# Boundary conditions for planar Stokes equations inducing vortices around concave corners

Filippo Gazzola - Gianmarco Sperone
Dipartimento di Matematica, Politecnico di Milano, Italy

**Abstract**

Fluid flows around an obstacle generate vortices which are difficult to locate and to describe. A variational formulation for a class of mixed and nonstandard boundary conditions on a smooth obstacle is discussed for the Stokes equations. Possible boundary data are then derived through separation of variables of biharmonic equations in a planar region having an internal concave corner. Explicit singular solutions show that, at least qualitatively, these conditions are able to reproduce vortices over the leeward wall of the obstacle.
**AMS Subject Classification:** 35G15, 76D07, 76D17.

## 1 Introduction

Experimental evidence (see, e.g. [5, 15, 16, 44]) shows that, when a fluid hits a bluff body, its flow is modified and creates vortices around the body (behind it), see Figure 1.1. Vortices may also be

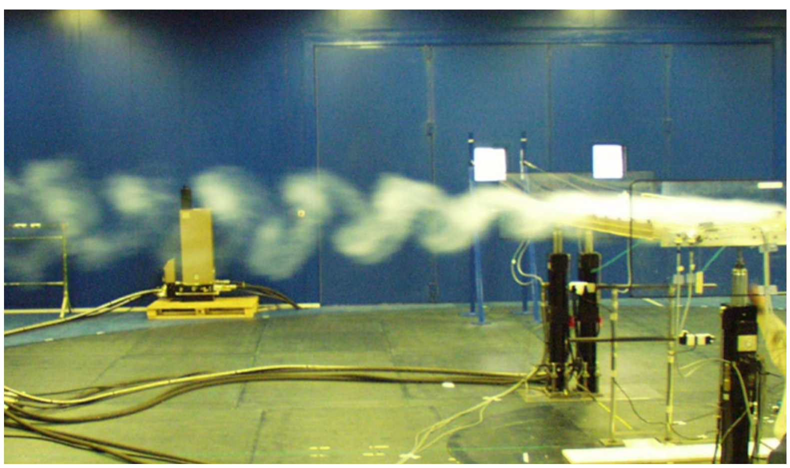

Figure 1.1: Vortices obtained in wind tunnel experiments at the Politecnico di Milano.

detected numerically [18, 19]. Depending on the geometry of the body, a symmetrical or asymmetrical rotating flow is periodically developed in that hidden part. A detailed description of this phenomenon is given in [42] but, even in the case of a perfectly circular cylinder, instabilities in the vortex shedding pattern may appear, see [40, Section 4.2.6]. In the vortex formation, flow separations and reattachments strongly depend on the Reynolds number through fairly complicated rules (see e.g. [18]), which makes the analytical and the numerical treatments very challenging. So far, the mathematical modeling of these phenomena is rather poor and totally unsatisfactory for engineers [27]. A viscous fluid past a rigid body immersed in the fluid is usually tackled under (no-slip) homogeneous Dirichlet boundary conditions for the velocity field on the surface of the obstacle, see [31, Chapter 2, Section 2] and [32, Chapter 2]. But since in the long term we have in mind to study an obstacle representing a suspension bridge which undergoes oscillations (moving obstacle) [6], an interactive motion of the obstacle should also be studied with different boundary conditions, as in [11]. Mixed boundary conditions (based on the normal velocity, tangential velocity, vorticity and pressure) arise naturally in a network of pipes [34], in fluid-structure models in hemodynamics [9] and in the thermoelectromagnetic flow of a viscous fluid [1]. These nonstandard boundary conditions for the Stokes and Navier-Stokes equations were introduced by Conca et al. [3, 4, 10] (see also [2, 8, 14, 21, 24] for subsequent developments), suggesting an alternative variational formulation for problems of fluids around an obstacle. Having in mind to explain the vortex shedding generated by the wind acting on the deck of an oscillating suspension bridge, in this paper we pursue a double objective: we discuss nonstandard variational formulations for the Stokes equations in domains with an obstacle and we determine possible boundary data which give rise to vortices.

For the first purpose we consider the stationary Stokes equations in a bounded domain $Q \subset \mathbb{R}^3$:

$$-\eta\Delta u + \nabla p = f\,, \quad \nabla \cdot u = 0 \qquad \text{in } Q\,, \tag{1.1}$$

where $\eta > 0$ denotes the kinematic viscosity, $u : Q \to \mathbb{R}^3$ is the velocity field, $p : Q \to \mathbb{R}$ is the scalar pressure, $f : Q \to \mathbb{R}^3$ is an external force. The domain $Q$ is not simply connected, it contains an obstacle $D \subset \mathbb{R}^3$ representing the bridge. It is clear that *linear equations* such as (1.1) may not be suitable to describe turbulent regimes and it is by far more realistic to stick to *nonlinear equations* such as the Euler equations or the full Navier-Stokes equations, see [33]. Therefore, this paper should just be seen as a first attempt to derive some information from the boundary behavior of the solution, possibly applicable to more sophisticated models: in fact, by assuming a constant transversal behavior of the force $f$, we further simplify the study by reducing to a planar domain. This paper is also a first contribution to a research project [6], submitted to the Thelam Fund (Belgium) in March 2018. The variational formulation in [4] is based on the vorticity instead of the gradient of the velocity and this suggests to impose boundary conditions on the vorticity itself. In the present paper, we revisit the procedure in [4] and we extend it to a slightly more general context, see Section 2.1 where we reduce the 3D problem to a 2D problem and we explain in detail the physical model. The well-posedness of the considered problem is established in Theorem 2.1 in Section 2.2. Its proof is based on a nontrivial application of the Lax-Milgram Theorem. The ellipticity of the related bilinear form depends on the regularity and the topology of the domain. If a domain with $\mathcal{C}^2$-boundary is not simply-connected, Foias-Temam [17] showed that the subspace of irrotational vector fields is nontrivial (its dimension is equal to the number of cuts needed to make the domain simply-connected). In [10, Appendix A], the authors managed to prove the ellipticity of the bilinear form when the domain is a convex polyhedron or when its boundary is of class $\mathcal{C}^{1,1}$. Although our domain is neither convex, nor a polyhedron, nor of class $\mathcal{C}^{1,1}$, nor simply-connected, we are still able to demonstrate the ellipticity of the bilinear form by combining some results contained in [25].

The second purpose of this paper is to determine boundary conditions and data which yield solutions of the Stokes equations (1.1) displaying vortices. Obviously, any change in the boundary data strongly modifies the behavior of the solution. We identify boundary conditions compatible with the considered variational formulation, although the "optimal choice" of the boundary data remains unclear. We focus our attention on an unbounded, simply connected planar region having a concave right angle. If on the one hand simple connectivity enables us to show the existence of a *stream function* satisfying a biharmonic equation (see [13] and Section 4), on the other hand it is known [26, 35] that existence and regularity results may fail in nonsmooth domains even if the data are smooth. A variety of methods have been developed in order to solve biharmonic equations in planar regions [12, 28, 29, 36, 37, 39]. The singularities of solutions in the neighborhood of a concave corner are described through functional spaces with weighted norms. In most cases, these singularities are of power type (see Borsuk-Kondrat'ev [7, Chapter 5]), but in the present article the singularity will be represented by the composition between trigonometric and logarithmic functions, see Theorem 3.1 below, and thus characterized by chaotic oscillations near the angle. In Section 5 we determine some boundary conditions and data that highlight vortices within the explicit solution, in separated-variable form, of (1.1). These boundary conditions impose a null normal component of the velocity field (in both faces of the concave angle) and some value for the scalar vorticity. Our boundary data are also justified by the regularity properties of the solution: we introduce *singular* solutions, see Definition 4.1, and we choose data giving rise to this kind of solutions.

This paper is organized as follows. In Section 2.1 we describe in detail the domain $Q$, together with its two-dimensional projection $\Omega$; the nonstandard boundary conditions for the Stokes equations to be solved in $\Omega$ are presented in Section 2.2, whose main core is the corresponding existence and uniqueness result, see Theorem 2.1. In Section 3, a review of the method of separation of variables is carried out for the biharmonic equation in polar coordinates in an unbounded domain $\Lambda \subset \mathbb{R}^2$ having a concave right angle. A class of separated-variable solutions is obtained in Theorem 3.1 that allows us to characterize, in Section 4, singular solutions of the Stokes equations in $\Lambda$, see Definition 4.1. Finally, in Section 5 we give explicit singular solutions of the Stokes system in $\Lambda$. This is done by considering two families of boundary conditions: for laminar inflow and for oriented velocity, see Sections 5.1 and 5.2, respectively. The results are complemented with some figures.

# 2 The Stokes equations with nonstandard boundary conditions

## 2.1 From the three-dimensional problem to the planar problem

In the space $\mathbb{R}^3$ we consider the deck of the bridge to be a thin plate defined by

$$D = (0, \pi) \times (-\ell, \ell) \times (-d, d), \tag{2.1}$$

where $d \ll \ell \ll \pi$. To have an idea, one could take $\ell = \pi/150$ and $d = \pi/1000$ (a deck of length 1km, with the width of about 13m, whose thickness is about 1m). Then we consider the region where the air surrounds the deck

$$Q = (0, \pi) \times (-L, L)^2 \setminus D, \tag{2.2}$$

where $L \gg \pi$, for instance $L = 100\pi$ (100km, an approximation of an unbounded region). The domains $Q$ and $D$, as well as their intersections $\Omega$ and $K$ with the plane $x = \frac{\pi}{2}$, are represented in Figure 2.1 (not in scale).

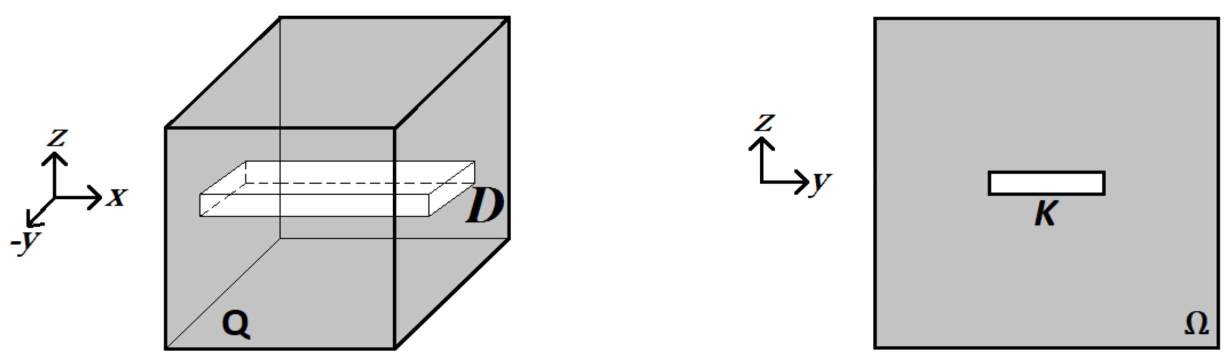


Figure 2.1: The domains $Q$ and $D$ (left) and their intersections $\Omega$ and $K$ with the plane $x = \frac{\pi}{2}$.

We are interested in solving (1.1) with nonstandard and mixed boundary conditions on the different parts of $\partial Q$, depending on the velocity, vorticity and pressure. This is a particular inflow-outflow problem in a rectangular cylinder (with obstacle) [22]. We model the case where the wind is blowing only in the $y$-direction, so that it is reasonable to analyze the planar section of this configuration, as represented in the right picture of Figure 2.1. Neither the 3D domain $Q$ nor the 2D domain $\Omega$ are simply connected. From now on, all the two-dimensional vector fields will be considered as three-dimensional vector fields, assuming that they do not depend on the first variable and that their first component is identically null. We are so led to study the planar problem of a flow around the rectangle $K$.

In fact, in this planar setting we consider a *smooth rectangular obstacle* of width $2\ell$ and height $2d$, still denoted by $K$, whose corners are smoothened by small quarters of circles, as in the left picture of Figure 2.2. Therefore, $K \subset (-\ell, \ell) \times (-d, d)$, while the open domain $\Omega = (-L, L)^2 \setminus \overline{K}$ is the region where the air surrounds the obstacle $K$: they are both represented, not in scale, in the right picture of Figure 2.2.

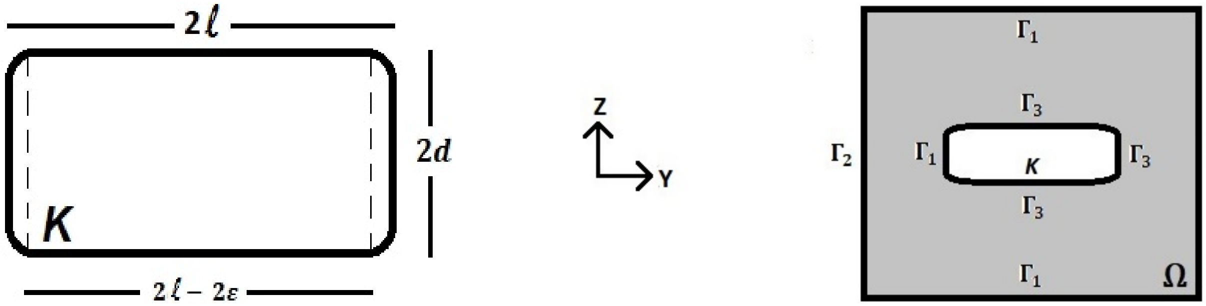


Figure 2.2: The obstacle $K$ (left) and the domain $\Omega$ (right).

