Steady Navier-Stokes equations in planar domains with obstacle and explicit bounds for unique solvability

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Abstract

Fluid flows around an obstacle generate vortices which, in turn, generate forces on the obstacle. This phenomenon is studied for planar viscous flows governed by the stationary Navier-Stokes equations with inhomogeneous Dirichlet boundary data in a (virtual) square containing an obstacle. In a symmetric framework the appearance of forces is strictly related to multiplicity of solutions. Precise bounds on the data ensuring uniqueness are then sought and several functional inequalities (concerning relative capacity, Sobolev embedding, solenoidal extensions) are analyzed in detail: explicit bounds are obtained for constant boundary data. The case of "almost symmetric" frameworks is also considered. A universal threshold on the Reynolds number ensuring that the flow generates no lift is obtained regardless of the shape and the nature of the obstacle. Based on the asymmetry/multiplicity principle, the performance of different obstacle shapes is then compared numerically. Finally, connections of the results with elasticity and mechanics are emphasized.
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Contents

1 Introduction

The whole science of flight is based on the understanding and control of the *lift force*, the resistance component orthogonal to the aircraft direction of motion, see e.g. [3, Chapter 1]. The modern theory of lift, developed in the fundamental works of Kutta [54] and Zhukovsky [76] at the beginning of the 20th century (see also [3] for the English translation), relies on the principle that a cambered surface produces lift through its ability to generate vortices about itself, see Figure 1.1 for a wind tunnel experiment.

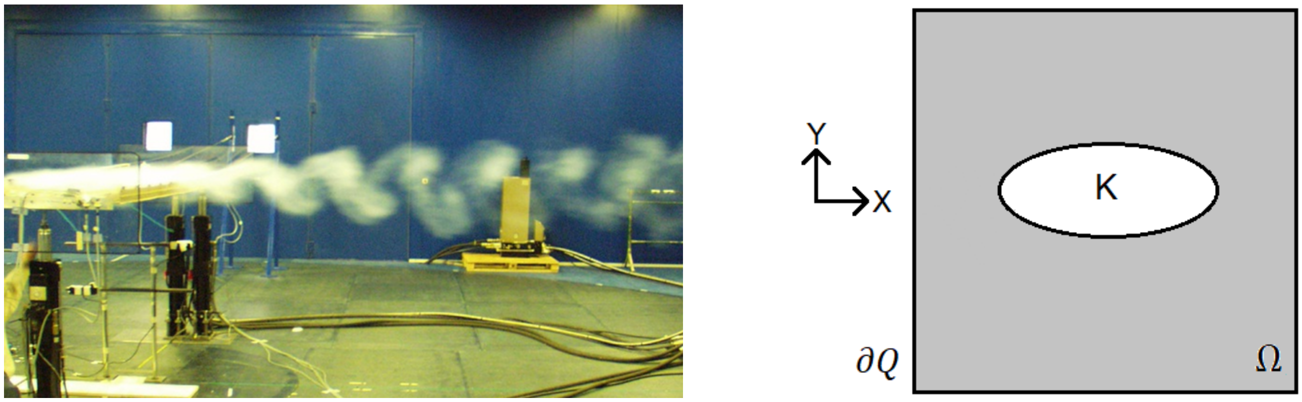

Figure 1.1: Left: vortices around a plate obtained in wind tunnel experiments at the Politecnico di Milano. Right: the planar domain Ω in (1.1) with a smooth obstacle K.

The celebrated d'Alembert paradox [60] shows that the lift is characteristic of viscous fluids so that the full evolution of aerodynamics was possible only after a precise comprehension of viscosity. Vortices in fluid dynamics appear both for turbulent flows with large Reynolds number and whenever a fluid surrounds an obstacle. The vortices generate a lift force acting on the obstacle orthogonally to the direction of the flow so that, if one considers a rigid obstacle having the shape of a 3D cylinder (the cartesian product of a planar compact set $\overline{K}$ with a bounded interval, as in the left picture of Figure 1.1), it is convenient to restrict the attention to the cross-section $\overline{K}$ of the cylinder.

In the plane $\mathbb{R}^2$ we consider an obstacle, represented by an open bounded simply connected domain K with Lipschitz boundary ∂K, and a big squared box Q containing the obstacle and such that $\partial Q \cap \partial K = \emptyset$. More precisely, we consider the domains

$$Q = (-L, L)^2\,, \qquad \Omega = Q \setminus \overline{K} \qquad \big(L \gg \mathrm{diam}(K)\big)\,, \tag{1.1}$$

where Ω should be seen as a sufficiently large (bounded) region surrounding $\overline{K}$. The boundary of Ω is split into two parts, $\partial\Omega = \partial K \cup \partial Q$, and the outward unit normal $\hat{n}$ is defined a.e. on $\partial\Omega$. This geometry appears to be the best choice to model, for instance, the motion of the wind around the cross-section of a bridge for which one needs a (squared) photo of the flow in a sufficiently large neighborhood, as in the left picture in Figure 1.1 but on a larger scale. A sketch of this geometry is illustrated in the right picture in Figure 1.1 (not in scale and with smooth ∂K).

In this paper we provide the tools for the full theory of planar stationary flows of viscous fluids around an obstacle, assuming that they are governed by the steady Navier-Stokes equations

$$-\eta\Delta u + (u\cdot\nabla)u + \nabla p = f, \qquad \nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{1.2}$$

where $u : \Omega \to \mathbb{R}^2$ is the velocity vector field, $p : \Omega \to \mathbb{R}$ is the scalar pressure, $f : \Omega \to \mathbb{R}^2$ denotes an external forcing term and $\eta > 0$ is the kinematic viscosity. To (1.2) we associate the boundary data

$$u = (U, V) \ \text{ on } \ \partial Q, \qquad u = (0, 0) \ \text{ on } \ \partial K, \tag{1.3}$$

for some given $(U, V) \in H^{1/2}(\partial Q)$ satisfying the compatibility condition (zero flux across ∂Q)

$$\int_{-L}^{L} [U(L, y) - U(-L, y)]\, dy + \int_{-L}^{L} [V(x, L) - V(x, -L)]\, dx = 0. \tag{1.4}$$

The boundary conditions (1.3) model the inflow/outflow of fluid across the boundary ∂Q with velocity (U, V), and with no-slip condition on the obstacle K where viscosity yields zero velocity of the flow. The inhomogeneous boundary datum (U, V) on ∂Q is mandatory since, as explained above, Q represents a virtual box (a planar region where the flow is analyzed) and *not* a region with solid boundary (contrary to the obstacle). For some of our results we focus the attention on the case where $(U, V) \in \mathbb{R}^2$ is constant on ∂Q; this choice is motivated by the fact that Q is much larger than K and possible effects of the vortex shedding created by the obstacle are not detectable far away from it.

It is well-known [39] that uniqueness for (1.2)-(1.3) is ensured only whenever the data f and (U, V) are "small" compared to the viscosity η, see also Theorem 3.1 below. The proof relies on a priori bounds which lead to a contradiction if one assumes the existence of multiple solutions of (1.2)-(1.3). While in the case of homogeneous Dirichlet boundary conditions $(U, V) = (0, 0)$ the a priori bounds may be obtained by testing the equation with the solution itself, in the inhomogeneous case $(U, V) \neq (0, 0)$ they are extremely delicate because the solution of (1.2)-(1.3) is not an admissible test function. The standard approach is to transform the inhomogeneous Dirichlet problem into a homogeneous one by determining a solenoidal extension of the boundary velocity, namely one needs to find a vector field w such that

$$\nabla \cdot w = 0 \ \text{ in } \ \Omega, \qquad w = (U, V) \ \text{ on } \ \partial Q, \qquad w = (0, 0) \ \text{ on } \ \partial K. \tag{1.5}$$

This problem, whose interest and applicability go far beyond fluid mechanics, has a long story, starting from the pioneering works by Cattabriga [18] and Ladyzhenskaya-Solonnikov [56, 57]; see also the book by Galdi [39, Section III.3]. Finding explicit bounds for solutions of (1.5) is an extremely difficult task and usually requires to introduce cutoff functions. Instead, when $(U, V) \in \mathbb{R}^2$, in Section 2.4 we construct a merely $\mathcal{C}^1$-extension by combining classical arguments [56, p.130] with suitable bounds for the relative capacity of the obstacle and repeated applications of the Maximum Principle for harmonic functions.

Finding explicit theoretical bounds for the critical Reynolds number, i.e. for the stability of the steady flow of a viscous fluid, constitutes a fundamental problem in fluid mechanics, see [59, Chapter III], closely related to the onset of turbulence from a laminar regime [58]. As we shall see in Section 3.3, in a symmetric framework the appearance of effective lift forces exerted by the fluid on the obstacle K is strictly related to non-uniqueness of solutions of (1.2)-(1.3). Therefore, for the uniqueness threshold of (1.2)-(1.3), explicit bounds are needed, as precise as possible. In turn, the uniqueness threshold is obtained through a priori bounds for the solutions of (1.5) but, so far, no such bounds are available in the literature. Obtaining explicit bounds for (1.5) and several related functional inequalities is precisely the first purpose of the present paper.

In Section 2 we obtain several bounds on the relative capacity of the obstacle K with respect to Q and on some Sobolev embedding constants; moreover, we suggest a new way to bound the solenoidal extension w in (1.5). For the relative capacity, we first prove a general statement (valid in any space dimension) that gives exact values for "weighted capacities", see Theorem 2.1. Then we seek bounds for the relative capacity of the obstacle. In [43], the first author defined the space of *web functions*, namely the subspace of $H^1_0(\Omega)$ comprising functions which only depend on the distance from the boundary $\partial\Omega$. These functions were previously introduced by Szegö [74] in a slightly different context. The main novelty in [43] was the possibility of obtaining bounds for some constants arising in variational problems, see [24, 26] and also [25] for bounds on the capacity. In our context of non simply connected domain, we cannot use web functions and we introduce instead the subset of *pyramidal functions*, see (2.8), in order to obtain bounds for the relative capacity of the obstacle. We also need to bound the Sobolev constant for the embedding $H^1(\Omega) \subset L^4(\Omega)$, which arises naturally due to the convective term in (1.2): here we have to face both the difficulties of dealing with a non simply connected domain and of inhomogeneous boundary data, especially because we seek precise estimates. For this reason, we use an optimal Gagliardo-Nirenberg inequality by del Pino-Dolbeault [27] with some adjustments: we combine it with Hölder and Poincaré inequalities in the case of zero traces and with a delicate ad hoc argument for nonzero traces, see Theorem 2.3. Nowadays numerics can give precise bounds, but only for given specific geometries. On the contrary, our theoretical bounds are independent of the geometry; we also show that they are fairly precise, see Remark 2.1 and Corollary 2.2. For this reason, and for possible further developments, we embed our results in a general theory which goes beyond the applications given in this paper.

The second main purpose of the present work is to obtain precise statements about the lift exerted by the solutions of (1.2)-(1.3) on the obstacle K. To this end, we need the bounds obtained in the first part: in particular, we use the pyramidal capacity approach in order to obtain bounds for the solutions of (1.5). The existence of symmetric solutions of the stationary Navier-Stokes equations has been proved in smooth symmetric domains in the pioneering work by Amick [5] and, subsequently, by several other authors [34, 35, 53, 61, 63]. As already mentioned, our focus is different, we connect symmetric solutions with uniqueness and with the computation of the lift. In Theorem 3.4 we study (1.2)-(1.3) in a perfectly symmetric situation, where a symmetric solution always exists and possible non-uniqueness is strictly related to the existence of asymmetric solutions. In Section 3.3 we define the drag and the lift, namely the forces exerted by the fluid governed by (1.2) on the bluff body represented by the obstacle K. We focus most of our attention on the lift force since it is responsible for the instability of K, as in civil engineering structures where it leads to dangerous oscillations. In regime of uniqueness, we prove that there is no lift in a symmetric situation and that the lift is small in an "almost symmetric" situation, see Theorem 3.7. This means that instability and/or non-uniqueness may appear only in asymmetric situations or with large data. Theorem 3.9 uses all the just mentioned results and gives an explicit universal bound such that, if a constant inflow velocity of the fluid is below this bound, then the obstacle is not subject to a lift force. In turn, this result also yields explicit bounds for the threshold of stability of a bluff body immersed in a viscous fluid.

While our bounds *do not* depend on the shape of the obstacle, one expects that the threshold of stability does depend on the shape. However, there is no available theory able to analyze the shape dependence of the lift, see [10] for related results about the drag. Therefore, in Section 3.5 we proceed through Computational Fluid Dynamics (CFD) by using the OpenFOAM toolbox. We use an asymmetry/multiplicity principle (see Corollary 3.2) in order to compute the performance of several obstacles having the same measure but different shapes. The idea is to numerically detect non-uniqueness for (1.2)-(1.3) by finding asymmetric solutions in a symmetric framework. The obtained numerical results give strong hints on which could be the best shape yielding the largest inflow velocity (U, V) ensuring that the lift is zero. They also strengthen a conjecture by Pironneau [68, 69] claiming that the inward face should look like a "rugby ball", see in particular [68, Figure 3], in order to minimize the drag. In fact, the numerical bounds for stability should not be compared with the theoretical ones obtained in Section 2, because the latter are found for a very large class of obstacles.

Finally, we mention that the functional inequalities discussed in Section 2, in particular the bound for solenoidal extensions, have several applications in different areas of mathematical physics. A whole bunch of inequalities arises both in fluid mechanics and elasticity [7, 23, 33, 50, 52], and they are all linked to each other. This is why Section 4 is devoted to some physical applications of our results. In Section 4.1 we embed our 2D results in a 3D framework where, in fact, the Navier-Stokes equations admit solutions depending only on two variables. We then apply our results to the stability of suspension bridges [44]: in Corollary 4.1 we state a sufficient condition on the wind velocity ensuring that the bridge will not oscillate. In Section 4.2 we show that the bifurcation phenomenon for the Navier-Stokes equations, related to the loss of symmetry, has a counterpart in a model of a buckled elastic plate.

This paper is organized as follows. In Section 2 we state and prove some functional inequalities with explicit constants, in particular: inequalities for the relative capacity, for the embedding $H^1(\Omega) \subset L^4(\Omega)$, and a priori bounds for (1.5). In Section 3 we set up the main tools for the study of (1.2)-(1.3), we analyze in detail symmetric and almost symmetric situations, we relate the appearance of lift with multiplicity of solutions; we provide numerical results giving some hints on which could be the most stable obstacle shape. Section 4 is devoted to some physical applications and interpretations of our results, while Section 5 contains some concluding remarks and several open problems.

2 Functional inequalities

Although we shall deal both with scalar and vector fields (or matrices), all the functional spaces will be denoted in the same way (except for Section 4.1).

2.1 Relative capacity and pyramidal functions

Let Ω be as in (1.1). The relative capacity of K with respect to Q is defined by

$$\mathrm{Cap}_Q(K) = \min_{\substack{v \in H^1_0(Q) \\ v=1 \text{ in } K}} \int_Q |\nabla v|^2 \tag{2.1}$$

and the relative capacity potential ψ, which achieves the minimum in (2.1), satisfies

$$\Delta\psi = 0 \;\text{ in }\; \Omega = Q \setminus K, \qquad \psi = 0 \;\text{ on }\; \partial Q, \qquad \psi = 1 \;\text{ in }\; K, \qquad \mathrm{Cap}_Q(K) = \|\nabla\psi\|^2_{L^2(\Omega)}. \tag{2.2}$$

We start with a general result concerning *weighted relative capacities*, that will be employed in Section 2.4. We state it in the framework of our model but the result remains true for all relative capacity problems, in any space dimension.

Theorem 2.1. *Let Ω be as in* (1.1) *and let $\psi \in H^1(\Omega)$ be the relative capacity potential of K with respect to Q, as in* (2.2). *For any function $g \in \mathcal{C}([0,1];\mathbb{R})$ we have*

$$\int_\Omega g(\psi)\, |\nabla\psi|^2 = \left(\int_0^1 g(t)\, dt\right) \mathrm{Cap}_Q(K). \tag{2.3}$$

Proof. Notice that an integration by parts yields

$$\mathrm{Cap}_Q(K) = -\int_\Omega \psi\Delta\psi + \int_{\partial\Omega} \psi \frac{\partial\psi}{\partial n} = \int_{\partial K} \nabla\psi \cdot \hat{n}\,, \tag{2.4}$$

where $\hat{n}$, the outward unit normal to $\partial\Omega$, is directed towards the interior of K. Consider any closed curve $\Gamma \subset \Omega$ that (strictly) encloses K, and define by $\Omega_\Gamma \subset \Omega$ the region delimited by ∂K and Γ. Since ψ is harmonic in Ω_Γ, the Divergence Theorem yields

$$0 = \int_{\Omega_\Gamma} \Delta\psi = \int_{\partial\Omega_\Gamma} \frac{\partial\psi}{\partial n} = -\int_\Gamma \nabla\psi \cdot \hat{n} + \int_{\partial K} \nabla\psi \cdot \hat{n}\,,$$

so that, in view of (2.4), we have

$$\mathrm{Cap}_Q(K) = \int_\Gamma \nabla\psi \cdot \hat{n}\,. \tag{2.5}$$

In particular, given $0 < \alpha < 1$ and denoting by Γ_α the α-level line of ψ (which, owing to the Maximum Principle, encloses K), thanks to (2.5) we have

$$\int_{\psi^{-1}([\alpha,1))} |\nabla\psi|^2 = -\int_{\psi^{-1}([\alpha,1))} \psi\Delta\psi - \alpha\int_{\Gamma_\alpha} \nabla\psi \cdot \hat{n} + \int_{\partial K} \nabla\psi \cdot \hat{n} = (1-\alpha)\mathrm{Cap}_Q(K).$$

In turn, this implies that

$$\int_{\psi^{-1}([\alpha,\beta])} |\nabla\psi|^2 = (\beta-\alpha)\, \mathrm{Cap}_Q(K) \qquad \text{for any }\; 0 < \alpha < \beta < 1. \tag{2.6}$$

Given $g \in \mathcal{C}([0,1];\mathbb{R})$ and $n \geq 2$, take any partition $\{a_1, \dots, a_n\}$ of the interval $[0,1]$, where $a_0 = 0$ and $a_n = 1$. In view of (2.6) we then have

$$\int_\Omega g(\psi)\,|\nabla\psi|^2 = \sum_{i=1}^{n} \left(\int\limits_{\psi^{-1}([a_{i-1},a_i])} g(\psi)\,|\nabla\psi|^2 \right) \leq \sum_{i=1}^{n} \left(\max_{s\in[a_{i-1},a_i]} g(s) \right) \left(\int\limits_{\psi^{-1}([a_{i-1},a_i])} |\nabla\psi|^2 \right)$$

$$= \mathrm{Cap}_Q(K) \sum_{i=1}^{n} (a_i - a_{i-1}) \max_{s\in[a_{i-1},a_i]} g(s)$$

and, similarly,

$$\int_\Omega g(\psi)\,|\nabla\psi|^2 \geq \mathrm{Cap}_Q(K) \sum_{i=1}^{n} (a_i - a_{i-1}) \min_{s\in[a_{i-1},a_i]} g(s)\,.$$

The statement follows by letting $n \to \infty$ in the two last inequalities. □

The exact value of the relative capacity is in general not known. In the next result, which has its own interest regardless of the applications considered in the present work, we give lower and upper bounds of it in a particular situation. The same idea will also be used to bound the gradients of some solenoidal extensions, see Theorem 2.5 in Section 2.4.

Theorem 2.2. *Consider the square $Q = (-L,L)^2$ and the rectangle $\mathcal{R} = (-a,a) \times (-d,d)$, where $a, d \in (0,L)$. Then*

$$\frac{2\pi}{\log(L) - \log\left(\sqrt{ad}\right)} \leq \mathrm{Cap}_Q(\mathcal{R}) \leq 4\,\frac{(L-a)^2 + (L-d)^2}{(L-a)(L-d)} \left[\log\left(\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right)\right]^{-1}. \qquad (2.7)$$

Proof. Divide the domain $Q \setminus \mathcal{R}$ into four trapezia T_1, T_2, T_3, T_4 as in the left picture in Figure 2.1.

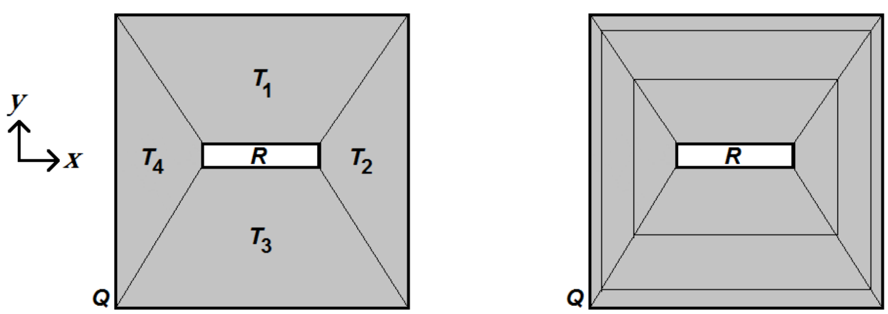

Figure 2.1: The domain $Q \setminus \mathcal{R}$ (left) and the level lines of pyramidal functions (right).

By *pyramidal function* we mean any function having the level lines as in the right picture of Figure 2.1, namely level lines parallel to ∂Q (and $\partial\mathcal{R}$) in each of the trapezia. In particular, pyramidal functions are constant on $\partial\mathcal{R}$ and constitute the following convex subset of $H^1_0(Q)$:

$$\mathcal{P}(Q) = \{u \in H^1_0(Q) \mid u = 1 \text{ in } \mathcal{R},\ u = u(y) \text{ in } T_1 \cup T_3,\ u = u(x) \text{ in } T_2 \cup T_4\}\,. \qquad (2.8)$$

Since $\mathcal{P}(Q) \subset H^1_0(Q)$, the relative capacity (2.1) may be upper bounded through the inequality

$$\mathrm{Cap}_Q(\mathcal{R}) \leq \min_{v\in\mathcal{P}(Q)} \int_Q |\nabla v|^2\,. \qquad (2.9)$$

We are so led to find the minimum in (2.9) and this is equivalent to solve a classical problem in calculus of variations. Precisely, any $V^\phi \in \mathcal{P}(Q)$ is fully characterized by a (continuous) function

$$\phi \in H^1([0,1];\mathbb{R}) \quad \text{such that} \quad \phi(0) = 1\,, \quad \phi(1) = 0\,, \qquad (2.10)$$

giving the values of V^ϕ on the oblique edges of the trapezia. For instance, consider the right trapezia $T_5, T_6 \subset Q$ being, respectively, half of the trapezia T_1 and T_2, defined by

$$T_5 = \left\{ (x,y) \in Q \;\middle|\; d < y < L,\ 0 < x < a + \frac{L-a}{L-d}(y-d) \right\}, \tag{2.11}$$

$$T_6 = \left\{ (x,y) \in Q \;\middle|\; a < x < L,\ 0 < y < d + \frac{L-d}{L-a}(x-a) \right\}. \tag{2.12}$$

Since V^ϕ is a function of y in T_1 and a function of x in T_2, ϕ and V^ϕ are linked through the formulas

$$V^\phi(x,y) = \phi\left(\frac{y-d}{L-d}\right) \qquad \forall (x,y) \in T_5, \qquad\qquad V^\phi(x,y) = \phi\left(\frac{x-a}{L-a}\right) \qquad \forall (x,y) \in T_6. \tag{2.13}$$

Whence,

$$\frac{\partial V^\phi}{\partial y}(x,y) = \frac{1}{L-d}\phi'\left(\frac{y-d}{L-d}\right) \quad \forall (x,y) \in T_5, \quad \frac{\partial V^\phi}{\partial x}(x,y) = \frac{1}{L-a}\phi'\left(\frac{x-a}{L-a}\right) \quad \forall (x,y) \in T_6. \tag{2.14}$$

We then seek the optimal ϕ minimizing the Dirichlet integral over Q of the pyramidal function V^ϕ.

For symmetry reasons, the contribution of $|\nabla V^\phi|$ over $T_1 \cup T_3$ is four times the contribution over T_5, whereas the contribution of $|\nabla V^\phi|$ over $T_2 \cup T_4$ is four times the contribution over T_6. By taking into account all these facts, in particular (2.14), we infer that

$$\begin{aligned}
\int_{Q\setminus\mathcal{R}} |\nabla V^\phi|^2 &= 4\int_d^L \int_0^{a+\frac{L-a}{L-d}(y-d)} \left|\frac{\partial V^\phi}{\partial y}\right|^2 dx\,dy + 4\int_a^L \int_0^{d+\frac{L-d}{L-a}(x-a)} \left|\frac{\partial V^\phi}{\partial x}\right|^2 dy\,dx \\
&= 4\int_d^L \left[a + \frac{L-a}{L-d}(y-d)\right] \left|\frac{\partial V^\phi}{\partial y}\right|^2 dy + 4\int_a^L \left[d + \frac{L-d}{L-a}(x-a)\right] \left|\frac{\partial V^\phi}{\partial x}\right|^2 dx \\
&= 4\int_0^1 \left(\frac{a+(L-a)s}{L-d} + \frac{d+(L-d)s}{L-a}\right) \phi'(s)^2\,ds \\
&= 4\,\frac{(L-a)^2+(L-d)^2}{(L-a)(L-d)} \int_0^1 \left[\frac{a(L-a)+d(L-d)}{(L-a)^2+(L-d)^2} + s\right] \phi'(s)^2\,ds\,.
\end{aligned} \tag{2.15}$$

Minimizing (2.15) among functions ϕ satisfying (2.10) yields the Euler-Lagrange equation

$$\frac{d}{ds}\left[\left(\frac{a(L-a)+d(L-d)}{(L-a)^2+(L-d)^2} + s\right)\phi'(s)\right] = 0 \quad \Longrightarrow \quad \phi'(s) = \frac{C}{\frac{a(L-a)+d(L-d)}{(L-a)^2+(L-d)^2} + s} \qquad \forall s \in [0,1]$$

so that

$$\phi(s) = C\log\left(s + \frac{a(L-a)+d(L-d)}{(L-a)^2+(L-d)^2}\right) + D \qquad \forall s \in [0,1],$$

for some constants C, D to be determined by imposing the conditions $\phi(0) = 1$ and $\phi(1) = 0$. We find

$$C = \left[\log\left(\frac{a(L-a)+d(L-d)}{L(L-a)+L(L-d)}\right)\right]^{-1} < 0$$

and, by inserting this into (2.15), we obtain

$$\min_{v\in\mathcal{P}(Q)} \int_Q |\nabla v|^2 = 4\,\frac{(L-a)^2+(L-d)^2}{(L-a)(L-d)} \left[\log\left(\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right)\right]^{-1}. \tag{2.16}$$

The upper bound in (2.7) follows from (2.9) and (2.16).

