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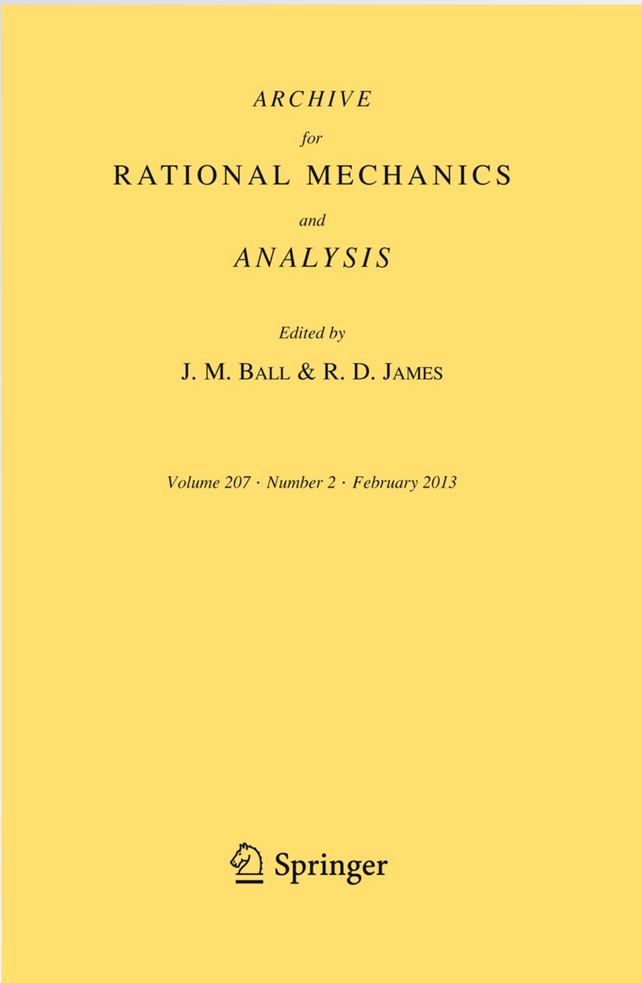
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Wide Oscillation Finite Time Blow Up for Solutions to Nonlinear Fourth Order Differential Equations

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Abstract

We give sufficient conditions for local solutions to some fourth order semilinear ordinary differential equations to blow up in finite time with wide oscillations, a phenomenon not visible for lower order equations. The result is then applied to several classes of semilinear partial differential equations in order to characterize the blow up of solutions including, in particular, its applications to a suspension bridge model. We also give numerical results which describe this oscillating blow up and allow us to suggest several open problems and to formulate some related conjectures.

1. Introduction

In this paper we are interested in finite time blow up of solutions to the ordinary differential equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad (s \in \mathbb{R}), \tag{1}$$

where $k \in \mathbb{R}$, and f is a locally Lipschitz function. This equation arises in several contexts. With no hope of being exhaustive, let us mention some models which lead to (1). When k is negative (1) is known as the extended Fisher–Kolmogorov equation, whereas when k is positive it is referred to as the Swift–Hohenberg equation, see [27]. For $f(t) = t - t^2$, (1) arises in the dynamic phase-space analogy of a nonlinearly supported elastic strut [20]. In [1] the existence of even homoclinics to $w \equiv 0$ was proved whenever $k \leqq 0$. When $f(t) = t^3 - t$, (1) serves as a model of pattern formation in many physical, chemical or biological systems, see [4,5] and references therein. The slightly different nonlinearity $f(t) = t - t^3 + t^5$ was used by Peletier [28] in order to investigate localization and spreading of deformation of a strut confined by an elastic foundation. Last but not least, we mention the important book by PELETIER–TROY [27], where one can find many other physical models, a survey of existing results, and further references.

The primary purpose of the present paper is to contribute to a better understanding of the qualitative properties of solutions to (1) when the nonlinearity f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(t)\,t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}. \tag{2}$$

Further assumptions on f are needed in the sequel, although the prototype nonlinearity we have in mind is

$$f(t) = \alpha|t|^{q-1}t + |t|^{p-1}t \qquad (p > q \geqq 1,\ \alpha \geqq 0). \tag{3}$$

The second, and probably most ambitious, purpose of the present paper is to connect the phenomena which hold for (1) with several classes of fourth order *partial differential equations*. The first example concerns a nonlinear fourth order wave equation. Under suitable boundary and initial conditions, the following nonlinear beam equation was proposed by LAZER–MCKENNA [22] as a model for a suspension bridge:

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x,t), \qquad x \in (0, L), \quad t > 0, \tag{4}$$

where $L > 0$ denotes the length of the bridge, $u^+ = \max\{u, 0\}$, γu^+ represents the force due to the cables which are considered as a spring with a one-sided restoring force (equal to γu if u is downward positive and to 0 if u is upward negative), and W represents the forcing term acting on the bridge (including its own weight per unit length and the wind or other external sources). The solution u represents the vertical displacement when the beam is bending. After some normalization, MCKENNA–WALTER [26] reduce the problem of finding traveling wave solutions of (4) to solving (1) with $k \in (0, 2)$ and $f(t) = (t+1)^+ - 1$. In Section 3.1 we discuss this model in more detail and we analyze a variant that we recently suggested in [18,19]; this new variant perfectly fits with our results.

When $k = -4$, equation (1) with $f(t) = e^t - 1$ arises while seeking radial solutions to the biharmonic PDE

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \quad \text{in} \quad \mathbb{R}^4 \setminus \{0\}; \tag{5}$$

we refer to [3] for the transformation of this equation, which leads to (1), and for further semilinear biharmonic PDEs which can be transformed into (1) with the same change of variables. Moreover, with a different change of variables, radial solutions to biharmonic PDEs both at critical growth (in the sense of Sobolev exponent) and degenerate, such as

$$\begin{aligned} &\Delta^2 u + |u|^{8/(n-4)}u = 0 \quad \text{in} \quad \mathbb{R}^n\ (n \geqq 5),\\ &\Delta\Big(|x|^2\Delta u\Big) + |x|^2|u|^{8/(n-2)}u = 0 \quad \text{in} \quad \mathbb{R}^n\ (n \geqq 3), \end{aligned} \tag{6}$$

can also be reduced to (1), see [14,18] and further results in Section 3.2. In particular, for the critical growth equation, our results are connected with some Liouville

Theorems, see [9]. Our results enable us to prove that radial solutions to these equations blow up at some finite radius with wide oscillations.

Finally, for $n \geqq 2$, consider the Cauchy problem for the nonlinear fourth order parabolic equation

$$\begin{cases} u_t + \Delta^2 u = |u|^{p-1}u & \text{in} \quad \mathbb{R}^{n+1}_+ \\ u\,(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^n \end{cases}, \tag{7}$$

where $p > 1 + 4/n$ and u_0 satisfies suitable assumptions; the exponent $1 + 4/n$ is the analogue of the Fujita-exponent (see [13,33] and references therein), arising in second order semilinear Cauchy problems. The existence of global solutions to (7) was proved in [15] for initial data u_0 sufficiently small in a suitable sense, see also [12] for decay and positivity properties of the solution. The problem of possible blow up for large initial data was left open, and only a partial result such as [15, Theorem 2] is known at present. In Section 3.3 we explain how the results of this paper may shed some light on the finite time blow up of the solutions to (7).

Let us now briefly explain our main result and how it can be applied to the just mentioned PDEs. We first recall the following statements proved in [3]:

Proposition 1. *Let $k \in \mathbb{R}$ and assume that f satisfies* (2).

(i) *If a local solution w to* (1) *blows up at some finite $R \in \mathbb{R}$, then*

$$\liminf_{s \to R} w(s) = -\infty \quad \textit{and} \quad \limsup_{s \to R} w(s) = +\infty. \tag{8}$$

(ii) *If f also satisfies*

$$\limsup_{t \to +\infty} \frac{f(t)}{t} < +\infty \quad \textit{or} \quad \limsup_{t \to -\infty} \frac{f(t)}{t} < +\infty, \tag{9}$$

then any local solution to (1) *exists for all $s \in \mathbb{R}$.*

If *both* the conditions in (9) are satisfied, then global existence follows from classical theory of ODEs; but (9) merely requires f to be "one-sided at most linear" so that statement (ii) is far from being trivial and, as shown in [18], it does not hold for differential equations of order at most 3. On the other hand, Proposition 1 (i) states that, under the sole assumption (2), the only way that finite time blow up can occur is with "wide and thinning oscillations" of the solution w; again, in [18] we showed that this kind of blow up is a phenomenon typical of (at least) fourth order problems such as (1), since it does not occur in related lower order equations. Note that assumption (9) includes, in particular, the cases where f is either concave or convex.

Although in [18] we gave strong evidence that (8) holds whenever $k \leqq 0$ and f is superlinear in a suitable sense, a full proof of this result is not yet available. In this paper (see Theorem 2) we fill this gap by determining a quite general sufficient condition for the validity of (8), and we describe in some detail the way the solutions blow up. This result is complemented by several comments and numerical experiments. In the rest of the paper we will try to convince the reader that this phenomenon occurs in most fourth order equations, including partial differential equations. We believe not only that this oscillating blow up is not visible for lower order equations, but also that it is somehow present in many fourth order equations. There are several reasons for this feeling. Firstly, as already mentioned, assumption (9) shows that if the positive (respectively, negative) part of the solution is controlled, then its negative (respectively, positive) part is also controlled, see also the proof of [3, Lemma 23]. Secondly, the energy functions used in the present paper seem to show an increasing chaotic behavior in (1). Finally, the PDEs which can be reduced to (1) also exhibit the same oscillating behavior. In particular, since this phenomenon is visible for suspension bridges, see Section 3.1, this means that it is a phenomenon which occurs in real life.

This paper is organized as follows. In Section 2 we state our main results about (1): a sufficient condition for finite time oscillating blow up (Theorem 2), plus a detailed description of how this blow up occurs (see the items in Theorem 2 and the subsequent Theorem 3). Section 2 also provides several related remarks and open problems. In Section 3 we show how our results can be applied to several PDEs such as (4), (5), (6) and (7). In Section 4.1 we discuss the case in which $k > 0$. In Section 4.2 we numerically study the dependence of the blow up time in terms of the parameters involved in (1). In Section 4.3 we numerically test the validity of some theoretically found blow up estimates. In Section 4.4 we numerically analyze the blow up of solutions for nonlinearities f which are quite different from (3), namely superlinearities with fairly different growths at $\pm\infty$; we also numerically analyze the blow up rate found theoretically in (72). In Section 5 we study the linearized equation in detail, and we show how different behaviors appear for different values of k. Although the linearized problem is an approximation of (1) for *small* values of the solution w, we examine whether these behaviors can also justify what happens when w blows up. In Section 6 we introduce the energy functions and tools needed to study (1). Sections 7 and 8 are devoted to the proofs of Theorems 2 and 3.

2. Main Results

Assume that f satisfies the regularity conditions

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}), \quad f''(t)t > 0 \quad \forall t \neq 0, \quad \liminf_{t\to\pm\infty} |f''(t)| > 0 \tag{10}$$

and the growth conditions

$$\begin{aligned} &\exists p > q \geqq 1,\ \alpha \geqq 0,\ 0 < \rho \leqq \beta, \quad \text{s.t.} \\ &\rho|t|^{p+1} \leqq f(t)t \leqq \alpha|t|^{q+1} + \beta|t|^{p+1} \quad \forall t \in \mathbb{R}. \end{aligned} \tag{11}$$

Notice that (10)–(11) strengthen (2) and that f in (3) satisfies both (10) and (11). Let

$$F(t) := \int_0^t f(\tau)\, \mathrm{d}\tau$$

denote an antiderivative of f.