For our purposes, it will be convenient to decompose the boundary of $\Omega$ as:

$$\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3},$$

where:

$$\begin{aligned} \Gamma_1 &= \{(y,z) \in \partial K \mid -\ell \le y < -\ell + \varepsilon\} \bigcup \{(y,z) \in \mathbb{R}^2 \mid y \in (-L,L)\,,\ z \in \{-L,L\}\ \}, \\ \Gamma_2 &= \{(y,z) \in \mathbb{R}^2 \mid y \in \{-L,L\}\,,\ z \in (-L,L)\,\}, \\ \Gamma_3 &= \{(y,z) \in \partial K \mid -\ell + \varepsilon < y \le \ell\}, \end{aligned} \tag{2.3}$$

so that the rounded corners on the left side of $K$ belong to $\Gamma_1$. Therefore, $\Omega$ is an open, bounded and connected set, with a locally Lipschitz boundary and with the interior boundary $\partial K$ of class $\mathcal{C}^{1,1}$. Consequently, the outward unit normal $\hat{n}$ is defined almost everywhere on $\partial\Omega$, as a Lipschitz-function on each connected component of $\partial\Omega$. This model was first suggested in the research project [6, 23].

## 2.2 An existence and uniqueness result

We model the situation in which a constant wind blows in the $y$-direction, so that the forcing term $f$ and its potential $F$ read

$$f = f(y,z) = (w,0)\,, \qquad F(y,z) = wy \qquad \forall (y,z) \in \Omega, \tag{2.4}$$

being $w > 0$ the scalar wind velocity. In this setting, the cross product of two planar vectors (in the plane spanned by $\{\hat{\mathbf{j}}, \hat{\mathbf{k}}\}$) and the curl of a two-dimensional vector field is a three-dimensional vector field whose only non-null component is the one parallel to $\hat{\mathbf{i}}$:

$$u(y,z) = u^1(y,z)\hat{\mathbf{j}} + u^2(y,z)\hat{\mathbf{k}} \qquad \Longrightarrow \qquad \nabla \times u = \left(\frac{\partial u^2}{\partial y} - \frac{\partial u^1}{\partial z}\right)\hat{\mathbf{i}} \qquad \forall u \in \mathcal{C}^1(\Omega)^2.$$

The stationary Stokes equations are analyzed over the domain $\Omega$:

$$-\eta\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \tag{2.5}$$

where, again, $u : \Omega \to \mathbb{R}^3$ is the velocity field (but with null first component) and $p : \Omega \to \mathbb{R}$ is the scalar pressure while $\eta > 0$ is the kinematic viscosity. Given $u_0 \in H^{1/2}(\Gamma_1)^2$, $p_0 \in H^{-1/2}(\Gamma_2)$ and $h \in H^{-1/2}(\Gamma_3)^2$, to (2.5) we associate the following boundary conditions:

$$u = u_0 \ \text{ on } \Gamma_1, \qquad u \times \hat{n} = 0,\ p = p_0 \ \text{ on } \Gamma_2, \qquad u \cdot \hat{n} = 0,\ (\nabla \times u) \times \hat{n} = h \times \hat{n} \ \text{ on } \Gamma_3. \tag{2.6}$$

The first condition in (2.6) prescribes the velocity field in all parts of $\Gamma_1$ (nonhomogeneous Dirichlet boundary condition), according to the expected physical properties of the problem. The second condition in (2.6) imposes that the flow is normal on the parts of $\partial\Omega$ where the flow is entering and exiting and, therefore, the pressure will be constant with opposite signs in correspondence of the inflow or outflow parts. The third condition in (2.6) states that the flow is tangential on $\Gamma_3$. From a physical point of view, it would be reasonable to assume that $h = 0$ on the upper and lower faces of the obstacle $K$; nevertheless, we will consider general data $h \in H^{-1/2}(\Gamma_3)^2$. On the opposite side of the deck (leeward wall), the velocity $u$ is tangential as well as the vorticity. This last boundary condition is crucial if one intends to model the shedding of vortices.

Next, we introduce two functional spaces:

$$V(\Omega) \doteq \{v \in H^1(\Omega)^2 \mid \nabla \cdot v = 0 \text{ in } \Omega; \quad v = 0 \text{ on } \Gamma_1; \quad v \times \hat{n} = 0 \text{ on } \Gamma_2; \quad v \cdot \hat{n} = 0 \text{ on } \Gamma_3\},$$

$$H(\Delta, \Omega) \doteq \{q \in L^2(\Omega) \mid \Delta q \in L^2(\Omega)\}.$$

Since the trace operator is linear and continuous from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$, we infer that $V(\Omega)$ is a closed subspace of $H^1(\Omega)^2$ and therefore it constitutes a Hilbert space under the usual scalar product of $H^1(\Omega)^2$, defined as:

$$(u, v)_{H^1(\Omega)^2} \doteq (u^1, v^1)_{H^1(\Omega)} + (u^2, v^2)_{H^1(\Omega)} \qquad \forall u, v \in H^1(\Omega)^2.$$

As explained in [4, Théorème 1.9], all the functions of $H(\Delta, \Omega)$ possess a trace belonging to $H^{-1/2}(\partial\Omega)$.

We also need to introduce the continuous bilinear form $A : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$A(u, v) = \eta \int_\Omega (\nabla \times u) \cdot (\nabla \times v)\, dx \qquad \forall u, v \in H^1(\Omega)^2; \tag{2.7}$$

note that the last two components of $\nabla \times u$ and $\nabla \times v$ are identically null and that

$$(\nabla \times u) \cdot (\nabla \times v) = \left(\frac{\partial u^2}{\partial y} - \frac{\partial u^1}{\partial z}\right)\left(\frac{\partial v^2}{\partial y} - \frac{\partial v^1}{\partial z}\right) \qquad \forall u, v \in H^1(\Omega)^2.$$

Finally, we will also need the continuous linear functional $L : H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$L(v) = \int_\Omega f(x) \cdot v(x)\, dx - \langle p_0, v \cdot \hat{n}\rangle_{\Gamma_2} + \eta \langle h \times \hat{n}, v\rangle_{\Gamma_3} \qquad \forall v \in H^1(\Omega)^2, \tag{2.8}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_i}$ denotes the duality product between $H^{-1/2}(\Gamma_i)$ and $H^{1/2}(\Gamma_i)$ $(i \in \{1, 2, 3\})$.

As in [4] (see also [10]), for the boundary datum $u_0$ we assume that:

$$\exists U_0 \in H^1(\Omega)^2 \quad \text{such that} \quad \begin{cases} \nabla \cdot U_0 = 0 & \text{in } \Omega \\ U_0 = u_0 & \text{on } \Gamma_1 \\ U_0 \times \hat{n} = 0 & \text{on } \Gamma_2 \\ U_0 \cdot \hat{n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{2.9}$$

Notice that the existence of such $U_0$ depends on $u_0 \in H^{1/2}(\Gamma_1)^2$ through the Divergence Theorem: hence, (2.9) is an assumption on $u_0$. More precisely, consider the space $X$ of solutions of the (incomplete) problem

$$\nabla \cdot V = 0 \text{ in } \Omega, \qquad V \times \hat{n} = 0 \text{ on } \Gamma_2, \qquad V \cdot \hat{n} = 0 \text{ on } \Gamma_3.$$

Then the space of admissible $u_0$ coincides with the traces over $\Gamma_1$ of functions $V \in H^1(\Omega)^2 \cap X$.

In this functional framework, and under assumption (2.9), we consider the following variational formulation (suggested in [10, (1.25)]) for the boundary-value problem (2.5)-(2.6):

$$\text{find } u \in H^1(\Omega)^2 \text{ such that: } (u - U_0) \in V(\Omega), \quad A(u, v) = L(v) \text{ for every } v \in V(\Omega). \tag{2.10}$$

The next result is the main core of the present section: the existence of a unique solution of the variational (or weak) problem (2.10) is stated, together with the equivalence between this variational formulation and the boundary-value problem (2.5)-(2.6), which justifies the validity of the weak formulation (2.10):

**Theorem 2.1.** *If $p_0 \in H^{-1/2}(\Gamma_2)$ and $h \in H^{-1/2}(\Gamma_3)^2$, the variational problem* (2.10) *has a unique solution $u \in H^1(\Omega)^2$. The solution $u$ is such that $(\nabla \times u) \in H(\Delta,\Omega)^3$ and there exists $p \in H(\Delta,\Omega)/\mathbb{R}$ such that $u$ and $p$ are solutions of the boundary-value problem* (2.5)-(2.6) *in the following sense:*

- $-\eta\Delta u + \nabla p = f$ *and* $\nabla \cdot u = 0$ *in* $\Omega$*, in distributional sense.*
- *$u$ satisfies* (2.6) *over $\Gamma_1, \Gamma_2$ and $\Gamma_3$ in the sense of traces of functions belonging to $H^1(\Omega)^2$, whereas $p$ and $(\nabla \times u)$ satisfy* (2.6) *over $\Gamma_2$ and $\Gamma_3$ in the following sense:*

$$\int_\Omega (-\eta\Delta u + \nabla p)\cdot v(x)\,dx - \eta\int_\Omega (\nabla \times u)\cdot(\nabla \times v)\,dx = \langle p_0, v\cdot\hat{n}\rangle_{\Gamma_2} - \eta\langle h\times\hat{n}, v\rangle_{\Gamma_3} \quad \forall v \in V(\Omega). \tag{2.11}$$

*Furthermore, if $\nabla \times (\nabla \times u) \in L^2(\Omega)^3$, then* (2.11) *implies that:*

- *$p = p_0$ over $\Gamma_2$ in the sense of $H^{-1/2}(\Gamma_2)/\mathbb{R}$ and $(\nabla \times u)\times\hat{n} = h\times\hat{n}$ over $\Gamma_3$, in the sense of $H^{-1/2}(\Gamma_3)^3$.*

*Finally, if $u \in \mathcal{C}^2(\overline{\Omega};\mathbb{R}^2)$ and $p \in \mathcal{C}^1(\overline{\Omega};\mathbb{R})$ are classical solutions of the boundary-value problem* (2.5)-(2.6)*, then $u$ is also a solution of the variational problem* (2.10)*.*

The proof of Theorem 2.1 follows closely [10] and a hint is given below. We first make precise what we intend by $V(\Omega)$-ellipticity:

**Definition 2.1.** *We say that the bilinear form $A : H^1(\Omega)^2 \times H^1(\Omega)^2 \to \mathbb{R}$ is $V(\Omega)$-**elliptic** if there exists $\gamma > 0$ (which depends only on the domain $\Omega$ and its boundary) such that:*

$$A(v,v) = \eta\int_\Omega |\nabla \times v|^2\,dx \geq \gamma\|v\|^2_{H^1(\Omega)^2} \qquad \forall v \in V(\Omega). \tag{2.12}$$

If the bilinear form $A$ is $V(\Omega)$-elliptic, then the Lax-Milgram Theorem allows us to deduce that the variational problem (2.10) has a unique solution. To this end, the following results contained in [25, Chapter I, Section 2 and Section 3] need to be recalled:

**Lemma 2.1.** *Let $\Phi \subset \mathbb{R}^2$ be an open bounded set, with a locally Lipschitz boundary. If $\Phi$ is a convex polygon or if its boundary is of class $\mathcal{C}^{1,1}$, then:*

- *the space*

$$U(\Phi) := \{v \in L^2(\Phi)^2 \mid \nabla\cdot v \in L^2(\Phi);\quad \nabla\times v \in L^2(\Phi)^3;\quad v\cdot\hat{n} = 0 \text{ on } \partial\Phi\}, \tag{2.13}$$

*is continuously embedded into $H^1(\Phi)^2$ and there exists $C > 0$, depending only on $\Phi$, such that:*

$$\|v\|_{H^1(\Phi)^2} \leq C\{\|v\|^2_{L^2(\Phi)^2} + \|\nabla\cdot v\|^2_{L^2(\Phi)} + \|\nabla\times v\|^2_{L^2(\Phi)^3}\}^{1/2} \quad \forall v \in U(\Phi); \tag{2.14}$$

- *the space*

$$W(\Phi) := \{v \in L^2(\Phi)^2 \mid \nabla\cdot v \in L^2(\Phi);\quad \nabla\times v \in L^2(\Phi)^3;\quad v\times\hat{n} = 0 \text{ on } \partial\Phi\}, \tag{2.15}$$

*is continuously embedded into $H^1(\Phi)^2$ and there exists $C > 0$, depending only on $\Phi$, such that:*

$$\|v\|_{H^1(\Phi)^2} \leq C\{\|v\|^2_{L^2(\Phi)^2} + \|\nabla\cdot v\|^2_{L^2(\Phi)} + \|\nabla\times v\|^2_{L^2(\Phi)^3}\}^{1/2} \quad \forall v \in W(\Phi). \tag{2.16}$$

As in [4, 10], we consider the following subspace of $L^2(\Omega)^2$:

$$\Psi(\Omega) = \{v \in L^2(\Omega)^2 \mid \nabla\times v \in L^2(\Omega)^3;\ \nabla\cdot v = 0 \text{ in } \Omega;\ v\times\hat{n} = 0 \text{ on } \Gamma_1\cup\Gamma_2;\ v\cdot\hat{n} = 0 \text{ on } \Gamma_1\cup\Gamma_3\}. \tag{2.17}$$

This functional space is well-defined since, if $v \in L^2(\Omega)^2$ is such that $\nabla \times v \in L^2(\Omega)^3$ and $\nabla \cdot v \in L^2(\Omega)$, then its tangential and normal traces exist, respectively, in the boundary spaces

$$v \times \hat{n} \in H^{-1/2}(\partial\Omega)^3; \qquad v \cdot \hat{n} \in H^{-1/2}(\partial\Omega).$$

Moreover, $\Psi(\Omega)$ is a Hilbert space when endowed with the scalar product

$$\langle v, w\rangle_{\Psi(\Omega)} = \int_\Omega v \cdot w \, dx + \int_\Omega (\nabla \times v) \cdot (\nabla \times w) \, dx \quad \forall v, w \in \Psi(\Omega),$$

with corresponding norm $\|v\|_{\Psi(\Omega)} = \langle v, v\rangle_{\Psi(\Omega)}^{1/2}$. Note that if $v \in \Psi(\Omega)$, then $v = 0$ on $\Gamma_1$, since $v \times \hat{n} = v \cdot \hat{n} = 0$ on $\Gamma_1$. It is also clear that the space $V(\Omega)$ (endowed with the standard norm of $H^1(\Omega)^2$) is continuously embedded into $\Psi(\Omega)$. Actually, the following result holds; the proof follows the same line as [10, Theorem A.1] in a slightly different geometric context (a planar domain, neither convex nor with a $\mathcal{C}^{1,1}$ boundary).