The lower bound in (2.7) is obtained through symmetrization. Let $\psi \in H_0^1(Q)$ be the relative capacity potential of $\mathcal{R}$ with respect to Q (see (2.2)), that is:

$$\Delta\psi = 0 \text{ in } Q \setminus \mathcal{R}, \qquad \psi = 0 \text{ on } \partial Q, \qquad \psi = 1 \text{ in } \mathcal{R}, \qquad \mathrm{Cap}_Q(\mathcal{R}) = \|\nabla\psi\|^2_{L^2(Q)}. \tag{2.17}$$

From the maximum principle we know that $0 \le \psi \le 1$ in $Q \setminus \mathcal{R}$, and hence in Q. Let $Q^* \subset \mathbb{R}^2$ be the disk centered at the origin of radius $r_2 = 2L/\sqrt{\pi}$, and $\mathcal{R}^* \subset \mathbb{R}^2$ be the disk centered at the origin of radius $r_1 = 2\sqrt{ad/\pi}$ (so that $|Q^*| = |Q|$ and $|\mathcal{R}^*| = |\mathcal{R}|$). The *symmetric decreasing rearrangement* $\psi^* \in H_0^1(Q^*)$ of ψ satisfies $\psi^* = 0$ on ∂Q^*, $\psi^* = 1$ in $\mathcal{R}^*$, $\|\nabla\psi^*\|_{L^2(Q^*)} \le \|\nabla\psi\|_{L^2(Q)}$ (see [70] for more details), so that, by (2.17),

$$\mathrm{Cap}_{Q^*}(\mathcal{R}^*) \le \|\nabla\psi^*\|^2_{L^2(Q^*)} \le \mathrm{Cap}_Q(\mathcal{R}). \tag{2.18}$$

The relative capacity potential of $\mathcal{R}^*$ with respect to Q^*, denoted by $\varphi \in H_0^1(Q^*)$, is the radial function

$$\varphi(\rho) = \frac{\log(\rho) - \log(r_2)}{\log(r_1) - \log(r_2)} \qquad \forall \rho \in [r_1, r_2], \qquad \varphi(\rho) = 1 \qquad \forall \rho \in [0, r_1],$$

so that

$$\mathrm{Cap}_{Q^*}(\mathcal{R}^*) = \|\nabla\varphi\|^2_{L^2(Q^*)} = \frac{2\pi}{\log(L) - \log\left(\sqrt{ad}\right)}.$$

Combined with (2.18), this concludes the proof of the lower bound. □

Remark 2.1. *When $d = a$, the inequalities in* (2.7) *become*

$$\frac{2\pi}{\log(L) - \log(a)} \le \mathrm{Cap}_Q(\mathcal{R}) \le \frac{8}{\log(L) - \log(a)},$$

so that $\mathrm{Cap}_Q(\mathcal{R})$ is estimated with a relative error of $(8 - 2\pi)/(2\pi) \approx 0.27$. Moreover, by using the same symmetrization method as in the proof of Theorem 2.2 we see that, for a general obstacle $K \subset Q$, one obtains the following lower bound for the relative capacity:

$$\mathrm{Cap}_Q(K) \ge \frac{4\pi}{\log(|Q|) - \log(|K|)}\,. \tag{2.19}$$

2.2 Bounds for some Sobolev constants

Let Ω be as in (1.1). We consider both the Sobolev space $H_0^1(\Omega)$ and the space of functions vanishing only on ∂K, which is a proper connected part of $\partial\Omega$ having positive 1D-measure:

$$H_*^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \ \text{ on } \ \partial K\}\,.$$

This space is the closure of $C_c^\infty(\overline{Q} \setminus \overline{K})$ with respect to the norm $v \mapsto \|\nabla v\|_{L^2(\Omega)}$: since $|\partial K|_1 > 0$ (the 1D-Hausdorff measure), the Poincaré inequality holds in $H_*^1(\Omega)$, which means that $v \mapsto \|\nabla v\|_{L^2(\Omega)}$ is indeed a norm on $H_*^1(\Omega)$, see [28]. Then we introduce the following proper subspace of $H_*^1(\Omega)$:

$$H_c^1(\Omega) = \{v \in H_*^1(\Omega) \mid v \text{ is constant on } \partial Q\}\,.$$

This space may be rigorously characterized by using the relative capacity potential ψ of K with respect to Q, see (2.2); it has the geometric characterization

$$H_c^1(\Omega) = H_0^1(\Omega) \oplus \mathbb{R}(\psi - 1)\,, \quad H_0^1(\Omega) \perp \mathbb{R}(\psi - 1)\,, \tag{2.20}$$

so that $H_0^1(\Omega)$ has codimension 1 within $H_c^1(\Omega)$ and the "missing dimension" is spanned by the function $\psi - 1$. To see this, determine the orthogonal complement of $H_0^1(\Omega)$ within $H_c^1(\Omega)$ as follows:

$$v \in H_0^1(\Omega)^\perp \ \Leftrightarrow \ v \in H_c^1(\Omega)\,, \ \int_\Omega \nabla v \cdot \nabla w = 0 \ \ \forall w \in H_0^1(\Omega) \ \Leftrightarrow \ v \in H_c^1(\Omega)\,, \ \langle \Delta v, w\rangle_\Omega = 0 \ \ \forall w \in H_0^1(\Omega)$$

so that v is weakly harmonic and, since $v \in H^1_c(\Omega)$, it is necessarily a real multiple of $\psi - 1$.

For later use, let us introduce

$$\mu_0 = \text{the first zero of the Bessel function of first kind of order zero} \approx 2.40483\,. \tag{2.21}$$

Then we define the three Sobolev constants

$$\mathcal{S} = \min_{v \in H^1_*(\Omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^4(\Omega)}}, \qquad \mathcal{S}_0 = \min_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^4(\Omega)}}, \qquad \mathcal{S}_1 = \min_{v \in H^1_c(\Omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^4(\Omega)}}. \tag{2.22}$$

Since $H^1_0(\Omega) \subset H^1_c(\Omega) \subset H^1_*(\Omega)$, we have $\mathcal{S} \le \mathcal{S}_1 \le \mathcal{S}_0$. Our first result in this section provides explicit lower bounds for these embedding constants.

Theorem 2.3. *Let Ω be as in (1.1). For any $u \in H^1_0(\Omega)$ one has*

$$\|u\|^2_{L^4(\Omega)} \le \frac{2L}{\sqrt{3}\pi^{3/2}} \min\left\{1, \frac{\sqrt{2\pi}}{\mu_0}\sqrt{1 - \frac{|K|}{|Q|}}\right\} \|\nabla u\|^2_{L^2(\Omega)}\,. \tag{2.23}$$

For any $u \in H^1_c(\Omega)$ one has

$$\begin{aligned} \|u\|^2_{L^4(\Omega)} \le{}& \frac{4L}{3\pi}\sqrt{1 - \frac{|K|}{|Q|}} \left(1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}\right)^{3/2} \\ &\times \left[1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)} + \frac{3\sqrt{3}}{4\sqrt{2}}\frac{|K|}{|Q| - |K|}\log^{3/2}\left(\frac{|Q|}{|K|}\right)\right]^{1/2} \|\nabla u\|^2_{L^2(\Omega)}. \end{aligned} \tag{2.24}$$

The inequalities (2.23) and (2.24) hold both for scalar functions and for vector fields.

Proof. We first show that it suffices to prove the inequalities for scalar functions. Indeed, assume that (2.23) has been proved for scalar functions and let $u = (u_1, u_2) \in H^1_0(\Omega)$ be a vector filed. Then, by the Hölder inequality and the scalar version of (2.23), we obtain

$$\begin{aligned} \|u\|^4_{L^4(\Omega)} &= \int_\Omega \left(|u_1|^2 + |u_2|^2\right)^2 = \int_\Omega |u_1|^4 + 2\int_\Omega |u_1|^2|u_2|^2 + \int_\Omega |u_2|^4 \\ &\le \left(\|u_1\|^2_{L^4(\Omega)} + \|u_2\|^2_{L^4(\Omega)}\right)^2 \le \frac{4L^2}{3\pi^3}\left(\|\nabla u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}\right)^2 = \frac{4L^2}{3\pi^3}\|\nabla u\|^4_{L^2(\Omega)}, \end{aligned}$$

which proves the first inequality (2.23) also for vector fields. One proceeds similarly for the second inequality in (2.23) and for (2.24). Therefore, from now on, we assume that u is a scalar function.

For scalar functions $w \in H^1_0(Q)$, we start by recalling that del Pino-Dolbeault [27, Theorem 1] obtained the optimal constant for the following Gagliardo-Nirenberg inequality in $\mathbb{R}^2$:

$$\|w\|^2_{L^4(Q)} \le \left(\frac{2}{3\pi}\right)^{1/4} \|\nabla w\|^{1/2}_{L^2(Q)}\|w\|^{3/2}_{L^3(Q)} \qquad \forall w \in H^1_0(Q). \tag{2.25}$$

Since functions in $H^1_0(Q)$ may be extended by zero outside Q, they can be seen as functions defined over the whole plane. We point out that (2.25) follows from a somehow "magic combination" of exponents: for general exponents, the optimal constant in the Gagliardo-Nirenberg inequality is not known, this is why the L^3-norm appears. By combining (2.25) with the following form of the Hölder inequality

$$\|w\|^3_{L^3(Q)} \le \|w\|_{L^2(Q)}\|w\|^2_{L^4(Q)} \qquad \forall w \in L^4(Q)\,,$$

we obtain

$$\|w\|_{L^4(Q)}^2 \le \left(\frac{2}{3\pi}\right)^{1/2} \|\nabla w\|_{L^2(Q)}\|w\|_{L^2(Q)} \qquad \forall w \in H_0^1(Q)\,. \tag{2.26}$$

Then we observe that $\cos(\frac{\pi x}{2L})\cos(\frac{\pi y}{2L})$ is an eigenfunction of the eigenvalue problem $-\Delta v = \lambda v$ in Q under Dirichlet boundary conditions. Since it is positive, it is associated to the least eigenvalue which is then given by $\lambda = \pi^2/2L^2$. Therefore, the Poincaré inequality reads

$$\|w\|_{L^2(Q)}^2 \le \frac{2L^2}{\pi^2}\|\nabla w\|_{L^2(Q)}^2 \qquad \forall w \in H_0^1(Q)$$

which, combined with (2.26), yields the first bound in (2.23) since any function $u \in H_0^1(\Omega)$ can be extended by 0 in K, thereby becoming a function in $H_0^1(Q)$.

In order to obtain the second bound in (2.23), we go back to (2.26) and we use the Faber-Krahn inequality, see [70]. We point out that the same extension argument as above enables us to compute all the norms in (2.26) in Ω instead of Q. Therefore, we may bound the $L^2(\Omega)$-norm in terms of the gradient by using the Poincaré inequality in Ω^*, namely a disk having the same measure as Ω. Since $|\Omega| = |Q| - |K|$, the radius of Ω^* is given by

$$R = \frac{2L}{\sqrt{\pi}}\sqrt{1 - \frac{|K|}{|Q|}}$$

that we write in this "strange form" for later use. Since the Poincaré constant (least eigenvalue) in the unit disk is given by μ_0^2, see (2.21), the Poincaré constant in Ω^* is given by μ_0^2/R^2, which means that

$$\min_{w \in H_0^1(\Omega)} \frac{\|\nabla w\|_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}} \ge \min_{w \in H_0^1(\Omega^*)} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}} = \frac{\mu_0}{R}\,.$$

Therefore,

$$\|w\|_{L^2(\Omega)} \le \frac{R}{\mu_0}\|\nabla w\|_{L^2(\Omega)} = \frac{2L}{\mu_0\sqrt{\pi}}\sqrt{1 - \frac{|K|}{|Q|}}\,\|\nabla w\|_{L^2(\Omega)} \qquad \forall w \in H_0^1(\Omega)$$

which, inserted into (2.26) (with Q replaced by Ω), gives the second bound in (2.23).

Let us now prove (2.24) and we restrict our attention to functions $u \in H_c^1(\Omega)\setminus H_0^1(\Omega)$: this restriction will be justified *a posteriori* because, if we manage proving (2.24) for these functions, then it will also hold for functions in $H_0^1(\Omega)$ since the constant in (2.23) is smaller, see also Figure 2.2 below. For functions $u \in H_c^1(\Omega)\setminus H_0^1(\Omega)$, it suffices to analyze the case where $u \ge 0$ in Ω (by replacing u with $|u|$), $u = 1$ on ∂Q (by homogeneity), and we define a.e. in Q the function

$$v(x,y) = \begin{cases} 1 - u(x,y) & \text{if } (x,y) \in \Omega \\ 1 & \text{if } (x,y) \in K, \end{cases}$$

so that $v \in H_0^1(Q)$ and v satisfies (2.25). Let us put

$$A = A(u) \doteq \left(\frac{2}{3\pi}\right)^{1/2} \|\nabla v\|_{L^2(Q)} = \left(\frac{2}{3\pi}\right)^{1/2} \|\nabla u\|_{L^2(\Omega)},$$

so that (2.25) reads

$$\int_Q |v|^4 \le A \int_Q |v|^3 \implies \int_\Omega \left[|1-u|^4 + \frac{|K|}{|\Omega|} - A\left(|1-u|^3 + \frac{|K|}{|\Omega|}\right)\right] \le 0. \tag{2.27}$$

The next step consists in finding $\alpha \in (0,1)$ and $\beta > 0$ (having ratio independent of u) for which

$$(1-s)^4 - A|1-s|^3 + (1-A)\frac{|K|}{|\Omega|} \ge \alpha s^4 - \beta A^4 \qquad \forall s \ge 0. \tag{2.28}$$

Since $s \mapsto (1-s)^4 - A|1-s|^3 + \gamma$ is symmetric with respect to $s=1$, for any $\gamma \in \mathbb{R}$, it suffices to find $\alpha \in (0,1)$ and $\beta > 0$ ensuring (2.28) for every $s \geq 1$. Thus, for all such α and β we define the function

$$\varphi(s) = (s-1)^4 - A(s-1)^3 - \alpha s^4 + (1-A)\frac{|K|}{|\Omega|} + \beta A^4 \qquad \forall s \geq 1,$$

and we seek $\alpha \in (0,1)$ and $\beta > 0$ in such a way that φ has a non-negative minimum value at some $s > 1$. Equivalently, we seek $\gamma > 3/4$ such that $\varphi(s)$ attains its minimum at $s_0 = 1 + \gamma A$, that is,

$$\varphi'(s_0) = A^3\gamma^2(4\gamma - 3) - 4\alpha(1+\gamma A)^3 = 0 \qquad \Longleftrightarrow \qquad \alpha = \frac{A^3}{4}\frac{\gamma^2(4\gamma-3)}{(1+\gamma A)^3} \in (0,1), \tag{2.29}$$

which fixes α in dependence of u. By imposing $\varphi(s_0) \geq 0$ and (2.29), we obtain a lower bound for β:

$$\beta \geq \frac{\gamma^3}{4} + \frac{\gamma^2(4\gamma-3)}{4A} + \frac{A-1}{A^4}\frac{|K|}{|\Omega|}.$$

This condition is certainly satisfied if we choose

$$\beta = \frac{\gamma^3}{4} + \frac{\gamma^2(4\gamma-3)}{4A} + \frac{1}{A^3}\frac{|K|}{|\Omega|}. \tag{2.30}$$

With the above choices of α and β we obtain the ratio

$$\frac{\beta}{\alpha} = \frac{4}{A^3}\frac{(1+\gamma A)^3}{\gamma^2(4\gamma-3)}\left[\frac{\gamma^3}{4} + \frac{\gamma^2(4\gamma-3)}{4A} + \frac{1}{A^3}\frac{|K|}{|\Omega|}\right], \tag{2.31}$$

which depends on u and on $\gamma > 3/4$. If we choose $\gamma = 1$ we obtain

$$\frac{\beta}{\alpha} = \left(1 + \frac{1}{A(u)} + \frac{4}{A(u)^3}\frac{|K|}{|\Omega|}\right)\left(1 + \frac{1}{A(u)}\right)^3, \tag{2.32}$$

where we emphasized the dependence of A on u. In order to obtain an upper bound for the ratio β/α independent of u, we use (2.19) which states that

$$A(u) \geq \sqrt{\frac{2}{3\pi}\mathrm{Cap}_Q(K)} \geq \sqrt{\frac{8}{3}}\frac{1}{\sqrt{\log\left(\frac{|Q|}{|K|}\right)}} \qquad \forall u \in H^1_c(\Omega) \text{ s.t. } u = 1 \text{ on } \partial Q,\ u \geq 0 \text{ in } \Omega.$$

Hence, from (2.32) we obtain the following uniform bound (independent of u)

$$\frac{\beta}{\alpha} \leq \left(1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}\right)^3 \left[1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)} + \frac{3\sqrt{3}}{4\sqrt{2}}\frac{|K|}{|\Omega|}\log^{3/2}\left(\frac{|Q|}{|K|}\right)\right].$$

In turn, from (2.27), by replacing s with u in (2.28) and integrating, we obtain

$$\begin{aligned}\|u\|^4_{L^4(\Omega)} &\leq \frac{\beta}{\alpha}A(u)^4|\Omega| \\ &\leq \frac{4|\Omega|}{9\pi^2}\left(1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}\right)^3 \left[1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)} + \frac{3\sqrt{3}}{4\sqrt{2}}\frac{|K|}{|\Omega|}\log^{3/2}\left(\frac{|Q|}{|K|}\right)\right]\|\nabla u\|^4_{L^2(\Omega)},\end{aligned}$$

for every $u \in H^1_c(\Omega)$ such that $u = 1$ on ∂Q and $u \geq 0$ in Ω. The bound (2.24) follows by taking the squared roots in the last inequality. $\square$

Several remarks about Theorem 2.3 are in order.

Remark 2.2. *The interpolation inequality by Ladyzhenskaya [55] (or [56, Lemma 1, p.8]) states that*

$$\|w\|^2_{L^4(\Omega)} \le \sqrt{2}\|\nabla w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \qquad \forall w \in H^1_0(\Omega).$$

Subsequently, Galdi [39, (II.3.9)] improved this Gagliardo-Nirenberg-type inequality by showing that

$$\|w\|^2_{L^4(\Omega)} \le \frac{1}{\sqrt{2}}\|\nabla w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \qquad \forall w \in H^1_0(\Omega).$$

Thanks to the result by del Pino-Dolbeault [27], with (2.26) *we improved further the constant of this inequality by around* 35%*: indeed,* $\sqrt{2/3\pi} \approx 0.65/\sqrt{2}$*. Finally, consider the entire function* $w(x,y) = (1+x^2+y^2)^{-1}$*; by computing its norms, we see that the optimal constant in this inequality is larger than* $(2\pi)^{-1/2}$*, showing that* (2.26) *cannot be improved by more than* 15%*.*

Remark 2.3. *The "break even" in the bound* (2.23) *occurs when* $|K|/|Q| = 1 - \mu_0^2/2\pi \approx 0.08$*: for smaller* $|K|$ *the first bound is better, for larger* $|K|$ *the second bound is better. Note that the constant in* (2.23) *tends to 0 whenever* $|K| \to |Q|$ *(the obstacle tends to fill the box) and remains uniformly bounded when* $|K| \to 0$*. On the contrary, the constant in* (2.24) *blows up when* $|K| \to 0$*: this is not just a consequence of our proof, also the optimal constant blows up, see Theorem 2.4 below.*

Remark 2.4. *The constant in* (2.23) *depends on the size of the surrounding box* Q *but it is mostly independent of the obstacle* K *(of its shape and of its position inside the box), it only weakly depends on its measure (in fact, its relative measure within* Q*); for this reason, we conjecture that it can be improved. The constant in* (2.24) *does not depend on the shape of* K*, nor on its position inside* Q *but it strongly depends on its measure; we believe that if* K *is close to* ∂Q*,* (2.24) *can be significantly improved. However, for our fluid-obstacle model to be reliable, we need to avoid "boundary effects" and maintain the obstacle* K *far away from* ∂Q *(the boundary of the photo, see the Introduction).*

Remark 2.5. *Some steps in the proof of* (2.24) *may be performed differently. For instance, one could have noticed that* $\max_{A>0}(A-1)/A^4 = 27/256$*, yielding a different bound for* β *in* (2.30)*. Also the choice of* $\gamma = 1$ *could be slightly modified. Nevertheless, the overall (small) improvements would not justify the great effort required and the final form of* (2.24) *would have a more unpleasant form. Moreover, these variants would not improve the bounds in Theorem 3.9 below.*

Theorem 2.3 yields the following lower bounds for the Sobolev constants:

Corollary 2.1. *Let* Ω *be as in (1.1). Let* $\mathcal{S}_0$ *and* $\mathcal{S}_1$ *be as in* (2.22)*. Then:*

$$\mathcal{S}_0 \ge \frac{\sqrt{3}\pi^{3/2}}{2L} \max\left\{1, \frac{\mu_0}{\sqrt{2\pi}}\sqrt{\frac{\frac{|Q|}{|K|}}{\frac{|Q|}{|K|}-1}}\right\},$$

$$\mathcal{S}_1 \ge \frac{3\pi}{4L}\sqrt{\frac{\frac{|Q|}{|K|}}{\frac{|Q|}{|K|}-1}}\left(1+\sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}\right)^{-3/2}\left[1+\sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}+\frac{3\sqrt{3}}{4\sqrt{2}}\frac{1}{\frac{|Q|}{|K|}-1}\log^{3/2}\left(\frac{|Q|}{|K|}\right)\right]^{-1/2}.$$

By dropping the multiplicative term $1/L$, the remainder of the lower bound for $\mathcal{S}_1$ in Corollary 2.1 can be treated as a function of $|Q|/|K| \in [1,\infty)$. This function vanishes like $[\log(|Q|/|K|)]^{-1}$ as $|Q|/|K| \to \infty$, see its plot in Figure 2.2 where we also compare it with the (larger) lower bound for $\mathcal{S}_0$, that becomes constant when $|Q|/|K| \approx 12.5$, see Remark 2.3.

It is then natural to wonder whether the lower bounds obtained in Corollary 2.1 are meaningful. This can be verified through suitable upper bounds. For $\mathcal{S}_0$ we take the function $w(x,y) = \cos(\frac{\pi x}{2L})\cos(\frac{\pi y}{2L})$, defined for $(x,y) \in \overline{Q}$, so that $w \in H^1_0(Q)$ and

$$\|w\|^2_{L^4(Q)} = \frac{3L}{4}, \quad \|\nabla w\|^2_{L^2(Q)} = \frac{\pi^2}{2} \quad \Longrightarrow \quad \mathcal{S}_0 \le \frac{2\pi^2}{3L},$$

showing that the first lower bound for $\mathcal{S}_0$ is quite accurate. An upper bound for $\mathcal{S}_1$ is given in the next statement.

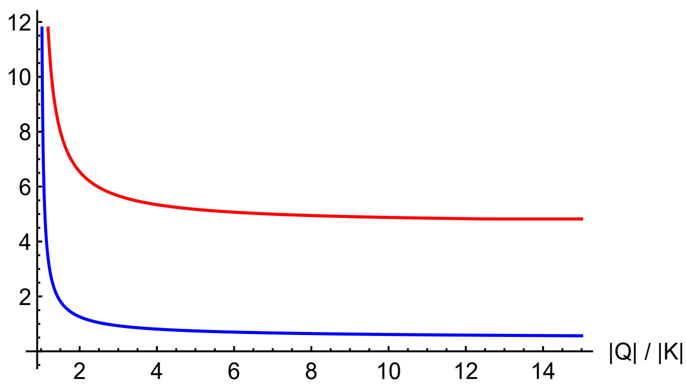

Figure 2.2: Behavior of the lower bounds for $\mathcal{S}_0$ (red) and $\mathcal{S}_1$ (blue) as functions of $|Q|/|K|$.

Theorem 2.4. *Let Ω be as in (1.1) and assume that*

$$\exists\ 0 < d \le a < L \text{ such that } \mathcal{R} = (-a,a) \times (-d,d) \supset K\,. \tag{2.33}$$

Then

$$\mathcal{S}_1 \le \frac{2\sqrt{2}\left[(L-a)^2+(L-d)^2\right]^2}{(L-a)(L-d)\sqrt{a(L-a)+d(L-d)}}\,\frac{\theta}{\sqrt{2L(2L-a-d)(d-a)^2\theta_1+(L-a)(L-d)\left[a(L-a)+d(L-d)\right]\theta_2}},$$

where

$$\theta = \left[\log\left(\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right)\right]^{-1}, \qquad \theta_1 = (1-4\theta+12\theta^2-24\theta^3+24\theta^4)\left[\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right]-24\theta^4,$$

$$\theta_2 = (2-4\theta+6\theta^2-6\theta^3+3\theta^4)\left[\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right]^2-3\theta^4.$$

Proof. Let $\mathcal{P}(Q)$ be as in (2.8), let $V^\phi \in \mathcal{P}(Q)$ be defined by (2.13) with

$$\phi(s) = \log\left(\frac{(L-a)^2+(L-d)^2}{L(L-a)+L(L-d)}s + \frac{a(L-a)+d(L-d)}{L(L-a)+L(L-d)}\right) \Bigg/ \log\left(\frac{a(L-a)+d(L-d)}{L(L-a)+L(L-d)}\right) \qquad \forall s \in [0,1],$$

with V^ϕ extended by 1 in $\mathcal{R}\setminus K$. From (2.15) and (2.16) we know that:

$$\|\nabla V^\phi\|^2_{L^2(\Omega)} = 4\,\frac{(L-a)^2+(L-d)^2}{(L-a)(L-d)}\ \left[\log\left(\frac{L(L-a)+L(L-d)}{a(L-a)+d(L-d)}\right)\right]^{-1}.$$

For symmetry reasons, the contribution of $|1-V^\phi|^4$ over $T_1\cup T_3$ is four times the contribution over the trapezium T_5 defined in (2.11), whereas the contribution of $|1-V^\phi|^4$ over $T_2\cup T_4$ is four times the contribution over the trapezium T_6 defined in (2.12). Then

$$\begin{aligned}
\int_{Q\setminus\mathcal{R}} |1-V^\phi|^4 &= 4\int_d^L \int_0^{a+\frac{L-a}{L-d}(y-d)} |1-V^\phi(y)|^4 dx\,dy + 4\int_a^L \int_0^{d+\frac{L-d}{L-a}(x-a)} |1-V^\phi(x)|^4 dy\,dx \\
&= 4\int_d^L \left[a+\tfrac{L-a}{L-d}(y-d)\right]\,|1-V^\phi(y)|^4\,dy + 4\int_a^L \left[d+\tfrac{L-d}{L-a}(x-a)\right]\,|1-V^\phi(x)|^4\,dx \\
&= 4\int_0^1 \left[a(L-d)+d(L-a)+2(L-a)(L-d)s\right]|1-\phi(s)|^4\,ds.
\end{aligned}$$

Using that $V^\phi \equiv 1$ in $\mathcal{R}\setminus K$ and the change of variable $t = 1-\phi(s)$, for $s\in[0,1]$, we then obtain

$$\|1-V^\phi\|^4_{L^4(\Omega)} = 2\frac{a(L-a)+d(L-d)}{\left[(L-a)^2+(L-d)^2\right]^2}\left\{2L(2L-a-d)(d-a)^2\theta_1+(L-a)(L-d)\left[a(L-a)+d(L-d)\right]\theta_2\right\}.$$

We finally notice that if $v \in \mathcal{P}(Q)$, then $1 - v \in H^1_c(\Omega)$ with $v = 1$ on ∂Q. Therefore,

$$\mathcal{S}_1 \leq \min_{v \in \mathcal{P}(Q)} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|1 - v\|^2_{L^4(\Omega)}} \leq \frac{\|\nabla V^\phi\|^2_{L^2(\Omega)}}{\|1 - V^\phi\|^2_{L^4(\Omega)}},$$

which concludes the proof. □

In the case where the obstacle is a square, Theorem 2.4 enables us to evaluate the precision of the lower bound for $\mathcal{S}_1$ given in Corollary 2.1.