We now state our main result, namely a sufficient condition for the finite time blow up of local solutions to (1).

Theorem 2. *Let $k \leqq 0$ and assume that f satisfies (10) and (11). Assume that $w = w(s)$ is a local solution to (1) in a neighborhood of $s = 0$, which satisfies*

$$w'(0)w''(0) - w(0)w'''(0) - kw(0)w'(0) > 0. \tag{12}$$

Then, w blows up in finite time for $s > 0$, that is, there exists $R < +\infty$ such that (8) holds. Therefore, there exists an increasing sequence $\{z_j\}_{j\in\mathbb{N}}$ such that:

(i) *$z_j \nearrow R$ as $j \to \infty$;*
(ii) *$w(z_j) = 0$ and w has constant sign in (z_j, z_{j+1}) for all $j \in \mathbb{N}$.*
Furthermore, in each interval (z_j, z_{j+1}) where $w(s) > 0$, the following facts hold:
(iii) *$0 < w'(z_j) < -w'(z_{j+1})$ and there exists a unique $m_j \in (z_j, z_{j+1})$ such that $w'(m_j) = 0$;*
(iv) *$w''(z_{j+1}) < 0 < w''(z_j)$, there exists a unique $r_j \in (z_j, z_{j+1})$ where w'' changes sign, and $r_j < m_j$.*
Facts similar to (iii)–(iv) (with obvious changes) occur in intervals (z_j, z_{j+1}), where $w(s) < 0$. Finally, with the notations of (iii),
(v) *$|w(m_j)| \to +\infty$ as $j \to \infty$ and $F(w(m_{j+1})) > F(w(m_j))$ for all j;*
(vi) *there exist $\kappa_1, \kappa_2 > 0$, depending only on the parameters in (11), such that*

$$m_{j+1} - m_j \leqq \frac{\kappa_1}{|w(m_j)|^{(p-1)/4}}, \quad z_{j+1} - z_j \geqq \frac{\kappa_2}{|w(m_j)|^{(p-1)/4}} \quad \forall j. \tag{13}$$

Theorem 2 deserves several comments and suggests some open problems which we summarize as follows.

• It would be interesting to have a similar statement when $k > 0$, since this would allow us to prove Conjecture 4, below. However, if $k > 0$, there are a couple of important tools which are missing and the proof of Theorem 2 cannot be extended in a simple way, see Section 4.1. In any case, numerical results suggest that a result similar to Theorem 2 also holds for $k > 0$, see again Section 4.1.

• Assumption (11) is a superlinearity assumption. Nevertheless, we believe that the restriction that f be bounded from both above and below by the same power $p > 1$ can be removed. Does Theorem 2 hold for more general kinds of superlinear functions f? In Section 4.4 we study numerically the behavior of the solutions when f has different growths at $\pm\infty$.

• What is the role played by the parameter k? Are the critical values of k for the linear problem (see Section 5) also important thresholds for the nonlinear problem (1)? Roughly speaking, these values of k play a role for small solutions w, but it is not clear whether they also influence the solution in cases where blow up occurs.

• Can assumption (12) be relaxed? We believe that it might be relaxed, although it cannot be completely removed since the trivial solution $w(s) \equiv 0$ is globally defined, that is, $R = +\infty$. Closely related is the question as to whether nontrivial global solutions to (1) exist. We performed many numerical experiments, but we could not detect any such solution.

• In Section 4.2 we give numerical evidence that the blow up time depends increasingly on $k \in \mathbb{R}$. It would be interesting to have an analytical proof of this fact, also in view of the application to the suspension bridge model, see Section 3.1. □

We now compare the rate of blow up of the displacement to that of the acceleration. The next result holds for any k and without assuming (10).

Theorem 3. *Let $k \in \mathbb{R}$ and assume that f satisfies (2) and (11). Assume that a local solution $w = w(s)$ to (1) blows up (in finite time) as $s \nearrow R < +\infty$. Denote by $\{z_j\}$ the increasing sequence of zeros of w, such that $z_j \nearrow R$ as $j \to +\infty$, see Proposition 1. Then*

$$\int_{z_j}^{z_{j+1}} \Big(f(w(s))w(s) + F(w(s)) \Big)\, ds \sim \frac{1}{2} \int_{z_j}^{z_{j+1}} w''(s)^2\, ds \tag{14}$$

as $j \to \infty$. Here, $g(j) \sim \psi(j)$ means that $g(j)/\psi(j) \to 1$ as $j \to \infty$.

In the particular case where f has the form (3), (14) becomes

$$\int_{z_j}^{z_{j+1}} |w(s)|^{p+1}\, ds \sim \frac{p+1}{2(p+2)} \int_{z_j}^{z_{j+1}} w''(s)^2\, ds.$$

Note that the term $w''(s)$ describes the vertical acceleration, whereas $f(w(s))w(s)+F(w(s))$ is a measure of the vertical displacement. Hence, by the superlinearity assumption (11), (14) means that the vertical acceleration has a higher rate of blow up than does the vertical displacement.

3. Applications to Fourth Order Partial Differential Equations

3.1. *Suspension Bridges: A Fourth Order Wave Equation*

In this section we discuss the behavior of traveling waves to (4) and some alternative models for suspension bridges. Following [26], we normalize (4) by setting $\gamma = 1$ and $W \equiv 1$. Then, seeking traveling waves $u(x, t) = 1 + w(x - ct)$ to (4) leads to the equation

$$w''''(s) + kw''(s) + [w(s) + 1]^+ - 1 = 0 \qquad (s \in \mathbb{R},\ k = c^2).$$

In order to maintain the same behavior but with a smooth nonlinearity, CHEN-MCKENNA [7] suggest considering the equation

$$w''''(s) + kw''(s) + e^{w(s)} - 1 = 0 \qquad (s \in \mathbb{R}), \tag{15}$$

which is exactly of the same kind as (1), with $f(t) = e^t - 1$ satisfying (2) but not (11). As pointed out by MCKENNA [25, Section 6], according to historical sources, one of the most interesting behaviors for suspension bridges (including the Golden Gate and the Tacoma Narrows Bridges) is the following:

> *large vertical oscillations can rapidly change, almost instantaneously, to a torsional oscillation.*

Our explanation for this fact is that

> *since the motion cannot be continued downwards due to the cables, when the bridge reaches its equilibrium position, the existing energy generates a crossing wave, namely a torsional oscillation.*

Because the Tacoma Bridge collapse (November 1940) was due to a wide torsional motion of the bridge (see [32]), the bridge cannot be considered as a one-dimensional beam. This problem was overcome in [10, Section 2.3] by introducing the *deflection from horizontal* as a second unknown function (in addition to the vertical displacement). In [18] we suggested maintaining the one-dimensional model, provided that one also allows displacements *below the equilibrium position* and that these displacements replace the deflection from horizontal; in other words, the unknown function w now represents the upwards vertical displacement when $w > 0$ and the deflection from horizontal when $w < 0$. Instead of (4), one should then consider the more general semilinear fourth order wave equation

$$u_{tt} + u_{xxxx} + f(u) = 0, \qquad x \in (0, L), \quad t > 0, \tag{16}$$

with a nonlinearity f, which should be superlinear and unbounded when both $u \to \pm\infty$. The superlinearity is justified by the fact that the farther the position of the bridge from the horizontal equilibrium position, the greater the action of the wind becomes relevant, because the wind hits the surface of the bridge transversally. If the bridge were ever to reach the limit vertical position, the wind would hit it orthogonally. This means that the forcing term f becomes more powerful for large displacements from the horizontal position.

Of course, traveling waves to (16) which propagate at some velocity $c > 0$ (depending on the elasticity of the material of the beam), solve (1) with $k = c^2 > 0$. On the other hand, the equation of the elastic combined vertical/torsional oscillation motion of suspension bridges in the wind seem to be well-known among engineers. In a simplified form, the stationary equation may be written as (1), where $k = -H < 0$ (H being the tension force of the cables due to the deadloads), and where the nonlinearity $f(w)$ is replaced by a nonlocal term, see for example (1) and (2) in [8]. Hence, (1) also arises in the description of bridge oscillations when $k < 0$. In any case, our numerical results suggest that a statement similar to Theorem 2 also holds for $k > 0$, see Section 4.1. We are thus led to formulate the following

Conjecture 4. *Assume that f satisfies (10) and (11). Then traveling waves $w(s) = u(s + ct, t)$ to (16) blow up at some finite time R where (8) holds.*

The Tacoma collapse is just the most celebrated and dramatic evidence of bridge's oscillations. The very day on which London's Millennium Bridge opened (April 2007), the crowd streamed on it and the bridge started to sway from side to side, see [23]. According to Sanderson [30], the bridge swaying was due to the way people balanced themselves, rather than the timing of their steps. Therefore, the pedestrians acted as negative dampers, adding energy to the bridge's natural sway. Macdonald [24, p.1056] explains this phenomenon by writing *above a certain critical number of pedestrians, this negative damping overcomes the positive structural damping, causing the onset of exponentially increasing vibrations.*

This description corresponds to a typical superlinear behavior, which justifies our assumption (11) and the particular shape of the nonlinearity f as in (3). It is not yet clear whether $\alpha = 0$ or $\alpha > 0$, that is, whether the superlinear behavior also occurs close to equilibrium. The Millennium Bridge was made secure by adding some (unaesthetic) positive dampers. These dampers correspond to taking a smaller coefficient β in (11), in such a way to delay the effect of the superlinear behavior of the forcing term f.

Another pedestrian bridge, the Assago metro Bridge in Milan, had a similar problem. In February 2011, just after a concert, the public crossed the bridge and, suddenly, the oscillations were so strong that people could hardly stand, see [11] and also the video from [2]. Even worse was the subsequent panic effect when the crowd started running in order to escape a possible collapse; this amplified oscillations. This problem was also solved by adding positive dampers, see [31].

3.2. Semilinear Elliptic Biharmonic Equations

In this section we show how our results apply to semilinear elliptic partial differential equations involving the biharmonic operator. For $n \geqq 2$ and for any $p > 1$, put

$$\Theta_{n,p} := 2(p+3)\Big((n-4)p-(n+4)\Big)\Big((n-2)p-(n+6)\Big).$$

Then, for any $\mu \in \mathbb{R}$, we consider the equation

$$\begin{aligned}&\Delta^2 u + 2\left(4-n+\frac{8}{p-1}\right)\frac{x\cdot\nabla\Delta u}{|x|^2}+\mu\frac{\Delta u}{|x|^2}\\ &-\left(\frac{(n-2)p-(n+6)}{p-1}\mu+\frac{\Theta_{n,p}}{(p-1)^3}\right)\frac{x\cdot\nabla u}{|x|^4}+|u|^{p-1}u=0 \quad \text{in} \quad \mathbb{R}^n. \quad (17)\end{aligned}$$

In spite of its unpleasant form, (17) has a couple of interesting particular cases.