**Lemma 2.2.** *The space $\Psi(\Omega)$ is continuously embedded into $V(\Omega)$, and therefore $\Psi(\Omega) = V(\Omega)$ (algebraically and topologically).*

**Proof.** We employ a localization argument, similar to the one in [10, Appendix A]. Since $\overline{\Omega} \subset \mathbb{R}^2$ is compact, it can covered by a finite number of open disks $\{\theta_i\}_{i=1}^m$, for some $m \geq 1$:

$$\overline{\Omega} \subset \bigcup_{i=1}^m \theta_i.$$

By reducing the radius of the disks $\{\theta_i\}_{i=1}^m$ (if necessary), we may assume that, if $i \in \{1, \ldots, m\}$ is such that $\theta_i \cap \partial K \neq \emptyset$, then $\theta_i$ does not intersect any of the faces of $\overline{\Omega}$ contained in the lines $y = \pm L$ or $z = \pm L$. Next, we introduce a partition of unity subordinate to the open cover $\{\theta_i\}_{i=1}^m$, that is, we consider a family of functions $\{\alpha_i\}_{i=1}^m \subset \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that:

$$\alpha_i \in \mathcal{C}_0^\infty(\theta_i), \quad 0 \leq \alpha_i(x) \leq 1 \text{ for every } x \in \overline{\Omega}, \qquad \forall i \in \{1, \ldots, m\},$$
$$\sum_{i=1}^m \alpha_i(x) = 1 \text{ for every } x \in \overline{\Omega}.$$

Therefore, for every function $v \in \Psi(\Omega)$ we can write:

$$v(x) = \sum_{i=1}^m \alpha_i(x) v(x) \quad \forall x \in \overline{\Omega},$$

and $\Psi(\Omega)$ is continuously embedded into $V(\Omega)$ provided that:
- $\alpha_i v \in H^1(\Omega)^2$, for every $v \in \Psi(\Omega)$ and for every $i \in \{1, \ldots, m\}$;
- there exist $C > 0$ (depending only on $\Omega$, $\{\theta_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$), such that:

$$\|\alpha_i v\|_{H^1(\Omega)^2} \leq C\|v\|_{\Psi(\Omega)} \quad \forall v \in \Psi(\Omega), \quad \forall i \in \{1, \ldots, m\}.$$

Having these targets in mind, let $v \in \Psi(\Omega)$ and $i \in \{1, \ldots, m\}$ and let us distinguish two different cases.

• **Case (A):** $\theta_i \cap \partial\Omega = \emptyset$, or $\theta_i \cap \partial\Omega \neq \emptyset$ but $\theta_i \cap \overline{K} = \emptyset$. In this case, since $\partial\Omega$ is a union of sets having Lipschitz-continuous boundaries and since $\alpha_i$ has compact support in $\theta_i$, it is not restrictive to assume that the function $\alpha_i v$ is defined in an open and convex subset of $\theta_i \cap \Omega$, which we shall denote by $\zeta_i$ (see Figure 2.3). Then $\zeta_i$ is a convex polygon and $\text{supp}(\alpha_i) \cap \Omega \subset \zeta_i$. On the other hand, since $v \in \Psi(\Omega)$, we infer that $\alpha_i v \in L^2(\zeta_i)^2$, $\nabla \cdot (\alpha_i v) \in L^2(\zeta_i)$, $\nabla \times (\alpha_i v) \in L^2(\zeta_i)^3$ and that $(\alpha_i v) \times \hat{n} = 0$ in $\partial\zeta_i$. By

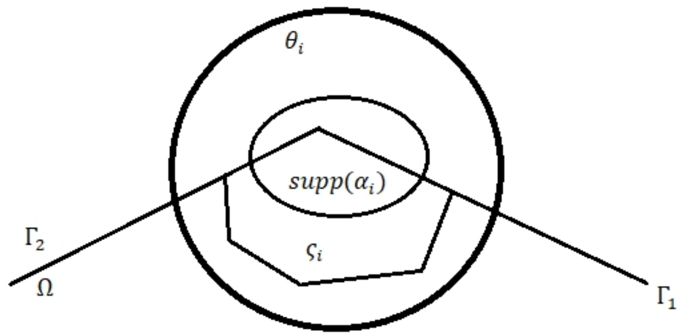


Figure 2.3: Construction of the open set $\zeta_i \subset (\theta_i \cap \Omega)$.

applying Lemma 2.1, we then deduce that $\alpha_i v \in H^1(\zeta_i)^2$ and that there exists $C_i > 0$ (depending only on $\zeta_i$) such that

$$\|\alpha_i v\|_{H^1(\zeta_i)^2} \le C_i\{\|\alpha_i v\|^2_{L^2(\zeta_i)^2} + \|\nabla \cdot (\alpha_i v)\|^2_{L^2(\zeta_i)} + \|\nabla \times (\alpha_i v)\|^2_{L^2(\zeta_i)^3}\}^{1/2}$$

which, in particular, implies that

$$\|\alpha_i v\|_{H^1(\theta_i\cap\Omega)^2} \le C_i\{\|\alpha_i v\|^2_{L^2(\theta_i\cap\Omega)^2} + \|\nabla \cdot (\alpha_i v)\|^2_{L^2(\theta_i\cap\Omega)} + \|\nabla \times (\alpha_i v)\|^2_{L^2(\theta_i\cap\Omega)^3}\}^{1/2}. \tag{2.18}$$

• **Case (B):** $\theta_i \cap \overline{K} \neq \emptyset$. In this case, since $\partial\Omega$ is a union of sets with Lipschitz-continuous boundaries and since $\alpha_i$ has compact support in $\theta_i$, it is not restrictive to assume that the function $\alpha_i v$ is defined in an open subset of $\theta_i \cap \Omega$, which we shall denote by $\zeta_i$ (see Figure 2.4). In the present situation, since

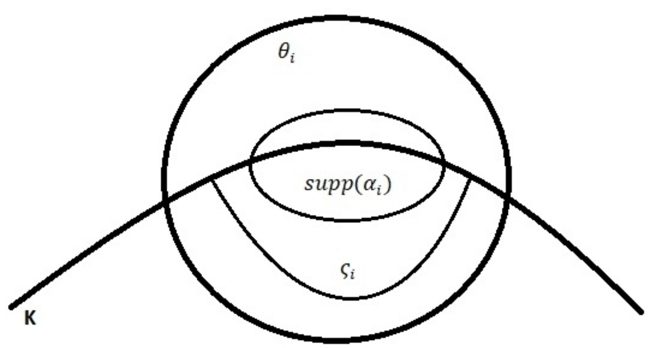


Figure 2.4: Construction of the open set $\zeta_i \subset (\theta_i \cap \Omega)$.

the domain $K$ is smooth, we may establish that $\zeta_i$ has a $\mathcal{C}^{1,1}$ boundary and that $\text{supp}(\alpha_i) \cap \Omega \subset \zeta_i$. On the other hand, as $v \in \Psi(\Omega)$, we deduce that $\alpha_i v \in L^2(\zeta_i)^2$, $\nabla \cdot (\alpha_i v) \in L^2(\zeta_i)$, $\nabla \times (\alpha_i v) \in L^2(\zeta_i)^3$ and that $(\alpha_i v) \cdot \hat{n} = 0$ in $\partial\zeta_i$. Then, applying again Lemma 2.1, we infer that $\alpha_i v \in H^1(\zeta_i)^2$ and that there exists a constant $C_i > 0$ (depending only on $\zeta_i$) such that (2.18) holds.
Hence, (2.18) holds in both cases (A) and (B), that is, it holds for every $i \in \{1, \dots, m\}$ and, therefore, there exists $C > 0$ (depending on $\Omega$, $\{\theta_i\}_{i=1}^m$ and $\{\alpha_i\}_{i=1}^m$) such that

$$\|\alpha_i v\|_{H^1(\Omega)^2} \le C\{\|v\|^2_{L^2(\Omega)^2} + \|\nabla \cdot v\|^2_{L^2(\Omega)} + \|\nabla \times v\|^2_{L^2(\Omega)^3}\}^{1/2}.$$

Since $\nabla \cdot v = 0$ in $\Omega$, we finally have that

$$\|\alpha_i v\|_{H^1(\Omega)^2} \le C\{\|v\|^2_{L^2(\Omega)^2} + \|\nabla \times v\|^2_{L^2(\Omega)^3}\}^{1/2}$$

for every $v \in \Psi(\Omega)$ and $i \in \{1, \dots, m\}$. This concludes the proof of the lemma. □

Next, we recall that [10, Lemma A.2] implies that

$$\text{the map defined by } V(\Omega) \ni v \mapsto \|\nabla \times v\|_{L^2(\Omega)^3} \text{ is a norm in } V(\Omega). \tag{2.19}$$

**Proof of Theorem 2.1**. We now have all the ingredients to demonstrate that the bilinear form $A(\cdot,\cdot)$ is $V(\Omega)$-elliptic. Lemma 2.2 implies the existence of a constant $C_1 > 0$ (depending on $\Omega$) such that:

$$\|v\|_{H^1(\Omega)^2} \leq C_1\{\|v\|^2_{L^2(\Omega)^2} + \|\nabla \times v\|^2_{L^2(\Omega)^3}\}^{1/2} \quad \forall v \in V(\Omega). \tag{2.20}$$

Therefore, in order to prove the $V(\Omega)$-ellipticity of $A(\cdot,\cdot)$, it suffices to show the existence of another constant $C_2 > 0$ (also depending only on $\Omega$) such that:

$$\|v\|_{L^2(\Omega)^2} \leq C_2\|\nabla \times v\|_{L^2(\Omega)^3} \quad \forall v \in V(\Omega). \tag{2.21}$$

For contradiction, assume that (2.21) does not hold. Then, there exists a sequence $\{v_n\} \subset V(\Omega)$ such that

$$\|v_n\|_{L^2(\Omega)^2} = 1, \quad \|\nabla \times v_n\|_{L^2(\Omega)^3} \leq \frac{1}{n} \qquad \forall n \in \mathbb{N}. \tag{2.22}$$

Using inequality (2.20), we see that (2.22) implies that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $V(\Omega)$ (with the standard $H^1(\Omega)^2$-norm). We may then extract a subsequence $\{v_{\varphi(n)}\}_{n\in\mathbb{N}}$ such that

$$v_{\varphi(n)} \rightharpoonup v \text{ in } H^1(\Omega)^2 \ \text{ and } \ v_{\varphi(n)} \to v \text{ in } L^2(\Omega)^2 \quad \text{as } n \to \infty \tag{2.23}$$

for some $v \in V(\Omega)$. But, in this situation, (2.22) implies that $\nabla \times v = 0$ in $\Omega$. Therefore, since $v \in V(\Omega)$, (2.19) allows us to conclude that $v = 0$, and hence, (2.22) contradicts (2.23). As a consequence, we conclude that the bilinear form $A(\cdot,\cdot)$ is $V(\Omega)$-elliptic, and the Lax-Milgram Theorem then ensures that the variational problem (2.10) has a unique solution.

Finally, the proofs of the statements related with the equivalence between the variational formulation (2.10) and the boundary-value problem (2.5)-(2.6) are omitted, since they can be found in [10, Section 1.5]. This concludes the proof of Theorem 2.1.

## 3 An overview of the separation of variables for biharmonic equations

In this section we survey the method of separation of variables for the biharmonic equation

$$\Delta^2\psi = 0 \quad \text{in } \Lambda \doteq \{(y,z) \in \mathbb{R}^2 \mid y > 0 \text{ or } z > 0\} = \left\{(\rho,\theta) \in \mathbb{R}^2 \mid \rho > 0,\ -\frac{\pi}{2} < \theta < \pi\right\}. \tag{3.1}$$

We emphasize that, with an abuse of notation, $\Lambda$ represents the domain both in cartesian and polar coordinates. In polar coordinates, equation (3.1) becomes:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \psi(\rho,\theta) = 0 \qquad \forall(\rho,\theta) \in \Lambda. \tag{3.2}$$

We here seek a solution of (3.2) having a separated-variable form $\psi(\rho,\theta) = h(\rho)g(\theta)$, for every $(\rho,\theta) \in \Lambda$, for some differentiable functions $h : (0,\infty) \to \mathbb{R}$ and $g : (-\frac{\pi}{2},\pi) \to \mathbb{R}$. The complete description of the solutions of (3.2) in separated variables is contained in Theorem 3.1, the main result of this section. We initially follow the method employed by Stampouloglou-Theotokoglou [43], which extends the work by Michell [38], allowing the appearance of solutions of the biharmonic equation having oscillatory forms. Nevertheless, since in Theorem 3.1 we obtain more separated-variable solutions of (3.2) than in [43, Section 2], the whole (lengthy and delicate) proof is included for the sake of completeness.