Corollary 2.2. *If $0 < a < L$ and $\Omega = (-L, L)^2 \setminus (-a, a)^2$, then*

$$\mathcal{S}_1 \geq \frac{1}{L} \frac{\frac{3\pi}{4} \frac{L}{a}}{\sqrt{\left(\frac{L}{a}\right)^2 - 1}} \left(1 + \frac{\sqrt{3}}{2} \log^{1/2}\left(\frac{L}{a}\right)\right)^{-3/2} \left[1 + \frac{\sqrt{3}}{2} \log^{1/2}\left(\frac{L}{a}\right)\left(1 + \frac{3}{\left(\frac{L}{a}\right)^2 - 1} \log\left(\frac{L}{a}\right)\right)\right]^{-1/2},$$

$$\mathcal{S}_1 \leq \frac{1}{L} \frac{4\sqrt{2}\, \frac{L}{a} \log\left(\frac{L}{a}\right)}{\sqrt{\left[2\log^4\left(\frac{L}{a}\right) - 4\log^3\left(\frac{L}{a}\right) + 6\log^2\left(\frac{L}{a}\right) - 6\log\left(\frac{L}{a}\right) + 3\right]\left(\frac{L}{a}\right)^2 - 3}}.$$

By dropping the multiplicative term $1/L$, the remainder of the lower and upper bounds for $\mathcal{S}_1$ in Corollary 2.2 can be treated as a function of $L/a \in (1, \infty)$. The ratio between the bounds tends to $4/\pi \approx 1.273$ as $L/a \to \infty$ so that, since we are interested in small obstacles compared to the size of the photo ($a \ll L$), Corollary 2.2 shows that the obtained bounds are quite precise. The plots in Figure 2.3 describe the overall behavior.

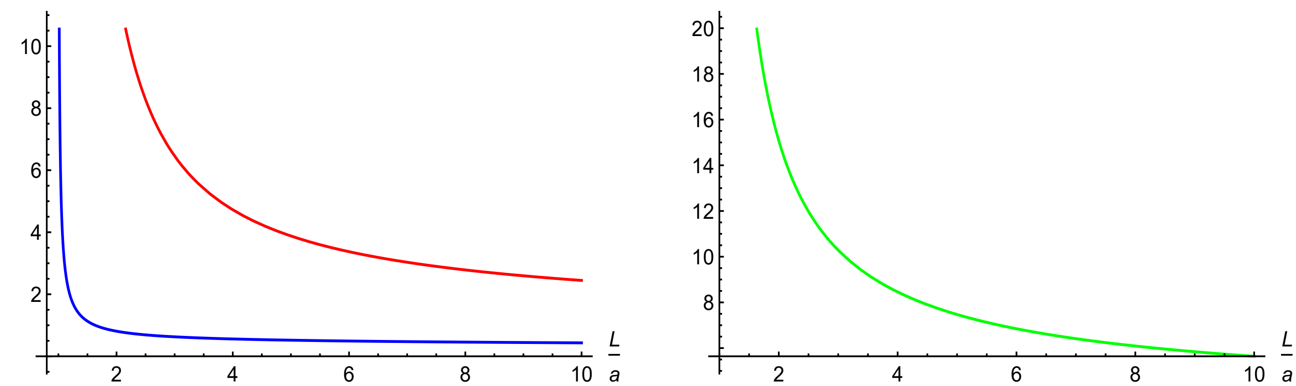

Figure 2.3: On the left: behavior of the lower and upper bounds for $\mathcal{S}_1$ from Corollary 2.2, as a function of L/a. On the right: ratio between the upper and lower bounds for $\mathcal{S}_1$ as a function of L/a.

2.3 Functional inequalities for the Navier-Stokes equations

In this section we quickly recall some well-known functional spaces and inequalities, by adapting them to our context. Let us introduce the two functional spaces of vector fields

$$\mathcal{V}_*(\Omega) = \{v \in H^1_*(\Omega) \mid \nabla \cdot v = 0 \text{ in } \Omega\} \quad \text{and} \quad \mathcal{V}(\Omega) = \{v \in H^1_0(\Omega) \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$

which are Hilbert spaces if endowed with the scalar product $(u, v) \mapsto (\nabla u, \nabla v)_{L^2(\Omega)}$. We also introduce the trilinear form

$$\beta(u, v, w) = \int_\Omega (u \cdot \nabla) v \cdot w \qquad \forall u, v, w \in H^1(\Omega), \tag{2.34}$$

which is continuous in $H^1_*(\Omega)\times H^1_*(\Omega)\times H^1_*(\Omega)$ and satisfies (see e.g. [39, Section IX.2])

$$|\beta(u,v,w)|\le \frac{1}{\mathcal{S}}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} \qquad \forall u,v,w\in H^1_*(\Omega), \tag{2.35}$$

$$|\beta(u,v,w)|\le \frac{1}{\sqrt{\mathcal{S}\mathcal{S}_0}}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} \qquad \forall u,v\in H^1_*(\Omega), w\in H^1_0(\Omega), \tag{2.36}$$

where $\mathcal{S}$ and $\mathcal{S}_0$ are as in (2.22). Moreover,

$$\begin{aligned}
&\beta(u,v,w)=-\beta(u,w,v) && \text{for any}\quad u\in\mathcal{V}_*(\Omega),\ v\in H^1(\Omega),\ w\in H^1_0(\Omega),\\
&\beta(u,v,v)=0 && \text{for any}\quad u\in\mathcal{V}_*(\Omega),\ v\in H^1_0(\Omega).
\end{aligned}\tag{2.37}$$

Since integration by parts will be performed repeatedly in the course, we recall a generalized Gauss identity from [39, Theorem III.2.2]. Since Ω in (1.1) is a bounded Lipschitz domain, its boundary $\partial\Omega$ has in a.e. point an outward unit normal $\hat{n}$. Then, for every $r,s\in(1,\infty)$ such that $\frac{1}{r}+\frac{1}{s}=1$ one has

$$\int_\Omega u(\nabla\cdot v)\,dx+\int_\Omega \nabla u\cdot v\,dx=\langle v\cdot\hat{n},u\rangle_{\partial\Omega} \qquad \forall u\in W^{1,s}(\Omega)\,,\ v\in E_r(\Omega), \tag{2.38}$$

where $E_r(\Omega)\doteq\{v\in L^r(\Omega)\mid \nabla\cdot v\in L^r(\Omega)\}$ and the "boundary term" $\langle\cdot,\cdot\rangle_{\partial\Omega}$ represents the duality between $W^{-\frac{1}{r},r}(\partial\Omega)$ and $W^{\frac{1}{r},s}(\partial\Omega)$; it is well-defined because

$$v\cdot\hat{n}|_{\partial\Omega}\in W^{-\frac{1}{r},r}(\partial\Omega) \quad\text{ and }\quad u|_{\partial\Omega}\in W^{\frac{1}{r},s}(\partial\Omega).$$

For later use, we remark that for constant boundary data one has

$$(U,V)\in\mathbb{R}^2 \implies \|(U,V)\|_{H^{1/2}(\partial Q)}=\|(U,V)\|_{L^2(\partial Q)}=2\sqrt{2L}\,\sqrt{U^2+V^2}\,. \tag{2.39}$$

We now recall a combination of results by Hopf [49] and Ladyzhenskaya-Solonnikov [57] (see also [39, Lemma IX.4.2]), that we also state for domains Ω that are symmetric with respect to the x-axis, namely $(x,y)\in\Omega$ if and only if $(x,-y)\in\Omega$.

Proposition 2.1. *Let Ω be as in* (1.1) *and let $\hat{n}$ be the a.e.-defined outward unit normal to $\partial\Omega$. Let $W\in H^{1/2}(\partial\Omega)$ be such that*

$$\int_{\partial Q} W\cdot\hat{n}\,ds=\int_{\partial K} W\cdot\hat{n}\,ds=0. \tag{2.40}$$

Then for all $\varepsilon>0$ there exists a solenoidal extension $A_\varepsilon\in H^1(\Omega)$ satisfying

$$A_\varepsilon=W\ \text{ on }\partial\Omega, \qquad \|A_\varepsilon\|_{H^1(\Omega)}\le M_\varepsilon\|W\|_{H^{1/2}(\partial\Omega)}, \qquad |\beta(v,A_\varepsilon,v)|\le\varepsilon\|\nabla v\|^2_{L^2(\Omega)} \quad \forall v\in\mathcal{V}(\Omega), \tag{2.41}$$

for some constant $M_\varepsilon>0$ that depends on ε and Ω. If Ω is symmetric with respect to the x-axis and $W=(W_1,W_2)$ is such that W_1 is y-even and W_2 is y-odd, then the solenoidal extension $A_\varepsilon=(A^1_\varepsilon,A^2_\varepsilon)$ can be chosen so that A^1_ε is y-even and A^2_ε is y-odd, with no increment of the H^1-norm.

Proof. Given $\varepsilon>0$ and a boundary datum $W\in H^{1/2}(\partial\Omega)$ satisfying (2.40), the existence of a vector field $A_\varepsilon\in H^1(\Omega)$ verifying (2.41) is proved (e.g.) in [39, Lemma IX.4.2]; indeed, (2.40) assumes "no separated sinks and sources of fluid inside Q", see [39, Formula (IX.4.7)].

Under the symmetry assumptions given in the statement, it can be seen that the vector field

$$B_\varepsilon(x,y)\doteq\frac{1}{2}\big(A^1_\varepsilon(x,y)+A^1_\varepsilon(x,-y),A^2_\varepsilon(x,y)-A^2_\varepsilon(x,-y)\big) \qquad \text{for a.e. } (x,y)\in\Omega,$$

is y-even in its first component, y-odd in its second component and still verifies (2.41). Indeed, the solenoidal condition is readily verified, as well as the boundary condition. The H^1-bound follows from the fact that $\|B_\varepsilon\|_{H^1(\Omega)} \le \|A_\varepsilon\|_{H^1(\Omega)}$; in turn, this follows from a direct computation (and using the Young inequality) or by observing that B_ε is the "symmetrized" of A_ε. Finally, the bound on β follows by arbitrariness of v: in particular, it holds for the symmetric and/or skew-symmetric parts of any $v \in \mathcal{V}(\Omega)$. □

As usual, the pressure p in (1.2) is defined up to an additive constant; therefore, we take it to have zero mean value and we introduce the space

$$L^2_0(\Omega) = \left\{ g \in L^2(\Omega) \;\Big|\; \int_\Omega g = 0 \right\}.$$

For any $g \in L^2_0(\Omega)$ we define its gradient $\nabla g \in H^{-1}(\Omega)$ as follows:

$$\langle \nabla g, \psi \rangle_\Omega = -\int_\Omega g\,(\nabla \cdot \psi) \qquad \forall \psi \in H^1_0(\Omega).$$

Bogovskii [15] showed that, given any $q \in L^2_0(\Omega)$, there exists $\psi \in H^1_0(\Omega)$ such that $\nabla \cdot \psi = q$ in Ω and

$$\|\nabla \psi\|_{L^2(\Omega)} \le C_B(\Omega)\|q\|_{L^2(\Omega)}, \tag{2.42}$$

where the constant $C_B(\Omega) > 0$ depends only on Ω. Then we obtain the bound

$$\|\nabla g\|_{H^{-1}(\Omega)} = \sup_{\substack{\psi \in H^1_0(\Omega) \\ \|\nabla\psi\|_{L^2(\Omega)}=1}} \left| \int_\Omega g\,(\nabla \cdot \psi) \right| \ge \frac{1}{C_B(\Omega)} \sup_{\substack{q \in L^2_0(\Omega) \\ \|q\|_{L^2(\Omega)}=1}} \left| \int_\Omega gq \right| = \frac{1}{C_B(\Omega)} \|g\|_{L^2(\Omega)},$$

that is,

$$\|g\|_{L^2(\Omega)} \le C_B(\Omega)\|\nabla g\|_{H^{-1}(\Omega)} \qquad \forall g \in L^2_0(\Omega). \tag{2.43}$$

2.4 Gradient bounds for solenoidal extensions

The presence of inhomogeneous boundary conditions in (1.3) constitutes a major difficulty when trying to obtain a priori bounds for the solutions of (1.2) and a quantitative statement for its uniqueness. Furthermore, as will be apparent in the proof of Theorem 3.1 below, the fundamental step lies in the determination of a solenoidal extension v_0 of the data $(U, V) \in H^{1/2}(\partial Q)$, namely a solution of (1.5) and a bound for its norm. The choice of v_0 influences the explicit form of the uniqueness bound and, therefore, what is needed is precisely an explicit form of v_0.

A classical way to build solenoidal extensions in case of constant velocity at infinity, in the unbounded region outside an obstacle, consists in adding to the constant vector the curl of some cutoff function, see [56, p.130] and also [39, Section IX.4]. Nevertheless, since we aim to obtain explicit extensions, this is not precise enough. In this section we are not considering a general cutoff function but, instead, we construct by hand a suitable $\mathcal{C}^1$-extension by using repeatedly the Maximum Principle for harmonic functions, combined with the pyramidal capacity approach developed in Section 2.1.

Assume (2.33) and consider the stadium

$$\mathcal{J} = \mathcal{R} \cup \left\{(x,y) \in \mathbb{R}^2 \mid (x-a)^2 + y^2 < d^2,\ x \ge a\right\} \cup \left\{(x,y) \in \mathbb{R}^2 \mid (x+a)^2 + y^2 < d^2,\ x \le -a\right\},$$

see Figure 2.4. If $a + d < L$, then $\mathcal{J} \subset Q$ and we put $\Omega_{\mathcal{J}} = Q \setminus \overline{\mathcal{J}}$ and let $\Psi \in H^1_0(Q)$ be the (scalar)

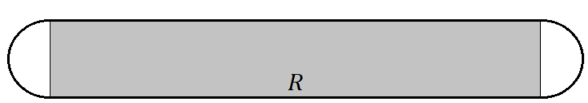

Figure 2.4: Stadium-shaped region $\mathcal{J}$ enclosing the rectangle $\mathcal{R}$.

relative capacity potential of $\mathcal{J}$ with respect to Q, that is,

$$\Delta\Psi = 0 \ \text{ in } \ \Omega_{\mathcal{J}}, \qquad \Psi = 0 \ \text{ on } \ \partial Q, \qquad \Psi = 1 \ \text{ in } \ \mathcal{J}, \qquad \mathrm{Cap}_Q(\mathcal{J}) = \|\nabla\Psi\|^2_{L^2(Q)}. \tag{2.44}$$

Since Q is a square and $\partial\mathcal{J}$ is of class $\mathcal{C}^{1,1}$, elliptic regularity arguments show that $\Psi \in H^2(\Omega_{\mathcal{J}}) \cap \mathcal{C}^{1,1}(\overline{\Omega_{\mathcal{J}}})$. Since Ψ is harmonic, we also know that $\Psi \in \mathcal{C}^\infty(\Omega_{\mathcal{J}})$.

Take a function $\varphi \in H^2(\Omega_{\mathcal{J}})$ and, for $(U,V) \in \mathbb{R}^2$, define the solenoidal vector field

$$W \in H^1(\Omega_{\mathcal{J}})\,, \qquad W(x,y) = \begin{pmatrix} U\varphi(x,y) + (Uy - Vx)\dfrac{\partial\varphi}{\partial y}(x,y) \\ \\ V\varphi(x,y) - (Uy - Vx)\dfrac{\partial\varphi}{\partial x}(x,y) \end{pmatrix} \qquad \forall (x,y) \in \Omega_{\mathcal{J}}.$$

Then we have

$$\begin{aligned} |\nabla W|^2 = {} & 4(Uy - Vx)\left[\frac{\partial^2\varphi}{\partial x\partial y}\left(U\frac{\partial\varphi}{\partial x} - V\frac{\partial\varphi}{\partial y}\right) + U\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2} - V\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}\right] \\ & + 2\left(U\frac{\partial\varphi}{\partial x} - V\frac{\partial\varphi}{\partial y}\right)^2 + 4U^2\left(\frac{\partial\varphi}{\partial y}\right)^2 + 4V^2\left(\frac{\partial\varphi}{\partial x}\right)^2 \\ & + (Uy - Vx)^2\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\right] \quad \text{in } \Omega_{\mathcal{J}}. \end{aligned} \tag{2.45}$$

We impose restrictions on φ for the vector field W to satisfy some symmetry properties and the boundary conditions $W = (0,0)$ on $\partial\mathcal{J}$ and $W = (U,V)$ on ∂Q. We take a function φ such that

$$\varphi = 1 \ \text{ on } \ \partial Q\,, \qquad \varphi = 0 \ \text{ on } \ \partial\mathcal{J}\,, \qquad \nabla\varphi = 0 \ \text{ on } \ \partial\Omega_{\mathcal{J}}\,, \qquad \varphi \text{ is both } x\text{-even and } y\text{-even}. \tag{2.46}$$

For the first term in (2.45) we note that

$$\begin{aligned} & 4\int_{\Omega_{\mathcal{J}}}(Uy - Vx)\left[\frac{\partial^2\varphi}{\partial x\partial y}\left(U\frac{\partial\varphi}{\partial x} - V\frac{\partial\varphi}{\partial y}\right) + U\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y^2} - V\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}\right] \\ & = 4\int_{\Omega_{\mathcal{J}}}(Uy - Vx)\left[\frac{U}{2}\frac{\partial}{\partial y}|\nabla\varphi|^2 - \frac{V}{2}\frac{\partial}{\partial x}|\nabla\varphi|^2\right] = -2(U^2+V^2)\int_{\Omega_{\mathcal{J}}}|\nabla\varphi|^2\,, \end{aligned}$$

where the second equality follows from (2.46) and an integration by parts. Thus, integrating (2.45) and putting together the first two lines, yields

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})} = 2\int_{\Omega_{\mathcal{J}}}\left[U\frac{\partial\varphi}{\partial y} - V\frac{\partial\varphi}{\partial x}\right]^2 + \int_{\Omega_{\mathcal{J}}}(Uy - Vx)^2\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\right].$$

Concerning the second integral, we notice that, by the symmetry assumption in (2.46) we deduce

$$\int_{\Omega_{\mathcal{J}}} xy\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\right] = 0\,.$$

Hence, we obtain

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})} = 2\int_{\Omega_{\mathcal{J}}}\left[U\frac{\partial\varphi}{\partial y} - V\frac{\partial\varphi}{\partial x}\right]^2 + \int_{\Omega_{\mathcal{J}}}(U^2y^2 + V^2x^2)\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\right].$$

Then, we bound the second integral by

$$L^2(U^2+V^2)\int_{\Omega_{\mathcal{J}}}\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2\right].$$

Moreover, if we had the additional regularity that $\varphi\in\mathcal{C}^3(\Omega_{\mathcal{J}})\cap\mathcal{C}^1(\overline{\Omega_{\mathcal{J}}})$, then a double integration by parts and (2.46) (vanishing of the gradient on the boundary) would show that

$$\int_{\Omega_{\mathcal{J}}}\left(\frac{\partial^2\varphi}{\partial x\partial y}\right)^2=\int_{\Omega_{\mathcal{J}}}\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}\,;$$

by density, this identity is also verified under the sole regularity assumption $\varphi\in H^2(\Omega_{\mathcal{J}})$. Whence,

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})}\le 2\int_{\Omega_{\mathcal{J}}}\left[U\frac{\partial\varphi}{\partial y}-V\frac{\partial\varphi}{\partial x}\right]^2+L^2\left(U^2+V^2\right)\int_{\Omega_{\mathcal{J}}}(\Delta\varphi)^2\,. \tag{2.47}$$

We now make a specific choice of the function φ satisfying all the restrictions in (2.46). For this, we consider a function $h\in\mathcal{C}^4([0,1];\mathbb{R})$ and take $\varphi(x,y)=h(\Psi(x,y))$, for all $(x,y)\in\overline{\Omega_{\mathcal{J}}}$, where Ψ is as in (2.44). The vector field W then becomes

$$W(x,y)=\begin{pmatrix}Uh(\Psi(x,y))+(Uy-Vx)h'(\Psi(x,y))\dfrac{\partial\Psi}{\partial y}(x,y)\\[2mm] Vh(\Psi(x,y))-(Uy-Vx)h'(\Psi(x,y))\dfrac{\partial\Psi}{\partial x}(x,y)\end{pmatrix}\qquad\forall(x,y)\in\Omega_{\mathcal{J}}.$$

By imposing the boundary conditions $W=(0,0)$ on $\partial\mathcal{J}$ and $W=(U,V)$ on ∂Q, we find

$$\begin{cases}Uh(0)+(Uy-Vx)h'(0)\dfrac{\partial\Psi}{\partial y}(x,y)=U\\[2mm] Vh(0)-(Uy-Vx)h'(0)\dfrac{\partial\Psi}{\partial x}(x,y)=V\end{cases}\qquad\text{on }\ \partial Q,$$

$$\begin{cases}Uh(1)+(Uy-Vx)h'(1)\dfrac{\partial\Psi}{\partial y}(x,y)=0\\[2mm] Vh(1)-(Uy-Vx)h'(1)\dfrac{\partial\Psi}{\partial x}(x,y)=0\end{cases}\qquad\text{on }\ \partial\mathcal{J}.$$

As a consequence, we take $h\in\mathcal{C}^4([0,1];\mathbb{R})$ such that $h(0)=1$ and $h(1)=h'(0)=h'(1)=0$. In particular, this implies that $\nabla\varphi=0$ on $\partial\Omega_{\mathcal{J}}$. Moreover, by symmetry and uniqueness, the capacity potential Ψ is both x-even and y-even and, hence, $\varphi=h(\Psi)$ inherits the same properties. Therefore, all the conditions in (2.46) are fulfilled and (2.47) holds. Moreover, in view of (2.44), we notice that

$$\Delta\varphi=h''(\Psi)|\nabla\Psi|^2+h'(\Psi)\Delta\Psi=h''(\Psi)|\nabla\Psi|^2\qquad\text{in }\Omega_{\mathcal{J}}$$

so that (2.47) becomes

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})}\le 2\int_{\Omega_{\mathcal{J}}}h'(\Psi)^2\left(U\frac{\partial\Psi}{\partial y}-V\frac{\partial\Psi}{\partial x}\right)^2+L^2\left(U^2+V^2\right)\int_{\Omega_{\mathcal{J}}}h''(\Psi)^2|\nabla\Psi|^4\,. \tag{2.48}$$

By the symmetry properties of Ψ, we have that Ψ_x is y-even, Ψ_y is y-odd and $h'(\Psi)^2$ is even, so that

$$\int_{\Omega_{\mathcal{J}}}h'(\Psi)^2\,\frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y}=0.$$

This fact, together with Theorem 2.1, shows that the first integral in (2.48) can be estimated as

$$\int_{\Omega_{\mathcal{J}}} h'(\Psi)^2 \left(U\frac{\partial \Psi}{\partial y} - V\frac{\partial \Psi}{\partial x} \right)^2 \leq \max\{U^2, V^2\} \, \|h'\|^2_{L^2([0,1])} \operatorname{Cap}_Q(\mathcal{J}) \,. \tag{2.49}$$

For the second integral in (2.48), we notice that $-\Delta(|\nabla\Psi|^2) \leq 0$ in $\Omega_{\mathcal{J}}$, since Ψ is harmonic. Therefore $|\nabla\Psi|$ attains it maximum value on $\partial\Omega_{\mathcal{J}} = \partial Q \cup \partial \mathcal{J}$ and one finds that

$$\int_{\Omega_{\mathcal{J}}} h''(\Psi)^2 |\nabla\Psi|^4 \leq \|\nabla\Psi\|^2_{L^\infty(\partial\Omega_{\mathcal{J}})} \int_{\Omega_{\mathcal{J}}} h''(\Psi)^2 |\nabla\Psi|^2 = \|\nabla\Psi\|^2_{L^\infty(\partial\Omega_{\mathcal{J}})} \, \|h''\|^2_{L^2([0,1])} \operatorname{Cap}_Q(\mathcal{J}) \,,$$

where we used again Theorem 2.1. Combined with (2.48) and (2.49), the last inequality yields

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})} \leq \left(U^2 + V^2\right) \left[2\, \|h'\|^2_{L^2([0,1])} + L^2 \, \|\nabla\Psi\|^2_{L^\infty(\partial\Omega_{\mathcal{J}})} \, \|h''\|^2_{L^2([0,1])} \right] \operatorname{Cap}_Q(\mathcal{J}) \,. \tag{2.50}$$

The task is now to lower as much as possible the right hand side of (2.50). Of course, we need also to estimate the gradient of Ψ on the boundary of $\Omega_{\mathcal{J}}$. Once this will be done, see below, it will be apparent that the larger term in (2.50) is the second and, therefore, we are led to minimize the quantity $\|h''\|_{L^2([0,1])}$ among functions $h \in \mathcal{C}^4([0,1];\mathbb{R})$ such that $h(0) = 1$ and $h(1) = h'(0) = h'(1) = 0$. The Euler-Lagrange equation for this minimization problem reads $h^{(4)} = 0$ in $(0,1)$ and we find $h(t) = 2t^3 - 3t^2 + 1$, for $t \in [0,1]$, which yields

$$\|h'\|^2_{L^2([0,1])} = \frac{6}{5} \,, \qquad \|h''\|^2_{L^2([0,1])} = 12 \,,$$

so that (2.50) becomes

$$\|\nabla W\|^2_{L^2(\Omega_{\mathcal{J}})} \leq 12 \left(U^2 + V^2\right) \left[\frac{1}{5} + L^2 \, \|\nabla\Psi\|^2_{L^\infty(\partial\Omega_{\mathcal{J}})} \right] \operatorname{Cap}_Q(\mathcal{J}) \,. \tag{2.51}$$

Let us now estimate $\|\nabla\Psi\|_{L^\infty(\partial\Omega_{\mathcal{J}})}$. By the Maximum Principle we have that $0 \leq \Psi \leq 1$ in $\overline{\Omega_{\mathcal{J}}}$, so that the function $\frac{L-x}{L-a-d} - \Psi$, which is harmonic in $\Omega_{\mathcal{J}}$, is non-negative on $\partial\Omega_{\mathcal{J}}$. A further application of the Maximum Principle then yields the inequality

$$0 \leq \Psi(x,y) \leq \frac{L-x}{L-a-d} \qquad \forall (x,y) \in \Omega_{\mathcal{J}}.$$

By comparison we then deduce

$$-\frac{1}{L-a-d} \leq \frac{\partial \Psi}{\partial x}(L,y) \leq 0 \qquad \text{for a.e. } y \in (-L,L).$$

Since the tangential derivative of Ψ is zero on ∂Q, the last inequality and the symmetry of Ψ imply

$$|\nabla\Psi(\pm L, y)| \leq \frac{1}{L-a-d} \qquad \text{for a.e. } y \in (-L,L).$$

Applying the same comparison method to the harmonic function $\frac{L-y}{L-d} - \Psi$ one also obtains the bound

$$|\nabla\Psi(x, \pm L)| \leq \frac{1}{L-d} < \frac{1}{L-a-d} \qquad \text{for a.e. } x \in (-L,L)$$

and, therefore, we finally infer

$$\|\nabla\Psi\|_{L^\infty(\partial Q)} \leq \frac{1}{L-a-d} \,. \tag{2.52}$$

In order to estimate $\|\nabla\Psi\|_{L^\infty(\partial\mathcal{J})}$, we consider, for any $x_0 \in [-a,a]$, the harmonic function

$$H_{x_0}(x,y) = \frac{\log\left((L-|x_0|)^2\right) - \log\left((x-x_0)^2 + y^2\right)}{\log\left((L-|x_0|)^2\right) - \log\left(d^2\right)} \qquad \forall (x,y) \in \overline{\Omega_{\mathcal{J}}}, \tag{2.53}$$

which equals 1 on the circle $(x-x_0)^2+y^2=d^2$ and vanishes on the circle $(x-x_0)^2+y^2=(L-|x_0|)^2$ that is contained in Q. Then $\Psi - H_{x_0}$ is harmonic in $\Omega_{\mathcal{J}}$ and $\Psi - H_{x_0} \geq 0$ on $\partial\Omega_{\mathcal{J}}$. Moreover, we also have $\Psi - H_{x_0} = 0$ in every point of the circle $(x-x_0)^2+y^2=d^2$ tangent to $\partial\mathcal{J}$. Therefore, again by the Maximum Principle, we infer first that $\Psi \geq H_{x_0}$ in $\Omega_{\mathcal{J}}$ and then that

$$|\nabla\Psi(x_0,y)| \leq |\nabla H_{x_0}(x_0,y)| = \left[d\log\left(\frac{L-a}{d}\right)\right]^{-1} \qquad \forall x_0 \in (-a,a)\,,\ |y|=d\,,$$

$$|\nabla\Psi(x,y)| \leq |\nabla H_{x_0}(x,y)| = \left[d\log\left(\frac{L-a}{d}\right)\right]^{-1} \qquad \text{if} \quad x_0 \in \{-a,a\}\,,\ (x-x_0)^2+y^2=d^2\,,\ |x|\geq a.$$

These two bounds cover the whole $\partial\mathcal{J}$ and, therefore,

$$\|\nabla\Psi\|_{L^\infty(\partial\mathcal{J})} \leq \left[d\log\left(\frac{L-a}{d}\right)\right]^{-1}\,,$$

which, together with (2.52), implies

$$\|\nabla\Psi\|_{L^\infty(\partial\Omega_{\mathcal{J}})} \leq \max\left\{\frac{1}{L-a-d}\,,\left[d\log\left(\frac{L-a}{d}\right)\right]^{-1}\right\} = \left[d\log\left(\frac{L-a}{d}\right)\right]^{-1}\,, \tag{2.54}$$

since $L > a+d$. We plug (2.54) into (2.51) and, since $\mathcal{J} \subset [-a-d,a+d]\times[-d,d]$, by the monotonicity of the capacity, we may apply Theorem 2.2 to $[-a-d,a+d]\times[-d,d]$ and state the following result.