If $n \geqq 5$, $p = \frac{n+4}{n-4}$ and $\mu = 0$, (17) becomes

$$\Delta^2 u + |u|^{8/(n-4)}u = 0 \qquad \text{in } \mathbb{R}^n, \tag{18}$$

that is, a semilinear critical growth equation (in the sense of Sobolev embedding). If $n \geqq 3$, $p = \frac{n+6}{n-2}$ and $\mu = 2n$, (17) becomes the degenerate equation

$$\Delta\Big(|x|^2\Delta u\Big) + |x|^2|u|^{8/(n-2)}u = 0 \qquad \text{in } \mathbb{R}^n. \tag{19}$$

We are interested in determining the behavior of radial solutions to (17). In its radial form, with $u = u(r)$ $(r = |x|)$, (17) reads

$$u''''(r) + \frac{2(3p+5)}{p-1}\frac{u'''(r)}{r} + \left(\frac{16(n-1)}{p-1} - (n-1)(n-5) + \mu\right)\frac{u''(r)}{r^2} + \left(\frac{p+7}{p-1}\mu - \frac{16(n-1)}{p-1} - \frac{\Theta_{n,p}}{(p-1)^3} + (n-1)(n-5)\right)\frac{u'(r)}{r^3} + |u(r)|^{p-1}u(r) = 0 \tag{20}$$

for all $r > 0$. As in [14] we set

$$u(r) = r^{-4/(p-1)}\, w(\log r) \quad (r > 0), \qquad w(s) = e^{4s/(p-1)}\, u(e^s) \quad (s \in \mathbb{R}). \tag{21}$$

Tedious calculations then show that

$$\begin{aligned}
\frac{u'(r)}{r^3} &= r^{-4p/(p-1)}\left[w'(s) - \frac{4}{p-1}w(s)\right],\\
\frac{u''(r)}{r^2} &= r^{-4p/(p-1)}\left[w''(s) - \frac{p+7}{p-1}w'(s) + \frac{4(p+3)}{(p-1)^2}w(s)\right],\\
\frac{u'''(r)}{r} &= r^{-4p/(p-1)}\left[w'''(s) - \frac{3(p+3)}{p-1}w''(s) + \frac{2(p^2+10p+13)}{(p-1)^2}w'(s)\right.\\
&\qquad\left. - \frac{8(p+1)(p+3)}{(p-1)^3}w(s)\right]\\
u''''(r) &= r^{-4p/(p-1)}\left[w''''(s) - \frac{2(3p+5)}{p-1}w'''(s) + \frac{11p^2+50p+35}{(p-1)^2}w''(s)\right.\\
&\qquad - \frac{2(3p^3+35p^2+65p+25)}{(p-1)^3}w'(s)\\
&\qquad\left. + \frac{8(p+1)(p+3)(3p+1)}{(p-1)^4}w(s)\right].
\end{aligned}$$

Therefore, after the change of variables (21), equation (20) reads $(s \in \mathbb{R})$

$$\begin{aligned}
&w''''(s) - \left(\frac{(n^2-6n+12)p^2-2(n^2+2n-20)p+n^2+10n+44}{(p-1)^2} - \mu\right)w''(s)\\
&+\frac{16}{(p-1)^2}\left(\frac{(n^2-6n+12)p^2-2(n^2+2n-20)p+n^2+10n+28}{(p-1)^2} - \mu\right)w(s)\\
&+|w(s)|^{p-1}w(s) = 0.
\end{aligned}$$

Therefore, if

$$\mu \leqq \frac{(n^2-6n+12)p^2-2(n^2+2n-20)p+n^2+10n+28}{(p-1)^2}, \tag{22}$$

then the coefficient of $w''(s)$ is negative, whereas the coefficient of $w(s)$ is non-negative and Theorem 2 applies. Hence, we have

Corollary 5. *Let $n \geqq 2$, $p > 1$ and $\mu \in \mathbb{R}$ satisfy (22). Let $u = u(r)$ be a nontrivial radially symmetric solution to the equation (17) in a neighborhood of the origin and such that $u(0)u''(0) < 0$. Then there exists $\rho \in (0, \infty)$ such that*

$$\liminf_{r \nearrow \rho} u(r) = -\infty \quad \text{and} \quad \limsup_{r \nearrow \rho} u(r) = +\infty.$$

In particular, this result applies to (18) and (19) since (22) is satisfied. After the reduction to the radial form and after the change of variables (21), they become, respectively,

$$w''''(s) - \frac{n^2 - 4n + 8}{2} w''(s) + \left(\frac{n(n-4)}{4}\right)^2 w(s) + |w(s)|^{8/(n-4)} w(s) = 0,$$

$$w''''(s) - \frac{(n-2)^2}{2} w''(s) + \frac{(n-2)^4}{16} w(s) + |w(s)|^{8/(n-2)} w(s) = 0$$

which are both like (1), with $k < 0$ and f satisfying (3). In the particular cases where $n = 8$ (first equation) and $n = 4$ (second equation), they become

$$w''''(s) - 20w''(s) + 64w(s) + w(s)^3 = 0,$$
$$w''''(s) - 2w''(s) + w(s) + w(s)^5 = 0.$$

Note also that the condition $u(0)u''(0) < 0$ replaces (12). Once again we point out that this condition cannot be completely dropped, since otherwise we could have the trivial solution $u(r) \equiv 0$.

3.3. Parabolic Biharmonic Equations

As already mentioned, when $p > 1 + 4/n$, global existence results for (7) were obtained in [12,15] under smallness assumptions on the initial data u_0. More precisely, [12, Theorem 1.5] states that there exists $\alpha > 0$ such that if $u_0 \in C^0(\mathbb{R}^n)$ satisfies

$$|u_0(x)| \leqq \frac{\alpha}{1 + |x|^\beta} \qquad \forall x \in \mathbb{R}^n$$

for some $\beta \geqq 4/(p-1)$, then the solution to (7) is global in time and converges uniformly to 0 as $t \to +\infty$. On the other hand, the possible finite time blow up in the presence of large initial data u_0 seems to be related to the sign changing properties of the biharmonic heat kernels. It is shown in [12,16] that the linear biharmonic heat operator has an "eventual local positivity" property; by this we mean that, for positive initial data u_0, the solution to the linear problem (with no source) is eventually positive on compact subsets of $\mathbb{R}^n$ but negativity can appear at any time far away from the origin. We also refer to [17] for possible extensions to higher order polyharmonic heat equations. This eventual local positivity property is also available for (7) for suitable initial data u_0, see [12].

The problem of understanding if the solution to (7) may blow up in finite time for large data u_0 is still open. Let

$$\underline{\omega} := \liminf_{|x| \to \infty} |x|^{4/(p-1)} u_0(x), \qquad \overline{\omega} := \limsup_{|x| \to \infty} |x|^{4/(p-1)} u_0(x). \tag{23}$$

Then [15, Theorem 2] states that there exists $\Lambda > 0$ such that if $\underline{\omega} > \Lambda$ or $\overline{\omega} < -\Lambda$, then the solution u to (7) may be global only if its negative part u^- and its positive part u^+ are "perfectly balanced", that is, their masses (computed in a suitable form) have the same weight. But our crucial estimate (72) seems to say that

$$\int_{\mathbb{R}^n} |u^+(x)|^{p+1}\,\mathrm{d}x > 2\int_{\mathbb{R}^n} |u^-(x)|^{p+1}\,\mathrm{d}x.$$

If this were true, then we would have blow up in finite time t with wide oscillations, also, for solutions to (7).

The fact that the noncoercive equation (7) has been considered should not change this point of view. As we have seen, finite time blow up for fourth order equations seems to occur with wide oscillations regardless of the signs of the terms. In any case, for the coercive equation $u_t + \Delta^2 u + |u|^{p-1}u = 0$ in $\mathbb{R}^{n+1}_+$, if one multiplies it by the solution u and formally integrates by parts, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -\|\Delta u(t)\|_2^2 - \|u(t)\|_{p+1}^{p+1} < 0.$$

This says that the L^2-norm of the solution is decreasing and, therefore, one expects that the solution could be global and should not blow up in finite time.

Summarizing, although our results do not apply directly to (7), they suggest the following

Conjecture 6. *Assume that $n \geqq 2$ and that $p > 1 + 4/n$. Let $u_0 \in C^0(\mathbb{R}^n)$ and let $\underline{\omega}$ and $\overline{\omega}$ be as in (23). There exists $\Lambda > 0$ such that if $\underline{\omega} > \Lambda$ or $\overline{\omega} < -\Lambda$, then the solution to (7) blows up in finite time with wide oscillations. That is, there exists $T \in (0, +\infty)$ such that*

$$\limsup_{t\nearrow T}\ \sup_{x\in\mathbb{R}^n} u(x,t) = +\infty \quad and/or \quad \liminf_{t\nearrow T}\ \inf_{x\in\mathbb{R}^n} u(x,t) = -\infty.$$

We do have some doubts about the and/or statement. We believe that blow up with oscillations (the "and " case) might occur, for instance, whenever

$$\liminf_{|x_1|\to\infty} |x_1|^{4/(p-1)}u_0(x) > \Lambda, \qquad \limsup_{|x_2|\to\infty} |x_2|^{4/(p-1)}u_0(x) < -\Lambda.$$

4. Numerical Results

Here we used a class of symmetric non-symplectic methods, which are improved versions of methods previously introduced in [6], called *block-Boundary Value Methods* (block-BVMs) [21]. They could also be rewritten in the form of implicit collocation Runge–Kutta methods, so they share all the nice properties of symmetric Runge–Kutta schemes. The block-BVMs are defined by a set of linear multi-step formulas combined in a suitable way. In our implementation, the time integration interval is discretized by using *two different meshes*: a coarser, equispaced mesh and a finer, nonequispaced mesh. This method enjoys excellent numerical stability properties [29] and, if a first integral exists, it is numerically conserved both on the coarse mesh and on the finer mesh, provided that a suitable timestep is used. It is known that if the considered differential problem is not Hamiltonian but still has a

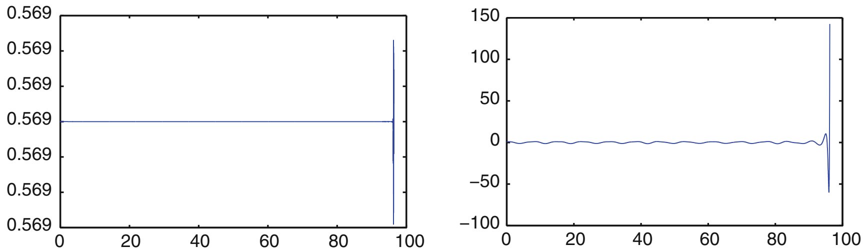

Fig. 1. Energy and solution for $k = 3.6$, $[w(0), w'(0), w''(0), w'''(0)] = [0.9, 0, 0, 0]$, $f(t) = t + t^3$

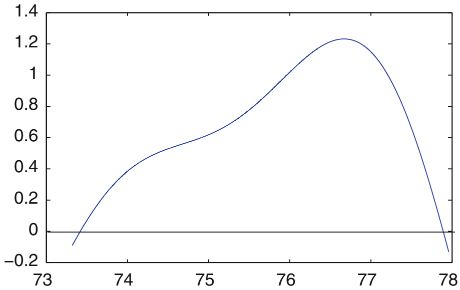

Fig. 2. Qualitative behavior of the solution w in the interval $[z_j, z_{j+1}]$

first integral, then a symmetric integrator is the most suitable. This fact motivates our choice, since our problem is not Hamiltonian, but enjoys the conservation of the energy function in (28). Here we define numerically the blow up time R as the last value of the independent variable s when the numerical algorithm stops for convergence reasons.