**Theorem 3.1.** *Let $\psi : \Lambda \to \mathbb{R}$ be a solution of (3.2) having a separated-variable form $\psi(\rho,\theta) = h(\rho)g(\theta)$, for some smooth functions $h : (0,\infty) \to \mathbb{R}$ and $g : \left(-\frac{\pi}{2},\pi\right) \to \mathbb{R}$. Then, $g$ is a combination (sum, product or linear combination) of trigonometric functions, exponentials and polynomials and, therefore, it is globally bounded over $\left(-\frac{\pi}{2},\pi\right)$. Moreover, $h$ may take one of the following forms (for $\rho > 0$):*

$$\begin{cases} h(\rho) = \rho[C_1\rho^a + C_2\rho^{-a} + C_3\rho^b + C_4\rho^{-b}], \\ h(\rho) = C_1\rho^3 + C_2\rho^{-1} + \rho[C_3 + C_4\log(\rho)], \\ h(\rho) = C_1 + C_2\log(\rho) + \rho^2[C_3 + C_4\log(\rho)], \\ h(\rho) = \rho^2[C_1\cos(\mu\log(\rho)) + C_2\sin(\mu\log(\rho))] + C_3\cos(\mu\log(\rho)) + C_4\sin(\mu\log(\rho)), \\ h(\rho) = \rho[C_1\cos(\mu\log(\rho)) + C_2\sin(\mu\log(\rho))], \end{cases} \tag{3.3}$$

*for some $a, b, \mu > 0$ and some arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$.*

**Proof.** After replacing into (3.2) the ansatz $\psi(\rho,\theta) = h(\rho)g(\theta)$, we observe that, in order for the equation to be fulfilled, the following identity must be satisfied

$$g^{(4)}(\theta) + 2\chi(\rho)g''(\theta) + M(\rho)g(\theta) = 0 \qquad \forall(\rho,\theta) \in \Lambda, \tag{3.4}$$

where

$$\chi(\rho) \doteq \rho^2\frac{h''(\rho)}{h(\rho)} - \rho\frac{h'(\rho)}{h(\rho)} + 2, \qquad M(\rho) \doteq \rho^4\frac{h^{(4)}(\rho)}{h(\rho)} + 2\rho^3\frac{h'''(\rho)}{h(\rho)} - \rho^2\frac{h''(\rho)}{h(\rho)} + \rho\frac{h'(\rho)}{h(\rho)}, \tag{3.5}$$

for every $\rho \in (0,\infty)$ such that $h(\rho) \neq 0$. By differentiating (3.4) with respect to $\rho$, we obtain

$$2\chi'(\rho)g''(\theta) + M'(\rho)g(\theta) = 0 \qquad \forall(\rho,\theta) \in \Lambda. \tag{3.6}$$

At this point, we distinguish two cases.

• **Case (I):** $\chi'(\rho) \neq 0$**.** In this situation, assuming that the function $g$ is not identically null over $(-\frac{\pi}{2},\pi)$ and, after dividing equation (3.6) by $\chi'(\rho)g(\theta)$, we infer that

$$\frac{g''(\theta)}{g(\theta)} = -\frac{M'(\rho)}{2\chi'(\rho)} \qquad \forall(\rho,\theta) \in \Lambda, \tag{3.7}$$

so that there exists $\lambda \in \mathbb{R}$ such that

$$g''(\theta) - \lambda g(\theta) = 0 \qquad \forall\theta \in \left(-\frac{\pi}{2},\pi\right). \tag{3.8}$$

In this precise point our procedure differs from that in [43] since we do not assume that $-\lambda$ is a squared integer (a "physical number", see [43, Section 2]). The reason is that, since the angular region $\Lambda$ does not cover the full range $[0,2\pi]$ for $\theta$, the function $g$ may not be periodic. Nevertheless, (3.8) shows that $g$ is a linear combination of trigonometric functions, exponentials and first-order polynomials and, therefore, it is bounded over $(-\frac{\pi}{2},\pi)$.

Note that (3.8) implies both $g''(\theta) = \lambda g(\theta)$ and $g^{(4)}(\theta) = \lambda^2 g(\theta)$ which, inserted into (3.4), yields

$$M(\rho) + 2\lambda\chi(\rho) + \lambda^2 = 0 \qquad \forall\rho \in (0,\infty). \tag{3.9}$$

By combining (3.5) with (3.9) we obtain:

$$\rho^4 h^{(4)}(\rho) + 2\rho^3 h'''(\rho) + \rho^2(2\lambda - 1)h''(\rho) + \rho(1-2\lambda)h'(\rho) + \lambda(4+\lambda)h(\rho) = 0 \qquad \forall\rho \in (0,\infty). \tag{3.10}$$

After the change of variables $t = \log(\rho)$, this equation becomes

$$h^{(4)}(t) - 4h'''(t) + (2\lambda+4)h''(t) - 4\lambda h'(t) + \lambda(4+\lambda)h(t) = 0 \qquad \forall t \in \mathbb{R}. \tag{3.11}$$

For a given $\lambda \in \mathbb{R}$, the characteristic polynomial of the ODE (3.11) reads

$$P(z) = z^4 - 4z^3 + (2\lambda + 4)z^2 - 4\lambda z + \lambda(4 + \lambda) = (z-1)^4 + 2(\lambda - 1)(z-1)^2 + (\lambda + 1)^2$$

so that $P(z) = 0$ if and only if $(z-1)^2 = 1 - \lambda \pm 2\sqrt{-\lambda}$. In the following list, according to the sign of $\lambda$, the roots $z_1, z_2, z_3, z_4 \in \mathbb{C}$ of $P$ are computed, together with the explicit formula of the corresponding solution of the ODE (3.10).

- *Case (I.1):* $\lambda < 0$, $\lambda \neq -1$. Therefore $1 - \lambda \pm 2\sqrt{|\lambda|} > 0$, and $P$ has four real distinct roots, given by:

$$z_1 = 1 + \sqrt{1 - \lambda + 2\sqrt{|\lambda|}}, \quad z_2 = 1 - \sqrt{1 - \lambda + 2\sqrt{|\lambda|}},$$
$$z_3 = 1 + \sqrt{1 - \lambda - 2\sqrt{|\lambda|}}, \quad z_4 = 1 - \sqrt{1 - \lambda - 2\sqrt{|\lambda|}}.$$

Hence, the solutions of (3.10) are as in $(3.3)_1$ with $a \doteq \sqrt{1 - \lambda + 2\sqrt{|\lambda|}}$, $b \doteq \sqrt{1 - \lambda - 2\sqrt{|\lambda|}}$ and any constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$ such that $\chi' \neq 0$.

- *Case (I.2):* $\lambda = -1$. $P$ has the real roots: $z_1 = 3$, $z_2 = -1$ and $z_3 = z_4 = 1$. Accordingly, solutions of (3.10) are as in $(3.3)_2$ for some arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$ such that $\chi' \neq 0$.
- *Case (I.3):* $\lambda = 0$. $P$ has two real double roots: $z_1 = z_3 = 2$ and $z_2 = z_4 = 0$. Accordingly, solutions of (3.10) are as in $(3.3)_3$ for some arbitrary constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$ such that $\chi' \neq 0$.
- *Case (I.4):* $\lambda > 0$. $P$ has two pairs of complex-conjugate roots: $z_1 = 2 + i\sqrt{\lambda}$, $z_2 = 2 - i\sqrt{\lambda}$, $z_3 = i\sqrt{\lambda}$, $z_4 = -i\sqrt{\lambda}$. Accordingly, solutions of (3.10) are as in $(3.3)_4$, where $\mu = \sqrt{\lambda}$ and $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants such that $\chi' \neq 0$.

- **Case (II):** $\chi'(\rho) = 0$. Then $\chi$ is constant and, by (3.6), also $M$ is constant: $\chi(\rho) \equiv \alpha$ and $M(\rho) \equiv \beta$ for some $\alpha, \beta \in \mathbb{R}$. Hence, if $g : (-\frac{\pi}{2}, \pi) \to \mathbb{R}$ is not identically null, from (3.4) we infer

$$g^{(4)}(\theta) + 2\alpha g''(\theta) + \beta g(\theta) = 0 \qquad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right), \tag{3.12}$$

whose solutions are combinations (sum, product or linear combination) of trigonometric functions, exponentials and polynomials. Furthermore, from (3.5) we obtain the following equations:

$$\begin{cases} \rho^2 h''(\rho) - \rho h'(\rho) + (2 - \alpha)h(\rho) = 0 \\ \rho^4 h^{(4)}(\rho) + 2\rho^3 h'''(\rho) - \rho^2 h''(\rho) + \rho h'(\rho) - \beta h(\rho) = 0 \end{cases} \qquad \forall \rho \in (0, \infty), \tag{3.13}$$

that can be solved through the change of variables $t = \log(\rho)$. Equation $(3.13)_1$ yields the following families of solutions (for $\rho > 0$):

$$\alpha < 1 \implies h(\rho) = \rho[C_1 \cos(\sqrt{1-\alpha}\log(\rho)) + C_2 \sin(\sqrt{1-\alpha}\log(\rho))], \tag{3.14}$$
$$\alpha = 1 \implies h(\rho) = \rho[C_1 + C_2 \log(\rho)], \tag{3.15}$$
$$\alpha > 1 \implies h(\rho) = C_1 \rho^{1+\sqrt{\alpha-1}} + C_2 \rho^{1-\sqrt{\alpha-1}}, \tag{3.16}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants.

Concerning $(3.13)_2$, the change of variables $t = \log(\rho)$ leads to $h^{(4)}(t) - 4h'''(t) + 4h''(t) - \beta h(t) = 0$ ($t \in \mathbb{R}$) whose characteristic polynomial is

$$H(z) \doteq z^4 - 4z^3 + 4z^2 - \beta = (z-1)^4 - 2(z-1)^2 + 1 - \beta,$$

so that $H(z) = 0$ if and only if $(z-1)^2 = 1 \pm \sqrt{\beta}$. In the following list, according to the values of $\beta$, the roots $z_1, z_2, z_3, z_4 \in \mathbb{C}$ of $H$ are computed, together with the corresponding solutions of $(3.13)_2$.

• *Case (II.1):* $\beta < 0$. Here, $H$ has two pairs of complex-conjugate roots:

$$z_1 = 1 + \sqrt{\frac{\sqrt{1-\beta}+1}{2}} + i\sqrt{\frac{\sqrt{1-\beta}-1}{2}}, \quad z_2 = 1 - \sqrt{\frac{\sqrt{1-\beta}+1}{2}} - i\sqrt{\frac{\sqrt{1-\beta}-1}{2}},$$
$$z_3 = 1 + \sqrt{\frac{\sqrt{1-\beta}+1}{2}} - i\sqrt{\frac{\sqrt{1-\beta}-1}{2}}, \quad z_4 = 1 - \sqrt{\frac{\sqrt{1-\beta}+1}{2}} + i\sqrt{\frac{\sqrt{1-\beta}-1}{2}},$$

and the general solution of $(3.13)_2$ is:

$$h(\rho) = \rho^{1+a}[C_1\cos(\mu\log(\rho)) + C_2\sin(\mu\log(\rho))] + \rho^{1-a}[C_3\cos(\mu\log(\rho)) + C_4\sin(\mu\log(\rho))] \qquad \forall\rho > 0, \tag{3.17}$$

where $a \doteq \sqrt{\sqrt{1-\beta}+1}/\sqrt{2}$, $\mu \doteq \sqrt{\sqrt{1-\beta}-1}/\sqrt{2}$ and $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants.

• *Case (II.2):* $\beta = 0$. Here $H$ has two double real roots: $z_1 = z_3 = 2$ and $z_2 = z_4 = 0$. The general solution of $(3.13)_2$ is then:

$$h(\rho) = C_1 + C_2\log(\rho) + \rho^2[C_3 + C_4\log(\rho)] \qquad \forall\rho > 0, \tag{3.18}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants.

• *Case (II.3):* $0 < \beta < 1$. Therefore $1 \pm \sqrt{\beta} > 0$ and $H$ has four real distinct roots, given by:

$$z_1 = 1 + \sqrt{1+\sqrt{\beta}}, \quad z_2 = 1 - \sqrt{1+\sqrt{\beta}}, \quad z_3 = 1 + \sqrt{1-\sqrt{\beta}}, \quad z_4 = 1 - \sqrt{1-\sqrt{\beta}}.$$

Accordingly, the general solution of $(3.13)_2$ is:

$$h(\rho) = C_1\rho^{1+\sqrt{1+\sqrt{\beta}}} + C_2\rho^{1-\sqrt{1+\sqrt{\beta}}} + C_3\rho^{1+\sqrt{1-\sqrt{\beta}}} + C_4\rho^{1-\sqrt{1-\sqrt{\beta}}} \qquad \forall\rho > 0, \tag{3.19}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants.

• *Case (II.4):* $\beta = 1$. Here $H$ has the real roots: $z_1 = 1 + \sqrt{2}$, $z_2 = 1 - \sqrt{2}$ and $z_3 = z_4 = 1$. The general solution of $(3.13)_2$ is then:

$$h(\rho) = C_1\rho^{1+\sqrt{2}} + C_2\rho^{1-\sqrt{2}} + \rho[C_3 + C_4\log(\rho)] \qquad \forall\rho > 0, \tag{3.20}$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants.

• *Case (II.5):* $\beta > 1$. Since $1-\sqrt{\beta} < 0 < 1+\sqrt{\beta}$, in this case $H$ has two real and two complex-conjugate roots, given by:

$$z_1 = 1 + \sqrt{\sqrt{\beta}+1}, \quad z_2 = 1 - \sqrt{\sqrt{\beta}+1}, \quad z_3 = 1 + i\sqrt{\sqrt{\beta}-1}, \quad z_4 = 1 - i\sqrt{\sqrt{\beta}-1}.$$

Accordingly, the general solution of $(3.13)_2$ is:

$$h(\rho) = \rho[C_1\cos(\mu\log(\rho)) + C_2\sin(\mu\log(\rho))] + C_3\rho^{1+\sqrt{\sqrt{\beta}+1}} + C_4\rho^{1-\sqrt{\sqrt{\beta}+1}} \qquad \forall\rho > 0, \tag{3.21}$$

where $\mu \doteq \sqrt{\sqrt{\beta}-1}$ and $C_1, C_2, C_3, C_4 \in \mathbb{R}$ are arbitrary constants.