Theorem 2.5. *Assume* (2.33) *with* $L > a+d$ *and let* $(U,V)\in\mathbb{R}^2$. *Then, there exists a vector field* $W \in H^1_c(\Omega)$ *satisfying*

$$\nabla\cdot W = 0 \quad in \quad \Omega, \qquad W = (U,V) \quad on \quad \partial Q,$$

together with the estimate

$$\|\nabla W\|^2_{L^2(\Omega)} \leq 48\left(U^2+V^2\right)\frac{\left[(L-a-d)^2+(L-d)^2\right]\left[\frac{1}{5}+L^2\left(d\log\left(\frac{L-a}{d}\right)\right)^{-2}\right]}{(L-a-d)(L-d)\log\left(\frac{L(L-a-d)+L(L-d)}{(a+d)(L-a-d)+d(L-d)}\right)}\,.$$

3 The planar Navier-Stokes equations around an obstacle

3.1 Existence, uniqueness and regularity

Let us first define what is meant by weak solution of problem (1.2)-(1.3).

Definition 3.1. *Given* $f \in H^{-1}(\Omega)$ *and* $(U,V)\in H^{1/2}(\partial Q)$ *satisfying* (1.4)*, we say that a vector field* $u \in \mathcal{V}_*(\Omega)$ *is a weak solution of* (1.2)-(1.3) *if* u *verifies* (1.3) *in the trace sense and*

$$\eta(\nabla u,\nabla\varphi)_{L^2(\Omega)} + \beta(u,u,\varphi) = \langle f,\varphi\rangle_\Omega \qquad \forall\varphi\in\mathcal{V}(\Omega). \tag{3.1}$$

Then we state a result which is essentially known, see e.g. [39, Section IX.4]. Nevertheless, for three important reasons we give here a proof by emphasizing several steps. First we are concerned with both nonzero forcing and boundary data, second the a priori bounds are needed in the proof of Theorem 3.6, third the quantitative bounds for uniqueness will play a crucial role in Section 3.4.

Theorem 3.1. *Let Ω be as in (1.1). For any $f \in H^{-1}(\Omega)$ and $(U,V) \in H^{1/2}(\partial Q)$ satisfying (1.4) there exists a weak solution $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (1.2)-(1.3) and any weak solution (u,p) satisfies the a priori bound*

$$\begin{cases} \|\nabla u\|_{L^2(\Omega)} \le C_1 \left(\|(U,V)\|^2_{H^{1/2}(\partial Q)} + \|(U,V)\|_{H^{1/2}(\partial Q)} + \|f\|_{H^{-1}(\Omega)} \right), \\ \quad \|p\|_{L^2(\Omega)} \le C_2 \left(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)} \right), \end{cases} \tag{3.2}$$

for some $C_1, C_2 > 0$ that depend on Ω and η. Moreover, there exists $\delta = \delta(\eta, \Omega) > 0$ such that if

$$\|(U,V)\|_{H^{1/2}(\partial Q)} + \|f\|_{H^{-1}(\Omega)} < \delta, \tag{3.3}$$

then the weak solution (u,p) of (1.2)-(1.3) is unique and also satisfies the estimate $\|\nabla u\|_{L^2(\Omega)} < \mathcal{S}_0 \eta$.

Proof. Existence of a weak solution $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (1.2)-(1.3) satisfying the a priori bounds (3.2) follows from [39, Theorem IX.4.1]. We give here the proof of the a priori bounds and uniqueness for small data because we need to make explicit the dependence of C_1 and C_2 appearing in (3.2) on the Sobolev constants $\mathcal{S}$ and $\mathcal{S}_0$ in (2.22), and on the solenoidal extension of the boundary datum. Indeed, Proposition 2.1 ensures the existence of a solenoidal vector field $u_0 \in \mathcal{V}_*(\Omega)$ satisfying

$$u_0 = (U,V) \quad \text{on} \quad \partial Q, \qquad \|\nabla u_0\|_{L^2(\Omega)} \le M \|(U,V)\|_{H^{1/2}(\partial Q)}, \qquad |\beta(v, u_0, v)| \le \frac{\eta}{2} \|\nabla v\|^2_{L^2(\Omega)} \quad \forall v \in \mathcal{V}(\Omega),$$

for some $M > 0$ depending on Ω. Define $\xi = u - u_0 \in \mathcal{V}(\Omega)$, and replace $u = \xi + u_0$ into (1.2) to obtain

$$-\eta \Delta \xi + [(\xi + u_0) \cdot \nabla](\xi + u_0) + \nabla p = \eta \Delta u_0 + f, \tag{3.4}$$

with $\eta \Delta u_0 + f \in H^{-1}(\Omega)$. Here, (3.4) is understood in the weak sense, see (3.1); we test it with ξ and we integrate by parts over Ω in order to obtain

$$\eta \|\nabla \xi\|^2_{L^2(\Omega)} \le (\eta \|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}) \|\nabla \xi\|_{L^2(\Omega)} - \beta(\xi + u_0, \xi + u_0, \xi). \tag{3.5}$$

By (2.36)-(2.37) we have $\beta(\xi + u_0, \xi + u_0, \xi) = \beta(\xi + u_0, u_0, \xi)$ and the estimate

$$|\beta(\xi + u_0, u_0, \xi)| \le \frac{\eta}{2} \|\nabla \xi\|^2_{L^2(\Omega)} + \frac{1}{\sqrt{\mathcal{S}\mathcal{S}_0}} \|\nabla \xi\|_{L^2(\Omega)} \|\nabla u_0\|^2_{L^2(\Omega)}, \tag{3.6}$$

where we have used the definition of $\mathcal{S}$ and $\mathcal{S}_0$ given in (2.22). By plugging (3.6) into (3.5) we deduce

$$\|\nabla u\|_{L^2(\Omega)} \le \|\nabla \xi\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega)} \le \frac{2}{\sqrt{\mathcal{S}\mathcal{S}_0}\,\eta} \|\nabla u_0\|^2_{L^2(\Omega)} + 3\|\nabla u_0\|_{L^2(\Omega)} + \frac{2}{\eta} \|f\|_{H^{-1}(\Omega)},$$

and then the inequality $\|\nabla u_0\|_{L^2(\Omega)} \le M \|(U,V)\|_{H^{1/2}(\partial Q)}$ yields $(3.2)_1$ in the following way:

$$\|\nabla u\|_{L^2(\Omega)} \le \frac{2M^2}{\sqrt{\mathcal{S}\mathcal{S}_0}\,\eta} \|(U,V)\|^2_{H^{1/2}(\partial Q)} + 3M \|(U,V)\|_{H^{1/2}(\partial Q)} + \frac{2}{\eta} \|f\|_{H^{-1}(\Omega)}. \tag{3.7}$$

The a priori bound for the pressure in $(3.2)_2$ is obtained after noticing that

$$\nabla p = \eta \Delta u - (u \cdot \nabla) u + f \quad \text{in the sense of} \quad H^{-1}(\Omega),$$

and applying (2.43) with some embedding inequalities.

The quantitative uniqueness statement relies on a different kind of a priori bound, based on a *given* solenoidal extension, that is,

$$v_0 \in \mathcal{V}_*(\Omega), \qquad v_0 = (U,V) \quad \text{on} \quad \partial Q, \qquad \|\nabla v_0\|_{L^2(\Omega)} \le C \|(U,V)\|_{H^{1/2}(\partial Q)}, \tag{3.8}$$

where the constant $C = C(\Omega) > 0$ is independent on the boundary data, see [57]. Then we seek solutions u of (1.2)-(1.3) in the form $u = \xi + v_0$ so that $\xi \in \mathcal{V}(\Omega)$ satisfies

$$-\eta\Delta\xi + [(\xi + v_0)\cdot\nabla](\xi + v_0) + \nabla p = \eta\Delta v_0 + f, \tag{3.9}$$

with $\eta\Delta v_0 + f \in H^{-1}(\Omega)$. Here, (3.9) is intended in the weak sense, see (3.1); we test it with ξ and we integrate by parts in Ω in order to obtain

$$\eta\|\nabla\xi\|^2_{L^2(\Omega)} \leq (\eta\|\nabla v_0\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)})\|\nabla\xi\|_{L^2(\Omega)} - \beta(\xi + v_0, \xi + v_0, \xi). \tag{3.10}$$

In view of (2.36)-(2.37) we have $\beta(\xi + v_0, \xi + v_0, \xi) = \beta(\xi + v_0, v_0, \xi)$ and the estimate

$$\begin{aligned} |\beta(\xi + v_0, v_0, \xi)| &\leq \|\xi\|_{L^4(\Omega)}\|\nabla v_0\|_{L^2(\Omega)}\big(\|\xi\|_{L^4(\Omega)} + \|v_0\|_{L^4(\Omega)}\big) \\ &\leq \frac{\|\nabla\xi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\|\nabla v_0\|_{L^2(\Omega)}\left(\frac{\|\nabla\xi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}} + \|v_0\|_{L^4(\Omega)}\right), \end{aligned} \tag{3.11}$$

where we used the definition of $\mathcal{S}_0$ given in (2.22). Inserting (3.11) into (3.10) yields

$$\eta\|\nabla\xi\|_{L^2(\Omega)} \leq \frac{\|\nabla v_0\|_{L^2(\Omega)}}{\mathcal{S}_0}\|\nabla\xi\|_{L^2(\Omega)} + \frac{\|\nabla v_0\|_{L^2(\Omega)}\|v_0\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_0}} + \eta\|\nabla v_0\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}.$$

Let C be as in (3.8); if the boundary datum is small enough so that

$$C\|(U, V)\|_{H^{1/2}(\partial Q)} < \mathcal{S}_0\eta\,, \tag{3.12}$$

then, for the chosen extension v_0, one also has $\|\nabla v_0\|_{L^2(\Omega)} < \mathcal{S}_0\eta$ and we infer that

$$\|\nabla\xi\|_{L^2(\Omega)} \leq \frac{\dfrac{\|\nabla v_0\|_{L^2(\Omega)}\|v_0\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_0}} + \eta\|\nabla v_0\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}}{\eta - \dfrac{\|\nabla v_0\|_{L^2(\Omega)}}{\mathcal{S}_0}}. \tag{3.13}$$

This is the sought a priori bound for solutions of (3.1), up to the additive solenoidal extension v_0 of the boundary data. We emphasize that it has been obtained under the smallness assumption (3.12).

Assuming (3.12), take two weak solutions $u, v \in H^1_*(\Omega)$ of (1.2)-(1.3), with possibly different pressures that are, however, ruled out by L^2-orthogonality of the gradients with $\mathcal{V}(\Omega)$. Indeed, subtract the equations (3.1) corresponding to u and v in order to obtain

$$\eta(\nabla w, \nabla\varphi)_{L^2(\Omega)} + \beta(u, w, \varphi) + \beta(w, v, \varphi) = 0 \qquad \forall\varphi \in \mathcal{V}(\Omega),$$

where $w \doteq u - v \in \mathcal{V}(\Omega)$. By taking $\varphi = w$, defining $\xi = v - v_0$ and using (2.36) and (3.13), we derive

$$\begin{aligned} \eta\|\nabla w\|^2_{L^2(\Omega)} &= -\beta(w, v, w) = \beta(w, w, v) \leq \|w\|_{L^4(\Omega)}\|\nabla w\|_{L^2(\Omega)}\|v\|_{L^4(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\|v\|_{L^4(\Omega)} \\ &\leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\big(\|\xi\|_{L^4(\Omega)} + \|v_0\|_{L^4(\Omega)}\big) \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\left(\frac{\|\nabla\xi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}} + \|v_0\|_{L^4(\Omega)}\right) \\ &\leq \|\nabla w\|^2_{L^2(\Omega)}\,\frac{\eta(\|\nabla v_0\|_{L^2(\Omega)} + \sqrt{\mathcal{S}_0}\,\|v_0\|_{L^4(\Omega)}) + \|f\|_{H^{-1}(\Omega)}}{\eta\mathcal{S}_0 - \|\nabla v_0\|_{L^2(\Omega)}}, \end{aligned} \tag{3.14}$$

which shows that $w = 0$ provided that

$$\eta\big(2\|\nabla v_0\|_{L^2(\Omega)} + \sqrt{\mathcal{S}_0}\,\|v_0\|_{L^4(\Omega)}\big) + \|f\|_{H^{-1}(\Omega)} < \mathcal{S}_0\eta^2\,. \tag{3.15}$$

In conclusion, unique solvability of (1.2)-(1.3) is achieved whenever both (3.12) and (3.15) hold. Since the most restrictive is the latter, and since $\|v_0\|_{L^4(\Omega)} \leq \|\nabla v_0\|_{L^2(\Omega)}/\sqrt{\mathcal{S}}$, uniqueness is ensured whenever

$$\eta\left(2+\sqrt{\frac{\mathcal{S}_0}{\mathcal{S}}}\right)\|\nabla v_0\|_{L^2(\Omega)}+\|f\|_{H^{-1}(\Omega)}<\mathcal{S}_0\eta^2\,. \tag{3.16}$$

In turn, by (3.8), (3.16) certainly holds if

$$\eta C\,\frac{2\sqrt{\mathcal{S}}+\sqrt{\mathcal{S}_0}}{\sqrt{\mathcal{S}}}\,\|(U,V)\|_{H^{1/2}(\partial Q)}+\|f\|_{H^{-1}(\Omega)}<\mathcal{S}_0\eta^2\,. \tag{3.17}$$

Therefore, an explicit expression for δ in (3.3) is given by

$$\delta(\eta,\Omega)=\min\left\{\frac{\eta}{C}\,\frac{\mathcal{S}_0\sqrt{\mathcal{S}}}{2\sqrt{\mathcal{S}}+\sqrt{\mathcal{S}_0}},\ \mathcal{S}_0\eta^2\right\}. \tag{3.18}$$

Finally, we have to prove the gradient bound for the unique solution whenever the inequality

$$\|(U,V)\|_{H^{1/2}(\partial Q)}+\|f\|_{H^{-1}(\Omega)}<\min\left\{\frac{\eta}{C}\,\frac{\mathcal{S}_0\sqrt{\mathcal{S}}}{2\sqrt{\mathcal{S}}+\sqrt{\mathcal{S}_0}},\ \mathcal{S}_0\eta^2\right\}$$

holds. This inequality implies (3.17) which, together with (3.8), implies

$$\|\nabla v_0\|_{L^2(\Omega)}<\eta\sqrt{\mathcal{S}\mathcal{S}_0}\,; \tag{3.19}$$

we point out that (3.19) slightly improves (3.12) since $\mathcal{S}\leq\mathcal{S}_0$. For the same reason, and since (3.19) holds, we may write a "slightly worse" bound than (3.13), namely

$$\|\nabla\xi\|_{L^2(\Omega)}\leq\frac{\dfrac{\|\nabla v_0\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}\mathcal{S}_0}}+\eta\|\nabla v_0\|_{L^2(\Omega)}+\|f\|_{H^{-1}(\Omega)}}{\eta-\dfrac{\|\nabla v_0\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}\mathcal{S}_0}}}.$$

Hence, recalling that $u=\xi+v_0$, by (3.16) we have that

$$\|\nabla u\|_{L^2(\Omega)}\leq\|\nabla\xi\|_{L^2(\Omega)}+\|\nabla v_0\|_{L^2(\Omega)}\leq\frac{2\eta\|\nabla v_0\|_{L^2(\Omega)}+\|f\|_{H^{-1}(\Omega)}}{\eta-\dfrac{\|\nabla v_0\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}\mathcal{S}_0}}}<\mathcal{S}_0\eta\,.$$

This proves the gradient bound and completes the proof. □

Remark 3.1. *Theorem 3.1 guarantees unique solvability of* (1.2)-(1.3) *under a smallness assumption on the data, which in turn yields the bound* $\|\nabla u\|_{L^2(\Omega)}<\mathcal{S}_0\eta$. *Conversely, the existence of such a "small" solution ensures unique solvability, see [39, Theorem IX.2.1].*

The constant δ in (3.3) depends on Ω through the embedding constants $\mathcal{S}$ and $\mathcal{S}_0$ and through the solenoidal extension constant C in (3.8). Theorem 3.1 guarantees the uniqueness of the solution whenever the data (U,V) and f are small also with respect to the kinematic viscosity η. If this smallness assumption is violated one expects multiplicity results, see [75] and also [39, Theorem IX.2.2] for a slightly more general situation: at a certain Reynolds number a bifurcation occurs.

What is left open in the proof of Theorem 3.1 is the choice of the particular solenoidal extension v_0. We can find an explicit form of v_0 in the case where the boundary data are constant (so that (1.4) is automatically fulfilled). To this end, for $0 < d \leq a < L$ such that $L > a + d$, we introduce the constants

$$\gamma_0 = \frac{\sqrt{3}\pi^{3/2}}{2L} \max\left\{1, \frac{\mu_0}{\sqrt{2\pi}} \sqrt{\frac{|Q|}{|Q| - |K|}}\right\},$$

$$\gamma_1 = \frac{3\pi}{4L}\sqrt{\frac{|Q|}{|Q|-|K|}} \left[1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)}\right]^{-\frac{3}{2}} \left[1 + \sqrt{\frac{3}{8}\log\left(\frac{|Q|}{|K|}\right)} + \frac{3\sqrt{3}}{4\sqrt{2}} \frac{|K|}{|Q|-|K|} \log^{3/2}\left(\frac{|Q|}{|K|}\right)\right]^{-\frac{1}{2}},$$

$$\gamma_2 = 48\, \frac{\left[(L-a-d)^2 + (L-d)^2\right] \left[\dfrac{1}{5} + L^2 \left(d \log\left(\dfrac{L-a}{d}\right)\right)^{-2}\right]}{(L-a-d)(L-d)\log\left(\dfrac{L(L-a-d) + L(L-d)}{(a+d)(L-a-d) + d(L-d)}\right)},$$

with $\mu_0 > 0$ as in (2.21). Notice that γ_0 and γ_1 represent, respectively, lower bounds for the Sobolev constants $\mathcal{S}_0$ and $\mathcal{S}_1$, see Corollary 2.1. On the other hand, γ_2 controls the norm of the solenoidal extension given in Theorem 2.5. Then, if we additionally assume that $f = 0$, Theorem 3.1 may be strengthened as follows.

Theorem 3.2. *Let Ω be as in* (1.1) *and assume* (2.33) *with $L > a+d$. For any $(U, V) \in \mathbb{R}^2$ there exists a weak solution $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of* (1.2)-(1.3) *with $f = 0$. If, moreover,*

$$\sqrt{U^2 + V^2} < \frac{\eta}{\sqrt{\gamma_2}} \frac{\gamma_0\sqrt{\gamma_1}}{\sqrt{\gamma_0} + 2\sqrt{\gamma_1}},$$

then the weak solution of (1.2)-(1.3) *is unique.*

Proof. Existence of a weak solution $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (1.2)-(1.3) with $f = 0$ follows from Theorem 3.1, noticing that the compatibility condition (1.4) is automatically fulfilled. Also, Theorem 2.5 guarantees the existence of a vector field $W \in H^1_c(\Omega)$ satisfying

$$\nabla \cdot W = 0 \;\text{ in }\; \Omega, \qquad W = (U, V) \;\text{ on }\; \partial Q, \qquad \|\nabla W\|_{L^2(\Omega)} \leq \sqrt{\gamma_2\,(U^2 + V^2)}. \tag{3.20}$$

We go back to the proof of Theorem 3.1, where the expression for δ in (3.15) now becomes

$$2\|\nabla W\|_{L^2(\Omega)} + \sqrt{\mathcal{S}_0}\, \|W\|_{L^4(\Omega)} < \mathcal{S}_0 \eta\,. \tag{3.21}$$

By (2.22) we observe that (3.21) is certainly fulfilled if

$$\left(2 + \sqrt{\frac{\mathcal{S}_0}{\mathcal{S}_1}}\right) \|\nabla W\|_{L^2(\Omega)} < \mathcal{S}_0 \eta\,.$$

In turn, thanks to (3.20) and Corollary 2.1 we see that the latter inequality is implied by

$$\sqrt{U^2 + V^2} < \frac{\eta}{\sqrt{\gamma_2}} \frac{\mathcal{S}_0\sqrt{\gamma_1}}{\sqrt{\mathcal{S}_0} + 2\sqrt{\gamma_1}}\,. \tag{3.22}$$

The proof is complete after noticing that the right-hand side of (3.22) is increasing with respect to $\mathcal{S}_0$, and using the lower bound for $\mathcal{S}_0$ given in Corollary 2.1. $\square$

Remark 3.2. *Theorem 3.2 not only gives a lower bound for δ in terms of η and Ω; since η and K are fixed, it also estimates the critical Reynolds number ensuring unique solvability of* (1.2)-(1.3) *with zero external forcing. Nevertheless, the method provided in the proof of Theorem 3.2 leads to an overestimation of the critical boundary velocity, since some of the inequalities employed are far from being sharp. Similar considerations, following a different approach for the computation of the critical Reynolds number ensuring the stability of a steady laminar flow, were already pointed out by Landau-Lifshitz in 1959, see [59, Chapter III].*

Regularity results for (1.2)-(1.3) are usually presented under the no-slip boundary condition on the whole boundary $\partial\Omega$, that is, when $U = V = 0$ on ∂Q. In this case, if $f \in L^2(\Omega)$, the regularity of a weak solution $(u,p) \in H^1_0(\Omega) \times L^2(\Omega)$ of (1.2)-(1.3) can be upgraded up to $\left[H^2(\Omega) \cap H^1_0(\Omega)\right] \times H^1(\Omega)$ whenever Ω is of class $\mathcal{C}^2$ (see [39, Theorem IX.5.2]). If Ω were a convex polygon, the same result holds, see [51]. But since we consider obstacles K having a merely Lipschitz boundary, the domain Ω may possess reentrant corners, a fact that introduces singularities in the solution, which may exhibit blow-up of the pressure and of the vorticity near the non-convex vertices, see [20]. Nevertheless, even if we remain with the minimal regularity $H^1(\Omega)\times L^2(\Omega)$, the normal component of the trace of functions in $E_r(\Omega)$ can be treated through (2.38). Furthermore, standard elliptic regularity arguments show that the solution of (1.2)-(1.3) is more regular far from K, a property that we make precise in the next statement. Since we were unable to find a unique reference for its proof, in particular because of the use of solenoidal extensions, for the sake of completeness we include it below by combining several known results adapted to the particular geometry of Ω in (1.1).