From Fig. 1 it is clear that the energy function is conserved until $s = 96.35$ with a maximum absolute error less than 10^{-8}, whereas the blow up time is $R = 96.59$. This means that, numerically, the energy function is conserved *almost* until $s = R$. This behavior was found in all the examples we ran. The corresponding solution is reported in the second plot in Fig. 1. It is worth noticing that the solution exhibits the same qualitative behavior as reported in [18, Figure 3], where we used numerical methods which do not conserve the energy function. In spite of this fact, the blow up time was computed there to be $R = 96.59$, as well.

4.1. Some Remarks on the Case $k > 0$

Let us start by explaining which parts of the proof of Theorem 2 cannot be extended to the case where $k > 0$. First, the energy functions G and H, see (30) and (31), do not possess nice monotonicity properties as in the case where $k \leqq 0$. Second, Lemma 11 does not hold and w may have very complicated behaviors in its positivity intervals. This makes it more difficult to obtain an estimate like (76) below. Figure 2 refers to the case in which $k = 3.5$, $[w(0), w'(0), w''(0), w'''(0)] = [0.8, 0, 0, 0]$, with $f(t) = t + t^3$.

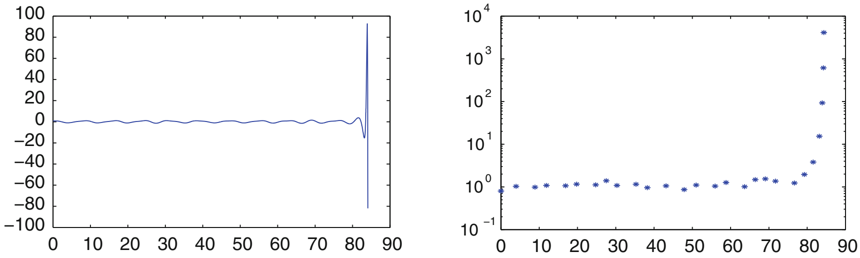

Fig. 3. Critical levels are not monotone when $k = 3.5$

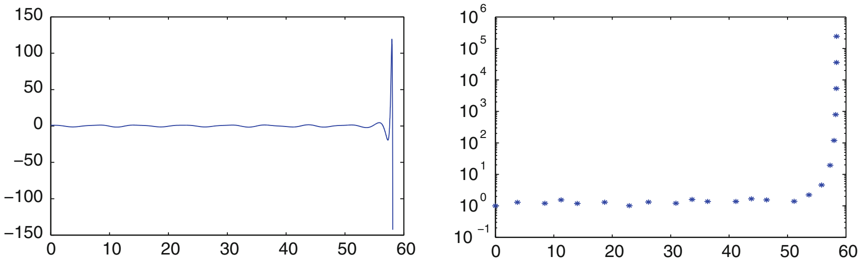

Fig. 4. Critical levels are not monotone when $k = 3.8$

In Figs. 3 and 4 we exhibit a couple of plots of solutions to (1) which show, however, that Theorem 2 probably also holds true when $k > 0$; we display the plot of the solution and the dependence $j \mapsto |M_j|$ for the first critical points of the solution to (1) in the two following cases:

$$-k = 3.5,\ f(t) = t + t^3 \quad \text{and} \quad [w(0), w'(0), w''(0), w'''(0)] = [0.8, 0, 0, 0];$$
$$-k = 3.8,\ f(t) = t + t^3 \quad \text{and} \quad [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0].$$

4.2. Dependence of the Blow Up Time on the Parameters of the Equation

In this section we give some numerical results which show how the blow up time R for (1) depends on α, p, k, and on the initial data; here α is the coefficient in (3). We ran many tests and we always obtained the same behaviors, so here we report just a few of them, to illustrate their general appearance.

On the whole, these figures (Figs. 5, 6, 7, 8, 9) enable us to make the following

Conjecture 7. *When all the other parameters remain fixed, the maps* $R = R(k)$ *and* $R = R(w''(0))$ *are strictly increasing, whereas the maps* $R = R(p)$, $R = R(\alpha)$, $R = R(w(0))$ *are strictly decreasing.*

4.3. Tests for the Theoretical Blow Up Estimate

For $f(t) = t + t^3$, we tested the validity of (72) which, in this case, asymptotically becomes $M_j^4/M_{j-1}^4 > 2$. In the next table we show the results related to the case $p = 3, k = -1, [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$. Condition (12) is satisfied at the second integration step. It clearly appears that $M_j^4/M_{j-1}^4 > 2$ always occurs (with no doubts!).

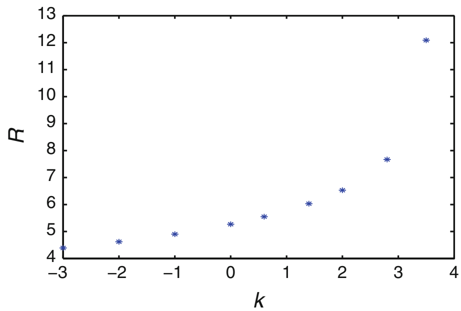

Fig. 5. $R = R(k)$ with $p = 3, \alpha = 1, [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$

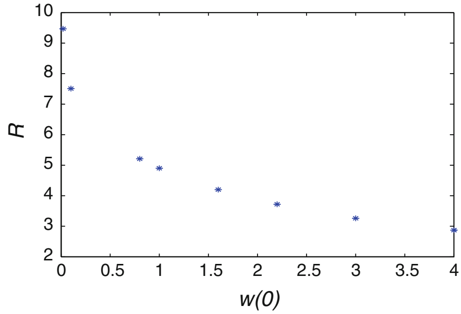

Fig. 6. $R = R(w(0))$ with $p = 3, \alpha = 1, k = -1, [w(0), w'(0), w''(0), w'''(0)] = [w(0), 0, 0, 0]$

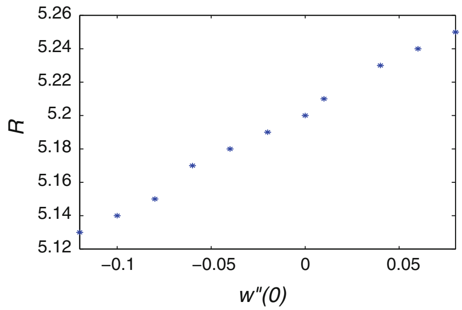

Fig. 7. $R = R(w''(0))$ with $p = 3, \alpha = 1, k = -1, [w(0), w'(0), w''(0), w'''(0)] = [0.8, 0, w''(0), 0]$

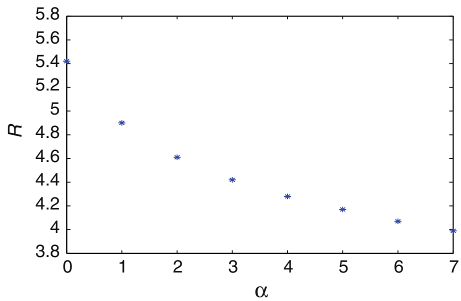

Fig. 8. $R = R(\alpha)$ with $p = 3, k = -1, [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$

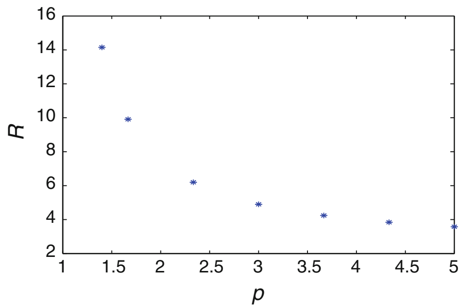

Fig. 9. $R = R(p)$ with $\alpha = 1$, $k = -1$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$

j	1	2	3	4	5	6
$\lvert M_{j+1}\rvert$	$1.00e+1$	$7.08e+1$	$4.80e+2$	$3.24e+3$	$2.18e+4$	$1.47e+5$
M_{j+1}^4/M_j^4	$1.00e+4$	$2.51e+3$	$2.11e+3$	$2.07e+3$	$2.05e+3$	$2.07e+3$

Next, we performed the same test in the case $f(t) = t + t^{5/3}, k = -1, [w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$. In this case, (72) asymptotically becomes $M_j^{8/3}/M_{j-1}^{8/3} > 2$. The results are shown in the next table and, again, show that (72) can probably be improved.

j	1	2	3	4	5	6	7
$\lvert M_{j+1}\rvert$	$2.62e+1$	$4.59e+2$	$6.57e+3$	$8.78e+4$	$1.14e+6$	$1.47e+7$	$1.87e+8$
$M_{j+1}^{8/3}/M_j^{8/3}$	$6.05e+3$	$2.07e+3$	$1.21e+3$	$1.00e+3$	$9.31e+2$	$9.14e+2$	$8.81e+2$

Although the above tables suggest several comments, we do not offer any conjectures in this situation.

4.4. A Nonlinearity with Different Growths at Infinity

In this section we numerically study equation (1) with $k = -2$ and $f(t) = e^t - 1 + t^3$, that is,

$$w''''(s) - 2w''(s) + e^{w(s)} - 1 + w(s)^3 = 0 \qquad (s \in \mathbb{R}). \tag{24}$$

Notice that f satisfies (2) and (9) except for the sign condition on f'', but, at least, we have $f'(t) > 0$ for all $t \neq 0$. Therefore, Lemma 12 still holds. Also, Lemma 11 holds since it merely requires (2). Hence, we may obtain (72), which reads

$$e^{M_{j+1}} + 1 - M_{j+1} + M_{j+1}^4 > 2e^{M_j} - 2M_j + 2M_j^4;$$

here we assume that $M_j > 0$ and $M_{j+1} < 0$. Assuming that the solution to (24) blows up in finite time, so that (8) holds, the latter asymptotically becomes

$$M_{j+1}^4 \, e^{-M_j} > 2. \tag{25}$$

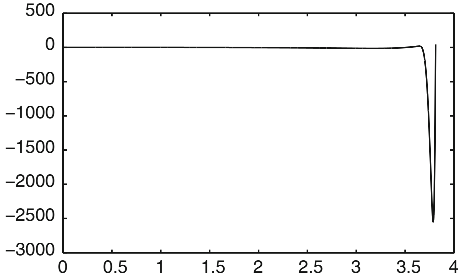

Fig. 10. Solution of (24) for initial data $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$

In Fig. 10, we plot a solution to (24).
Then we test the validity of (25), see the next table.

j	1	3
$M_j > 0$	$1.00e + 0$	$1.99e + 1$
$M_{j+1} < 0$	$-1.36e + 1$	$-2.55e + 3$
$M_{j+1}^4 e^{-M_j}$	$1.26e + 4$	$9.63e + 4$

Therefore, it seems that Theorem 2 also holds true for nonlinearities f satisfying (2) plus some superlinearity conditions at $\pm\infty$, but fairly different from (3).

5. The Linear Problem

For a better understanding of (1), we are interested in the case where $f(t) = t$, thereby complementing the analysis in [3]. Note that this function f satisfies (2)–(10) but not (11). It turns out that several different critical values of k appear.

In this case, (1) reads

$$w''''(s) + kw''(s) + w(s) = 0 \qquad (s \in \mathbb{R}). \tag{26}$$

This is also the linearized equation at 0 if we assume that $f'(0) = 1$. More generally, if f is a function such that $f'(0) = A > 0$ and w is a solution to (1) then $z(s) = w(s/\sqrt[4]{A})$ solves the new equation

$$z''''(s) + \frac{k}{\sqrt{A}} z''(s) + \widetilde{f}(z(s)) = 0,$$

where $\widetilde{f}(t) = \frac{1}{A} f(t)$ and $\widetilde{f}'(0) = 1$. Hence, up to scaling k, we may always assume that $f'(0) = 1$. Of course, if $f'(0) = 0$, this trick is no longer available.