From (3.14) until (3.21), the functions that simultaneously solve both the equations in (3.13) are:
– functions (3.19) with $C_3 = C_4 = 0$ and $\alpha = 2 + \sqrt{\beta}$, or with $C_1 = C_2 = 0$ and $\alpha = 2 - \sqrt{\beta}$, that coincide with (3.16), a form included in $(3.3)_1$;
– functions (3.20) with $C_3 = C_4 = 0$, that coincide with (3.16) if $\alpha = 3$, a form included in $(3.3)_1$;
– functions (3.20) with $C_1 = C_2 = 0$, that coincide with (3.15), a form included in $(3.3)_2$;
– functions (3.21) with $C_1 = C_2 = 0$, that coincide with (3.16) if $\alpha = 2 + \sqrt{\beta}$, a form included in $(3.3)_1$;
– functions (3.21) with $C_3 = C_4 = 0$, that coincide with (3.14) if $\alpha = 2 - \sqrt{\beta} < 1$, giving $(3.3)_5$. □

# 4 Singular Stokes flows around a right angle

We consider here the stationary Stokes equations over the domain $\Lambda$ defined in (3.1):

$$-\eta\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Lambda. \tag{4.1}$$

The region $\Lambda \subset \mathbb{R}^2$ is open and simply-connected, with a Lipschitz boundary. The origin $O$ of the reference system lies in its corner, as depicted in Figure 4.1.

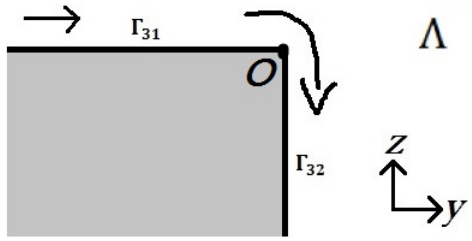


Figure 4.1: Schematic representation of the domain $\Lambda$ and of the expected flow pattern.

We denote by $(u^1(y,z), u^2(y,z))$ the components of the velocity field, so that the *scalar vorticity* $\omega : \Lambda \to \mathbb{R}$ is:

$$\omega(y,z) = \frac{\partial u^2}{\partial y}(y,z) - \frac{\partial u^1}{\partial z}(y,z) \quad \forall (y,z) \in \Lambda.$$

Since $\Lambda$ is simply-connected, the incompressibility condition implies the existence of a *stream function* $\psi : \Lambda \to \mathbb{R}$ such that

$$u^1(y,z) = \frac{\partial \psi}{\partial z}(y,z), \quad u^2(y,z) = -\frac{\partial \psi}{\partial y}(y,z), \quad \omega(y,z) = -\Delta\psi(y,z) \quad \forall (y,z) \in \Lambda. \tag{4.2}$$

Moreover, if we assume $f$ to be constant, the equation of conservation of momentum in (4.1) can be rewritten as the biharmonic equation (3.1) for the stream function $\psi$; see, e.g., [31, Chapter 2]. Then, the pressure can be found by solving $\nabla p = f + \eta\Delta u$. In the present section, (3.1) will be tackled using the separation of variables method developed in Section 3, with two main targets:
– to find the boundary conditions that could be imposed on the faces of the obstacle;
– to give a precise local description of the solution obtained with these boundary conditions.

Usually, the second target is the first step if one aims to propose a variational formulation of the Stokes equation (complemented with some boundary conditions) in a *weighted* Sobolev space, following the ideas contained in [35]. However, this is beyond the scopes of this article. Instead, we are here interested in classifying the solutions of (4.1), according to the following characterization:

**Definition 4.1.** *Let $u = (u^1, u^2)$ be a solution of (4.1) in $\mathcal{C}^2(\Lambda)^2$. We say that $u$ has a **separated-variable form** if its stream function $\psi : \Lambda \to \mathbb{R}$, defined by (4.2), has the form $\psi(\rho,\theta) = h(\rho)g(\theta)$, for some smooth functions $h : (0,\infty) \to \mathbb{R}$ and $g : \left(-\frac{\pi}{2}, \pi\right) \to \mathbb{R}$. We also say that $u$ is:*
- *a **physical solution** if $u \in L^\infty_{\mathrm{loc}}(\overline{\Lambda})^2$,*
- *a **finite-energy solution** if $u \in H^1_{\mathrm{loc}}(\overline{\Lambda})^2$,*
- *a **singular solution** if $u \in L^\infty_{\mathrm{loc}}(\overline{\Lambda})^2 \setminus H^1_{\mathrm{loc}}(\overline{\Lambda})^2$.*

We emphasize that there is a small abuse of language in Definition 4.1. If the stream function has the separated-variable form $\psi(\rho,\theta) = h(\rho)g(\theta)$ (in polar coordinates), for some smooth $h : (0,\infty) \to \mathbb{R}$ and $g : \left(-\frac{\pi}{2}, \pi\right) \to \mathbb{R}$, then the components of the velocity field can be recovered through (4.2):

$$u^1(\rho,\theta) = h'(\rho)g(\theta)\sin(\theta) + \frac{h(\rho)}{\rho}g'(\theta)\cos(\theta), \quad u^2(\rho,\theta) = -h'(\rho)g(\theta)\cos(\theta) + \frac{h(\rho)}{\rho}g'(\theta)\sin(\theta), \tag{4.3}$$

for $(\rho,\theta) \in \Lambda$. But, strictly speaking, (4.3) *do not* have a separated-variable form of the kind $H(\rho)G(\theta)$. However, in order to avoid more complicated definitions, we still call it in separated-form.

We point out that one has $H^1_{\rm loc}(\overline{\Lambda}) \subset L^p_{\rm loc}(\overline{\Lambda})$ for all $1 \leq p < \infty$, but the embedding $H^1_{\rm loc}(\overline{\Lambda}) \subset L^\infty_{\rm loc}(\overline{\Lambda})$ fails since $\Lambda$ is a planar domain. Therefore, not all finite-energy solutions will be physical solutions (nor the vice-versa). Since we have to deal with the singularity of vortices, we are here interested in singular solutions (physical infinite-energy solutions), namely bounded solutions of (4.1) with non-$L^2$ vorticity $\omega$. In view of (4.2), this also means that the stream function $\psi$ does not belong to $H^2_{\rm loc}(\overline{\Lambda})$. Together with Theorem 3.1, the following result is then obtained:

**Theorem 4.1.** *Consider the radial component $h : (0,\infty) \to \mathbb{R}$ of the stream function of a separated-variable solution $u$ of (4.1) in $\Lambda$. If $u$ is a singular solution, then $h$ is necessarily given by:*

$$h_{\mathcal{S}}(\rho) = \rho[C_1 \cos(\mu \log(\rho)) + C_2 \sin(\mu \log(\rho))] \qquad \forall \rho > 0, \tag{4.4}$$

*for some coefficient $\mu \geq 0$ and arbitrary constants $C_1, C_2 \in \mathbb{R}$.*

**Proof.** Let us write as $\psi(\rho,\theta) = h(\rho)g(\theta)$, for some smooth $h : (0,\infty) \to \mathbb{R}$ and $g : \left(-\frac{\pi}{2}, \pi\right) \to \mathbb{R}$, the stream function of a separated-variable solution $u$ of the Stokes equations (4.1) in $\Lambda$. From Theorem 3.1 we know that $h$ must have one of the forms in (3.3), while the function $g$ belongs to $C^\infty\left(-\frac{\pi}{2}, \pi\right)$.

If we require that $u \in L^\infty_{\rm loc}(\overline{\Lambda})^2$, identities (4.3) imply that

$$\limsup_{\rho \to 0} \left[ |h'(\rho)g(\theta)| + \frac{1}{\rho}|h(\rho)g'(\theta)| \right] < \infty. \tag{4.5}$$

When $g$ is not a constant function, (4.5) is equivalent to the condition

$$\limsup_{\rho \to 0} \left( |h'(\rho)| + \frac{|h(\rho)|}{\rho} \right) < \infty, \tag{4.6}$$

which immediately allows us to rule out the forms $\rho^{1-\alpha}$, $\log(\rho)$, $\rho\log(\rho)$, $\cos(\mu\log(\rho))$ and $\sin(\mu\log(\rho))$ (for any $\alpha, \mu > 0$) appearing in (3.3), since they violate (4.6). On the other hand, when $g$ is constant, there must exist $C_1, C_2, C_3, C_4 \in \mathbb{R}$ (see the proof of Theorem 3.1) such that $h(\rho) = C_1 + C_2\log(\rho) + \rho^2[C_3 + C_4\log(\rho)]$, $\forall \rho > 0$, which fulfills (4.5) if and only if $C_2 = 0$. Nevertheless, as we will see in the next item, if $g$ is constant and $h(\rho) = C_1 + \rho^2[C_3 + C_4\log(\rho)]$, then $u$ belongs to $H^1_{\rm loc}(\overline{\Lambda})^2$ so that it is not a singular solution.

If $u \notin H^1_{\rm loc}(\overline{\Lambda})^2$, then there exists $\delta > 0$ such that

$$\begin{aligned} &\int_{-\frac{\pi}{2}}^{\pi} \int_0^{\delta} \rho(|\nabla u^1(\rho,\theta)|^2 + |\nabla u^2(\rho,\theta)|^2)\, d\rho\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\pi} \int_0^{\delta} \rho \left[ h''(\rho)^2 g(\theta)^2 + 2\left(\frac{\rho h'(\rho) - h(\rho)}{\rho^2}\right)^2 g'(\theta)^2 + \left(\frac{h'(\rho)}{\rho}g(\theta) + \frac{h(\rho)}{\rho^2}g''(\theta)\right)^2 \right] d\rho\, d\theta = \infty. \end{aligned} \tag{4.7}$$

Then, the case when $g$ is constant and $h(\rho) = C_1 + \rho^2[C_3 + C_4\log(\rho)]$ is ruled out. Moreover, the condition

$$\int_0^r \left( \rho h''(\rho)^2 + \frac{h'(\rho)^2}{\rho} + \frac{h(\rho)^2}{\rho^3} \right) d\rho < \infty \qquad \forall r \in (0,\infty) \tag{4.8}$$

implies that $u \in H^1_{\rm loc}(\overline{\Lambda})^2$. This allows us to exclude the terms $\rho^{1+\alpha}$, $\rho^2\log(\rho)$, $\rho^2\cos(\mu\log(\rho))$ and $\rho^2\sin(\mu\log(\rho))$ (for any $\alpha, \mu > 0$) appearing in (3.3), since they verify (4.8).

Summarizing, the only expression in (3.3) not being ruled out by criteria (4.5)-(4.7) is precisely $h_{\mathcal{S}}$ in (4.4). In this case, we infer that $g$ cannot be constant, so that (4.6) may be used to show that $u$ belongs to $L^\infty_{loc}(\overline{\Lambda})^2$. Indeed, $h_\mathcal{S}(\rho)/\rho$ and $h'_\mathcal{S}(\rho)$ are bounded in any interval $(0, r)$ with finite $r > 0$ so that (4.6) is verified. Moreover, when $\mu > 0$ one has that

$$\int_0^1 \frac{\cos^2(\mu \log(\rho))}{\rho}\, d\rho = \int_0^{+\infty} \cos^2(\mu t)\, dt = \infty \qquad \text{and} \qquad \int_0^1 \frac{\sin^2(\mu \log(\rho))}{\rho}\, d\rho = \infty.$$

so that (4.7) is verified for $\delta = 1$, and then $u \notin H^1_{loc}(\overline{\Lambda})^2$ for any $\mu > 0$.