Theorem 3.3. *Let Ω be as in* (1.1)*. For $f \in L^2(\Omega)$ and $(U,V) \in \mathbb{R}^2$, let $(u,p) \in \mathcal{V}_*(\Omega) \times L^2(\Omega)$ be a weak solution of* (1.2)-(1.3)*. Then, for any open set $\Omega_0 \subset \Omega$ such that $\partial\Omega \cap \partial\Omega_0 = \partial Q$ and with an internal boundary of class $\mathcal{C}^2$, one has $(u,p) \in H^2(\Omega_0) \times H^1(\Omega_0)$. Moreover, there exists a constant $C > 0$, depending on η and Ω_0, such that:*

$$\|u\|_{H^2(\Omega_0)} + \|p\|_{H^1(\Omega_0)} \leq C\left(|(U,V)|^4 + |(U,V)| + \|f\|^2_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right). \tag{3.23}$$

Proof. From (2.39) and (3.2) we know that

$$\begin{cases} \|\nabla u\|_{L^2(\Omega)} \leq C\left(|(U,V)|^2 + |(U,V)| + \|f\|_{L^2(\Omega)}\right), \\ \|p\|_{L^2(\Omega)} \leq C\left(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right), \end{cases}$$

where, from now on, $C > 0$ will denote a generic constant depending on η and Ω_0. In particular, we have that $(u\cdot\nabla)u \in L^{3/2}(\Omega)$ with

$$\|(u\cdot\nabla)u\|_{L^{3/2}(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}\|u\|_{L^6(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)} \leq C\left(|(U,V)|^2 + |(U,V)| + \|f\|_{L^2(\Omega)}\right)^2, \tag{3.24}$$

by the embedding $H^1(\Omega) \subset L^6(\Omega)$ and the generalized Poincaré inequality from [28]. Then the couple (u,p) also weakly solves the Stokes equations

$$-\eta\Delta u + \nabla p = f - (u\cdot\nabla)u, \qquad \nabla\cdot u = 0 \quad \text{in } \Omega. \tag{3.25}$$

Consider a (non simply connected) $\mathcal{C}^2$-domain $\Omega_1 \subset \Omega$ such that $\overline{\Omega_1} \subset \Omega$ and with the exterior boundary "close" to ∂Q, while the interior boundary lies between $\partial\Omega_0$ and ∂K: roughly speaking, Ω_1 is wider than Ω_0 close to K and smaller than Ω_0 close to ∂Q. Clearly, the constants C that depend directly on Ω_1 also depend indirectly on Ω_0. The Stokes equations (3.25) are also satisfied in Ω_1, so from (3.24) and [39, Theorem IV.4.1] we know that

$$\|u\|_{W^{2,3/2}(\Omega_1)} + \|p\|_{W^{1,3/2}(\Omega_1)} \leq C\left(|(U,V)|^4 + |(U,V)| + \|f\|^2_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right).$$

With this additional regularity of u, we infer that $(u \cdot \nabla)u \in L^2(\Omega_1)$ and, by repeating the above argument, we obtain

$$\|u\|_{H^2(\Omega_1)} + \|p\|_{H^1(\Omega_1)} \leq C \left(|(U,V)|^4 + |(U,V)| + \|f\|^2_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right) . \tag{3.26}$$

This gives the required bound in Ω_1, namely far away from ∂Q and from the obstacle. In order to reach ∂Q, we employ a localization argument which covers the residual domain $\Omega_* \doteq \Omega_0 \setminus \overline{\Omega_1}$: since it is precompact, it can be covered by a finite number of open disks $\{\theta_i\}_{i=1}^m$, for some $m \geq 1$:

$$\overline{\Omega_*} \subset \bigcup_{i=1}^m \theta_i.$$

By reducing the radius of the disks $\{\theta_i\}_{i=1}^m$ (if necessary), we may assume that θ_i does not intersect the internal boundary of Ω_1, for all $i \in \{1, \ldots, m\}$ (in particular, $\theta_i \cap \partial K = \emptyset$).

Next, we introduce a partition of unity subordinate to the open cover $\{\theta_i\}_{i=1}^m$, that is, we consider a family of functions $\{\phi_i\}_{i=1}^m \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that:

$$\phi_i \in \mathcal{C}_0^\infty(\theta_i), \qquad 0 \leq \phi_i(x,y) \leq 1 \quad \forall (x,y) \in \overline{\Omega_*}, \quad \forall i \in \{1, \ldots, m\}; \qquad \sum_{i=1}^m \phi_i(x,y) = 1 \quad \forall (x,y) \in \overline{\Omega_*}.$$

Therefore, we have

$$u(x,y) = \sum_{i=1}^m \phi_i(x,y) u(x,y), \qquad p(x,y) = \sum_{i=1}^m \phi_i(x,y) p(x,y) \qquad \text{for a.e. } (x,y) \in \overline{\Omega_*},$$

and it suffices to prove that $\phi_i u \in H^2(\Omega_* \cap \theta_i)$ and $\phi_i p \in H^1(\Omega_* \cap \theta_i)$, for every $i \in \{1, \ldots, m\}$. In order to achieve this, we notice that, since Q is convex and ϕ_i has compact support in θ_i, there exists a convex polygon ζ_i such that $\text{supp}(\phi_i) \cap \Omega_* \subset \zeta_i$, see Figure 3.1.

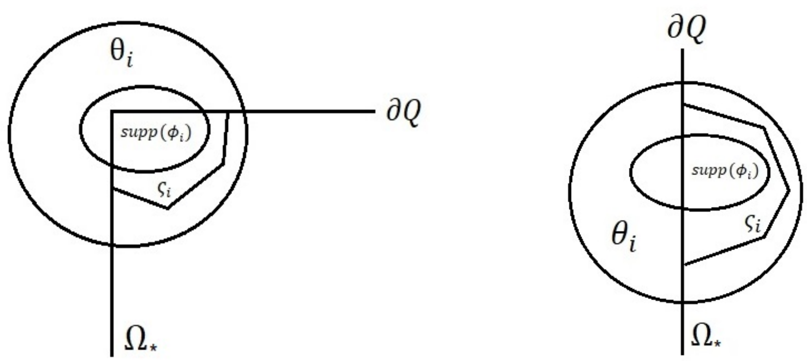

Figure 3.1: Construction of the open set $\zeta_i \subset (\theta_i \cap \Omega_*)$.

Defining $\overline{u} \doteq u - (U,V)$, one notices that $(\phi_i \overline{u}, \phi_i p) \in H_0^1(\zeta_i) \times L^2(\zeta_i)$ and $\nabla \cdot (\phi_i \overline{u}) = \nabla \phi_i \cdot \overline{u} \in H_0^1(\zeta_i)$. Thus, [39, Theorem III.3.3] guarantees the existence of a vector field $v_i \in H^2(\zeta_i) \cap H_0^1(\zeta_i)$ such that

$$\nabla \cdot v_i = \nabla \phi_i \cdot \overline{u} \quad \text{in } \zeta_i, \qquad \|v_i\|_{H^2(\zeta_i)} \leq c_i \|\nabla \phi_i \cdot \overline{u}\|_{H^1(\zeta_i)}, \tag{3.27}$$

for some constant $c_i > 0$ depending only on θ_i. Since (u,p) is a solution of (1.2)-(1.3), we deduce that the pair $(\phi_i \overline{u} - v_i, \phi_i p) \in H_0^1(\zeta_i) \times L^2(\zeta_i)$ satisfies the Stokes system

$$-\eta \Delta(\phi_i \overline{u} - v_i) + \nabla(\phi_i p) = \omega_i + \eta(\Delta \phi_i)(U,V) + \eta \Delta v_i, \qquad \nabla \cdot (\phi_i \overline{u} - v_i) = 0 \qquad \text{in } \zeta_i,$$

with $\omega_i \doteq \phi_i[f-(u\cdot\nabla)u]-\eta[(\Delta\phi_i)u+2(\nabla\phi_i\cdot\nabla)u]+p\nabla\phi_i \in L^{3/2}(\zeta_i)\subset H^{-1/3}(\zeta_i)$. Then, since ζ_i is a convex polygon, we may "interpolate" the basic regularity of the Stokes equation (H^{-1}-source implies $H^1\times L^2$-solution) with the improved regularity from [51, Theorem 2] (L^2-source implies $H^2\times H^1$-solution) to infer that $\phi_i\overline{u}\in H^{5/3}(\zeta_i)$. As $H^{5/3}(\zeta_i)\subset L^\infty(\zeta_i)$, we finally have $\omega_i\in L^2(\zeta_i)$. Applying again [51, Theorem 2] we infer that $(\phi_i\overline{u},\phi_i p)\in H^2(\zeta_i)\times H^1(\zeta_i)$ and the existence of $C_i>0$ (depending only on θ_i) such that

$$\|\phi_i\overline{u}-v_i\|_{H^2(\zeta_i)}+\|\phi_i p\|_{H^1(\zeta_i)}\le C_i\left(\|\omega_i\|_{L^2(\zeta_i)}+\eta|(U,V)|\,\|\Delta\phi_i\|_{L^2(\zeta_i)}+\eta\|\Delta v_i\|_{L^2(\zeta_i)}\right).$$

In view of (3.27), this implies

$$\|\phi_i u\|_{H^2(\Omega_*\cap\theta_i)}+\|\phi_i p\|_{L^2(\Omega_*\cap\theta_i)}\le C_i\left(\|\omega_i\|_{L^2(\Omega_*\cap\theta_i)}+|(U,V)|\,\|\phi_i\|_{H^2(\Omega_*\cap\theta_i)}+\|\nabla\phi_i\cdot\overline{u}\|_{H^1(\Omega_*\cap\theta_i)}\right),$$

where $C_i>0$ now denotes a constant depending on η and θ_i. By summing over $i\in\{1,\dots,m\}$ we get

$$\begin{aligned}\|u\|_{H^2(\Omega_*)}+\|p\|_{H^1(\Omega_*)}&\le\sum_{i=1}^m C_i\left(\|\omega_i\|_{L^2(\Omega_*\cap\theta_i)}+|(U,V)|\,\|\phi_i\|_{H^2(\Omega_*\cap\theta_i)}+\|\nabla\phi_i\cdot\overline{u}\|_{H^1(\Omega_*\cap\theta_i)}\right)\\&\le C\left[\|\nabla u\|_{L^2(\Omega_0)}\left(\|\nabla u\|_{L^2(\Omega_0)}+1\right)+\|p\|_{L^2(\Omega_0)}+\|f\|_{L^2(\Omega_0)}+|(U,V)|\right]\\&\le C\left(|(U,V)|^4+|(U,V)|+\|f\|^2_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}\right),\end{aligned}\tag{3.28}$$

after applying the Poincaré-type inequalities to $\overline{u}=u-(U,V)$, using that $\{\phi_i\}_{i=1}^m\subset\mathcal{C}_0^\infty(\mathbb{R}^2)$ and (3.2). The proof is complete after putting together (3.26) and (3.28). □

Remark 3.3. *If the obstacle K has a $\mathcal{C}^2$ boundary, then the arguments of Theorem 3.3 enable to prove that weak solutions of* (1.2)-(1.3) *in Ω belong to $H^2(\Omega)\times H^1(\Omega)$.*

3.2 Symmetry and almost symmetry

Turbulence in fluids with large Reynolds number may be detected by refined numerical simulations using Computational Fluid Dynamics [36], see Figure 3.2 where the dependence of the flow on the Reynolds number is emphasized in a symmetric domain.

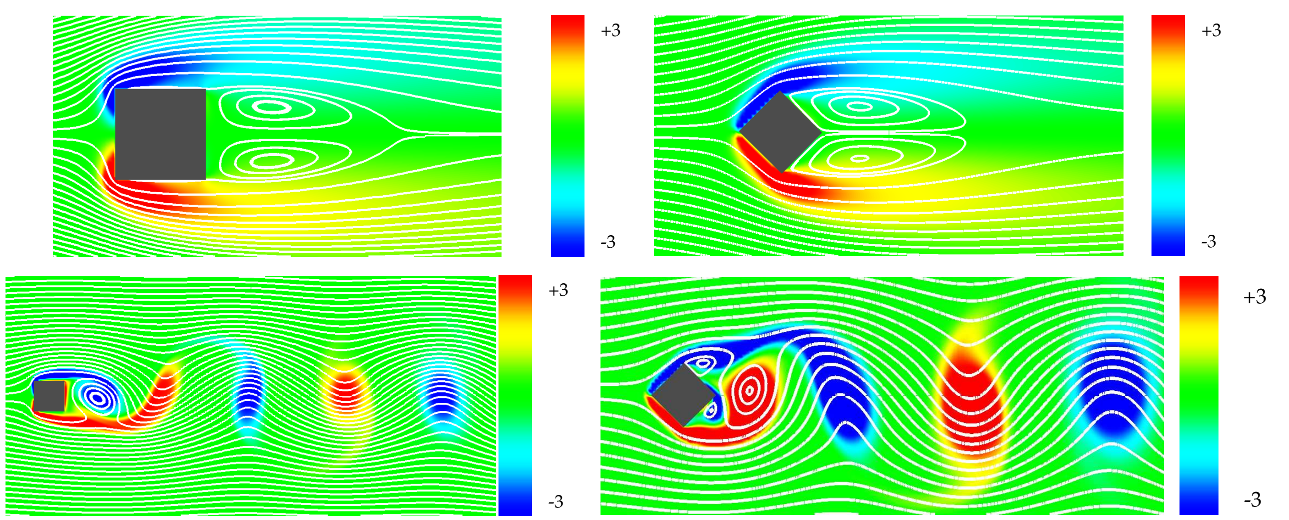

Figure 3.2: CFD simulation of a flow around a square cylinder (top line Re= 30, bottom line Re= 200) by Fuka-Brechler [36], reproduced with courtesy of the authors.

The pattern displayed in Figure 3.2 will be essential to comment the results throughout the paper.

We consider here domains Ω being symmetric with respect to the x-axis. Moreover, we initially assume that the boundary data in (1.3) satisfy

$$U(x,-y) = U(x,y) \quad \text{and} \quad V(x,-y) = -V(x,y) \qquad \forall (x,y) \in \partial Q. \tag{3.29}$$

Concerning the source $f = (f_1, f_2) \in H^{-1}(\Omega)$, we recall that a distribution is called even (resp. odd) if its kernel contains the space of odd (resp. even) test functions. In this symmetric framework, we complement Theorem 3.1 with the following result; see [38] for a related work in unbounded domains.

Theorem 3.4. *Let Ω be as in* (1.1)*, K being symmetric with respect to the x-axis. Suppose that $f = (f_1, f_2) \in H^{-1}(\Omega)$ and that $(U, V) \in H^{1/2}(\partial Q)$ satisfy* (1.4)*. Assume moreover that f_1 is y-even, f_2 is y-odd, and (U, V) verifies* (3.29)*. Then:*

- *there exists (at least) one weak solution $(u_1, u_2, p) \in \mathcal{V}_*(\Omega)^2 \times L^2_0(\Omega)$ of* (1.2)-(1.3) *satisfying the symmetry property*

$$u_1(x,-y) = u_1(x,y), \quad u_2(x,-y) = -u_2(x,y), \quad p(x,-y) = p(x,y) \qquad \textit{for a.e. } (x,y) \in \Omega; \tag{3.30}$$

- *if $(u_1, u_2, p) \in H^1(\Omega)^2 \times L^2_0(\Omega)$ is a weak solution of* (1.2)-(1.3)*, then also (v_1, v_2, q) with*

$$v_1(x,y) = u_1(x,-y), \quad v_2(x,y) = -u_2(x,-y), \quad q(x,y) = p(x,-y) \qquad \textit{for a.e. } (x,y) \in \Omega \tag{3.31}$$

solves (1.2)-(1.3)*;*

- *if* (3.3) *holds, then the unique weak solution of* (1.2)-(1.3) *satisfies* (3.30).

Proof. By Proposition 2.1, there exists a symmetric solenoidal extension $\hat{v} \in H^1(\Omega)$ of the boundary data $(U, V) \in H^{1/2}(\partial Q)$ such that

$$\begin{cases} \nabla \cdot \hat{v} = 0 \quad \text{in } \Omega, \qquad \hat{v} = (U, V) \text{ on } \partial Q, \qquad \hat{v} = (0,0) \text{ on } \partial K \\ |\beta(z, \hat{v}, z)| \leq \dfrac{\eta}{2} \|\nabla z\|^2_{L^2(\Omega)} \quad \forall z \in \mathcal{V}(\Omega); \quad \hat{v}_1 \text{ is } y\text{-even}, \quad \hat{v}_2 \text{ is } y\text{-odd}. \end{cases} \tag{3.32}$$

We introduce the space

$$\mathcal{Z}(\Omega) = \{ v \in \mathcal{V}(\Omega) \mid v \text{ satisfies the symmetry property (3.30)} \},$$

which is a closed subspace of $\mathcal{V}(\Omega)$ and therefore it constitutes a Hilbert space under the Dirichlet scalar product. To prove the existence of a weak symmetric solution $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (1.2)-(1.3) amounts to show the existence of $(\hat{u}, p) \in \mathcal{Z}(\Omega) \times L^2_0(\Omega)$ such that

$$-\eta \Delta \hat{u} + (\hat{u} \cdot \nabla)\hat{u} + (\hat{u} \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)\hat{u} + \nabla p = f + \eta \Delta \hat{v} - (\hat{v} \cdot \nabla)\hat{v} \quad \text{in } \Omega \tag{3.33}$$

in weak sense, then the solution will be given by $u = \hat{u} + \hat{v}$ and p will have the required symmetry property as a consequence of (3.33). Fix $v_0 \in \mathcal{Z}(\Omega)$ and consider the linearized version of (3.33), namely

$$-\eta \Delta \hat{u} + (v_0 \cdot \nabla)\hat{u} + (\hat{u} \cdot \nabla)\hat{v} + (\hat{v} \cdot \nabla)\hat{u} + \nabla p = f + \eta \Delta \hat{v} - (\hat{v} \cdot \nabla)\hat{v}, \qquad \nabla \cdot \hat{u} = 0 \text{ in } \Omega.$$

By a symmetric weak solution of this problem we understand a function $\hat{u} \in \mathcal{Z}(\Omega)$ such that

$$\eta(\nabla \hat{u}, \nabla \varphi)_{L^2(\Omega)} + \beta(v_0, \hat{u}, \varphi) + \beta(\hat{u}, \hat{v}, \varphi) + \beta(\hat{v}, \hat{u}, \varphi) = \langle F, \varphi \rangle_\Omega \qquad \forall \varphi \in \mathcal{Z}(\Omega), \tag{3.34}$$

where $F \doteq f + \eta \Delta \hat{v} - (\hat{v} \cdot \nabla)\hat{v} \in H^{-1}(\Omega)$ is such that F_1 is y-even and F_2 is y-odd. It is quite standard for the Navier-Stokes equations to see that the bilinear form $A : \mathcal{Z}(\Omega) \times \mathcal{Z}(\Omega) \to \mathbb{R}$ defined by

$$A(v, w) = \eta(\nabla v, \nabla w)_{L^2(\Omega)} + \beta(v_0, v, w) + \beta(v, \hat{v}, w) + \beta(\hat{v}, v, w) \qquad \forall v, w \in \mathcal{Z}(\Omega),$$

is continuous and coercive (for the latter property, one needs the bound in (3.32)). Therefore, the Lax-Milgram Theorem ensures the existence of a unique function $\hat{u} \in \mathcal{Z}(\Omega)$ satisfying (3.34). Whence, in view of the compact embedding $\mathcal{Z}(\Omega) \subset L^4(\Omega)$, we have constructed a compact operator $T : L^4(\Omega) \to L^4(\Omega)$ such that, for any $v_0 \in L^4(\Omega)$, $T(v_0) = \hat{u}$ is the unique symmetric solution of (3.34). Moreover, after testing (3.34) with $\varphi = \hat{u}$ and using the bound in (3.32) we obtain

$$\|\nabla \hat{u}\|_{L^2(\Omega)} \leq \frac{2}{\eta}\|F\|_{H^{-1}(\Omega)},$$

so that T actually maps the (non-empty) convex compact set $\{v \in L^4(\Omega) \mid \eta\|\nabla v\|_{L^2(\Omega)} \leq 2\|F\|_{H^{-1}(\Omega)}\}$ into itself. Then the Schauder Fixed Point Theorem ensures the existence of $\hat{u} \in \mathcal{Z}(\Omega)$ such that $T(\hat{u}) = \hat{u}$, that is, $\hat{u}$ is a weak solution of (3.33) satisfying the symmetry property (3.30). By the symmetry properties of F, we infer that the resulting pressure $p \in L^2_0(\Omega)$, which arises as a consequence of [39, Lemma III.1.1], also satisfies the symmetry property given in (3.30).

Finally, under the assumptions of the statement, one can check that also (3.31) solves (1.2)-(1.3). Thus, in case of uniqueness, the solution satisfies the symmetry property (3.30). □

Remark 3.4. *Since δ in (3.18) depends increasingly on η, and therefore decreasingly on Re, Figure 3.2 is compatible with Theorem 3.1: as long as Re is small the flow is symmetric, while if Re is large, uniqueness is lost and asymmetric solutions may arise. Hence, in a symmetric framework, the existence of an asymmetric solution is a sufficient condition for non-uniqueness. Whether it is also a necessary condition is an open problem. For 2D symmetric conditions in a channel past a circular cylinder, Sahin-Owens [71, Fig.6] (see branches 1, 3, 5 therein), numerically found different symmetric solutions for suitable Reynolds numbers but for different proportions between the width of the channel and the diameter of the cylinder.*

In real life, perfect symmetry does not exist, there are no perfectly symmetric flows and any obstacle inevitably has small imperfections. It is therefore natural to wonder whether "almost symmetric" boundary data and obstacles give rise to "almost symmetric" solutions, in a suitable sense. Although some of the below results hold under milder assumptions, from now on we take

$$(U, V) \in H^{1/2}(\partial Q) \qquad f \in L^2(\Omega).$$

Firstly, we maintain the obstacle K fixed and we perturb the boundary velocity and the external force. For any $\varepsilon > 0$, $(U, V) \in H^{1/2}(\partial\Omega)$ and $f \in L^2(\Omega)$ we denote

$$\mathcal{B}_\varepsilon(U, V, f) = \left\{ (A, B, g) \in H^{1/2}(\partial\Omega)^2 \times L^2(\Omega) \;\middle|\; \begin{array}{c} (A, B) \text{ satisfies (1.4)}, \\ \|(A - U, B - V)\|_{H^{1/2}(\partial\Omega)} + \|g - f\|_{L^2(\Omega)} < \varepsilon \end{array} \right\}.$$

In this setting, we prove the following continuous dependence result.

Theorem 3.5. *Let Ω be as in (1.1), $f \in L^2(\Omega)$ and $(U, V) \in H^{1/2}(\partial\Omega)$ satisfying (1.4). There exists $\delta_0 = \delta_0(\eta, \Omega) > 0$ such that, if*

$$\|(U, V)\|_{H^{1/2}(\partial Q)} + \|f\|_{L^2(\Omega)} < \delta_0, \tag{3.35}$$

then (1.2)-(1.3) in Ω admits a unique weak solution $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ with data (U, V, f). Furthermore, there exists $\varepsilon_0 = \varepsilon_0(U, V, f) > 0$ such that, for all $\varepsilon < \varepsilon_0$ and all $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(U, V, f)$, problem (1.2)-(1.3) with data $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon$ admits a unique weak solution $(u_\varepsilon, p_\varepsilon) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$. Furthermore, the following limit holds:

$$\lim_{\varepsilon \to 0} \sup_{(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon} \left(\|\nabla(u - u_\varepsilon)\|_{L^2(\Omega)} + \|p - p_\varepsilon\|_{L^2(\Omega)} \right) = 0.$$

Proof. The quantitative uniqueness statement (3.35) follows directly from (3.3), since

$$\|f\|_{H^{-1}(\Omega)} \leq \lambda^{-1}\|f\|_{L^2(\Omega)},$$

with $\lambda > 0$ the Poincaré constant of Ω. Now, let $\delta_0 = \delta_0(\eta, \Omega) > 0$ be as in (3.35). Define

$$\varepsilon_0 = \varepsilon_0(U, V, f) = \delta_0 - \|(U, V)\|_{H^{1/2}(\partial\Omega)} - \|f\|_{L^2(\Omega)} > 0$$

so that, if $0 < \varepsilon < \varepsilon_0$ and $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon$, we have

$$\begin{aligned}\|(U_\varepsilon, V_\varepsilon)\|_{H^{1/2}(\partial\Omega)} + \|f_\varepsilon\|_{L^2(\Omega)} &\leq \|(U_\varepsilon - U, V_\varepsilon - V)\|_{H^{1/2}(\partial\Omega)} + \|f_\varepsilon - f\|_{L^2(\Omega)} + \|(U, V)\|_{H^{1/2}(\partial\Omega)} + \|f\|_{L^2(\Omega)} \\ &< \varepsilon + \delta_0 - \varepsilon_0 < \delta_0,\end{aligned}$$

and problem (1.2)-(1.3) with data $(U_\varepsilon, V_\varepsilon, f_\varepsilon)$ admits a unique weak solution $(u_\varepsilon, p_\varepsilon) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$, see Theorem 3.1. So, fix $\varepsilon \in (0, \varepsilon_0)$ and choose any $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon$. In view of Proposition 2.1, there exists a vector field $w_0 \in \mathcal{V}_*(\Omega)$ such that

$$w_0 = (U, V) - (U_\varepsilon, V_\varepsilon) \quad \text{on} \quad \partial Q, \qquad \|\nabla w_0\|_{L^2(\Omega)} \leq C\|(U, V) - (U_\varepsilon, V_\varepsilon)\|_{H^{1/2}(\partial Q)} \leq C\varepsilon, \tag{3.36}$$

for some constant $C > 0$ depending only on Ω. Let $\xi \doteq u - u_\varepsilon - w_0$, so that $\xi \in \mathcal{V}(\Omega)$. After subtracting the equations (1.2) satisfied by (u, p, f) and $(u_\varepsilon, p_\varepsilon, f_\varepsilon)$ in Ω we infer that:

$$-\eta\Delta\xi + [(\xi + w_0) \cdot \nabla](\xi + w_0) + [(u - u_\varepsilon) \cdot \nabla]u_\varepsilon + (u_\varepsilon \cdot \nabla)(u - u_\varepsilon) + \nabla(p - p_\varepsilon) = \eta\Delta w_0 + f - f_\varepsilon, \tag{3.37}$$

with $\eta\Delta w_0 + f - f_\varepsilon \in H^{-1}(\Omega)$. Here (3.37) is understood in the weak sense, see (3.1); we test it with ξ, we integrate by parts in Ω in order to obtain the upper bound (after applying (3.36)):

$$\eta\|\nabla\xi\|^2_{L^2(\Omega)} + \beta(\xi + w_0, \xi + w_0, \xi) + \beta(u - u_\varepsilon, u_\varepsilon, \xi) + \beta(u_\varepsilon, u - u_\varepsilon, \xi) \leq \varepsilon(C\eta + 1)\|\nabla\xi\|_{L^2(\Omega)}. \tag{3.38}$$

After applying property (2.37) repeatedly we deduce that:

$$\beta(\xi + w_0, \xi + w_0, \xi) + \beta(u - u_\varepsilon, u_\varepsilon, \xi) + \beta(u_\varepsilon, u - u_\varepsilon, \xi) = \beta(\xi + w_0, u, \xi) + \beta(u_\varepsilon, w_0, \xi). \tag{3.39}$$

In view of (2.35) and (3.36) we have

$$|\beta(\xi + w_0, u, \xi) + \beta(u_\varepsilon, w_0, \xi)| \leq \frac{1}{\mathcal{S}_0}\|\nabla u\|_{L^2(\Omega)}\|\nabla\xi\|^2_{L^2(\Omega)} + \frac{C\varepsilon}{\mathcal{S}}\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\right)\|\nabla\xi\|_{L^2(\Omega)},$$

inequality that, together with (3.39), can be inserted into (3.38) to yield

$$\left(\eta - \frac{1}{\mathcal{S}_0}\|\nabla u\|_{L^2(\Omega)}\right)\|\nabla\xi\|_{L^2(\Omega)} \leq \left[C\left(\frac{1}{\mathcal{S}}\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\right) + \eta\right) + 1\right]\varepsilon. \tag{3.40}$$