A crucial role in the study of (26) is played by the so-called characteristic equation

$$\lambda^4 + k\lambda^2 + 1 = 0, \tag{27}$$

whose solutions are formally given by

$$\lambda^2 = \frac{-k \pm \sqrt{k^2 - 4}}{2}.$$

We must distinguish different cases.

- **Case** $k < -2$**.** The solutions to (27) are all real and are given by

$$\lambda \in \left\{ \pm\sqrt{\frac{|k| + \sqrt{k^2-4}}{2}}, \pm\sqrt{\frac{|k| - \sqrt{k^2-4}}{2}} \right\} =: \{\pm\lambda_1, \pm\lambda_2\}.$$

  Moreover, two of them are positive whereas the two others are negative. All the solutions to (26) are given by linear combinations of the functions

$$e^{\lambda_1 s}, \quad e^{-\lambda_1 s}, \quad e^{\lambda_2 s}, \quad e^{-\lambda_2 s}.$$

- **Case** $k = -2$**.** The solutions to (27) are $\lambda \in \{\pm 1\}$, both with multiplicity 2. All the solutions to (26) are given by linear combinations of the functions

$$e^{s}, \quad se^{s}, \quad e^{-s}, \quad se^{-s}.$$

Remark 8. For later use, we remark that the solution $w(s) = se^{-s}$ has a maximum point at $s = 1$, where $H(1) = -2e^{-2} < 0$, and it converges to 0 as $s \to \infty$. This example shows that Lemma 10 does not hold if $H(m) \leqq 0$.

- **Case** $-2 < k < 2$**.** The solutions to (27) are all complex and are given by

$$\lambda \in \left\{ \pm\frac{\sqrt{2-k}}{2} \pm i\,\frac{\sqrt{2+k}}{2}, \pm\frac{\sqrt{2-k}}{2} \mp i\,\frac{\sqrt{2+k}}{2} \right\}$$
$$=: \{\pm\alpha \pm i\beta, \pm\alpha \mp i\beta\}.$$

  Hence, the real part of these solutions can be either positive or negative. All the solutions to (26) are given by linear combinations of the functions

$$e^{\alpha s}\cos(\beta s), \quad e^{\alpha s}\sin(\beta s), \quad e^{-\alpha s}\cos(\beta s), \quad e^{-\alpha s}\sin(\beta s).$$

- **Case** $k = 2$**.** The solutions to (27) are $\lambda \in \{\pm i\}$, both with multiplicity 2. All the solutions to (26) are given by linear combinations of the functions

$$\cos(s), \quad \sin(s), \quad s\cos(s), \quad s\sin(s).$$

- **Case** $k > 2$**.** The solutions to (27) are all purely imaginary and are given by

$$\lambda \in \left\{ \pm i\sqrt{\frac{k + \sqrt{k^2-4}}{2}}, \pm i\sqrt{\frac{k - \sqrt{k^2-4}}{2}} \right\} =: \{\pm i\lambda_1, \pm i\lambda_2\}.$$

  All the solutions to (26) are given by linear combinations of the functions

$$\cos(\lambda_1 s), \quad \sin(\lambda_1 s), \quad \cos(\lambda_2 s), \quad \sin(\lambda_2 s).$$

In the table below we summarize the behavior of (nontrivial) solutions to (26).

k	number of zeros	limit at $\pm\infty$
$k < -2$	finite	$\{0, +\infty, -\infty\}$
$k = -2$	finite	$\{0, +\infty, -\infty\}$
$-2 < k < 2$	infinite	$\{0, +\infty, -\infty\}$ or $-\infty = \liminf < \limsup = +\infty$
$k = 2$	infinite	$-\infty < \liminf < 0 < \limsup < +\infty$ or $-\infty = \liminf < \limsup = +\infty$
$k > 2$	infinite	$-\infty < \liminf < 0 < \limsup < +\infty$

6. Energy Functions and Preliminary Lemmas

In this section we introduce some useful tools (energy functions) and we prove some lemmas which will enable us to reach the proofs of our main results. We point out that in some of the following statements the function f is not required to satisfy assumptions (10) and (11), but only weaker assumptions. However, all the results hold under assumptions (10)–(11).

To equation (1) we associate the *energy function*

$$\mathcal{E}(s) := \frac{k}{2}\, w'(s)^2 + w'(s)w'''(s) + F(w(s)) - \frac{1}{2}\, w''(s)^2. \tag{28}$$

Then, if w solves (1), there holds

$$\mathcal{E}'(s) = 0 \quad \Longrightarrow \quad \mathcal{E}(s) = C, \tag{29}$$

for some $C \in \mathbb{R}$.

We also define

$$G(s) := w'(s)^2 - w(s)w''(s) - \frac{k}{2}\, w(s)^2 \tag{30}$$

so that

$$H(s) := G'(s) = w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s) \tag{31}$$

and

$$H'(s) = G''(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)). \tag{32}$$

If $k \leqq 0$ and (2) holds, by (32) we infer that

$$G''(s) = H'(s) \geqq 0 \qquad \text{so that} \qquad H \text{ is nondecreasing and } G \text{ is convex.} \tag{33}$$

All the above properties follow by repeatedly using (1) in the computations. Further energy functions will be introduced under additional assumptions on f, see Lemma 12 below. These energy functions are quite useful for proving qualitative properties of the solution to (1). The first of such properties reads:

Lemma 9. *Let $k \leqq 0$ and assume that $f \in \mathrm{Lip_{loc}}(\mathbb{R})$ satisfies (11). Let w be a solution to (1) defined on some maximal interval $[0, R)$. Then, for the function H defined in (31), the following alternative holds:*

(i) *If $H(s)$ is bounded as $s \nearrow R$, then $R = +\infty$, $H(s) \leqq 0$ for all s and*

$$\lim_{s\to+\infty} H(s) = \lim_{s\to+\infty} w(s) = 0.$$

(ii) *If $H(0) > 0$, then*

$$\lim_{s\to R} H(s) = +\infty, \quad \lim_{s\to R} G(s) = +\infty,$$

and $w(s)$ is unbounded as $s \to R$.

Proof. If $R < +\infty$, Proposition 1 states that there exists a sequence of local maxima m_j such that $m_j \nearrow R$ and $w(m_j) \to \infty$ as $j \to \infty$. Hence, (29) shows that

$$\lim_{j\to\infty} \frac{w''(m_j)^2}{2} = \lim_{j\to\infty} [F(w(m_j)) - C] = +\infty.$$

Since $w''(m_j) < 0$, we infer that $G(m_j) = -w(m_j)w''(m_j) - \frac{k}{2}w(m_j)^2 \to \infty$ and, subsequently, $H(m_j) = G'(m_j) \to \infty$ in view of (33). Thus, we have proved that if $H(s)$ remains bounded, then $R = +\infty$. For the remaining statements, we refer to [3, Theorem 8] for the case $k < 0$ and to [18, Lemma 9] for the case $k = 0$. In the case where $R < \infty$ (statement (ii)) one should once more invoke Proposition 1. $\square$

Next, we turn our attention to geometric properties of the solution, such as monotonicity and concavity. The next two statements are also obtained by exploiting the features of the energy functions. In particular, Remark 8 shows that the next result may not hold if the assumption $H(m) > 0$ is violated.

Lemma 10. *Let $k \leqq 0$ and assume that f satisfies (2). Assume that a solution $w = w(s)$ to (1) admits a local maximum at some m such that $w(m) > 0$ and $H(m) > 0$. Then w is strictly concave in some maximal interval (m, ξ). In particular, in such an interval the solution w is strictly decreasing. Moreover:*

- *if $\xi = +\infty$, then $\lim_{s\to\infty} w(s) = -\infty$;*
- *if $\xi < +\infty$, then $w(\xi) < 0$ and $F(w(m)) < F(w(\xi))$.*
  Therefore, the solution w vanishes exactly once in (m, ξ).

Proof. The assumptions $w(m) > 0$ and $H(m) > 0$ imply that $w'''(m) < 0$. Hence, $w'''(s) < 0$ in some maximal right neighborhood (m, σ) of m. Since $w''(m) \leqq 0$, we also have that $w''(s) < 0$ in some maximal interval (m, ξ) with $\xi \geqq \sigma$ (equality holds only in the case where $\sigma = +\infty$).

If $\xi = +\infty$, then $\lim_{s\to\infty} w(s) = -\infty$ (recall that w is strictly decreasing).

If $\xi < +\infty$, then $\sigma < +\infty$ and

$$s \mapsto w''(s)^2 \qquad \text{is strictly increasing in } [m, \sigma]. \tag{34}$$

Note that $\sigma > m$ is the first stationary point of $w''(s)^2$ and $w'''(\sigma) = 0$ so that, by (29),

$$F(w(m)) - \frac{w''(m)^2}{2} = \mathcal{E}(m) = \mathcal{E}(\sigma) = \frac{k}{2}\, w'(\sigma)^2 + F(w(\sigma)) - \frac{w''(\sigma)^2}{2}.$$

Since $w''(\sigma)^2 > w''(m)^2$ by (34) and since $k \leqq 0$, we then have

$$F(w(\sigma)) - F(w(m)) = -\frac{k}{2}\, w'(\sigma)^2 + \frac{w''(\sigma)^2}{2} - \frac{w''(m)^2}{2} > 0. \tag{35}$$

Since $w(\sigma) < w(m)$ and since (2) implies that $t \mapsto F(t)$ is increasing for $t \geqq 0$, we necessarily have $w(\sigma) < 0$. Finally, since $\xi > \sigma$ and w is strictly decreasing in (σ, ξ), we have $w(\xi) < w(\sigma) < 0$ and, by (2) and (35), $F(w(m)) < F(w(\sigma)) < F(w(\xi))$. $\square$

Let $w = w(s)$ be a local solution to (1) and assume that there exists an interval $[z_1, z_2] \subset (0, +\infty)$ such that

$$w(z_1) = w(z_2) = 0 \qquad \text{and} \qquad w(s) > 0 \quad \forall s \in (z_1, z_2). \tag{36}$$

We now prove very precise geometric properties of w in these intervals; what follows can be extended to intervals where w is negative.