If $\mu = 0$, then (4.4) reduces to $h_\mathcal{S}(\rho) = C_1\rho$, for every $\rho > 0$, a function that satisfies condition (4.6) and that fulfills (4.7) for $\delta = 1$ only when the associated angular function $g_\mathcal{S}$ does not belong to the span of $\{\cos(\theta), \sin(\theta)\}$. Within the proof of Theorem 3.1, the form $h_\mathcal{S}(\rho) = C_1\rho$ yields $\alpha = \beta = 1$ in Case (II) and, accordingly, $g_\mathcal{S}$ solves the ODE

$$g^{(4)}_\mathcal{S}(\theta) + 2g''_\mathcal{S}(\theta) + g_\mathcal{S}(\theta) = 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right), \tag{4.9}$$

whose general solution is given by

$$g_\mathcal{S}(\theta) = Q_1\theta\cos(\theta) + Q_2\theta\sin(\theta) + Q_3\cos(\theta) + Q_4\sin(\theta) \quad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right), \tag{4.10}$$

for some constants $Q_1, Q_2, Q_3, Q_4 \in \mathbb{R}$. The resulting functions $\rho\cos(\theta)$ and $\rho\sin(\theta)$ are not singular so that we may choose $Q_3 = Q_4 = 0$. Then we conclude that $h_\mathcal{S}(\rho)g_\mathcal{S}(\theta)$ fulfills (4.7) and, therefore, $u \notin H^1_{loc}(\overline{\Lambda})^2$ also when $\mu = 0$. □

If we **drop the separated-variable assumption**, as a consequence of Theorem 4.1 we have

**Corollary 4.1.** *Any stream function of the kind*

$$\begin{aligned}\psi_\mathcal{S}(\rho, \theta) &= \rho[Q_1\theta\cos(\theta) + Q_2\theta\sin(\theta)] \\ &+ \sum_{k=1}^{\infty} \rho[C_1\cos(k\log(\rho)) + C_2\sin(k\log(\rho))][e^{k\theta}(A_1\cos(\theta) + A_2\sin(\theta)) + e^{-k\theta}(A_3\cos(\theta) + A_4\sin(\theta))],\end{aligned} \tag{4.11}$$

*for every $(\rho, \theta) \in \Lambda$ and arbitrary constants $Q_1, Q_2 \in \mathbb{R}$, yields a singular and non-separated variable solution $u$ of (4.1), provided that the constants $A_1, A_2, A_3, A_4, C_1, C_2 \in \mathbb{R}$ (that depend on $k \in \mathbb{N}$) are properly chosen in order to ensure the convergence of the series.*

**Proof.** From Theorem 4.1 we know that the radial component $h_\mathcal{S} : (0, \infty) \to \mathbb{R}$ of the stream function of a singular and separated-variable solution $u$ of (4.1) in $\Lambda$ is given by (4.4), for some coefficient $\mu \geq 0$ and arbitrary constants $C_1, C_2 \in \mathbb{R}$. Furthermore, the functions $\chi_\mathcal{S}$ and $M_\mathcal{S}$ in (3.5) are given by:

$$\chi_\mathcal{S}(\rho) = 1 - \mu^2, \qquad M_\mathcal{S}(\rho) = (1 + \mu^2)^2 \quad \forall \rho > 0.$$

That is, according to the proof of Theorem 3.1, (4.4) corresponds to the form $(3.3)_5$, which is obtained in Case (II) where $\chi'_\mathcal{S} \equiv 0$. In turn, the associated angular function $g_\mathcal{S}$ must satisfy (3.4), which reads:

$$g^{(4)}_\mathcal{S}(\theta) + 2(1 - \mu^2)g''_\mathcal{S}(\theta) + (1 + \mu^2)^2 g_\mathcal{S}(\theta) = 0 \quad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right). \tag{4.12}$$

When $\mu > 0$, all the solutions of (4.12) may be written as

$$g_\mathcal{S}(\theta) = e^{\mu\theta}[A_1\cos(\theta) + A_2\sin(\theta)] + e^{-\mu\theta}[A_3\cos(\theta) + A_4\sin(\theta)] \quad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right), \tag{4.13}$$

for some constants $A_1, A_2, A_3, A_4 \in \mathbb{R}$ that may depend on $\mu$. On the other hand, when $\mu = 0$, the general solution of (4.12) is given by (4.10), for some $Q_1, Q_2, Q_3, Q_4 \in \mathbb{R}$. However, since we are only interested in singular solutions of (4.1), we take $Q_3 = Q_4 = 0$, as in the proof of Theorem 4.1. Recall that, by (4.4), the angular functions (4.13) and (4.10) (with $Q_3 = Q_4 = 0$) can only be coupled to the radial functions $\rho[C_1\cos(\mu\log(\rho)) + C_2\sin(\mu\log(\rho))]$ and $\rho$, respectively. If we restrict ourselves to integer values of the coefficient $\mu$, this enables us to drop the separated-variable assumption and to find physical and infinite-energy solutions of (3.1) in the form (4.11). □

# 5 Some boundary conditions leading to vortices

In this section we show that some singular solutions of (4.1) can successfully describe, both from an analytical and physical point of view, the rather chaotic dynamics of the vortex shedding pattern described in the Introduction. By adapting the first boundary condition over $\Gamma_3$ in (2.6) to the "localized" configuration in $\Lambda$, we see that the first (resp. second) component of the velocity field must vanish over the vertical face $\Gamma_{32}$ (resp. horizontal face $\Gamma_{31}$):

$$u^2 = 0 \quad \text{on} \quad \Gamma_{31} \qquad \text{and} \qquad u^1 = 0 \quad \text{on} \quad \Gamma_{32}. \tag{5.1}$$

In the next two subsections we complement "by hand" (5.1) with further boundary conditions in order to build solutions of (4.1) displaying vortices. We will exhibit singular solutions $u \in \mathcal{C}^2(\overline{\Lambda} \setminus \{(0,0)\})^2$ whose streamlines qualitatively describe vortices.

## 5.1 Boundary conditions for laminar inflow

In this subsection we exhibit an example in which the boundary conditions satisfied by the singular solution of (3.1) (in separated variables) lead to the formation of vortices around the corner and over the leeward wall of the domain $\Lambda$. The mechanical description of the vortex shedding given in [40, Section 4.2.6] and [42] suggest that the flow should be laminar over $\Gamma_{31}$. Therefore, taking into account (5.1), we will seek solutions $u = (u^1, u^2)$ of (4.1) in $\Lambda$ verifying

$$u^2 = \omega = 0 \quad \text{on} \quad \Gamma_{31} \qquad \text{and} \qquad u^1 = 0, \ \omega = \omega_0 \quad \text{on} \quad \Gamma_{32}. \tag{5.2}$$

We point out that boundary conditions such as (5.2) were considered by Kwon-Kweon [30, Section 2] for $\omega_0 = 0$, while our choice will be different, see (5.13) below.

We take $h_{\mathcal{S}}$ in (4.4), with $C_1 = C_2 = \mu = 1$, as the radial component of the singular stream function. Correspondingly, the angular component $g_{\mathcal{S}}$ must satisfy (3.12) with $\alpha = 0$ and $\beta = 4$:

$$g_{\mathcal{S}}^{(4)}(\theta) + 4 g_{\mathcal{S}}(\theta) = 0 \qquad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right), \tag{5.3}$$

that is, there exist constants $A_1, A_2, A_3, A_4 \in \mathbb{R}$ such that

$$g_{\mathcal{S}}(\theta) = A_1 \cosh(\theta)\cos(\theta) + A_2 \cosh(\theta)\sin(\theta) + A_3 \sinh(\theta)\cos(\theta) + A_4 \sinh(\theta)\sin(\theta) \qquad \forall \theta \in \left(-\frac{\pi}{2}, \pi\right). \tag{5.4}$$

Therefore, we take $\psi_{\mathcal{S}}(\rho, \theta) \doteq h_{\mathcal{S}}(\rho) g_{\mathcal{S}}(\theta)$ as singular solution of (3.1) and we use (4.3) to compute

$$\begin{cases} u_{\mathcal{S}}^1(\rho,\theta) = \cos(\theta)[\cos(\log(\rho)) + \sin(\log(\rho))][(A_2 + A_3)\cos(\theta)\cosh(\theta) + (A_4 - A_1)\sin(\theta)\cosh(\theta)] \\ \qquad\qquad + \cos(\theta)[\cos(\log(\rho)) + \sin(\log(\rho))][(A_1 + A_4)\cos(\theta)\sinh(\theta) + (A_2 - A_3)\sin(\theta)\sinh(\theta)] \\ \qquad\qquad + 2\sin(\theta)\cos(\log(\rho))\{[A_1\cos(\theta) + A_2\sin(\theta)]\cosh(\theta) + [A_3\cos(\theta) + A_4\sin(\theta)]\sinh(\theta)\} \\ u_{\mathcal{S}}^2(\rho,\theta) = \sin(\theta)[\cos(\log(\rho)) + \sin(\log(\rho))][(A_2 + A_3)\cos(\theta)\cosh(\theta) + (A_4 - A_1)\sin(\theta)\cosh(\theta)] \\ \qquad\qquad + \sin(\theta)[\cos(\log(\rho)) + \sin(\log(\rho))][(A_1 + A_4)\cos(\theta)\sinh(\theta) + (A_2 - A_3)\sin(\theta)\sinh(\theta)] \\ \qquad\qquad - 2\cos(\theta)\cos(\log(\rho))\{[A_1\cos(\theta) + A_2\sin(\theta)]\cosh(\theta) + [A_3\cos(\theta) + A_4\sin(\theta)]\sinh(\theta)\}, \end{cases} \tag{5.5}$$

for $(\rho, \theta) \in \Lambda$. Due to (5.1), the constants $A_1, A_2, A_3, A_4$ must be chosen in such a way that

$$\begin{cases} u_{\mathcal{S}}^1\left(\rho, -\frac{\pi}{2}\right) = 2\cos(\log(\rho))\left[A_2\cosh\left(\frac{\pi}{2}\right) - A_4\sinh\left(\frac{\pi}{2}\right)\right] = 0 \\ u_{\mathcal{S}}^2(\rho, \pi) = -2\cos(\log(\rho))[A_1\cosh(\pi) + A_3\sinh(\pi)] = 0, \end{cases} \tag{5.6}$$

for $\rho > 0$. Moreover, for the scalar vorticity we have

$$\begin{aligned}\omega_{\mathcal{S}}(\rho,\theta) = -\Delta\psi_{\mathcal{S}}(\rho,\theta) = &-\frac{2}{\rho}\cosh(\theta)\sin(\theta)[(A_2-A_3)\cos(\log(\rho)) - (A_2+A_3)\sin(\log(\rho))]\\ &-\frac{2}{\rho}\cosh(\theta)\cos(\theta)[(A_1+A_4)\cos(\log(\rho)) + (A_4-A_1)\sin(\log(\rho))]\\ &-\frac{2}{\rho}\sinh(\theta)\cos(\theta)[(A_2+A_3)\cos(\log(\rho)) + (A_2-A_3)\sin(\log(\rho))]\\ &+\frac{2}{\rho}\sinh(\theta)\sin(\theta)[(A_1-A_4)\cos(\log(\rho)) + (A_1+A_4)\sin(\log(\rho))],\end{aligned} \tag{5.7}$$

for $(\rho,\theta) \in \Lambda$. In view of (5.2), the constants $A_1, A_2, A_3, A_4$ must also satisfy

$$\begin{aligned}\omega_{\mathcal{S}}(\rho,\pi) = &\frac{2}{\rho}\sin(\log(\rho))[(A_4-A_1)\cosh(\pi) + (A_2-A_3)\sinh(\pi)]\\ &+\frac{2}{\rho}\cos(\log(\rho))[(A_1+A_4)\cosh(\pi) + (A_2+A_3)\sinh(\pi)] = 0,\end{aligned} \tag{5.8}$$

for $\rho > 0$. Thus, after combining (5.6) and (5.8), we infer that the constants must satisfy the following algebraic system in matrix form:

$$\begin{bmatrix} \cosh(\pi) & 0 & \sinh(\pi) & 0 \\ -\cosh(\pi) & \sinh(\pi) & -\sinh(\pi) & \cosh(\pi) \\ \cosh(\pi) & \sinh(\pi) & \sinh(\pi) & \cosh(\pi) \\ 0 & \cosh\left(\frac{\pi}{2}\right) & 0 & -\sinh\left(\frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.9}$$

Notice that the matrix of the left-hand side of (5.9) is singular, since the first and third columns are proportional. Consequently, system (5.9) has infinitely many solutions (as expected, because we are only imposing three boundary conditions out of possible four) given by

$$A_2 = A_4 = 0\,, \qquad A_1 = -A_3\tanh(\pi)\,.$$

The resulting expressions for the singular stream function, velocity field and vorticity are given below.

- **Stream function:** $\psi_{\mathcal{S}}(\rho,\theta) = A\cos(\theta)\cosh(\theta)[\tanh(\theta) - \tanh(\pi)]\cdot\rho[\cos(\log(\rho)) + \sin(\log(\rho))]$, for $(\rho,\theta) \in \Lambda$ and any constant $A \in \mathbb{R}$.

- **Components of the velocity field:**

$$\begin{cases} u^1_{\mathcal{S}}(\rho,\theta) = A\cosh(\theta)\cos(\theta)[\cos(\theta) + \tanh(\pi)\sin(\theta)][\cos(\log(\rho)) + \sin(\log(\rho))] \\ \qquad\qquad - A\sinh(\theta)\cos(\theta)[\sin(\theta) + \tanh(\pi)\cos(\theta)][\cos(\log(\rho)) + \sin(\log(\rho))] \\ \qquad\qquad - 2A\cos(\theta)\sin(\theta)\dfrac{\sinh(\pi-\theta)}{\cosh(\pi)}\cos(\log(\rho)) \\ u^2_{\mathcal{S}}(\rho,\theta) = A\cosh(\theta)\sin(\theta)[\cos(\theta) + \tanh(\pi)\sin(\theta)][\cos(\log(\rho)) + \sin(\log(\rho))] \\ \qquad\qquad - A\sinh(\theta)\sin(\theta)[\sin(\theta) + \tanh(\pi)\cos(\theta)][\cos(\log(\rho)) + \sin(\log(\rho))] \\ \qquad\qquad + 2A\cos^2(\theta)\dfrac{\sinh(\pi-\theta)}{\cosh(\pi)}\cos(\log(\rho)), \end{cases} \tag{5.10}$$

  for $(\rho,\theta) \in \Lambda$ and any constant $A \in \mathbb{R}$. Notice that $(u^1_{\mathcal{S}}, u^2_{\mathcal{S}}) \in \mathcal{C}^2(\overline{\Lambda} \setminus \{(0,0)\})^2$.