The uniqueness assumption ensures that $\|\nabla u\|_{L^2(\Omega)} < \mathcal{S}_0\eta$, and since $u - u_\varepsilon = \xi + w_0$, we have

$$\|\nabla(u - u_\varepsilon)\|_{L^2(\Omega)} \leq \frac{C\left[\left(\frac{\mathcal{S}_0}{\mathcal{S}} - 1\right)\|\nabla u\|_{L^2(\Omega)} + \frac{\mathcal{S}_0}{\mathcal{S}}\|\nabla u_\varepsilon\|_{L^2(\Omega)} + 2\mathcal{S}_0\eta\right] + \mathcal{S}_0}{\mathcal{S}_0\eta - \|\nabla u\|_{L^2(\Omega)}}\,\varepsilon. \tag{3.41}$$

On the other hand, by subtracting the equations of conservation of momentum (1.2) satisfied by (u, p, f) and $(u_\varepsilon, p_\varepsilon, f_\varepsilon)$ in Ω we infer that:

$$\nabla(p - p_\varepsilon) = \eta\Delta(u - u_\varepsilon) + [(u_\varepsilon - u) \cdot \nabla]u_\varepsilon + (u \cdot \nabla)(u_\varepsilon - u) + f - f_\varepsilon, \tag{3.42}$$

an identity that must also be intended in the weak sense of $H^{-1}(\Omega)$. In particular:

$$\begin{aligned}\|\nabla(p - p_\varepsilon)\|_{H^{-1}(\Omega)} &\leq \eta\|\nabla(u - u_\varepsilon)\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\|u - u_\varepsilon\|_{L^6(\Omega)} + \|\nabla(u - u_\varepsilon)\|_{L^2(\Omega)}\|u\|_{L^6(\Omega)} + \varepsilon \\ &\leq \left[\eta + C\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\right)\right]\|\nabla(u - u_\varepsilon)\|_{L^2(\Omega)} + \varepsilon,\end{aligned}$$

where $C > 0$ is the embedding constant for $H^1(\Omega) \subset L^6(\Omega)$. From (2.43) we can deduce the existence of a constant $M > 0$ (depending only on Ω) such that $\|p - p_\varepsilon\|_{L^2(\Omega)} \le M\|\nabla(p - p_\varepsilon)\|_{H^{-1}(\Omega)}$. This yields

$$\|\nabla(u-u_\varepsilon)\|_{L^2(\Omega)} + \|p-p_\varepsilon\|_{L^2(\Omega)} \le \left[1 + M\left(\eta + C\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega)}\right)\right)\right] \|\nabla(u-u_\varepsilon)\|_{L^2(\Omega)} + M\varepsilon,$$

which, together with (3.41) and (3.2), completes the proof. □

Theorem 3.5 is a continuous dependence result which shows, in particular, that if the first component of f is y-even, the second component of f is y-odd and the boundary data (U, V) satisfies the symmetry property (3.29), then the solution $(u_\varepsilon, p_\varepsilon)$ is "almost symmetric", since in this case (u, p) verifies (3.30). This is made precise in the following statement for which we introduce a further notation. For any function (or distribution) $\phi = \phi(x, y)$ we denote its even and odd parts by

$$\phi^E(x,y) = \frac{\phi(x,y) + \phi(x,-y)}{2}, \qquad \phi^O(x,y) = \frac{\phi(x,y) - \phi(x,-y)}{2}.$$

Then we have

Corollary 3.1. *Let Ω be as in* (1.1) *and K symmetric with respect to the x-axis. Let $f = (f_1, f_2) \in L^2(\Omega)^2$ with f_1 y-even and f_2 y-odd, let $(U, V) \in H^{1/2}(\partial Q)$ satisfy* (1.4), (3.3) *and* (3.29). *Then there exists $\varepsilon_0 = \varepsilon_0(U, V, f) > 0$ such that, for all $\varepsilon < \varepsilon_0$ and all $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(0,0,0)$, the problem*

$$-\eta\Delta v + (v\cdot\nabla)v + \nabla q = f + f_\varepsilon \quad \text{in} \quad \Omega, \qquad v = (U + U_\varepsilon, V + V_\varepsilon) \quad \text{on} \quad \partial Q, \qquad v = (0,0) \quad \text{on} \quad \partial K,$$

admits a unique weak solution $(v_1, v_2, q) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ and

$$\lim_{\varepsilon\to 0} \sup_{(U_\varepsilon, V_\varepsilon, f_\varepsilon)\in\mathcal{B}_\varepsilon} \left(\|\nabla(v_1^O, v_2^E)\|_{L^2(\Omega)} + \|q^O\|_{L^2(\Omega)}\right) = 0.$$

Theorem 3.5 and Corollary 3.1 make assumptions on the fluid flow, namely on the boundary data (U, V) and on the force f. This means that *two flows having almost the same boundary data and forcing behave quite similarly.* We are now interested in a second perturbation result, by considering *the same flow conditions but with possibly different obstacles*, that is, we fix $(U, V) \in H^{1/2}(\partial Q)$ and $f \in L^2(\Omega)$, and we allow K to vary. This is the problem that occurs if an object (the obstacle) has not been manufactured with enough precision. However, this second problem is extremely more delicate and we need first to make clear what kind of imprecisions are allowed.

Definition 3.2. *Given a $\mathcal{C}^2$-domain $K \subset Q$ such that $\partial K \cap \partial Q = \emptyset$, we say that the family of Lipschitz domains $\{K_\varepsilon\}_{\varepsilon>0}$ outer-approximates K as $\varepsilon \to 0$ if:*

– $K \subset K_{\varepsilon_2} \subset K_{\varepsilon_1} \subset Q$, for every $0 < \varepsilon_2 < \varepsilon_1$;

– $\mathrm{dist}_H(\overline{K_\varepsilon}, \overline{K}) \le \varepsilon$, for every $\varepsilon > 0$, where dist_H denotes the Hausdorff distance;

– there exists a finite number of disks $B_1, \dots, B_N \subset Q$ such that, for any $\varepsilon > 0$, $Q \setminus K_\varepsilon$ is contained in the union of N domains, each one being star-shaped with respect to one of these disks ($\star$).

The first two conditions in Definition 3.2 tell us that K_ε approximates K monotonically from outside. The third condition, that we denote by ($\star$), is a geometric assumption that yields uniform bounds for some constants depending on K_ε. We can now prove the following statement.

Theorem 3.6. *Let Ω be as in* (1.1), *K with $\mathcal{C}^2$-boundary. Let $f \in L^2(\Omega)$ and $(U, V) \in H^{1/2}(\partial Q)$ satisfy* (1.4) *and* (3.35), *and let $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ be the unique weak solution of* (1.2)-(1.3), *see Theorem 3.5. For any family of Lipschitz domains $\{K_\varepsilon\}_{\varepsilon>0}$ that outer-approximates K, there exists $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then* (1.2)-(1.3) *in $\Omega_\varepsilon = Q \setminus K_\varepsilon$ admits a unique solution $(u_\varepsilon, p_\varepsilon) \in \mathcal{V}_*(\Omega_\varepsilon) \times L^2_0(\Omega_\varepsilon)$ and*

$$\lim_{\varepsilon\to 0} \left(\|\nabla(u_\varepsilon - u)\|_{L^2(\Omega_\varepsilon)} + \|p_\varepsilon - p\|_{L^2(\Omega_\varepsilon)}\right) = 0. \tag{3.43}$$

Proof. From [62, Section 1.1.8], we know that condition $(\star)$ in Definition 3.2 implies

$$\begin{aligned} &- \text{the existence of } \Gamma > 0 \text{ (depending on } \Omega \text{ but not on } \varepsilon) \text{ such that } |\partial K_\varepsilon|_1 \le \Gamma^2,\ \forall \varepsilon > 0; \\ &- \text{a uniform } (\varepsilon\text{-independent) cone property for } \partial K_\varepsilon. \end{aligned} \tag{3.44}$$

Take an (open) smooth connected domain K_0 such that $\overline{K} \subset K_0 \subset \overline{K_0} \subset Q$; we have in mind a small neighborhood of K. Let $\Omega_0 = Q \setminus K_0$ and consider a solenoidal extension $\overline{v}$ of the data in Ω_0, that is,

$$\overline{v} \in \mathcal{V}_*(\Omega_0), \qquad \overline{v} = (U, V) \text{ on } \partial Q, \qquad \|\nabla \overline{v}\|_{L^2(\Omega_0)} \le C_0 \|(U, V)\|_{H^{1/2}(\partial Q)},$$

where $C_0 = C_0(\Omega_0) > 0$ is independent on the boundary data, see [57]. Then the function

$$v_0(x, y) = \begin{cases} \overline{v}(x, y) & \text{if } (x, y) \in \Omega_0 \\ 0 & \text{if } (x, y) \in K_0 \setminus K \end{cases}$$

is a solenoidal extension of the data (U, V) in Ω and also in Ω_ε, provided ε is small enough in such a way that $K_\varepsilon \subset K_0$. Hence, the constant C_0 can be used to compute the uniqueness threshold (3.18) (and also (3.35)) in Ω and Ω_ε, for any small enough $\varepsilon > 0$.

For all $\varepsilon > 0$ the existence of a weak solution $(u_\varepsilon, p_\varepsilon)$ of (1.2)-(1.3) in Ω_ε follows from Theorem 3.1 applied to Ω_ε. Theorem 3.5 ensures uniqueness whenever

$$\|(U, V)\|_{H^{1/2}(\partial Q)} + \|f\|_{L^2(\Omega_\varepsilon)} < \delta_\varepsilon, \tag{3.45}$$

where $\delta_\varepsilon = \delta_\varepsilon(\eta, \Omega_\varepsilon)$ is as in (3.35), but relative to Ω_ε. A careful look at the proof of Theorem 3.5 and formula (3.18) show that δ_ε depends on $\mathcal{S}^\varepsilon$ and $\mathcal{S}_0^\varepsilon$, namely the Sobolev constants defined in (2.22) but relative to Ω_ε, and on the Poincaré constant of Ω_ε (by the above construction, C_0 is independent of ε). Since K_ε outer-approximates K, one has $\mathcal{S}^\varepsilon \to \mathcal{S}$ and $\mathcal{S}_0^\varepsilon \to \mathcal{S}_0$ as $\varepsilon \to 0$, and also the Poincaré constants converge, due to the continuity of these functionals with respect to the Hausdorff convergence of domains. Therefore, by (3.35) and by continuity, we know that (3.45) holds provided ε is small enough, say $\varepsilon < \varepsilon_0$. Not only this proves the uniqueness of the solution $(u_\varepsilon, p_\varepsilon)$ but, according to Theorem 3.1, it also proves the uniform bound

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} < \mathcal{S}_0^\varepsilon \eta \le B \qquad \forall \varepsilon > 0, \tag{3.46}$$

for some $B > 0$ (independent of ε) since $\mathcal{S}_0^\varepsilon \to \mathcal{S}_0$ as $\varepsilon \to 0$.

To complete the proof we have to show that (3.43) holds. To this end, we first claim that there exist positive constants $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ such that $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ and

$$\|u\|_{H^{1/2}(\partial K_\varepsilon)} = \|u\|_{L^2(\partial K_\varepsilon)} + \left(\int\limits_{\partial K_\varepsilon} \int\limits_{\partial K_\varepsilon} \frac{|u(z_1) - u(z_2)|^2}{|z_1 - z_2|^2} \, ds_{z_1} \, ds_{z_2} \right)^{1/2} \le \sigma_\varepsilon \qquad \forall \varepsilon > 0. \tag{3.47}$$

Indeed, as in the proof of Theorem 3.3, by localizing in a neighborhood of K one may deduce that $(u, p) \in H^2(\mathcal{O} \setminus \overline{K}) \times H^1(\mathcal{O} \setminus \overline{K})$, for any $\mathcal{C}^2$-domain $\mathcal{O} \subset Q$ such that $\mathcal{O} \cap \partial\Omega = \partial K$. In particular, $u \in \mathcal{C}^{0,\nu}\left(\overline{\mathcal{O}}\right)$ for any $0 < \nu < 1$. Then, since u vanishes on ∂K, the uniform continuity of u in $\overline{\mathcal{O}}$ ensures the existence of positive constants $\{\theta_\varepsilon\}_{\varepsilon > 0}$ such that $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$ and $\|u\|_{L^\infty(\partial K_\varepsilon)} \le \theta_\varepsilon$ for every $\varepsilon > 0$. By combining this with $(3.44)_1$, we infer

$$\|u\|_{L^2(\partial K_\varepsilon)} \le \|u\|_{L^\infty(\partial K_\varepsilon)} \sqrt{|\partial K_\varepsilon|_1} \le \Gamma \, \theta_\varepsilon \qquad \forall \varepsilon > 0\,. \tag{3.48}$$

Moreover, if $M > 0$ denotes the Hölder constant of u in $\overline{\mathcal{O}}$ for $\nu = 4/5$, we have

$$|u(z_1) - u(z_2)|^2 = |u(z_1) - u(z_2)|^{1/8} |u(z_1) - u(z_2)|^{15/8} \le (2\theta_\varepsilon)^{1/8} M^{15/8} |z_1 - z_2|^{3/2}$$

and, in turn,

$$\int\limits_{\partial K_\varepsilon}\int\limits_{\partial K_\varepsilon} \frac{|u(z_1)-u(z_2)|^2}{|z_1-z_2|^2}\, ds_{z_1}\, ds_{z_2} \le (2M^{15}\theta_\varepsilon)^{1/8} \int\limits_{\partial K_\varepsilon}\int\limits_{\partial K_\varepsilon} \frac{ds_{z_1}\, ds_{z_2}}{|z_1-z_2|^{1/2}}. \tag{3.49}$$

From $(3.44)_2$ we know that the Lipschitz constant of ∂K_ε is independent of ε. Thus, the integral on the right hand side of (3.49) is uniformly bounded, that is, there exists $\Lambda > 0$ (independent of ε) such that

$$\left(\int\limits_{\partial K_\varepsilon}\int\limits_{\partial K_\varepsilon} \frac{|u(z_1)-u(z_2)|^2}{|z_1-z_2|^2}\, ds_{z_1}\, ds_{z_2} \right)^{1/2} \le \Lambda\, \theta_\varepsilon^{1/16} \qquad \forall \varepsilon > 0\,.$$

By combining this bound with (3.48) we obtain (3.47) with $\sigma_\varepsilon = \Gamma\theta_\varepsilon + \Lambda\theta_\varepsilon^{1/16}$.

We then claim that there exists a solenoidal vector field $w_\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$w_\varepsilon = (0,0) \ \text{ on } \ \partial Q, \qquad w_\varepsilon = u|_{\partial K_\varepsilon} \ \text{ on } \ \partial K_\varepsilon, \qquad \|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\sigma_\varepsilon, \tag{3.50}$$

for some constant $C > 0$ that depends on Ω but is independent of ε. The construction of such w_ε is performed in two steps:

(1) In view of [37, Teorema 1.I], there exists $W_{1,\varepsilon} \in H^1(\Omega_\varepsilon)$ (not necessarily solenoidal) such that

$$W_{1,\varepsilon} = (0,0) \ \text{ on } \ \partial Q, \qquad W_{1,\varepsilon} = u|_{\partial K_\varepsilon} \ \text{ on } \ \partial K_\varepsilon, \qquad \|\nabla W_{1,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \le C_{1,\varepsilon}\|u\|_{H^{1/2}(\partial K_\varepsilon)}, \tag{3.51}$$

for some constant $C_{1,\varepsilon} > 0$ that depends on Ω_ε. From [37] (see also [39, Theorem II.4.3]) we know that $C_{1,\varepsilon}$ depends on Ω_ε through the number of domains R_m ($m = 1, ..., \overline{m}$) needed to cover $\partial\Omega_\varepsilon$ (namely, $\partial\Omega_\varepsilon \subset \cup_m R_m$) in such a way that each R_m may be transformed into a rectangle through a bijective uniformly bi-Lipschitz map ϕ_m, see [37, p.285]. In view of condition ($\star$) in Definition 3.2, the number $\overline{m} = \overline{m}(\varepsilon)$ of such rectangles R_m remains bounded as $\varepsilon \to 0$. Then, invoking again [37], $C_{1,\varepsilon} > 0$ depends on Ω_ε through the Lipschitz constants of the maps ϕ_m; by (3.44), also this constant remains bounded as $\varepsilon \to 0$. Hence, there exists $C > 0$ (depending only on the family of disks $\{B_1, \dots, B_N\}$ in Definition 3.2) such that $C_{1,\varepsilon} \le C$ for every $\varepsilon > 0$.

(2) We notice that the incompressibility condition and (1.4) imply that

$$\int_{\partial\Omega_\varepsilon} u \cdot \hat{n}\, ds = \int_{\partial K_\varepsilon} u \cdot \hat{n}\, ds = 0. \tag{3.52}$$

From (3.51), (3.52) and the Divergence Theorem we then have $\nabla \cdot W_{1,\varepsilon} \in L^2_0(\Omega_\varepsilon)$. Hence, as in [15], we deduce the existence of $W_{2,\varepsilon} \in H^1_0(\Omega_\varepsilon)$ such that

$$\nabla \cdot W_{2,\varepsilon} = -\nabla \cdot W_1 \ \text{ in } \ \Omega_\varepsilon, \qquad \|\nabla W_{2,\varepsilon}\|_{L^2(\Omega_\varepsilon)} \le C_{2,\varepsilon}\|\nabla \cdot W_{1,\varepsilon}\|_{L^2(\Omega_\varepsilon)}, \tag{3.53}$$

for some constant $C_{2,\varepsilon} > 0$ that depends on Ω_ε. Condition ($\star$) in Definition 3.2 together with [39, Theorem III.3.1] imply that $C_{2,\varepsilon} \le C$, for all $\varepsilon > 0$, where C depends only on the family $B_1, \dots, B_N$.

Finally, we take $w_\varepsilon \doteq W_{1,\varepsilon} + W_{2,\varepsilon}$, which satisfies (3.50) after combining (3.47), (3.51) and (3.53).

We now follow the procedure of the proof of Theorem 3.5, taking into account that the functions involved belong to Sobolev spaces over different domains. For every $\varepsilon > 0$ let $\xi = u - u_\varepsilon - w_\varepsilon \in \mathcal{V}(\Omega_\varepsilon)$. After subtracting the equations (1.2) satisfied by (u, p, f) and $(u_\varepsilon, p_\varepsilon, f)$ in Ω_ε we infer

$$-\eta\Delta\xi + [(\xi + w_\varepsilon)\cdot\nabla](\xi + w_\varepsilon) + [(u - u_\varepsilon)\cdot\nabla]u_\varepsilon + (u_\varepsilon\cdot\nabla)(u - u_\varepsilon) + \nabla(p - p_\varepsilon) = \eta\Delta w_\varepsilon,$$

with $\eta\Delta w_\varepsilon \in H^{-1}(\Omega_\varepsilon)$, so that the equation is understood in the weak sense, see (3.1). We test it with ξ, we integrate by parts in Ω_ε order to obtain the upper bound (after applying (3.47) and (3.50))

$$\eta\|\nabla\xi\|^2_{L^2(\Omega_\varepsilon)} + \beta_\varepsilon(\xi + w_\varepsilon, \xi + w_\varepsilon, \xi) + \beta_\varepsilon(u - u_\varepsilon, u_\varepsilon, \xi) + \beta_\varepsilon(u_\varepsilon, u - u_\varepsilon, \xi) \le C\eta\sigma_\varepsilon\|\nabla\xi\|_{L^2(\Omega_\varepsilon)}, \tag{3.54}$$

where $\beta_\varepsilon : H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}$ denotes the trilinear form (2.34) with the integral computed over Ω_ε. Since $\xi \in \mathcal{V}(\Omega_\varepsilon)$, we have

$$\beta_\varepsilon(\xi+w_\varepsilon,\xi+w_\varepsilon,\xi)+\beta_\varepsilon(u-u_\varepsilon,u_\varepsilon,\xi)+\beta_\varepsilon(u_\varepsilon,u-u_\varepsilon,\xi) = \beta_\varepsilon(\xi,u,\xi)+\beta_\varepsilon(w_\varepsilon,u,\xi)+\beta_\varepsilon(u_\varepsilon,w_\varepsilon,\xi). \quad (3.55)$$

Since $\Omega_\varepsilon \subset \Omega$, every function in $H^1_*(\Omega_\varepsilon)$ may be extended by zero in $\overline{K_\varepsilon} \setminus K$, becoming an element of $H^1_*(\Omega)$. Therefore, $\mathcal{S}$ and $\mathcal{S}_0$, defined in (2.22) for Ω, may also be used as embedding constants in Ω_ε. By combining this fact with (3.50), we obtain the estimates

$$\begin{aligned}
&\left|\beta_\varepsilon(\xi,u,\xi)\right| \leq \frac{1}{\mathcal{S}_0}\|\nabla u\|_{L^2(\Omega)}\|\nabla \xi\|^2_{L^2(\Omega_\varepsilon)}, \qquad \left|\beta_\varepsilon(u_\varepsilon,w_\varepsilon,\xi)\right| \leq \frac{C\sigma_\varepsilon}{\mathcal{S}}\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla \xi\|_{L^2(\Omega_\varepsilon)},\\
&\left|\beta_\varepsilon(w_\varepsilon,u,\xi)\right| \leq \frac{C\sigma_\varepsilon}{\mathcal{S}}\|\nabla u\|_{L^2(\Omega)}\|\nabla \xi\|_{L^2(\Omega_\varepsilon)}\,. \qquad (3.56)
\end{aligned}$$

We plug (3.55)-(3.56) into (3.54) to deduce that

$$\left(\eta - \frac{\|\nabla u\|_{L^2(\Omega)}}{\mathcal{S}_0}\right)\|\nabla \xi\|_{L^2(\Omega_\varepsilon)} \leq C\sigma_\varepsilon\left(\eta + \frac{\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}}{\mathcal{S}} + \frac{\|\nabla u\|_{L^2(\Omega)}}{\mathcal{S}}\right).$$

We have seen above that (3.35) implies (3.3) and, in turn, Theorem 3.1 ensures $\|\nabla u\|_{L^2(\Omega)} < \mathcal{S}_0\eta$. Hence, the latter inequality yields an upper bound for $\|\nabla \xi\|_{L^2(\Omega_\varepsilon)}$ which, combined with (3.46) and (3.50), yields

$$\|\nabla(u-u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \leq \|\nabla \xi\|_{L^2(\Omega_\varepsilon)} + \|\nabla w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \frac{2\mathcal{S}\eta + (1-\mathcal{S}/\mathcal{S}_0)\|\nabla u\|_{L^2(\Omega)} + B}{\mathcal{S}_0\eta - \|\nabla u\|_{L^2(\Omega)}}\,\frac{C\mathcal{S}_0}{\mathcal{S}}\sigma_\varepsilon \to 0,$$

as $\varepsilon \to 0$. In order to control the pressure terms, we note that the same extension argument of $H^1_*(\Omega_\varepsilon)$ into $H^1_*(\Omega)$ proves that the embedding constant of $H^1_*(\Omega) \subset L^6(\Omega)$ bounds the corresponding embedding constant in Ω_ε. Then, as in the proof of Theorem 3.5, but applying condition $(\star)$, (2.43) and (3.46), we can deduce the existence of a constant $A > 0$, depending on η and Ω, such that

$$\|p - p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq A\left(1 + \|\nabla u\|_{L^2(\Omega)} + \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}\right)\|\nabla(u-u_\varepsilon)\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0.$$

This shows (3.43) and completes the proof. □

3.3 Definition and computation of drag and lift

In this section we analyze the forces of a fluid flow in Ω over the obstacle K. The stress tensor of a viscous incompressible fluid governed by (1.2) is (see [59, Chapter 2])

$$\mathbb{T}(u,p) \doteq -p\mathbb{I}_2 + \eta[\nabla u + (\nabla u)^\intercal] \quad \text{in } \Omega, \qquad (3.57)$$

where $\mathbb{I}_2$ is the 2×2-identity matrix. Accordingly, the total force exerted by the fluid over the obstacle K is formally given by

$$F_K(u,p) = -\int_{\partial K} \mathbb{T}(u,p)\cdot \hat{n}\, ds, \qquad (3.58)$$

where the minus sign is due to the fact that the outward unit normal $\hat{n}$ to Ω is directed towards the interior of K. To be precise, (3.58) makes sense only if (u,p) are regular while if $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ is a weak solution of (1.2) with $f \in L^2(\Omega)$, a generalized formula is needed. Indeed, in such case, one has $u \in L^p(\Omega)$, for every $p < \infty$ so that, in particular,

$$\mathbb{T}(u,p) \in L^2(\Omega) \subset L^{3/2}(\Omega) \quad \text{and} \quad \nabla\cdot\mathbb{T}(u,p) = (u\cdot\nabla)u - f \in L^{3/2}(\Omega). \qquad (3.59)$$

Therefore, $\mathbb{T}(u,p) \in E_{3/2}(\Omega)$ and the normal component of the trace of $\mathbb{T}(u,p)$ belongs to $W^{-\frac{2}{3},\frac{3}{2}}(\partial\Omega)$, the dual space of $W^{\frac{2}{3},3}(\partial\Omega)$, see (2.38). Then, we can rigorously define the force as follows.

Definition 3.3. *Let $f \in L^2(\Omega)$ and let $(u,p) \in \mathcal{V}_*(\Omega) \times L^2(\Omega)$ be a weak solution of* (1.2)*. Then, the total force exerted by the fluid over the obstacle K is given by*

$$F_K(u,p) = -\langle \mathbb{T}(u,p) \cdot \hat{n}, 1 \rangle_{\partial K}, \tag{3.60}$$

where $\langle \cdot, \cdot \rangle_{\partial K}$ denotes the duality pairing between $W^{-\frac{2}{3},\frac{3}{2}}(\partial K)$ and $W^{\frac{2}{3},3}(\partial K)$.

The classical literature [2, Introduction] defines the *drag force* as the component of F_K parallel to the incoming stream and the *lift force* as the component of F_K perpendicular to the stream. This characterization is rigorous only if the direction of the inflow velocity is constant.

Definition 3.4. *For $(U,V) \in H^{1/2}(\partial Q) \backslash \{(0,0)\}$ such that $(U,V)/|(U,V)|$ is constant, the drag $\mathcal{D}_K(u,p)$ and the lift $\mathcal{L}_K(u,p)$, exerted by the fluid over the obstacle K, are given by*

$$\mathcal{D}_K(u,p) = F_K(u,p) \cdot \frac{(U,V)}{|(U,V)|} \qquad \text{and} \qquad \mathcal{L}_K(u,p) = F_K(u,p) \cdot \frac{(-V,U)}{|(U,V)|}.$$

In the case where $V = 0$ and $U \in H^{1/2}(\partial Q)$ is a strictly positive function on ∂Q, this reduces to

$$\mathcal{D}_K(u,p) = F_K(u,p) \cdot (1,0) \qquad \text{and} \qquad \mathcal{L}_K(u,p) = F_K(u,p) \cdot (0,1).$$

Clearly, the signs of $\mathcal{D}_K$ and $\mathcal{L}_K$ are just a matter of orientation and one could just take the absolute values, especially if one is merely interested in evaluating the strength of these forces.