Lemma 11. *Let $k \leqq 0$ and assume that f satisfies (2). Let w be a solution to (1) defined on $[0, +\infty)$ and satisfying $H(0) > 0$ and $G(0) \geqq 0$. Assume that there exists an interval $[z_1, z_2] \subset (0, +\infty)$ such that (36) holds. Then the following facts hold:*

- (i) *$0 < w'(z_1) < -w'(z_2)$ and there exists a unique $m \in (z_1, z_2)$ such that $w'(m) = 0$;*
- (ii) *$w''(z_2) < 0 < w''(z_1)$, there exists a unique $r \in (z_1, z_2)$ where w'' changes sign, moreover $r < m$.*

Proof. Since $H(0) > 0$ and $G(0) \geqq 0$, by (31) and (33), we know that $0 \leqq G(0) < w'(z_1)^2 = G(z_1) < G(z_2) = w'(z_2)^2$. Hence, $0 < w'(z_1) < -w'(z_2)$. Moreover, w cannot admit two critical points in view of Lemma 10. This proves Item (i). By (33) we infer that $0 < H(0) < w'(z_1)w''(z_1) = H(z_1) < H(z_2) = w'(z_2)w''(z_2)$ which, together with the just proved Item (i), shows that $w''(z_2) < 0 < w''(z_1)$ and the existence of a first $r \in (z_1, z_2)$ such that $w''(r) = 0$ and w'' changes sign in r. Lemma 10 states that $r < m$. So, we just have to prove uniqueness of the point r in (z_1, m). If not, there exists a second point $\sigma \in (r, m)$ such that $w''(\sigma) = 0$ and w'' changes sign in σ. Since in r the function w'' changes from positive to negative, we necessarily have $w'''(r) \leqq 0$. Similarly, we have $w'''(\sigma) \geqq 0$. Hence,

$$w'(r)w'''(r) \leqq 0 \leqq w'(\sigma)w'''(\sigma). \tag{37}$$

Moreover, since $w''(s) < 0$ for $s \in (r, \sigma)$, we have $0 < w'(\sigma) < w'(r)$ and, in turn,

$$0 \geqq \frac{k}{2}\, w'(\sigma)^2 \geqq \frac{k}{2}\, w'(r)^2, \tag{38}$$

with strict inequalities if $k < 0$. Finally, recalling that (2) implies the monotonicity of F in $[0, \infty)$, since $w'(s) > 0$ for $s \in (r, \sigma)$, we have $F(w(r)) < F(w(\sigma))$. Combined with (37) and (38), this gives

$$\begin{aligned}\mathcal{E}(r) = \frac{k}{2}\, w'(r)^2 + w'(r)w'''(r) + F(w(r)) &< \frac{k}{2}\, w'(\sigma)^2 \\ &+ w'(\sigma)w'''(\sigma) + F(w(\sigma)) = \mathcal{E}(\sigma),\end{aligned}$$

which is in contradiction with (29). $\square$

The simple geometric properties of the solution found in Lemma 11 are displayed in Fig. 11.

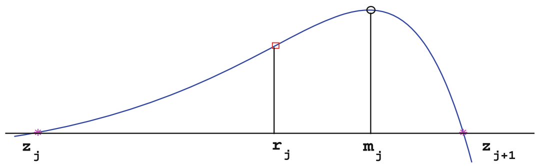

Fig. 11. Qualitative behavior of the solution w in the interval $[z_j, z_{j+1}]$

We now introduce two further energy functions. Let $w = w(s)$ be a local solution to (1) and let

$$\Phi(s) := \frac{w''(s)^2}{2} + F(w(s)), \qquad \Psi(s) := w''(s)^2 - \frac{k}{2}w'(s)^2 - w'(s)w'''(s). \tag{39}$$

Then, if f is increasing, we can prove

Lemma 12. *Assume* (2) *and* (10). *Assume that* $k \leqq 0$ *and let* $w = w(s)$ *be a nontrivial local solution to* (1). *Then* Φ *and* Ψ *are strictly convex functions. Moreover, if* w *admits a local maximum at some* m *such that* $w(m) > 0$ *and* $H(m) > 0$, *then* Φ *and* Ψ *are strictly increasing for* $s \geqq m$.

Proof. Note first that (2)–(10) imply that

$$f'(t) > 0 \qquad \forall t \neq 0. \tag{40}$$

By differentiating and by using (1), we obtain

$$\Phi'(s) = w''(s)w'''(s) + f(w(s))w'(s),$$
$$\Phi''(s) = w'''(s)^2 - kw''(s)^2 + f'(w(s))w'(s)^2.$$

By (40) and recalling that $k \leqq 0$, we obviously have $\Phi''(s) > 0$ for almost all s, except for at most some isolated s where $\Phi''(s) = 0$ or where Φ'' is not defined (when $w(s) = 0$). If the local maximum m exists, the assumptions $w(m) > 0$ and $H(m) > 0$ imply that $w'''(m) < 0$, that is, $\Phi'(m) \geqq 0$ and hence $\Phi'(s) > 0$ for all $s > m$. This proves the statements for Φ. Since $\mathcal{E}(s) = \Phi(s) - \Psi(s)$, by (29) we obtain $\Psi'(s) = \Phi'(s)$ and $\Psi''(s) = \Phi''(s)$, which prove the statements, also, for Ψ. $\square$

Finally, we prove a crucial and somewhat unexpected result. Roughly speaking, it states that (1) has no solutions eventually of one sign. If $k \geqq 0$, we recall from [3, Theorem 4] that a similar result holds by merely assuming (2) and

$$\liminf_{t \to \pm\infty} |f(t)| > 0. \tag{41}$$

Proposition 13. *Let* $k \geqq 0$ *and let* f *satisfy* (2) *and* (41). *If* w *is a nontrivial global solution to* (1), *then* $w(s)$ *changes sign infinitely many times as* $s \to +\infty$ *and as* $s \to -\infty$.

It is also known [3] that, under the sole assumption (2), this phenomenon may not occur when $k < 0$. Moreover, when $k \leqq 0$ the linear problem studied in Section 5 does have global solutions eventually of one sign. These are the reasons we believe that the next result is somehow surprising.

Lemma 14. *Let $k \leqq 0$ and assume that f satisfies (10) and (11). Let w be a local solution to (1) such that $H(0) > 0$ and $G(0) \geqq 0$. Then w cannot be continued on $[0, +\infty)$ as a solution eventually of one sign.*

Proof. If $k = 0$ this statement is known, see Proposition 13. So, take $k < 0$.

Assume first that there exists $\sigma \geqq 0$ such that $w(s) \geqq 0$ for $s \in [\sigma, +\infty)$. If w admits a local maximum at some $m > \sigma$, then $w(m) > 0$ and $H(m) > 0$, the latter in view of (33). Then, by Lemma 10, we would have that w changes sign, which is a contradiction. Therefore, w does not admit a local maximum and, in turn, w admits a limit $\ell \in [0, \infty]$ as $s \to \infty$. By Lemma 9 (ii), we necessarily have $\ell = +\infty$.

Thus we have shown that if $w(s)$ is eventually positive, then

$$\lim_{s \to +\infty} w(s) = +\infty. \tag{42}$$

In particular, since we have also just seen that w cannot admit a local maximum, this means that

$$w'(s) \geqq 0 \qquad \forall s \geqq \sigma. \tag{43}$$

Next, we study the second derivative. It cannot be that $w''(s) \leqq 0$ eventually, since otherwise from (1) and (42) we would obtain $w''''(s) = -kw''(s) - f(w(s)) \to -\infty$, implying that $w''(s) \to -\infty$ and, subsequently, that $w(s) \to -\infty$. Therefore, $w''(s) > 0$ on some interval (s_1, s_2) with $s_1 > \sigma$. If $w''(s_2) = 0$, then again by (1) we would get that $w''''(s_2) < 0$ and also that w'''' remains negative as long as w'' is negative. Hence, w'' is concave as long as it remains negative and therefore it is eventually negative, contradicting what we just said. This shows that there exists $\overline{s} \geqq \sigma$ such that

$$w''(s) > 0 \qquad \forall s > \overline{s}. \tag{44}$$

This also allows us to strengthen (43) with

$$w'(s) > 0 \qquad \forall s > \overline{s}. \tag{45}$$

By differentiating (1) twice we obtain

$$w^{(6)}(s) = -kw''''(s) - f''(w(s))w'(s)^2 - f'(w(s))w''(s). \tag{46}$$

It cannot be that $w''''(s) < 0$ eventually, since otherwise from (10)–(40)–(42)–(44)–(45)–(46) we would obtain $w^{(6)}(s) < -f''(w(s))w'(s)^2 < -c$ for some $c > 0$, implying that $w''''(s) \to -\infty$, against (44). Therefore, $w''''(s) > 0$ on some interval (s_1, s_2) with $s_1 > \overline{s}$. If $w''''(s_2) = 0$, then again by (46) we would get that $w^{(6)}(s_2) < 0$ and also that $w^{(6)}$ remains negative as long as w'''' is negative.

Hence, w'''' is concave as long as it remains negative and therefore it is eventually negative, contradicting what we just said. This shows that there exists $\overline{\sigma}$ such that

$$w''''(s) > 0 \qquad \forall s > \overline{\sigma}. \tag{47}$$

By (1) and (47), we readily obtain that $kw''(s) + f(w(s)) < 0$ for all $s > \overline{\sigma}$. By multiplying this inequality by $w'(s)$ and recalling (45), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{k}{2}w'(s)^2 + F(w(s))\right] = kw''(s)w'(s) + f(w(s))w'(s) < 0 \qquad \forall s > \overline{\sigma}.$$

By (11), this proves that there exists $c_1 \in \mathbb{R}$ such that

$$\frac{k}{2}w'(s)^2 + \frac{\rho}{p+1}w(s)^{p+1} \leqq \frac{k}{2}w'(s)^2 + F(w(s)) \leqq c_1 \qquad \forall s > \overline{\sigma}.$$

Hence, if we divide by $w(s)^{p+1}$ and we recall (42), we get

$$2c_2^2 := \frac{2\rho}{|k|(p+1)} \leqq \frac{w'(s)^2}{w(s)^{p+1}} + o(1) \qquad \forall s > \overline{\sigma}.$$

By taking the square root and choosing a sufficiently large $\sigma_0 > \overline{\sigma}$, this shows that

$$c_2 \leqq \frac{w'(s)}{w(s)^{(p+1)/2}} \qquad \forall s > \sigma_0.$$

By integrating over (σ_0, s) this gives

$$c_2(s - \sigma_0) \leqq \frac{2}{p-1}\left(\frac{1}{w(\sigma_0)^{(p-1)/2}} - \frac{1}{w(s)^{(p-1)/2}}\right),$$

and we get a contradiction by letting $s \to \infty$. This shows that w cannot be eventually positive.

Similarly, by reversing all signs, we can reach a contradiction if $w(s)$ is eventually negative. □

7. Proof of Theorem 2

Step 1. Organization of the proof.

Denote by $[0, R)$ the maximal interval of continuation of the local solution $w = w(s)$. In order to prove that $R < +\infty$, we need some delicate estimates, see Steps 2–3–4–5 below. Once these estimates are obtained, in Step 6 we prove that $R < +\infty$. However, before doing this, we need to remark on some preliminary facts, regardless of whether R is finite or infinite.

Items (i) and (ii) follow from Proposition 1 in the case where $R < +\infty$ and from Lemma 14 in the case where $R = +\infty$.

For all j let $m_j \in (z_j, z_{j+1})$ be the point where $|w(s)|$ attains its maximum on $[z_j, z_{j+1}]$ and let $M_j = w(m_j)$. If $R < +\infty$, by Proposition 1 we infer that

$$\limsup_{j\to\infty} |M_j| = +\infty.$$

Therefore, there exists a subsequence $\{m_h\} \subset \{m_j\}$ such that

$$\lim_{h\to\infty} |M_h| = +\infty.$$

In view of (11) and (29) we then infer

$$\lim_{h\to\infty} w''(m_h)^2 = 2 \lim_{h\to\infty} [F(M_h) - C] = +\infty.$$

Hence, recalling the definition of G in (30) and noticing that $w(m_h)w''(m_h) < 0$, we get

$$\lim_{h\to\infty} G(m_h) = -\lim_{h\to\infty} \left[M_h w''(m_h) + \frac{k}{2} M_h^2 \right] = +\infty.$$

By (33) we then deduce that $\lim_{s\nearrow R} G(s) = +\infty$ without extracting subsequences. In particular, we get that

$$\lim_{j\to\infty} G(m_j) = -\lim_{j\to\infty} \left[M_j w''(m_j) + \frac{k}{2} M_j^2 \right] = +\infty \tag{48}$$

on the whole sequence $\{m_j\}$ of maxima of $|w(s)|$. Using (29) again, we obtain that

$$|w''(m_j)| = \sqrt{2(F(M_j) - C)} \tag{49}$$

which, replaced into (48), proves that

$$\lim_{j\to+\infty} |M_j| = +\infty \tag{50}$$

whenever $R < +\infty$. If $R = +\infty$, in what follows we assume that the local solution $w = w(s)$ can be continued as $s \to +\infty$ so that w is defined (at least) on $[0, +\infty)$. By Lemma 14 we know that $w(s)$ changes sign infinitely many times as $s \to +\infty$. Note that by (30) and Lemma 9, we have again (48), whereas by (29), we again have (49). Hence, we obtain (50) in the case $R = +\infty$, also.

