-1$ we obtain infinitely many geometrically distinct critical orbits.

9. Appendix: examples and further remarks

Example 9.1. Assume that there exists $M > 0$ such that

$$\nabla a_{ij}(x,s) \equiv 0 \qquad \text{if } |s| \ge M$$

and, for all $\xi \in \mathbb{R}^n$,

$$\sum_k |\partial_k a_{ij}(x,s)\xi_i\xi_j| \le \frac{1}{2eM} a_{ij}(x,s)\xi_i\xi_j \qquad \text{if } |s| \le M.$$

Then (2.4) holds with $K = \frac{1}{4}$ and

$$\psi(t) = \begin{cases} \frac{t}{4M} & \text{if } 0 \le t \le M \\ \frac{1}{4} & \text{if } t \ge M. \end{cases}$$

Example 9.2. Take $a_{ij}(x,s) = a_{ij}(s) = (\nu + \arctan(|s|^2))\delta_{ij}$ with $\nu \ge e\sqrt{m}(\sqrt{3} + \pi)$. Then there exist $K > 0$ and a function ψ satisfying (2.4). Indeed by the assumption on ν we have

$$\frac{\sqrt{m}}{\nu}\left[\frac{\sqrt{3}}{4}+\int_{3^{-1/4}}^{\infty}\frac{\tau}{1+\tau^4}d\tau\right]\leq\frac{1}{4e}.$$

Therefore, there exists $K>0$ such that

$$\frac{\sqrt{m}}{\nu}\left[\frac{\sqrt{3}}{4}+\int_{3^{-1/4}}^{\infty}\frac{\tau}{1+\tau^4}d\tau\right]=Ke^{-4K}.$$

For such K define the function ψ such that

$$\psi(t)=\frac{\sqrt{m}e^{4K}}{\nu}\begin{cases}\frac{3^{3/4}}{4}t & \text{if } \ 0\leq t\leq 3^{-1/4}\\ \frac{\sqrt{3}}{4}+\int\limits_{3^{-1/4}}^{t}\frac{\tau}{1+\tau^4}d\tau & \text{if } \ t\geq 3^{-1/4}\end{cases}$$

Then, (i) (ii) of (2.4) are readily verified; moreover $\psi''(t)\leq 0$, which proves (iii); finally,

$$|\partial_k a_{ij}(s)\xi_i\xi_j|=\left|\frac{2s_k}{1+|s|^4}|\xi|^2\right|$$

which yields

$$\sum_k|\partial_k a_{ij}(s)\xi_i\xi_j|\leq\frac{2\sqrt{m}|s|}{1+|s|^4}|\xi|^2\leq 2e^{-4K}\psi'(|s|)(\nu+\arctan(|s|^2))|\xi|^2$$

and proves (iv) of (2.4).

Example 9.3. More generally, consider the (continuous) function $\beta:[0,+\infty)\to[0,+\infty)$ defined by

$$\beta(t)=\frac{1}{2\nu}\sum_{k=1}^{m}\left(\operatorname*{esssup}_{x\in\Omega}\max_{|u|=t}\max_{|\xi|=1}|\partial_k a_{ij}(x,u)\xi_i\xi_j|\right);$$

clearly, $\beta(0)=0$, β admits a global maximum point and $\lim_{t\to+\infty}\beta(t)=0$. Next, define the function $\alpha:[0,+\infty)\to[0,+\infty)$ by

$$\alpha(t)=\max_{\tau\geq t}\beta(\tau) \tag{9.1}$$

If

$$\int_0^{+\infty}\alpha(\tau)d\tau\leq\frac{1}{4e}, \tag{9.2}$$

then there exist $K>0$ and a function $\psi\in C^1(\mathbb{R})$ satisfying (2.4). Indeed, by (9.2) there exists $K>0$ such that

$$\int_0^{+\infty}\alpha(\tau)d\tau=Ke^{-4K};$$

for such K define the function

$$\psi(t) = e^{4K} \int_0^t \alpha(\tau) d\tau.$$

Conditions (i) and (ii) of (2.4) are readily verified; condition (iii) follows from the fact that α is non-increasing on $\mathbb{R}_+$. Finally, (iv) follows from the definition of α in (9.1) and from (2.2).

Example 9.4. An example of a sublinear term satisfying (F0) and (F1) is given by $F_1(x,s) = |s|^2(1+|s|^2)^{-1/3}e^{-|x|}$.

Remark 9.1. Under the assumptions of Theorem 2.1 the equation is said to be resonant when the corresponding linear operator at infinity has a nontrivial kernel; the resonant case is more difficult to handle because no a priori estimates are available. Our proof holds both in the resonant and in the nonresonant cases.

Remark 9.2. If Ω is a bounded set, then assumptions (2.12) and (2.14) may be weakened by requiring that they only hold for sufficiently large values of $|s|$.

Remark 9.3. If in Theorems 2.2 and 3.2 one takes assumption (F3) instead of the weaker (F2), then by Lemma 6.4 the solution(s) are essentially bounded.

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