The main purpose of this section is to discuss a well-known experimental fact: a bluff body immersed in a viscous fluid experiences no lift when its cross-section is symmetric with respect to the angle of attack of the fluid, as well illustrated in [66, Figure 2.6]. Moreover, any small symmetry-breaking angle of attack produces a lift on the obstacle. This was already observed by Kutta [54] in 1910 (see also [3, Chapter 12]): "With regard to dynamic lift effects, the most important types of a body immersed in a flowing fluid are long flat plates placed at an angle to the flow and slightly curved cylindrical shells, which experience lift forces even if the chord of their cross-section lies parallel to the flow".

For simplicity, we merely consider the case of constant positive horizontal data ($U \in \mathbb{R}_+$ and $V = 0$) and, if $\mathcal{B}_\varepsilon$ is as in Corollary 3.1, we prove

Theorem 3.7. *Let Ω be as in* (1.1)*, let K be symmetric with respect to the x-axis. Assume that $U > 0$ is constant, $V = 0$, and that $f = (f_1, f_2) \in L^2(\Omega)$ is such that f_1 is y-even and f_2 is y-odd. If*

$$2\sqrt{2L}\,|U| + \|f\|_{L^2(\Omega)} < \delta_0\,, \tag{3.61}$$

with δ_0 as in (3.35)*, then the fluid governed by* (1.2)*-*(1.3) *exerts no lift over K. Moreover, there exists $\varepsilon_0 = \varepsilon_0(U,f) > 0$ such that, for all $\varepsilon < \varepsilon_0$ and all $(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(0,0,0)$, the problem*

$$-\eta\Delta v + (v \cdot \nabla)v + \nabla q = f + f_\varepsilon \text{ in } \Omega, \qquad v = (U + U_\varepsilon, V_\varepsilon) \text{ on } \partial Q, \quad v = (0,0) \text{ on } \partial K, \tag{3.62}$$

admits a unique weak solution $(v_\varepsilon, q_\varepsilon) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ and

$$\lim_{\varepsilon \to 0} \sup_{(U_\varepsilon, V_\varepsilon, f_\varepsilon) \in \mathcal{B}_\varepsilon} \big|\mathcal{L}_K(v_\varepsilon, q_\varepsilon)\big| = 0.$$

Proof. The compatibility condition (1.4) is evidently satisfied, thus ensuring the existence of (at least) one solution $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (1.2)-(1.3), see Theorem 3.1. Furthermore, from (2.39) we see that (3.35) becomes (3.61) and then Theorem 3.5 ensures that the solution (u,p) is unique. Theorem 3.4 then states that (u,p) satisfies the symmetry properties (3.30).

From (3.60) and Definition 3.4 we have that

$$\mathcal{L}_K(u,p) = -\langle \mathbb{T}_2(u,p) \cdot \hat{n}, 1 \rangle_{\partial K} = \langle \mathbb{T}_2(u,p) \cdot \hat{n}, 1 \rangle_{\partial Q} - \langle \mathbb{T}_2(u,p) \cdot \hat{n}, 1 \rangle_{\partial \Omega}, \tag{3.63}$$

where

$$\mathbb{T}_2(u,p) = \left[\eta\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right), 2\eta\frac{\partial u_2}{\partial y} - p\right]^\intercal.$$

By Theorem 3.3 we know that the term over ∂Q in (3.63) can be treated as an integral. On the other hand, (2.38) allows us to manage the term over $\partial\Omega$ and we obtain

$$\begin{aligned}
\mathcal{L}_K(u,p) &= \int_{\partial Q} \mathbb{T}_2(u,p)\cdot\hat{n} - \int_\Omega \nabla\cdot\mathbb{T}_2(u,p) \\
&= \eta\int_{-L}^{L}\left[\frac{\partial u_1}{\partial y}(L,y) - \frac{\partial u_1}{\partial y}(-L,y) + \frac{\partial u_2}{\partial x}(L,y) - \frac{\partial u_2}{\partial x}(-L,y)\right]dy + 2\eta\int_{-L}^{L}\left[\frac{\partial u_2}{\partial y}(x,L) - \frac{\partial u_2}{\partial y}(x,-L)\right]dx \\
&\quad + \int_{-L}^{L}\big[p(x,-L) - p(x,L)\big]dx + \int_\Omega \big[f_2(x,y) - u(x,y)\cdot\nabla u_2(x,y)\big]dx\,dy = 0.
\end{aligned}$$

Let us explain in detail why all the above terms vanish. In the first integral, the terms with $\frac{\partial u_1}{\partial y}$ vanish because u_1 is constant on ∂Q. For the term with $\frac{\partial u_2}{\partial x}$ in the first integral we remark that with the change of variables $y \mapsto -t$ it becomes

$$\int_{-L}^{L}\left[\frac{\partial u_2}{\partial x}(L,-t) - \frac{\partial u_2}{\partial x}(-L,-t)\right]dt$$

while, by (3.30), we know that it is also equal to the same expression with opposite sign. The second integral vanishes because (3.30) implies that $\frac{\partial u_2}{\partial y}$ is y-even and the summands cancel. The integral $\int_\Omega f_2$ vanishes because f_2 is y-odd and Ω is y-symmetric. Finally, $u\cdot\nabla u_2 = u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y}$ and, again by (3.30), each summand is the product of a y-even and a y-odd function so that $u\cdot\nabla u_2$ is y-odd.

The number $\varepsilon_0 = \varepsilon_0(U,f) > 0$ can be chosen as in the proof of Theorem 3.5; then Theorem 3.1 guarantees the existence and uniqueness of a solution $(v_\varepsilon, q_\varepsilon) \in \mathcal{V}_*(\Omega)\times L^2_0(\Omega)$ of (3.62). By (2.38) and (3.60) (and by linearity), we infer

$$\mathbb{T}(v_\varepsilon,q_\varepsilon) = \mathbb{T}(v_\varepsilon - u, q_\varepsilon - p) + \mathbb{T}(u,p), \qquad F_K(v_\varepsilon,q_\varepsilon) = F_K(v_\varepsilon - u, q_\varepsilon - p) + F_K(u,p),$$

so that, by (3.63),

$$\mathcal{L}_K(v_\varepsilon,q_\varepsilon) = \mathcal{L}_K(v_\varepsilon - u, q_\varepsilon - p) = \langle\mathbb{T}_2(v_\varepsilon - u, q_\varepsilon - p)\cdot\hat{n}, 1\rangle_{\partial Q} - \langle\mathbb{T}_2(v_\varepsilon - u, q_\varepsilon - p)\cdot\hat{n}, 1\rangle_{\partial\Omega}.$$

By combining this with Theorem 3.5 (and by continuity of traces, see [37] or [39, Theorem II.4.3]) we obtain the statement. □

Next, in the spirit of Theorem 3.6, we estimate the difference between the forces exerted by a given flow over two nearby obstacles.

Theorem 3.8. *Let Ω be as in* (1.1)*, K with $\mathcal{C}^2$-boundary and symmetric with respect to the x-axis. Assume that $U > 0$ is constant, $V = 0$, and that $f \in L^2(\Omega)$ with f_1 y-even and f_2 y-odd; assume also that* (3.61) *holds. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of Lipschitz domains that outer-approximates K and let ε_0 be as in Theorem 3.6. For all $\varepsilon \in (0,\varepsilon_0)$ denote by $\Omega_\varepsilon = Q\setminus K_\varepsilon$ and by $(u_\varepsilon,p_\varepsilon) \in \mathcal{V}_*(\Omega_\varepsilon)\times L^2_0(\Omega_\varepsilon)$ the unique weak solution of* (1.2)-(1.3) *in Ω_ε. Then*

$$\lim_{\varepsilon\to 0}\mathcal{L}_{K_\varepsilon}(u_\varepsilon,p_\varepsilon) = 0.$$

Proof. Existence and uniqueness of (u,p) and $(u_\varepsilon,p_\varepsilon)$ as in the statement follow as in the proof of Theorem 3.6. Fix $\varepsilon\in(0,\varepsilon_0)$. From Theorem 3.7 we know that $\mathcal{L}_K(u,p)=0$ and, by arguing as in that proof, we obtain

$$
\begin{aligned}
\mathcal{L}_{K_\varepsilon}(u_\varepsilon,p_\varepsilon) &= \mathcal{L}_{K_\varepsilon}(u_\varepsilon,p_\varepsilon)-\mathcal{L}_K(u,p)\\
&=\eta\int_{-L}^{L}\left[\frac{\partial(u_\varepsilon)_1}{\partial y}(L,y)-\frac{\partial u_1}{\partial y}(L,y)-\frac{\partial(u_\varepsilon)_1}{\partial y}(-L,y)+\frac{\partial u_1}{\partial y}(-L,y)\right]dy\\
&\quad+\eta\int_{-L}^{L}\left[\frac{\partial(u_\varepsilon)_2}{\partial x}(L,y)-\frac{\partial u_2}{\partial x}(L,y)-\frac{\partial(u_\varepsilon)_2}{\partial x}(-L,y)+\frac{\partial u_2}{\partial x}(-L,y)\right]dy\\
&\quad+2\eta\int_{-L}^{L}\left[\frac{\partial(u_\varepsilon)_2}{\partial y}(x,L)-\frac{\partial u_2}{\partial y}(x,L)-\frac{\partial(u_\varepsilon)_2}{\partial y}(x,-L)+\frac{\partial u_2}{\partial y}(x,-L)\right]dx\\
&\quad+\int_{-L}^{L}[p_\varepsilon(x,-L)-p(x,-L)-p_\varepsilon(x,L)+p(x,L)]\,dx\\
&\quad+\int_{\Omega_\varepsilon}[u\cdot\nabla u_2-u_\varepsilon\cdot\nabla(u_\varepsilon)_2]+\int_{\Omega\setminus\Omega_\varepsilon}u\cdot\nabla u_2-\int_{\Omega\setminus\Omega_\varepsilon}f_2\,,
\end{aligned}
\tag{3.64}
$$

and we claim that all the terms after the equality sign in (3.64) vanish as $\varepsilon\to0$.

For the boundary integrals over ∂Q in (3.64), we fix an open set $\Omega_0\subset\Omega_{\varepsilon_0}\subset\Omega$ having an internal boundary of class $\mathcal{C}^2$ and such that $\partial\Omega_{\varepsilon_0}\cap\partial\Omega_0=\partial Q$; then Theorem 3.3 yields

$$(u,p),(u_\varepsilon,p_\varepsilon)\in H^2(\Omega_0)\times H^1(\Omega_0)\qquad\forall\varepsilon>0.$$

Indeed, this choice of Ω_0 also ensures that $\Omega_0\subset\Omega_\varepsilon$ and $\partial\Omega_\varepsilon\cap\partial\Omega_0=\partial Q$ for all $\varepsilon>0$ since K_ε outer-approximates K. In fact, (3.23) says more: $\|u_\varepsilon\|_{H^2(\Omega_0)}$ and $\|p_\varepsilon\|_{H^1(\Omega_0)}$ are bounded independently of ε since $\|\nabla u_\varepsilon\|_{L^2(\Omega_0)}$ and $\|p_\varepsilon\|_{L^2(\Omega_0)}$ are bounded by (3.43). Therefore, also $\|u_\varepsilon-u\|_{H^2(\Omega_0)}$ and $\|p_\varepsilon-p\|_{H^1(\Omega_0)}$ are bounded, a fact that, combined with an interpolation and with (3.43), shows that

$$u_\varepsilon\to u\text{ in }H^s(\Omega_0)\quad\forall s<2\quad\text{ and }\quad p_\varepsilon\to p\text{ in }H^r(\Omega_0)\quad\forall r<1\,,\quad\text{ as }\varepsilon\to0.$$

Then a result by Gagliardo [37] (see also [39, Theorem II.4.3]) states that

$$u_\varepsilon\to u\text{ in }H^s(\partial\Omega_0)\quad\forall s<\frac{3}{2}\quad\text{ and }\quad p_\varepsilon\to p\text{ in }H^r(\partial\Omega_0)\quad\forall r<\frac{1}{2}\,,\quad\text{ as }\varepsilon\to0.$$

In turn, this shows that all the boundary integrals in (3.64) tend to vanish. Concerning the last line in (3.64), we notice that the first integral tends to vanish thanks to Theorem 3.6 and (3.43) while the second and third integrals tend to vanish because of the Lebesgue Theorem and because $|\Omega\setminus\Omega_\varepsilon|\to0$ as $\varepsilon\to0$ (since K_ε outer-approximates K).

Summarizing, all the integrals in (3.64) tend to zero as $\varepsilon\to0$ and the result is proved. □

Remark 3.5. *A careful look at the proof of Theorem 3.7 shows that the lift is due to the asymmetric part of the solution, namely $\mathcal{L}_K(u_1,u_2,p)=\mathcal{L}_K(u_1^E,u_2^O,p^E)$. Moreover, assuming that K has a $\mathcal{C}^2$-boundary and under suitable assumptions on the boundary datum $(U,V)\in H^{3/2}(\partial Q)$ (needed to prove the $H^2\times H^1$-regularity of the solutions, see [48]), arguments similar to the ones employed in the proofs of Theorems 3.7 and 3.8 can be used to obtain the (respectively) stronger statements*

$$\lim_{\varepsilon\to0}\ \sup_{(U_\varepsilon,V_\varepsilon,f_\varepsilon)\in\mathcal{B}_\varepsilon}\big|F_K(v_\varepsilon,q_\varepsilon)\big|=0\,,\qquad\lim_{\varepsilon\to0}F_{K_\varepsilon}(u_\varepsilon,p_\varepsilon)=0.$$

This means that also the drag varies with continuity.

3.4 A universal threshold for the appearance of lift

In this section we restrict our attention to a simple case: we consider problem (1.2)-(1.3) assuming that K is symmetric with respect to the x-axis, that $f = 0$, $U \in \mathbb{R}_+$ and $V = 0$, thereby obtaining

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u + \nabla p = 0, \qquad \nabla\cdot u = 0 \qquad \text{in } \Omega, \\ u = (U,0) \ \text{ on } \ \partial Q, \qquad u = (0,0) \ \text{ on } \ \partial K. \end{cases} \tag{3.65}$$

Note that (1.4) is satisfied and that (3.65) models a horizontal flow as in Figure 1.1.

Our purpose is to study the transition in (3.65) from uniqueness to non-uniqueness regimes (or, similarly, from symmetric to asymmetric solutions). Then, in the next section, we numerically analyze how the obtained threshold depends on the shape of the (symmetric) obstacle K.

The advantage of (3.65) is that we focus our attention on a unique parameter. Indeed, u solves (3.65) for some $\eta > 0$ and $U = 1$ if and only if $v = ku$ (for some $k > 0$) solves (3.65) for a viscosity ηk and with $U = k$. Therefore, the transition of (3.65) from the uniqueness to the non-uniqueness regimes can be studied for fixed η and variable U. In order to make sure that we are in the uniqueness regime for (3.65) (see Theorem 3.1), we use the quantitative functional inequalities obtained in Section 2.

So, let us revisit Theorem 3.2 in this simplified context.

Theorem 3.9. *Let Ω be as in* (1.1) *and assume* (2.33) *with $L > a + d$. Define $\gamma_0, \gamma_1, \gamma_2 > 0$ as in Theorem 3.2. For any $U > 0$ there exists a weak solution $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of* (3.65)*. If, moreover,*

$$U < \frac{\eta}{\sqrt{\gamma_2}} \frac{\gamma_0\sqrt{\gamma_1}}{\sqrt{\gamma_0} + 2\sqrt{\gamma_1}}\,, \tag{3.66}$$

then the weak solution of (3.65) *is unique. Furthermore, if K is symmetric with respect to the x-axis and* (3.66) *holds, then the unique solution of* (3.65) *exerts no lift on K.*

Proof. Existence of a weak solution $(u,p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ of (3.65) follows from Theorem 3.1, noticing that the compatibility condition (1.4) is automatically fulfilled. On the other hand, the threshold (3.66) for the critical inflow velocity ensuring unique solvability of (3.65) is directly obtained from Theorem 3.2 by putting $V = 0$.

Finally, in the case when K is symmetric with respect to the x-axis, by combining Theorems 3.4 and 3.7, we infer that the unique solution of (3.65) exerts no lift on K. □

Combined with Theorem 3.1, Theorem 3.9 states that, for a given measure of the symmetric obstacle, but regardless of its shape, there is no lift on the obstacle as long as (at least) the horizontal boundary velocity U satisfies (3.66). Hence, we have obtained

an absolute bound on the fluid velocity under which any symmetric obstacle is subject to no lift.

This bound merely depends on the viscosity of the fluid and is independent of the nature of the obstacle (a flag, any elastic body, any structure in civil engineering). If we view the fluid as the air and U as the velocity of the wind, the drag force $\mathcal{D}$ is the force directly exerted from the wind on the obstacle and, therefore, it comes from where the wind is blowing; hence it is mostly concentrated windward (the part upwind). On the contrary, the lift force $\mathcal{L}$ is an indirect force generated by an instability of the obstacle for large drag forces; this is the reason why it is oriented orthogonally to the flow and it acts downwind, on the "hidden part" of the obstacle. This situation is depicted in Figure 3.3 for a "stadium-shaped" obstacle, namely a rectangle ended by two half circles, to be compared with Figure 1.1.

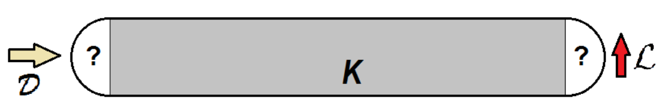

Figure 3.3: Drag $\mathcal{D}$ and lift $\mathcal{L}$ forces acting on a stadium-shaped obstacle K.

3.5 Multiplicity of solutions and numerical testing of shape performance

In the previous sections we gave sufficient conditions ensuring unique solvability of (1.2)-(1.3), but, as far as we are aware, there exist no sufficient conditions for the existence of multiple solutions. The first purpose of this section is precisely to give such condition in a symmetric framework, as a consequence of Theorem 3.4.

Corollary 3.2. *Let Ω be as in* (1.1)*, K being symmetric with respect to the x-axis. Suppose that $f=(f_1,f_2)\in H^{-1}(\Omega)$ and that $(U,V)\in H^{1/2}(\partial Q)$ satisfy* (1.4)*. Assume moreover that f_1 is y-even, f_2 is y-odd, and (U,V) verifies* (3.29)*. If* (1.2)-(1.3) *admits one asymmetric solution $(u,p)\in\mathcal{V}_*(\Omega)^2\times L^2_0(\Omega)$ (i.e., violating* (3.30)*), then there exist at least two more solutions of* (1.2)-(1.3)*: its reflection* (3.31) *and a symmetric solution satisfying* (3.30)*.*

Corollary 3.2 turns out to be extremely useful for numerical experiments, where one can visualize the streamlines of the solutions and determine possible asymmetries. In this section we use this principle to give hints on the shapes having better aerodynamic performances, namely, having smaller drag and lift. We choose adequately the size of the box $(-L,L)^2$ since we know from [16] that the drag decreases when L increases, and increases as the obstacle increases. In fact, this is the same monotonicity as for the Sobolev constant, see Section 2.2. Hence, imagine that one wishes to modify the shape of the obstacle in Figure 3.3 in such a way to lower both the drag and the lift forces: $\mathcal{D}$ has to be minimized in order to decrease as much as possible the input of energy from the wind into the obstacle whereas $\mathcal{L}$ has to be minimized in order to decrease as much as possible the vertical instability of the obstacle. As in any shape optimization problem, some common geometrical constraints need to be imposed.

⋄ The total area of the obstacle is unchanged. This means that if the rectangle has thickness $2d$ then each of the two "caps" (the white semicircles in Figure 3.3) needs to have an area of $\pi d^2/2$. This constraint is needed both to ensure that the obstacle maintains its total mass and that the mass itself remains balanced on the right and the left of the barycenter of the rectangle.
⋄ The obstacle is convex and symmetric with respect to the x-axis.
⋄ The two caps yield a nonsmooth obstacle; this appears as a "numerical constraint", since corners give some computational difficulties and it appears unfair to compare smooth and nonsmooth obstacles.

Note that horizontal symmetry is not required and, in fact, it should not be expected as we now explain. We need to replace the two circular caps with two planar regions. A careful look at Figure 3.2 shows that, for the same Re (same line), the drag is stronger in the left picture while the lift is stronger on the right picture. Therefore, one expects that the stability might increase with asymmetry, namely in obstacles with the upwind part different from the downwind part. Since in many geographical regions the wind has mostly a constant direction, if the fluid modeled by (1.2) is the air, the obstacle K should be planned asymmetric following the expected wind direction (U,V).

In order to determine the shape performance, we fix the geometry and measure of the square and the (symmetric) obstacle. Take a square Q with edges measuring $2L=30$ [m], and the gray rectangle of Figure 3.3 having thickness 0.25 [m] and width 3 [m]. After completing with the caps (each one having area equal to $\pi/128\approx 0.025$ [m^2], for a total area of approximately 0.8 [m^2]), all the considered obstacles can be enclosed by the rectangle $\mathcal{R}$ in (2.33) with $a=1.7$ [m] and $d=0.125$ [m]; the kinematic viscosity of air is about $\eta=1.5\times10^{-5}$ [m^2/s]. With these measures, Theorem 3.9 becomes

Corollary 3.3. *Let Ω be as in* (1.1) *with $L=15$* [m]*, and assume* (2.33) *with $a=1.7$* [m]*, $d=0.125$* [m]*. If $V=0$ and $U<5.52\times10^{-9}$* [m/s] *then the solution of* (1.2)-(1.3) *is unique and it exerts no lift on K.*

In order to determine the shape performance, we proceeded computationally by employing the OpenFOAM toolbox `http://openfoam.org`, through the use of the SIMPLE algorithm for the numerical resolution of the steady-state Navier-Stokes equations in laminar regime, see [17]. In Table 1 we quote some numerical results obtained with the above parameters: the flow goes from left to right on the obstacles depicted in the first column, all having two caps of total area $\pi d^2/8$.

Shape of the obstacle	$U_* \times 10^3$ [m/s]	$C^*_{\mathcal{D}} \times 10^6$	$C_{\mathcal{D}} \times 10^6$	$\lvert C_{\mathcal{L}} \rvert \times 10^6$
	3.7	0.635	139.35	72.55
	3.2	0.508	124.21	83.35
	1.9	0.239	191.33	306.62
	3.1	0.496	280.43	569.51

Table 1: Critical velocity U_*, drag coefficient $C^*_{\mathcal{D}}$ at critical velocity, drag $C_{\mathcal{D}}$ and lift $C_{\mathcal{L}}$ coefficients at velocity $U = 0.1$ [m/s].

In the second column we report the numerically found critical velocity U_* for which uniqueness for (3.65) fails: due to the symmetry of the problem and to the absence of lift for $U < U_*$, see Proposition 3.7, the number U_* should be seen as the critical velocity generating lift. The *drag coefficient* $C_{\mathcal{D}}$ and the *lift coefficient* $C_{\mathcal{L}}$ are, respectively, dimensionless forms of the drag $\mathcal{D}$ and lift $\mathcal{L}$ exerted by the fluid governed by (3.65). For the previously specified inlet velocities, they are computed numerically according to the following expressions (see [64, Chapter 9]):

$$C_{\mathcal{D}} = \frac{\mathcal{D}}{\frac{1}{2}\rho U^2 A_f}, \qquad C_{\mathcal{L}} = \frac{\mathcal{L}}{\frac{1}{2}\rho U^2 A_p}, \tag{3.67}$$

where $\rho = 1$ [kg/m^2] is the air density, A_f is the *frontal length* (the projected length seen by an observer looking towards the object from a direction parallel to the upstream velocity), and A_p is the *planform length* (the projected length seen by an observer looking towards the object from a direction normal to the upstream velocity). In the third column we report, for $U = U_*$, the corresponding drag coefficient $C^*_{\mathcal{D}}$. For a given a boundary velocity U larger than all the critical velocities U_*, the last two columns of Table 1 contain the resulting drag and lift coefficients $C_{\mathcal{D}}$ and $C_{\mathcal{L}}$. It turns out that U_* and $\mathcal{L}$ do not have the same behavior: the threshold of instability does not have the same monotonicity as the lift at $U = 0.1$ [m/s]. In fact, the most relevant results are contained in the fifth column: there we see the comparison between different shapes for the same flow velocity, ordered from top to bottom as the "best shape" towards the "worse shape", namely for increasing values of the lift coefficient. We tested several intermediate values of U, between U_* and $U = 0.1$ [m/s] and, as expected, for all the shapes we have noticed a clear monotonicity of the lift coefficient as U increases. Since the threshold of instability U_* has two orders of magnitude less, what really measures the performances of the shapes is the rate of increment of lift with respect to the velocity of the flow. Hence, by looking at the last column in Table 1 we see that, as far as the lift is concerned, the performance of the obstacle increases (lower lift) in presence of a convex angle on the upwind part and a flat face on the downwind part. Our interpretation is that the upwind part determines the separation of the flow and, therefore, the amount of energy around the obstacle. On the other hand, the downwind part quantifies how much of this energy is effectively able to lift vertically the obstacle and, hence, a flat boundary with less friction yields less lift.

Let us now turn to some numerical results which give strength to a conjecture by Pironneau [68, 69] about the optimal shape minimizing the drag. We consider a family of "rugby balls", that is, portion of ellipses glued together. More precisely, for $0 < \beta < 2\alpha$ we consider the family of functions ψ satisfying

$$\psi(x) = \alpha\sqrt{4 - x^2} - \beta\,, \quad 0 \le x \le \sqrt{4 - \frac{\beta^2}{\alpha^2}}\,, \qquad \int_0^{\sqrt{4-\beta^2/\alpha^2}} \psi(x)\,dx = \frac{A}{4}\,,$$

where A is the area of the obstacles represented in Table 1. The integral constraint yields

$$2\alpha \arcsin\sqrt{1-\frac{\beta^2}{4\alpha^2}} - \beta \arcsin\sqrt{1-\frac{\beta^2}{4\alpha^2}} = \frac{A}{4} \approx 0.2\,. \tag{3.68}$$

Then we extend by symmetry the graph of ψ, with respect to both the axes, obtaining a rugby ball as in Figure 3.4.

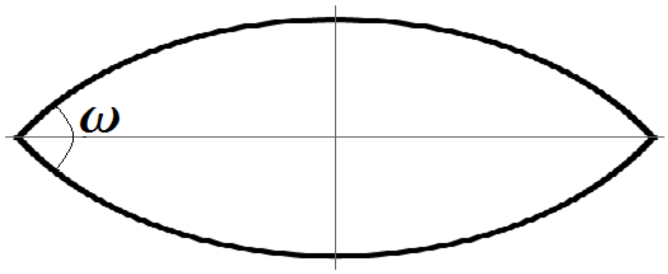

Figure 3.4: A rugby-ball-shaped obstacle.