Since (12) is equivalent to $H(0) > 0$, by Lemma 9 (ii) we know that there exists $\sigma \geqq 0$ such that $H(\sigma) > 0$ and $G(\sigma) \geqq 0$. Since (1) is autonomous, we may assume that $\sigma = 0$. Hence, Lemma 11 applies. We now prove some estimates related to the points found in Lemma 11. For sake of simplicity, we denote by (z_j, z_{j+1}) an interval where $w(s) > 0$ and by (z_{j-1}, z_j) an interval where $w(s) < 0$; moreover, we put $M_j = w(m_j) > 0$ and $M_{j-1} = w(m_{j-1}) < 0$. Clearly the estimates below can be reversed on intervals where w has the opposite sign.

Step 2. We prove that

$$\lim_{j\to\infty} (r_j - z_j) = 0. \tag{51}$$

Assume for contradiction that the claim is false, so that $\limsup_{j\to\infty}(r_j - z_j) > 0$. Then there exists $a > 0$ and a subsequence (still denoted in the same way) such that $(r_j - z_j) \geqq a$ for all j. By Lemma 9 we know that $w'(z_j)^2 = G(z_j) \to +\infty$ as $j \to \infty$. In turn, by Lemma 11, we know that $w'(s) \to +\infty$ for all $s \in [z_j, r_j]$ as $j \to \infty$. Finally, this proves that

$$w(z_j + \sigma) \to +\infty \qquad \forall \sigma \in (0, r_j - z_j] \quad \text{as } j \to \infty. \tag{52}$$

Let $h(s) := (s - z_j)^3 (r_j - s)^4$. By (52) and assumption (11), we infer that

$$\begin{aligned} &h(z_j + \sigma) f(w(z_j + \sigma)) + k h''(z_j + \sigma) w(z_j + \sigma) \\ &\quad + h''''(z_j + \sigma) w(z_j + \sigma) \to +\infty \\ &\quad \forall \sigma \in (0, r_j - z_j) \end{aligned} \tag{53}$$

as $j \to \infty$. Multiply (1) by $h(s)$ and integrate over $[z_j, r_j]$. Since h, h', h'' vanish in $\{z_j, r_j\}$ and $h'''(r_j) = 0$, four integration by parts yield

$$\int_{z_j}^{r_j} [h(s) f(w(s)) + k h''(s) w(s) + h''''(s) w(s)] \, ds = 0.$$

This contradicts (53) unless (51) holds and this contradiction proves the claim of Step 2.

Step 3. We prove that there exists $C_1 = C_1(\rho, p) > 0$ such that if j is sufficiently large, then

$$r_j - z_j \leqq \frac{C_1}{|M_{j-1}|^{(p-1)/4}}. \tag{54}$$

In what follows, c denotes a positive constant which depends on ρ and p and which may vary from line to line, and also within the same formula. Let

$$h(s) := \sin^4 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right)$$

so that

$$\begin{aligned} h'(s) &:= \frac{4\pi}{r_j - z_j} \sin^3 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \cos \left(\pi \, \frac{s - z_j}{r_j - z_j} \right), \\ h''(s) &:= \frac{4\pi^2}{(r_j - z_j)^2} \left[3 \sin^2 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) - 4 \sin^4 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \right], \\ h'''(s) &:= \frac{8\pi^3}{(r_j - z_j)^3} \left[3 \sin \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \cos \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \right. \\ &\qquad \left. - 8 \sin^3 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \cos \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \right], \\ h''''(s) &:= \frac{8\pi^4}{(r_j - z_j)^4} \left[3 - 30 \sin^2 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) + 32 \sin^4 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \right]. \end{aligned}$$

Multiply (1) by $h(s)$ and integrate over $[z_j, r_j]$. Since h, h', h'', h''' vanish in $\{z_j, r_j\}$, four integrations by parts yield

$$\int_{z_j}^{r_j} h(s) f(w(s))\,\mathrm{d}s = -\int_{z_j}^{r_j} [kh''(s) + h''''(s)]w(s)\,\mathrm{d}s. \tag{55}$$

Our purpose is now to estimate the terms in (55). Before doing this, we need some energy arguments. Since m_{j-1} is a minimum for w, we have $w''(m_{j-1}) > 0$ so that, by (29), we have $w''(m_{j-1}) = \sqrt{2(F(M_{j-1}) - C)}$. Hence, by (50),

$$\begin{aligned} G(m_{j-1}) &= |M_{j-1}| w''(m_{j-1}) - \frac{k}{2} M_{j-1}^2 \\ &\geqq |M_{j-1}| \sqrt{2(F(M_{j-1}) - C)} \geqq c |M_{j-1}|^{\frac{p+3}{2}}, \end{aligned}$$

where the last inequality follows from assumption (11). By taking into account (33), we then infer that

$$w'(z_j)^2 = G(z_j) > G(m_{j-1}) \geqq c|M_{j-1}|^{\frac{p+3}{2}}.$$

Since $w(s)$ is convex in $[z_j, r_j]$, see Lemma 11, we then deduce

$$w(s) \geqq c|M_{j-1}|^{\frac{p+3}{4}}(s - z_j) \qquad \forall s \in [z_j, r_j].$$

In particular, by (11) we also have

$$f(w(s)) \geqq \rho w(s)^p \geqq c|M_{j-1}|^{\frac{(p+3)(p-1)}{4}} (s - z_j)^{p-1} w(s) \qquad \forall s \in [z_j, r_j]. \tag{56}$$

Next, we estimate

$$\begin{aligned} &-kh''(s) - h''''(s) \\ &= -\frac{4k\pi^2}{(r_j - z_j)^2} \left[3\sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right) - 4\sin^4\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)\right] \\ &\quad -\frac{8\pi^4}{(r_j - z_j)^4} \left[3 - 30\sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right) + 32\sin^4\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)\right] \\ &\leqq \frac{12|k|\pi^2}{(r_j - z_j)^2} \sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right) + \frac{8\pi^4}{(r_j - z_j)^4}\left[-3 + 30\sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)\right] \\ &\leqq \frac{24\pi^4}{(r_j - z_j)^4}\left[-1 + 11\sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)\right], \end{aligned} \tag{57}$$

where the last inequality follows from (51), provided j is sufficiently large. By inserting (56) and (57) into (55), we obtain

$$\begin{aligned} &|M_{j-1}|^{\frac{(p+3)(p-1)}{4}} \int_{z_j}^{r_j} \sin^4\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)(s - z_j)^{p-1} w(s)\,\mathrm{d}s \\ &\qquad \leqq \frac{c}{(r_j - z_j)^4} \int_{z_j}^{r_j} \left[-1 + 11\sin^2\left(\pi\,\frac{s - z_j}{r_j - z_j}\right)\right] w(s)\,\mathrm{d}s. \end{aligned} \tag{58}$$

Let

$$\gamma := \frac{1}{\pi} \arcsin \frac{1}{\sqrt{11}} \simeq 0.0975$$

and notice that

$$11 \sin^2 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \leqq 1 \qquad \forall s \in [z_j, z_j + \gamma (r_j - z_j)] \cup [r_j - \gamma (r_j - z_j), r_j].$$

Therefore, from (58) we deduce

$$\begin{aligned} &|M_{j-1}|^{\frac{(p+3)(p-1)}{4}} \int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} \sin^4 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) (s - z_j)^{p-1} w(s) \, \mathrm{d}s \\ &\quad \leqq \frac{c}{(r_j - z_j)^4} \int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} \left[-1 + 11 \sin^2 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \right] w(s) \, \mathrm{d}s. \qquad (59) \end{aligned}$$

On the new interval of integration $[z_j + \gamma(r_j - z_j), r_j - \gamma(r_j - z_j)]$, we have uniform bounds such as

$$\sin^4 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \geqq \frac{1}{121}, \qquad -1 + 11 \sin^2 \left(\pi \, \frac{s - z_j}{r_j - z_j} \right) \leqq 10.$$

Hence, from (59) we may finally obtain

$$\begin{aligned} &|M_{j-1}|^{\frac{(p+3)(p-1)}{4}} \gamma^{p-1} (r_j - z_j)^{p-1} \int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} w(s) \, \mathrm{d}s \\ &\quad \leqq |M_{j-1}|^{\frac{(p+3)(p-1)}{4}} \int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} (s - z_j)^{p-1} w(s) \, \mathrm{d}s \\ &\quad \leqq \frac{c}{(r_j - z_j)^4} \int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} w(s) \, \mathrm{d}s, \end{aligned}$$

which we rewrite as (54) for some $C_1 > 0$ depending only on ρ and p which appear in (11).

Step 4. We prove that there exists $C_2 = C_2(\rho, p) > 0$ such that if j is sufficiently large, then

$$z_{j+1} - m_j \leqq \frac{C_2}{M_j^{(p-1)/4}}. \qquad (60)$$

Let $h(s) := (s - m_j)^2 (z_{j+1} - s)^3$ and note that

$$h(m_j) = h'(m_j) = h(z_{j+1}) = h'(z_{j+1}) = h''(z_{j+1}) = 0. \qquad (61)$$

For all $\ell \in \{0, 1, 2, 3, 4\}$ let $h^{(\ell)}$ denote the ℓ-th derivative of h. Clearly, $h^{(\ell)}$ is a linear combination of polynomials such as $(s - m_j)^a (z_{j+1} - s)^b$ with $a + b = 5 - \ell$. Therefore,

$$\begin{aligned} &\forall \ell \in \{0, 1, 2, 3, 4\} \quad \exists c_\ell > 0 \quad \text{such that } |h^{(\ell)}(s)| \leqq c_\ell (z_{j+1} - m_j)^{5-\ell} \\ &\forall s \in [m_j, z_{j+1}]. \qquad (62) \end{aligned}$$

Recall that $w(m_j) = M_j$ and $w(z_{j+1}) = w'(m_j) = 0$; then, by (61), four integrations by parts yield

$$-\int_{m_j}^{z_{j+1}} w''''(s)h(s)\,\mathrm{d}s = -\int_{m_j}^{z_{j+1}} w(s)h''''(s)\,\mathrm{d}s - h'''(m_j)M_j,$$

so that, by (62),

$$\begin{aligned}-\int_{m_j}^{z_{j+1}} w''''(s)h(s)\,\mathrm{d}s &\leqq c_4(z_{j+1}-m_j)\int_{m_j}^{z_{j+1}} w(s)\,\mathrm{d}s + c_3(z_{j+1}-m_j)^2 M_j\\ &\leqq (c_3+c_4)(z_{j+1}-m_j)^2 M_j. \end{aligned} \tag{63}$$

Similarly, two integrations by parts yield

$$-k\int_{m_j}^{z_{j+1}} w''(s)h(s)\,\mathrm{d}s = -k\int_{m_j}^{z_{j+1}} w(s)h''(s)\,\mathrm{d}s \leqq c_2|k|(z_{j+1}-m_j)^4 M_j. \tag{64}$$