- **Vorticity:**

$$\begin{aligned}\omega_{\mathcal{S}}(\rho,\theta) = &\frac{2A}{\rho}\sinh(\theta)\{\cos(\theta)[\sin(\log(\rho)) - \cos(\log(\rho))] - \tanh(\pi)\sin(\theta)[\sin(\log(\rho)) + \cos(\log(\rho))]\}\\ &+\frac{2A}{\rho}\cosh(\theta)\{[\sin(\theta) - \tanh(\pi)\cos(\theta)]\sin(\log(\rho)) + [\sin(\theta) + \tanh(\pi)\cos(\theta)]\cos(\log(\rho))\},\end{aligned} \tag{5.11}$$

for $(\rho,\theta)\in\Lambda$ and any constant $A\in\mathbb{R}$. As expected, the vorticity is not in $L^2(\Lambda)$. In particular, the restriction of the vorticity field to the vertical face $\Gamma_{32}$ is given by:

$$\omega_{\mathcal{S}}\left(\rho,-\frac{\pi}{2}\right)=-\frac{2A}{\rho}\frac{\cosh\left(\frac{3\pi}{2}\right)}{\cosh(\pi)}[\sin(\log(\rho))+\cos(\log(\rho))]\qquad\forall\rho>0. \tag{5.12}$$

Therefore, by selecting the value of $A=A_0\doteq-\frac{1}{2}\frac{\cosh(\pi)}{\cosh\left(\frac{3\pi}{2}\right)}$, we infer that the velocity field $u_{\mathcal{S}}$ in (5.10) satisfies (4.1) in $\Lambda$ and the boundary conditions (5.2) with

$$\omega_0(\rho)\doteq\frac{1}{\rho}[\sin(\log(\rho))+\cos(\log(\rho))]\qquad\forall\rho>0. \tag{5.13}$$

A contour plot of $\omega_{\mathcal{S}}$ in (5.11), with $A=A_0$, is presented in Figure 5.1, where such quantity is considered in a disk of radius 0.005 around the corner (compare with the *isovorticity* plots obtained in [15]).

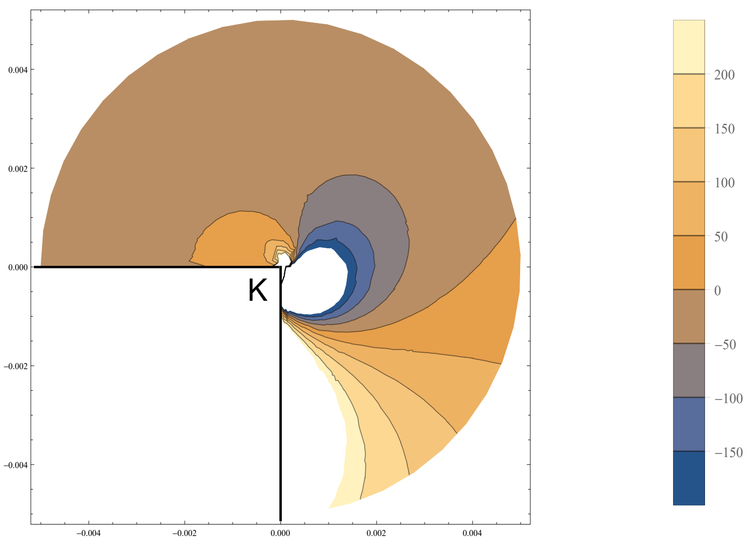


Figure 5.1: Contour plot of $\omega_{\mathcal{S}}$ in (5.11) for $A=A_0$, on a disk of radius 0.005 around the corner.

Furthermore, the streamline plot of the velocity field (5.10) (with $A=A_0$) in Figure 5.2 displays a noticeable vortex pattern around the corner (to be compared with [15, Figure 12] or [44, Figure 2]).

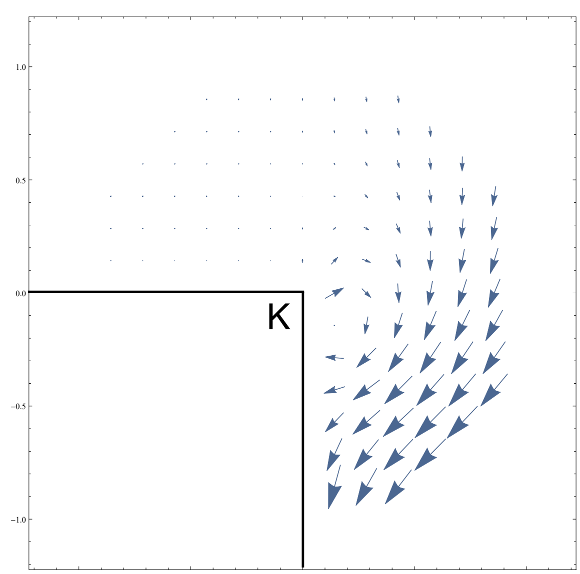


Figure 5.2: Streamline plot of $u_{\mathcal{S}}=(u^1_{\mathcal{S}},u^2_{\mathcal{S}})$ in (5.10) for $A=A_0$, on a disk of unitary radius.

### 5.2 Boundary conditions with oriented velocity

A further natural condition over the horizontal face $\Gamma_{31}$ concerns the first component of the velocity field, which must have a positive sign in order to follow the inflow direction. In [20] (see also [30, Corollary 1.1]) such positivity condition is imposed on some given part of the boundary as a constraint for a related drag-minimization problem. Therefore, given two functions $\xi : \Gamma_{31} \to (0, \infty)$ and $\omega_0 : \Gamma_{32} \to \mathbb{R}$, and taking into account (5.1), here we seek solutions $u = (u^1, u^2)$ of (4.1) in $\Lambda$ verifying

$$u^1 = \xi, \; u^2 = 0 \quad \text{on} \;\; \Gamma_{31} \qquad \text{and} \qquad u^1 = 0, \; \omega = \omega_0 \quad \text{on} \;\; \Gamma_{32}. \tag{5.14}$$

Explicit forms for $\xi$ and $\omega_0$ will be given in (5.27), in order to fit singular solutions of (4.1) into the boundary conditions (5.14). To this end, we need first to "adjust" the general expression (4.11) with some regular solution of (4.1), according to the following definition:

**Definition 5.1.** *We say that a solution $u$ of* (4.1) *is* **regular** *if $u \in C^2(\overline{\Lambda})^2$.*

Clearly, a regular solution is associated to a stream function $\psi \in C^4(\overline{\Lambda})$. Following [28], we then write the solutions of (3.2) as

$$\psi(\rho, \theta) = \psi_{\mathcal{S}}(\rho, \theta) + \psi_{\mathcal{R}}(\rho, \theta) \qquad \forall (\rho, \theta) \in \Lambda, \tag{5.15}$$

where $\psi_{\mathcal{S}}$ is as in (4.11) and $\psi_{\mathcal{R}} : \Lambda \to \mathbb{R}$ denotes the regular component of the stream function. The regular part $\psi_{\mathcal{R}}$ in (5.15) has much more freedom, since it only needs to "balance" $\psi_{\mathcal{S}}$ in order to match (5.14). Therefore, we consider a very simple form, we take $\psi_{\mathcal{R}}(y, z) = Ay + Bz$, for some $A, B \in \mathbb{R}$. Clearly, $\Delta^2 \psi_{\mathcal{R}} = 0$ in $\Lambda$ and

$$u^1_{\mathcal{R}}(y, z) = B \,, \quad u^2_{\mathcal{R}}(y, z) = -A, \quad \omega_{\mathcal{R}}(y, z) = 0 \qquad \forall (y, z) \in \overline{\Lambda}. \tag{5.16}$$

Notice that the dependence with respect to $\rho$ in the series (4.11) does not vanish at the boundaries $\Gamma_{31}$ and $\Gamma_{32}$. Therefore, such series cannot be equalized to a constant different from zero, and the expression $\rho[Q_1 \theta \cos(\theta) + Q_2 \theta \sin(\theta)]$ needs to act as the counterbalance part. After imposing the boundary conditions (5.14), and in view of (4.11)-(5.15)-(5.16), we see that

- $u^1\left(\rho, -\frac{\pi}{2}\right) = u^1_{\mathcal{R}}\left(\rho, -\frac{\pi}{2}\right) + u^1_{\mathcal{S}}\left(\rho, -\frac{\pi}{2}\right) = 0$, for $\rho > 0$, that is

$$B - \frac{\pi}{2} Q_2 + \sum_{k=1}^{\infty} e^{-\frac{k\pi}{2}} (A_2 + e^{k\pi} A_4)[(C_1 + kC_2)\cos(k \log(\rho)) + (C_2 - kC_1)\sin(k \log(\rho))] = 0 \qquad \forall \rho > 0. \tag{5.17}$$

  This implies that $B = \dfrac{\pi}{2} Q_2$ and $A_2(k) = -e^{k\pi} A_4(k)$, for any integer $k \geq 1$.

- $u^2(\rho, \pi) = u^2_{\mathcal{R}}(\rho, \pi) + u^2_{\mathcal{S}}(\rho, \pi) = 0$, for $\rho > 0$, that is

$$-A - \pi Q_1 - \sum_{k=1}^{\infty} e^{-k\pi} (A_3 + e^{2k\pi} A_1)[(C_1 + kC_2)\cos(k \log(\rho)) + (C_2 - kC_1)\sin(k \log(\rho))] = 0 \qquad \forall \rho > 0. \tag{5.18}$$

  This implies that $A = -\pi Q_1$ and $A_3(k) = -e^{2k\pi} A_1(k)$, for any integer $k \geq 1$.

- Regarding the positivity condition in (5.14), the relations derived from (5.17)-(5.18) imply that:

$$\begin{aligned} u^1(\rho, \pi) &= u^1_{\mathcal{R}}(\rho, \pi) + u^1_{\mathcal{S}}(\rho, \pi) \\ &= -\frac{A}{\pi} + 3B + \sum_{k=1}^{\infty} [(e^{-k\pi} - e^{2k\pi}) A_4(k) + 2k e^{k\pi} A_1(k)][C_1(k)\cos(k \log(\rho)) + C_2(k)\sin(k \log(\rho))], \end{aligned} \tag{5.19}$$

  for $\rho > 0$. Now, as stated in [41, Supplement 2], for every $a \in (-1, 1)$ and $x \in \mathbb{R}$ we have the following series

$$\sum_{k=0}^{\infty} a^k \cos(kx) = \frac{1 - a\cos(x)}{1 - 2a\cos(x) + a^2}, \qquad \sum_{k=0}^{\infty} a^k \sin(kx) = \frac{a\sin(x)}{1 - 2a\cos(x) + a^2},$$

and the sum of the first series is a strictly positive function in $\mathbb{R}$. This leads us to impose $-\frac{A}{\pi}+3B=1$ and to select the following values for the sequences of constants appearing in (5.19):

$$A_1(k)=\frac{e^{-3k\pi}}{2k},\quad A_4(k)=C_2(k)=0\quad\text{and}\quad C_1(k)=1\quad \forall k\geq 1,\ k\in\mathbb{N}. \tag{5.20}$$

After inserting (5.20) into (5.19) we obtain:

$$u^1(\rho,\pi)=1+\sum_{k=1}^{\infty}e^{-2k\pi}\cos(k\log(\rho))=\frac{1-e^{-2\pi}\cos(\log(\rho))}{1-2e^{-2\pi}\cos(\log(\rho))+e^{-4\pi}}\qquad\forall\rho>0. \tag{5.21}$$

As a particular case, we choose $A=2\pi$ and $B=1$, so that $Q_1=-2$ and $Q_2=2/\pi$. The resulting expressions for the stream function, velocity and vorticity fields are computed below.

- **Stream function:**

$$\psi(\rho,\theta)=\rho\left[2(\pi-\theta)\cos(\theta)+\left(\frac{2\theta}{\pi}+1\right)\sin(\theta)\right]+\frac{\rho}{2}\cos(\theta)\sum_{k=1}^{\infty}\frac{1}{k}[e^{-k(3\pi-\theta)}-e^{-k(\theta+\pi)}]\cos(k\log(\rho)), \tag{5.22}$$

for $(\rho,\theta)\in\Lambda$. Notice that the interval of definition of $\theta$ ensures the convergence of the series of functions in (5.22).

- **First component of the velocity field:**

$$\begin{aligned}u^1(\rho,\theta)&=\frac{1}{\pi}[2\theta+\sin(2\theta)]-\cos(2\theta)\\&+\frac{1}{2}\cos^2(\theta)\left[\frac{1-e^{-(\theta+\pi)}\cos(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}+\frac{1-e^{(\theta-3\pi)}\cos(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}-2\right]\\&+\frac{1}{2}\cos(\theta)\sin(\theta)\left[\frac{e^{-(\theta+\pi)}\sin(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}-\frac{e^{(\theta-3\pi)}\sin(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}\right],\end{aligned} \tag{5.23}$$

for $(\rho,\theta)\in\Lambda$.

- **Second component of the velocity field:**

$$\begin{aligned}u^2(\rho,\theta)&=2(\theta-\pi)-\sin(2\theta)+\frac{1}{\pi}[1-\cos(2\theta)]+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}[e^{-k(\theta+\pi)}-e^{-k(3\pi-\theta)}]\cos(k\log(\rho))\\&+\frac{1}{2}\cos(\theta)\sin(\theta)\left[\frac{1-e^{-(\theta+\pi)}\cos(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}+\frac{1-e^{(\theta-3\pi)}\cos(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}-2\right]\\&-\frac{1}{2}\cos^2(\theta)\left[\frac{e^{-(\theta+\pi)}\sin(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}-\frac{e^{(\theta-3\pi)}\sin(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}\right],\end{aligned} \tag{5.24}$$

for $(\rho,\theta)\in\Lambda$. Notice again that $(u^1,u^2)\in\mathcal{C}^2(\overline{\Lambda}\setminus\{(0,0)\})^2$.