The angle ω of the rugby ball can be computed through the derivative evaluated at the endpoint $\xi = \sqrt{4-\beta^2/\alpha^2}$ of the interval; for instance,

$$\left\{\begin{array}{c} \omega = \frac{2\pi}{3} \\ \omega = \frac{\pi}{2} \\ \omega = \frac{\pi}{3} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} \psi'(\xi) = -\sqrt{3} \\ \psi'(\xi) = -1 \\ \psi'(\xi) = -\frac{1}{\sqrt{3}} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} 4\alpha^2 = \alpha^2\beta^2 + 3\beta^2 \\ 4\alpha^2 = \alpha^2\beta^2 + \beta^2 \\ 4\alpha^2 = \alpha^2\beta^2 + \frac{\beta^2}{3} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} (\alpha,\beta) \approx (0.06696, 0.00517) \\ (\alpha,\beta) \approx (0.06986, 0.00973) \\ (\alpha,\beta) \approx (0.07638, 0.02002) \end{array}\right\},$$

where the last equivalence also accounts of (3.68). For these angles we obtained the numerical results reported in Table 2, now ordered increasingly with respect to the second and third columns.

ω	$U_* \times 10^3$ [m/s]	$C^*_{\mathcal{D}} \times 10^6$	$C_{\mathcal{D}} \times 10^6$	$\lvert C_{\mathcal{L}}\rvert \times 10^6$
$2\pi/3$	13	4.09	82.87	29.94
$\pi/3$	15	5.12	84.27	34.98
$\pi/2$	17	6.17	82.24	20.69

Table 2: Critical velocity U_*, drag coefficient $C^*_{\mathcal{D}}$ at critical velocity, drag $C_{\mathcal{D}}$ and lift $C_{\mathcal{L}}$ coefficients at velocity $U = 0.1$ [m/s].

Table 2 gives strength to a conjecture by Pironneau [68, 69] claiming that, not only rugby balls lower the drag compared to other obstacles but also that the rugby balls minimizing the drag threshold are the ones having angle $\omega = 2\pi/3$.

We conclude this section by emphasizing that the second and third columns in Tables 1 and 2 suggest that the map $U_* \mapsto C^*_{\mathcal{D}}$ is increasing and superlinear. Moreover, the data from these two columns interpolate so nicely that they seem to show that "the drag force does not depend on the shape of the obstacle". This would mean that

the shape of the obstacle has the full responsibility of transforming the drag forces into lift forces.

4 Two connections with elasticity and mechanics

4.1 A three-dimensional model: the deck of a bridge

The purpose of this section is to apply the results of the present paper to a bridge model that was first suggested in the research project [45], see also [47]. In the space $\mathbb{R}^3$ we consider the deck of a bridge to be a thin plate defined by

$$D = (-a, a) \times (-d, d) \times (-\Lambda, \Lambda) = K \times (-\Lambda, \Lambda), \tag{4.1}$$

where $d \ll a \ll \Lambda$. To have an idea, one could take $a = \Lambda/75$ and $d = \Lambda/1000$ (a deck of length 1km, with the width of about 13m, whose thickness is about 1m). Then we consider the region where the air surrounds the deck

$$B = (-L, L)^2 \times (-\Lambda, \Lambda) \setminus \overline{D} = \Omega \times (-\Lambda, \Lambda), \tag{4.2}$$

where $L \gg \Lambda$, for instance $L = 100\Lambda$ (100km, as a picture taken far away from the bridge). The domains B and D, as well as their intersections $\Omega = (-L, L)^2 \setminus \overline{K}$ and $K = (-a, a) \times (-d, d)$ with the plane $z = 0$, are represented in Figure 4.1 (not in scale).

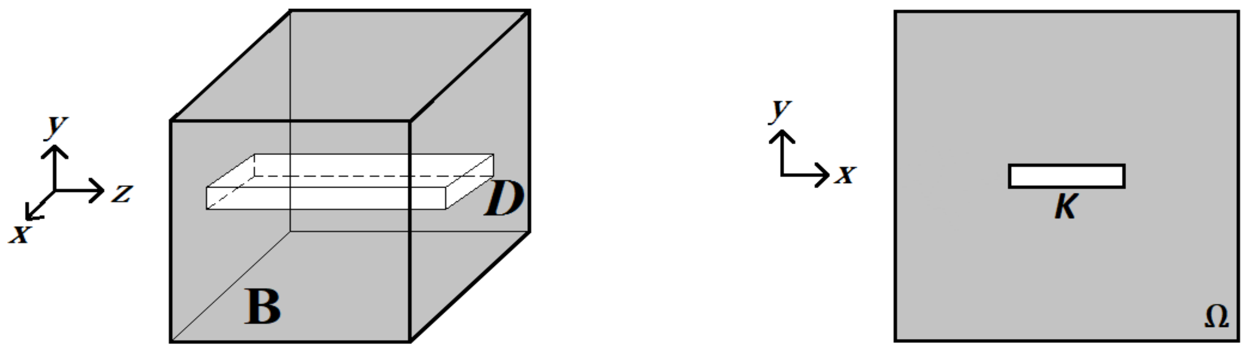

Figure 4.1: The domains B and D (left) and their intersections Ω and K with the plane $z = 0$.

The bridge is subject to a wind whose flow is governed by the Navier-Stokes equations. We model the case where the wind is blowing only in the x-direction, so that one has to analyze the planar section of this configuration, as represented in the right picture of Figure 4.1, leading us to study the planar problem of a flow around the obstacle K, governed by (1.2)-(1.3), as in Section 3.

In the three-dimensional configuration of the left picture in Figure 4.1, it is be convenient to decompose the boundary of B as

$$\partial B = \Sigma_1 \cup \Sigma_2 \cup \partial D,$$

where

$$\begin{aligned} \Sigma_1 &= \{(x, y, z) \in \partial B \mid x \in \{-L, L\}\ \} \bigcup \{(x, y, z) \in \partial B \mid y \in \{-L, L\}\ \}, \\ \Sigma_2 &= \{(x, y, z) \in \partial B \mid (x, y) \notin D,\ z \in \{-\Lambda, \Lambda\}\ \}. \end{aligned} \tag{4.3}$$

We then consider the three-dimensional Navier-Stokes equations in B, that is

$$-\eta\Delta v + (v \cdot \nabla)v + \nabla q = F, \quad \nabla \cdot v = 0 \quad \text{in} \quad B, \tag{4.4}$$

for some $F \in L^2(B)$, complemented with appropriate boundary conditions. Notice that the obstacle D and the domain B are symmetric with respect to the plane $y = 0$, in the sense that $(x, y, z) \in B$ if and only if $(x, -y, z) \in B$. It is therefore natural to wonder whether symmetry and bifurcation results also hold in this 3D setting.

Proposition 4.1. *For any $F = (f_1, f_2, f_3) \in L^2(B)$ and $(U, V, W) \in H^{1/2}(\Sigma_1 \cup \Sigma_2)$ satisfying*

$$\int_{\Sigma_1 \cup \Sigma_2} (U, V, W) \cdot \hat{n}\, dA = 0, \tag{4.5}$$

there exists a weak solution $(v, q) = (v_1, v_2, v_3, q) \in H^1(B)^3 \times L^2_0(B)$ of (4.4) in B complemented with the boundary conditions

$$v = (U, V, W) \text{ on } \Sigma_1 \cup \Sigma_2, \qquad v = (0, 0, 0) \text{ on } \partial D. \tag{4.6}$$

Moreover:

- *there exists* $\gamma = \gamma(\eta, B) > 0$ *such that if* $\|(U,V,W)\|_{H^{1/2}(\Sigma_1\cup\Sigma_2)} + \|F\|_{L^2(B)} < \gamma$, *then the weak solution of* (4.4)-(4.6) *is unique;*
- *if* f_1, f_3, U, W *are* y*-even and* f_2, V *are* y*-odd, then also* $(\xi_1, \xi_2, \xi_3, \pi)$ *with*

$$\xi_1(x,y,z) = v_1(x,-y,z), \quad \xi_2(x,y,z) = -v_2(x,-y,z), \quad \xi_3(x,y,z) = v_3(x,-y,z), \quad \pi(x,y,z) = q(x,-y,z),$$

for a.e. $(x,y,z) \in B$, *solves the problem* (4.4)-(4.6);
- *if* f_1, f_3, U, W *are* y*-even, if* f_2, V *are* y*-odd and if* $\|(U,V,W)\|_{H^{1/2}(\Sigma_1\cup\Sigma_2)} + \|F\|_{L^2(B)} < \gamma$, *then the weak solution of* (4.4)-(4.6) *is unique and satisfies the symmetry property*

$$v_1(x,y,z) = v_1(x,-y,z), \quad v_2(x,y,z) = -v_2(x,-y,z), \quad v_3(x,y,z) = v_3(x,-y,z), \quad q(x,y,z) = q(x,-y,z),$$

for a.e. $(x,y,z) \in B$.

The proof of this result is completely similar to that of Theorem 3.4 and therefore we omit it. A particular solution of (4.4) can be obtained by extending to B a solution of the corresponding planar problem in Ω, as the next result shows.

Proposition 4.2. *Let* $f = (f_1, f_2) \in L^2(\Omega)$ *and* $(U,V) \in H^{1/2}(\partial Q)$ *satisfy* (1.4). *Define* $F(x,y,z) = (f_1(x,y), f_2(x,y), 0)$ *for a.e.* $(x,y,z) \in B$. *There exists* $\overline{\gamma} = \overline{\gamma}(\eta, B) > 0$ *such that, if* $\|(U,V)\|_{H^{1/2}(\partial Q)} + \|f\|_{L^2(\Omega)} < \overline{\gamma}$, *then:*

- *problem* (1.2)-(1.3) *in* Ω *admits a unique weak (planar) solution* $(u_1, u_2, p) \in H^1(\Omega)^2 \times L^2_0(\Omega)$;
- *problem* (4.4), *complemented with the boundary conditions*

$$v = (U,V,0) \text{ on } \Sigma_1, \qquad v = (u_1, u_2, 0) \text{ on } \Sigma_2, \qquad v = (0,0,0) \text{ on } \partial D,$$

admits a unique weak solution $(v,q) \in H^1(B)^3 \times L^2_0(B)$, *which does not depend on* z *and is given by*

$$v(x,y,z) = (u_1(x,y), u_2(x,y), 0), \qquad q(x,y,z) = p(x,y) \quad \text{for a.e. } (x,y,z) \in B. \tag{4.7}$$

Proof. Take $\overline{\gamma} = \min\{\delta, \gamma\}$, with δ as in Theorem 3.1 and γ as in Proposition 4.1. Then problem (1.2)-(1.3) in Ω admits a unique weak (planar) solution $(u_1, u_2, p) \in H^1(\Omega)^2 \times L^2_0(\Omega)$. From (4.7) we infer that $(v,q) \in H^1(B)^3 \times L^2_0(B)$ is a weak solution of (4.4), where $(U,V,0) \in H^{1/2}(\Sigma_1)^3$ and $(u_1,u_2,0) \in H^{1/2}(\Sigma_2)^3$ (the definition of weak solution for the 3D problem (4.4) is naturally extended from Definition 3.1). The uniqueness of such solution is guaranteed by Proposition 4.1. □

The proof of Proposition 4.2, although simple, makes a connection between the uniqueness of the Navier-Stokes system in two and three dimensions. As a consequence of Theorem 3.9, and by putting together the results of the present paper, we obtain a sufficient condition for the stability of bridges.

Corollary 4.1. *Assume that the deck of a bridge coincides with the obstacle* D *in* (4.1) *and that the wind is blowing only in the* x*-direction with velocity* $U > 0$, *in absence of external forces. If* U *is sufficiently small, then the bridge does not oscillate.*

To see this, it suffices to take U sufficiently small so that we fall in both the uniqueness regimes for the 2D and 3D Navier-Stokes equations, see Theorem 3.9 and Proposition 4.1. Then the unique solution of (4.4) with $F = 0$ is two-dimensional, see Proposition 4.2. In view of the symmetry of the domain, Theorem 3.9 ensures that there is no lift on any of the two-dimensional cross-sections of the deck.

4.2 An impressive similitude with buckled plates

In this section we show that the bifurcation from uniqueness for the Navier-Stokes equations, related to loss of symmetry, has a counterpart in a model of a buckled elastic plate.

Consider a thin narrow rectangular plate with the two short edges hinged while the two long edges are free. In absence of forces, the plate lies horizontally flat and is represented by the planar domain

$\Omega = (0,\pi) \times (-\ell,\ell)$ with $0 < \ell \ll \pi$. The plate is only subject to compressive forces along the edges, the so-called buckling loads. Following the plate model suggested by Berger [11], the nonlocal equation modeling the deformation of the plate reads

$$\begin{cases} \Delta^2 u + \left[P - S\|u_x\|^2_{L^2(\Omega)}\right] u_{xx} = 0 & \text{in } \Omega \\ u = u_{xx} = 0 & \text{on } \{0,\pi\} \times [-\ell,\ell] \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 & \text{on } [0,\pi] \times \{-\ell,\ell\}\,, \end{cases} \tag{4.8}$$

where $\sigma \in (0,1)$ is the Poisson ratio, $S > 0$ depends on the elasticity of the material composing the plate, $S\|u_x\|^2_{L^2(\Omega)}$ measures the geometric nonlinearity of the plate due to its stretching, while P is the buckling constant: one has $P > 0$ if the plate is compressed and $P < 0$ if the plate is stretched in the x-direction.

Partially hinged rectangular plates governed by (4.8) were introduced in [30] as models for the deck of suspension bridges. For the variational characterization of (4.8) we introduce the functional space

$$H^2_*(\Omega) = \{v \in H^2(\Omega) \mid v = 0 \text{ on } \{0,\pi\} \times [-\ell,\ell]\}$$

and the inner product

$$(v,w)_{H^2_*(\Omega)} = \int_\Omega \big(\Delta v \Delta w - (1-\sigma)\big(v_{xx}w_{yy} + v_{yy}w_{xx} - 2v_{xy}w_{xy}\big)\big)\, dx\, dy$$

with corresponding norm $\|v\|^2_{H^2_*(\Omega)} = (v,v)_{H^2_*(\Omega)}$. Since $\sigma \in (0,1)$, this inner product defines a norm which makes $H^2_*(\Omega)$ a Hilbert space; see [30, Lemma 4.1]. The problem (4.8) is *variational* and this is the main crucial difference with (1.2): its solutions may be found as critical points of the "energy functional" defined by

$$J(v) = \frac{1}{2}\|v\|^2_{H^2_*(\Omega)} - \frac{P}{2}\|v_x\|^2_{L^2(\Omega)} + \frac{S}{4}\|v_x\|^4_{L^2(\Omega)} \qquad \forall v \in H^2_*(\Omega)\,.$$

It is proved in [29, 30] that the space $H^2_*(\Omega)$ is spanned by the eigenfunctions of the problem

$$\begin{cases} \Delta^2 u = -\lambda u_{xx} & \text{in } \Omega \\ u = u_{xx} = 0 & \text{on } \{0,\pi\} \times [-\ell,\ell] \\ u_{yy} + \sigma u_{xx} = u_{yyy} + (2-\sigma)u_{xxy} = 0 & \text{on } [0,\pi] \times \{-\ell,\ell\}\,, \end{cases} \tag{4.9}$$

that are given by

$$\mathcal{E}^k_m(x,y) = \varphi_{m,k}(y)\sin(mx)\,, \qquad \mathcal{O}^k_m(x,y) = \psi_{m,k}(y)\sin(mx)\,, \qquad (m,k = 1,2,...)\,, \tag{4.10}$$

where $\varphi_{m,k}$ and $\psi_{m,k}$ are explicit linear combinations of $\sin(y)$, $\cos(y)$, $\sinh(y)$, $\cosh(y)$; the former are even with respect to y, while the latter are odd. Dropping this distinction, let us order increasingly the eigenvalues of (4.9) along a sequence $\{\lambda_n\}$ $(n = 1,2,...)$ and let us denote by $\{w_n\}$ the associated sequence of normalized eigenfunctions, $\|(w_n)_x\|_{L^2(\Omega)} = 1$: the eigenvalue λ_1 is simple and w_1 has constant sign and, as a convention, we put $\lambda_0 = 0$.

By combining arguments from [4, 8, 29], we obtain the following statement.

Proposition 4.3. *For any $S > 0$ and $P \geq 0$, the function $u_0 = 0$ solves (4.8).*

• *If $P \in (\lambda_n, \lambda_{n+1}]$ for some $n \geq 0$ and all the eigenvalues smaller than or equal to λ_n have multiplicity 1, then (4.8) admits exactly $2n+1$ solutions which are explicitly given by*

$$u_0 = 0\,, \qquad \pm u_j = \pm\sqrt{\frac{P - \lambda_j}{S}}\; w_j \quad (j = 1,...,n)\,;$$

moreover, for each solution the energy is

$$J(u_0) = 0\,, \qquad J(\pm u_j) = -\frac{(P-\lambda_j)^2}{4S} \quad (j = 1, ..., n)$$

and the Morse index M is

$$M(u_0) = n\,, \qquad M(\pm u_j) = j - 1 \quad (j = 1, ..., n)\,.$$

• *If $P \in (\lambda_n, \lambda_{n+1}]$ for some $n \geq 1$ and at least one of the eigenvalues smaller than or equal to λ_n has multiplicity larger than* 1*, then* (4.8) *admits infinitely many solutions.*

In particular, Proposition 4.3 states that (4.8) admits a unique solution whenever $P \leq \lambda_1$, whereas Theorem 3.1 states that (1.2)-(1.3) admits a unique solution whenever $\|(U, V)\|_{H^{1/2}(\partial Q)} + \|f\|_{H^{-1}(\Omega)} < \delta$. At the value $P = \lambda_1$ a *bifurcation* in (4.8) occurs and, when P overcomes λ_1, two further solutions appear

$$\pm u_1 = \pm\sqrt{\frac{P - \lambda_1}{S}}\,w_1$$

and these solutions converge to u_0 as $P \searrow \lambda_1$. The counterpart of this phenomenon is the bifurcation which arises in (1.2)-(1.3) when the symmetric data overcome the critical threshold and multiple (asymmetric) solutions may appear.

As long as $\lambda_1 < P \leq \lambda_2$ only these three solutions exist and the statement about the Morse index tells us that $\pm u_1$ are stable while u_0 is unstable. Since $w_1(x, y) = \varphi(y)\sin(x)$ for some even function φ, see (4.10), the (positive) equilibrium solution of (4.8) has the shape as in Figure 4.2. The buckling load (black arrows in Figure 4.2) generates a lift (white arrow in Figure 4.2) which is orthogonal to its action. Clearly, this lift does not have the same meaning as in Section 3.3 but, still, we are in presence

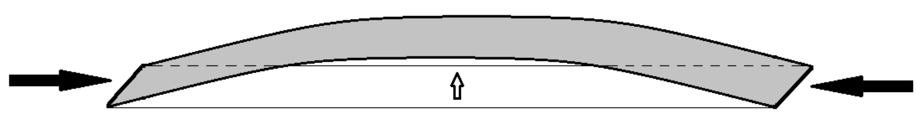

Figure 4.2: Buckling load (black) and consequent lift (white) in a partially hinged plate.

of a phenomenon where a force acting on an object has its effect in the orthogonal direction.

Letting P increase further beyond λ_2, each time P crosses an eigenvalue λ_n the number of solutions of (4.8) increases by 2, thereby their total number remains odd: the solution u_0 is symmetric, while the other solutions $\pm u_j$ are asymmetric but coupled (each asymmetric solution is coupled with its opposite), as in Theorem 3.4. The only stable solutions (with zero Morse index) are the asymmetric solutions $\pm u_1$. Since numerics (CFD) usually captures stable solutions, our feeling is that also the asymmetric solutions displayed in the second line of Figure 3.2 (large Re) are stable, while the symmetric ones are probably unstable since CFD is unable to detect them.

This pattern continues until P crosses some multiple eigenvalue, if any: in this case, the number of solutions becomes infinite because there are infinitely many possible linear combinations of the multiple eigenfunctions that solve (4.8). It is a generic property (with respect to the measures of the rectangular plate) that all the eigenvalues are simple and, in this situation, Proposition 4.3 shows that (4.8) admits a finite number of solutions (in fact, an odd number of solutions) for any $P \geq 0$. A similar result, obtained through an application of the Sard-Smale Lemma, holds for the Navier-Stokes equations: problem (3.65) admits a finite number of solutions, see Foias-Temam [31, 32], generically with respect to U and η.

5 Final comments and open problems

If we were forced to indicate just one main result among all the others obtained in this paper, we would select Theorem 3.9, which takes into account all the remaining results and gives an explicit universal bound such that if the boundary velocity of the fluid is below this bound, then the obstacle is not subject to a lift force. In order to reach this bound, in Section 2 we went through several functional inequalities. Most of these inequalities are stated in literature as "there exists a constant $C > 0$ such that..." with little information (or no information at all!) on the magnitude of C. Our purpose was to give bounds, as precise as possible, on these constants. With these bounds at hand, in Section 3 we tackled the problem of estimating the forces exerted by a viscous fluid on a bluff body. We showed how uniqueness and symmetry play a fundamental role and, in a simple situation, we managed giving fairly precise estimates as in Theorem 3.9. As shown in Section 4, these bounds have important applications in physics and engineering. We believe that the results of the present paper open new perspectives on fluid-structure interaction models [66], leading to a bunch of natural questions and open problems that we list here.

• In Section 2.2 we obtained several bounds for some embedding constants. As pointed out in Remark 2.4, we believe that they could be improved by taking into account the shape and the position of the obstacle. This would lead to a double shape/position optimization problem. We recall that, since the outer squared box Q is only virtual (it is the frame of a photo), one has the freedom of moving the obstacle inside the frame. In fact, the position of the obstacle within the flow plays a significant role, see [40] where, however, the boundary effects are important.

• As a pure functional-analytic curiosity, one could seek bounds for the embedding $H^1(\Omega) \subset L^p(\Omega)$ for any $p \geq 1$, and not just $p = 4$. For which p is our capacity approach giving better bounds? Moreover, the very same bounds could be sought in higher space dimension $n \geq 3$, where there are two crucial differences: the capacity potential behaves like $1/|x|^{n-2}$ (as the fundamental solution) and there exists a critical exponent ($p = 2n/(n-2)$) for the Sobolev embedding $H^1(\Omega) \subset L^p(\Omega)$. For the capacity, one should check if the pyramidal functions introduced in Section 2.1 still allow to obtain reliable bounds. For the critical exponent, it could be of some interest to investigate how the method developed in Section 2.2 allows to approximate the optimal embedding constant which, not only is known explicitly, but is independent of the domain.

• Quite interesting appears the 3D version of the stationary problem (1.2)-(1.3). By this we mean a non simply connected domain as in the right picture in Figure 4.1, which would model the deck of a bridge. As mentioned above, the functional inequalities in these domains appear quite different, as well as the computation of the lift. Indeed, a simple characterization as in Definition 3.4 is not available, since the directions orthogonal to the flow generate a plane and not just one line. Moreover, weaker embedding are available in 3D, which yields major difficulties in regularity results. For instance, for the perturbation of the obstacle (Theorem 3.6), we used the embedding $H^2(\Omega) \subset \mathcal{C}^{0,\nu}\left(\overline{\Omega}\right)$ for all $\nu < 1$ while in 3D one just has $\nu = 1/2$. Therefore, the 3D case is not just an extension of the 2D case, new issues will be needed.

• A result in the spirit of Theorem 3.9 could be of great interest also for other boundary conditions. For instance, conditions involving the pressure as in a network of pipes [13, 14, 21] or for the so-called Navier boundary conditions [65]. For the latter, we mention that they appear appropriate in many physically relevant cases [72], also for turbulent boundary layers [41, 67]. The Navier-Stokes equations under the Navier boundary conditions (with and without friction) have been studied by many authors, starting from Solonnikov-Shchadilov [73], see e.g. [1, 6, 9] and references therein; we mention in particular the work by Berselli [12] which appears relevant for our purposes since he considers flat 3D boundaries, in which cases the Navier boundary conditions reduce to combined Dirichlet-Neumann conditions.

• The evolution problem with constant data on ∂Q but with moving obstacle could be tackled from two different points of view. First, in the spirit of Galdi-Silvestre [42], one could seek periodic solutions by assuming that the obstacle is oscillating with given periodic law which maintains it far away from ∂Q: do periodic solutions exist, regardless of the magnitude of the (constant) inflow conditions? Second, in the spirit of Conca-San Martín-Tucsnak [22], one could set up a full fluid-structure interaction model. In this case, a major problem is to prevent collisions between the obstacle and ∂Q which, for our specific problem, *is not* a physical boundary. How does the non-collision condition vary with respect to the magnitude and the direction of the (constant) inflow?

• The appearance of violent lift forces creates serious problems in suspension bridges, possibly leading to disasters [44, Chapter 1]. The whole structure oscillates and both the cables and the hangers generate unexpected behaviors of the deck, such as torsional movements. It is therefore desirable to find a relationship between the fluid velocity, the resulting lift, and the attainment of the thresholds for hanger slackening and cable shortening, as obtained explicitly in [46] for a simplified model.

• We saw in Theorem 3.4 that, in a symmetric framework (both the domain and the data), the existence of asymmetric solutions implies non-uniqueness of solutions. The multiplicity of symmetric solutions is however an open problem, see Remark 3.4. It would be extremely important (and very challenging) to have a complete picture of the bifurcation diagram for multiple solutions of (1.2)-(1.3) in dependence of the Reynolds number.

• We have seen in Remark 2.1 and Corollary 2.2 that our bounds for the relative capacity and for the Sobolev constant of the embedding $H^1(\Omega) \subset L^4(\Omega)$ are quite accurate. Therefore, possible improvements of the threshold given in Theorem 3.9 may only be achieved through a different analysis of problem (1.5).

• Let $(u, p) \in \mathcal{V}_*(\Omega) \times L^2_0(\Omega)$ be a solution of (1.2)-(1.3) with $(U, V) \in \mathbb{R}^2$. Let $u = v + w$ be the decomposition according to (2.20) so that $v \in H^1_0(\Omega)$ and $w \in \mathbb{R}(\psi - 1)$. In fact, from the boundary conditions we know more, namely

$$w = (1-\psi)\begin{pmatrix} U \\ V \end{pmatrix} \implies \|\nabla w\|^2_{L^2(\Omega)} = (U^2+V^2)\|\nabla\psi\|^2_{L^2(\Omega)} = (U^2+V^2)\mathrm{Cap}_Q(K)\,.$$

By (2.37) we infer that $\beta(u, u, u) = \beta(u, w, w)$ and, therefore,

$$|\beta(u,u,u)| \leq (U^2+V^2)\,\frac{\mathrm{Cap}_Q(K)}{\mathcal{S}_1}\,\|\nabla u\|_{L^2(\Omega)}\,.$$

Is it possible to use this inequality to improve the bounds? In particular, in Theorem 3.2.

• Is it possible to set up a theoretical shape optimization able to compute the derivative of the lift with respect to variations of the shape of the obstacle? See [10, 19] for the case of drag derivative.

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