- **Scalar vorticity:**

$$\begin{aligned}\omega(\rho,\theta)&=-\frac{4}{\pi\rho}[\cos(\theta)+\pi\sin(\theta)]\\&+\frac{1}{\rho}\sin(\theta)\left[\frac{1-e^{-(\theta+\pi)}\cos(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}+\frac{1-e^{(\theta-3\pi)}\cos(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}-2\right]\\&-\frac{1}{\rho}\cos(\theta)\left[\frac{e^{-(\theta+\pi)}\sin(\log(\rho))}{1-2e^{-(\theta+\pi)}\cos(\log(\rho))+e^{-2(\theta+\pi)}}-\frac{e^{(\theta-3\pi)}\sin(\log(\rho))}{1-2e^{(\theta-3\pi)}\cos(\log(\rho))+e^{2(\theta-3\pi)}}\right],\end{aligned} \tag{5.25}$$

for $(\rho,\theta)\in\Lambda$. As expected, notice that the vorticity is clearly not in $L^2(\Lambda)$. In particular, the restriction of the vorticity to the vertical face $\Gamma_{32}$ is given by:

$$\omega\left(\rho,-\frac{\pi}{2}\right)=\frac{6}{\rho}-\frac{1}{\rho}\left[\frac{1-e^{-\frac{\pi}{2}}\cos(\log(\rho))}{1-2e^{-\frac{\pi}{2}}\cos(\log(\rho))+e^{-\pi}}+\frac{1-e^{-\frac{7\pi}{2}}\cos(\log(\rho))}{1-2e^{-\frac{7\pi}{2}}\cos(\log(\rho))+e^{-7\pi}}\right]\quad\forall\rho>0. \tag{5.26}$$

As a consequence of these computations, we infer that the $\psi$ (5.22) is a biharmonic function in $\Lambda$, whose velocity field (5.23)-(5.24) satisfies (4.1) in $\Lambda$ and (5.14) with:

$$\begin{cases} \xi(\rho) \doteq \dfrac{1-e^{-2\pi}\cos(\log(\rho))}{1-2e^{-2\pi}\cos(\log(\rho))+e^{-4\pi}}, \\ \omega_0(\rho) \doteq \dfrac{1}{\rho}\left[6-\dfrac{1-e^{-\frac{\pi}{2}}\cos(\log(\rho))}{1-2e^{-\frac{\pi}{2}}\cos(\log(\rho))+e^{-\pi}}-\dfrac{1-e^{-\frac{7\pi}{2}}\cos(\log(\rho))}{1-2e^{-\frac{7\pi}{2}}\cos(\log(\rho))+e^{-7\pi}}\right], \end{cases} \tag{5.27}$$

for $\rho > 0$. A contour plot of the vorticity (5.25) is presented in Figure 5.3. The chaotic dynamics of the vortex shedding process are properly illustrated and characterized by the increasing values of the vorticity (to be compared with Figure 5.1 and the *isovorticity* plots obtained in [15]).

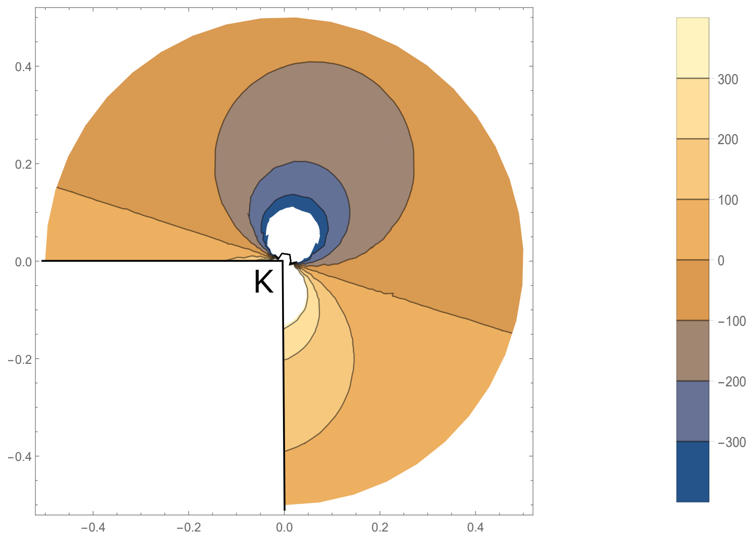


Figure 5.3: Contour plot of $\omega$ in (5.25) on a disk of radius 0.5 around the corner.

On the other hand, a streamline plot of the velocity field (5.23)-(5.24) (where the series of functions in (5.24) was numerically approximated with the sum of the ten first terms) in Figure 5.4 reproduces some of the typical geometrical patterns induced by low-Reynolds-number flows past square cylinders (see [15, Figure 10] or [44, Figure 5]).

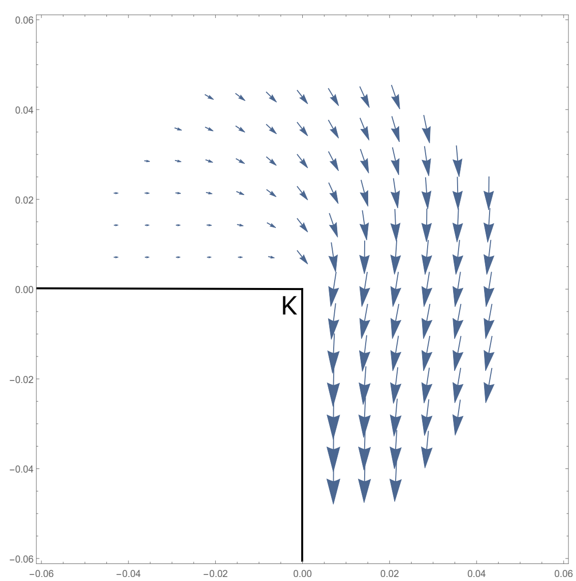


Figure 5.4: Streamline plot of $u = (u^1, u^2)$ in (5.23)-(5.24) on a disk of radius 0.05 around the corner.

**Acknowledgements.** The Authors are grateful to Andrei Fursikov (Moscow State University) for his valuable comments on a preliminary version of the present paper. The first Author is partially supported by the PRIN project *Equazioni alle derivate parziali di tipo ellittico e parabolico: aspetti geometrici, disuguaglianze collegate, e applicazioni* and by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

## References

[1] G. Alekseev. Mixed boundary value problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid. *Journal of Mathematical Fluid Mechanics*, 18(3):591–607, 2016.

[2] H. Bellout, J. Neustupa, and P. Penel. On the Navier-Stokes equation with boundary conditions based on vorticity. *Mathematische Nachrichten*, 269(1):59–72, 2004.

[3] C. Bègue, C. Conca, F. Murat, and O. Pironneau. A nouveau sur les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. *Comptes Rendus de l'Académie des Sciences. Série I, Mathématique*, 304(1):23–28, 1987.

[4] C. Bègue, C. Conca, F. Murat, and O. Pironneau. Les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. In *Nonlinear Partial Differential equations and their Applications, Collège de France Seminar*, volume 9, pages 179–264. Pitman, 1988.

[5] M. Boghosian and K. Cassel. On the origins of vortex shedding in two-dimensional incompressible flows. *Theoretical and Computational Fluid Dynamics*, 30(6):511–527, 2016.

[6] D. Bonheure, F. Gazzola, and E. Moreira dos Santos. Mathematical study of the stability of suspended bridges: focus on the fluid-structure interactions. *Research proposal FRB 2018-J1150080-210411 BONHEURE, submitted to the Thelam Fund (Fondation Roi Baudouin, Belgium)*, Available online at: http://homepages.ulb.ac.be/∼ dbonheur/Thelam.pdf, March 2018.

[7] M. Borsuk and V. Kondrat'ev. *Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains*, volume 69. Elsevier, 2006.

[8] D. Bucur, E. Feireisl, and S. Nečasová. Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions. *Arch. Ration. Mech. Anal.*, 197:117–138, 2010.

[9] T. Chacón Rebollo, V. Girault, F. Murat, and O. Pironneau. Analysis of a coupled fluid-structure model with applications to hemodynamics. *SIAM Journal on Numerical Analysis*, 54(2):994–1019, 2016.

[10] C. Conca, F. Murat, and O. Pironneau. The Stokes and Navier - Stokes equations with boundary conditions involving the pressure. *Japanese Journal of Mathematics. New series*, 20(2):279–318, 1994.

[11] C. Conca, J. San Martín, and M. Tucsnak. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Communications in Partial Differential Equations*, 25(5-6):1019–1042, 2000.

[12] C. D. Coster, S. Nicaise, and G. Sweers. Comparing variational methods for the hinged Kirchhoff plate with corners. To appear in *Mathematische Nachrichten.*

[13] D. G. Crowdy and S. J. Brzezicki. Analytical solutions for two-dimensional Stokes flow singularities in a no-slip wedge of arbitrary angle. *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, 473(2202):20170134, 2017.

[14] H. B. da Veiga and L. C. Berselli. Navier-Stokes equations: Green's matrices, vorticity direction, and regularity up to the boundary. *Journal of Differential Equations*, 246(2):597–628, 2009.

[15] R. W. Davis and E. Moore. A numerical study of vortex shedding from rectangles. *Journal of Fluid Mechanics*, 116:475–506, 1982.

[16] G. Diana, F. Resta, M. Belloli, and D. Rocchi. On the vortex shedding forcing on suspension bridge deck. *J. Wind Engineering and Industrial Aerodynamics*, 94:341–363, 2006.

[17] C. Foias and R. Temam. Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 5(1):29–63, 1978.

[18] S. Franzetti, M. Greco, S. Malavasi, and D. Mirauda. Flow induced excitation on basic shape structures. In *Vorticity and Turbulence Effects in Fluid Structure Interaction - An Application to Hydraulic Structure Design*, chapter 6, pages 131–156. WIT Press, 2006.

[19] V. Fuka and J. Brechler. Large eddy simulation of the stable boundary layer. In *Finite Volumes for Complex Applications VI - Problems & Perspectives*, pages 485–493. Springer, 2011. http://artax.karlin.mff.cuni.cz/~fukav1am/sqcyl.html.

[20] A. Fursikov and R. Rannacher. Optimal Neumann control for the two-dimensional steady-state Navier-Stokes equations. In *New Directions in Mathematical Fluid Mechanics*, pages 193–221. Springer, 2009.

[21] F. Gazzola. On a decomposition of the Hilbert space $L^2$ and its applications to Stokes problem. *Annali Univ. Ferrara*, 41(1):95–115, 1995.

[22] F. Gazzola and P. Secchi. Inflow-outflow problems for Euler equations in a rectangular cylinder. *Nonlinear Differential Equations and Applications*, 8:195–217, 2001.

[23] F. Gazzola and G. Sperone. Navier-Stokes equations interacting with plate equations. *Annual Report of the Politecnico di Milano PhD School*, 2017.

[24] V. Girault. Curl-conforming finite element methods for Navier-Stokes equations with non-standard boundary conditions in $\mathbb{R}^3$. In *The Navier-Stokes Equations: Theory and Numerical Methods*, pages 201–218. Springer, 1990.

[25] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, volume 5. Springer Science & Business Media, 2012.

[26] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman Advanced Publishing Program, 1985.

[27] S. Hansen. Vortex-induced vibrations of structures. In *Structural Engineers World Congress 2007, Bangalore, India*, pages 2–7, 2007. http://www.eurocodes.fi/1991/1991-1-4/background/Hansen_2007.pdf.

[28] V. A. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskovskogo Matematicheskogo Obshchestva*, 16:209–292, 1967.

[29] V. A. Kondrat'ev and O. A. Oleinik. Boundary-value problems for partial differential equations in non-smooth domains. *Russian Mathematical Surveys*, 38(2):1–86, 1983.

[30] O. S. Kwon and J. R. Kweon. For the vorticity–velocity–pressure form of the Navier-Stokes equations on a bounded plane domain with corners. *Nonlinear Analysis: Theory, Methods & Applications*, 75(5):2936–2956, 2012.

[31] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*, volume 76. Gordon and Breach New York, 1969.

[32] L. Landau and E. Lifshitz. *Theoretical Physics: Fluid Mechanics*, volume 6. Pergamon Press, 1987.

[33] A. Majda and A. Bertozzi. *Vorticity and Incompressible Flow*, volume 27. Cambridge University Press, 2002.

[34] E. Marušić-Paloka. Rigorous justification of the Kirchhoff law for junction of thin pipes filled with viscous fluid. *Asymptotic Analysis*, 33(1):51–66, 2003.

[35] V. Maz'ya and J. Rossmann. *Elliptic Equations in Polyhedral Domains*. Number 162. American Mathematical Society, 2010.

[36] V. Meleshko. Biharmonic problem in a rectangle. In *In Fascination of Fluid Dynamics*, pages 217–249. Springer, 1998.

[37] V. Meleshko. Selected topics in the history of the two-dimensional biharmonic problem. *Applied Mechanics Reviews*, 56(1):33–85, 2003.

[38] J. Michell. On the direct determination of stress in an elastic solid, with application to the theory of plates. *Proceedings of the London Mathematical Society*, 1(1):100–124, 1899.

[39] S. A. Nazarov, A. Stylianou, and G. Sweers. Hinged and supported plates with corners. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 63:929–960, 2012.

[40] H. Oertel. *Prandtl's Essentials of Fluid Mechanics*. Springer Science & Business Media, 2004.

[41] A. D. Polyanin and A. V. Manzhirov. *Handbook of Integral Equations*. Chapman & Hall/CRC, 2008.

[42] G. Schewe. Reynolds-number-effects in flow around a rectangular cylinder with aspect ratio 1:5. *Journal of Fluids and Structures*, 39:15–26, 2013.

[43] I. Stampouloglou and E. Theotokoglou. Additional separated-variable solutions of the biharmonic equation in polar coordinates. *Journal of Applied Mechanics*, 77(2):021003, 2010.

[44] S. C. Yen and C. W. Yang. Flow patterns and vortex shedding behavior behind a square cylinder. *Journal of Wind Engineering and Industrial Aerodynamics*, 99(8):868–878, 2011.

Filippo Gazzola and Gianmarco Sperone
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci 32
20133 Milano
Italy
**E-mail**: filippo.gazzola@polimi.it, gianmarcosilvio.sperone@polimi.it.