In view of Lemma 11 we know that w is concave over $[m_j, z_{j+1}]$ so that

$$w(s) \geqq \frac{M_j(z_{j+1}-s)}{z_{j+1}-m_j} \qquad \forall s \in [m_j, z_{j+1}].$$

Then, by assumption (11), we infer that

$$f(w(s)) \geqq \rho w(s)^p \geqq \rho M_j^p \frac{(z_{j+1}-s)^p}{(z_{j+1}-m_j)^p} \qquad \forall s \in [m_j, z_{j+1}].$$

Therefore,

$$\begin{aligned}\int_{m_j}^{z_{j+1}} f(w(s))h(s)\,\mathrm{d}s &\geqq \frac{\rho M_j^p}{(z_{j+1}-m_j)^p}\int_{m_j}^{z_{j+1}} (s-m_j)^2(z_{j+1}-s)^{p+3}\,\mathrm{d}s\\ (\sigma = z_{j+1}-s) &= \frac{\rho M_j^p}{(z_{j+1}-m_j)^p}\int_0^{z_{j+1}-m_j} \sigma^{p+3}(z_{j+1}-m_j-\sigma)^2\,\mathrm{d}s\\ &= \frac{2\rho M_j^p}{(p+4)(p+5)(p+6)}(z_{j+1}-m_j)^6. \end{aligned} \tag{65}$$

Multiply (1) by $h(s)$, and integrate over $[m_j, z_{j+1}]$ to obtain

$$\int_{m_j}^{z_{j+1}} f(w(s))h(s)\,\mathrm{d}s = -\int_{m_j}^{z_{j+1}} w''''(s)h(s)\,\mathrm{d}s - k\int_{m_j}^{z_{j+1}} w''(s)h(s)\,\mathrm{d}s.$$

By plugging the estimates (63), (64), (65) into this identity, we get

$$\begin{aligned}&\frac{2\rho M_j^p}{(p+4)(p+5)(p+6)}(z_{j+1}-m_j)^6\\ &\quad\leqq (c_3+c_4)(z_{j+1}-m_j)^2 M_j + c_2|k|(z_{j+1}-m_j)^4 M_j\end{aligned}$$

that is,

$$\frac{2\rho M_j^{p-1}}{(p+4)(p+5)(p+6)}(z_{j+1}-m_j)^4 - c_2|k|(z_{j+1}-m_j)^2 - (c_3+c_4) \leqq 0.$$

By solving this biquadratic algebraic inequality, we infer that

$$(z_{j+1}-m_j)^2 \leqq \frac{c_2|k| + \sqrt{c_2^2k^2 + 8\rho M_j^{p-1}(c_3+c_4)/\omega}}{4\rho M_j^{p-1}}\,\omega, \quad \text{with}$$
$$\omega = (p+4)(p+5)(p+6).$$

Finally, this yields (60) for some $C_2 > 0$ depending only on ρ and p which appear in (11).

Step 5. We prove that there exists $C_3 = C_3(\rho, p) > 0$ such that if j is sufficiently large, then

$$m_j - r_j \leqq \frac{C_3}{M_j^{(p-1)/4}}. \tag{66}$$

We proceed as in Step 4 but with a different test function.
Let $h(s) := (s-r_j)^4(m_j-s)^2$ and note that

$$h(m_j) = h'(m_j) = h(r_j) = h'(r_j) = h''(r_j) = h'''(r_j) = 0. \tag{67}$$

For all $\ell \in \{0, 1, 2, 3, 4\}$ let $h^{(\ell)}$ denote the ℓ-th derivative of h. Clearly, $h^{(\ell)}$ is a linear combination of polynomials such as $(s-r_j)^a(m_j-s)^b$ with $a+b = 6-\ell$. Therefore,

$$\forall \ell \in \{0,1,2,3,4\} \quad \exists c_\ell > 0 \quad \text{such that } |h^{(\ell)}(s)| \leqq c_\ell (m_j - r_j)^{6-\ell}$$
$$\forall s \in [r_j, m_j]. \tag{68}$$

Recall that $w(m_j) = M_j$ and $w'(m_j) = 0$; then, by (67), four integrations by parts yield

$$-\int_{r_j}^{m_j} w''''(s)h(s)\,\mathrm{d}s = -\int_{r_j}^{m_j} w(s)h''''(s)\,\mathrm{d}s + h'''(m_j)M_j,$$

so that, by (68),

$$-\int_{r_j}^{m_j} w''''(s)h(s)\,\mathrm{d}s \leqq c_4(m_j-r_j)^2$$
$$\times \int_{r_j}^{m_j} w(s)\,\mathrm{d}s + c_3(m_j-r_j)^3 M_j \leqq (c_3+c_4)(m_j-r_j)^3 M_j. \tag{69}$$

Similarly, two integrations by parts yield

$$-k\int_{r_j}^{m_j} w''(s)h(s)\,\mathrm{d}s = -k\int_{r_j}^{m_j} w(s)h''(s)\,\mathrm{d}s \leqq c_2|k|(m_j-r_j)^5 M_j. \tag{70}$$

By Lemma 11 we know that w is concave over $[r_j, m_j]$, so that (this inequality is far from being optimal!)

$$w(s) \geqq \frac{M_j(s-r_j)}{m_j - r_j} \qquad \forall s \in [r_j, m_j].$$

Then, by assumption (11), we infer that

$$f(w(s)) \geqq \rho w(s)^p \geqq \rho M_j^p \frac{(s-r_j)^p}{(m_j-r_j)^p} \qquad \forall s \in [r_j, m_j].$$

Therefore,

$$\begin{aligned}\int_{r_j}^{m_j} f(w(s))h(s)\,\mathrm{d}s &\geqq \frac{\rho M_j^p}{(m_j-r_j)^p}\int_{r_j}^{m_j}(s-r_j)^{p+4}(m_j-s)^2\,\mathrm{d}s\\ (\sigma = s-r_j) &= \frac{\rho M_j^p}{(m_j-r_j)^p}\int_0^{m_j-r_j}\sigma^{p+4}(m_j-r_j-\sigma)^2\,\mathrm{d}s\\ &= \frac{2\rho M_j^p}{(p+5)(p+6)(p+7)}(m_j-r_j)^7. \qquad (71)\end{aligned}$$

Multiply (1) by $h(s)$ and integrate over $[r_j, m_j]$ to obtain

$$\int_{r_j}^{m_j} f(w(s))h(s)\,\mathrm{d}s = -\int_{r_j}^{m_j} w''''(s)h(s)\,\mathrm{d}s - k\int_{r_j}^{m_j} w''(s)h(s)\,\mathrm{d}s.$$

By plugging the estimates (69), (70), (71) into this identity, we get

$$\begin{aligned}&\frac{2\rho M_j^p}{(p+5)(p+6)(p+7)}(m_j-r_j)^7\\ &\qquad \leqq (c_3+c_4)(m_j-r_j)^3 M_j + c_2|k|(m_j-r_j)^5 M_j,\end{aligned}$$

that is,

$$\frac{2\rho M_j^{p-1}}{(p+5)(p+6)(p+7)}(m_j-r_j)^4 - c_2|k|(m_j-r_j)^2 - (c_3+c_4) \leqq 0.$$

By solving this biquadratic algebraic inequality, we infer that

$$\begin{aligned}(m_j-r_j)^2 &\leqq \frac{c_2|k| + \sqrt{c_2^2k^2 + 8\rho M_j^{p-1}(c_3+c_4)/\omega}}{4\rho M_j^{p-1}}\,\omega, \qquad \text{with}\\ \omega &= (p+5)(p+6)(p+7).\end{aligned}$$

Finally, this yields (66) for some $C_3 > 0$ depending only on ρ and p, which appear in (11).

Step 6. We show that $R < +\infty$.

By (12) we may apply Lemma 12 to obtain, for all $j \in \mathbb{N}$,

$$\begin{aligned} F(M_j) &= F(w(m_j)) > F(w(r_j)) = \Phi(r_j) \\ &> \Phi(m_{j-1}) = \frac{w''(m_{j-1})^2}{2} + F(w(m_{j-1})) = 2F(M_{j-1}) - C, \end{aligned} \tag{72}$$

the latter equality being a consequence of

$$\mathcal{E}(m_{j-1}) = F(w(m_{j-1})) - \frac{w''(m_{j-1})^2}{2} = C,$$

which holds in view of (29). In particular, (72) shows that

$$j \mapsto F(M_j) \quad \text{is strictly increasing and} \quad \lim_{j\to\infty} F(M_j) = +\infty. \tag{73}$$

By iterating (72) we find $F(M_j) > 2^j[F(M_0) - C] + C$ for all $j \geqq 1$. In turn, by (73) we may relabel the indices j (in such a way that $F(M_0) > 2C$) and obtain

$$F(M_j) > 2^{j-1} F(M_0) \qquad \forall j \in \mathbb{N}. \tag{74}$$

Moreover, by using (11), (72) gives

$$\begin{aligned} \frac{\alpha}{q+1} M_j^{q+1} + \frac{\beta}{p+1} M_j^{p+1} &\geqq F(M_j) > 2F(M_{j-1}) - C \\ &\geqq \frac{2\rho}{p+1} |M_{j-1}|^{p+1} - C, \end{aligned}$$

so that, by (73) and by possibly relabeling j, we infer that

$$|M_j|^{p+1} \geqq c(\beta, \rho) \, |M_{j-1}|^{p+1} \qquad \forall j \in \mathbb{N}. \tag{75}$$

By combining (75) with (54)–(60)–(66)–(74), we readily obtain that

$$\begin{aligned} m_j - m_{j-1} &\leqq \frac{\kappa_1}{|M_{j-1}|^{(p-1)/4}} \leqq \frac{c}{[F(M_{j-1})]^{(p-1)/4(p+1)}} \\ &\leqq c \left(\frac{1}{2^{(p-1)/4(p+1)}} \right)^{j-1} \qquad \forall j \geqq 1 \end{aligned} \tag{76}$$

for some $\kappa_1 = \kappa_1(\beta, \rho, p) > 0$. This proves the first part of (13). Finally, by combining (75) with (76), we obtain

$$R - m_0 = \sum_{j=0}^{\infty} (m_{j+1} - m_j) \leqq c \sum_{j=0}^{\infty} \left(\frac{1}{2^{(p-1)/4(p+1)}} \right)^j < +\infty,$$

since the geometric series has ratio $\left(\frac{1}{2}\right)^{\frac{p-1}{4(p+1)}} < 1$. Therefore, $R < +\infty$ and the solution blows up in finite time.

Step 7. We prove the second part of (13). First of all, we recall the well-known Poincaré-type inequalities

$$\|u\|_2 \leqq (z_{j+1}-z_j)\|u'\|_2 \leqq (z_{j+1}-z_j)^2\|u''\|_2 \qquad \forall u \in H^2 \cap H^1_0(z_j, z_{j+1}), \tag{77}$$

where $\|\cdot\|_2$ denotes the $L^2(z_j, z_{j+1})$-norm, whereas H^2 and H^1_0 represent the usual Sobolev spaces. Next, we observe that, upon integration, (11) yields

$$F(t) \leqq \frac{\alpha}{q+1}|t|^{q+1} + \frac{\beta}{p+1}|t|^{p+1} \qquad \forall t \in \mathbb{R}. \tag{78}$$

We may now start the proof of the estimate. Some integrations by parts and (1) yield

$$\begin{aligned}\int_{z_j}^{z_{j+1}} w'(s)w'''(s)\,\mathrm{d}s &= -\int_{z_j}^{z_{j+1}} w(s)w''''(s)\,\mathrm{d}s\\ &= \int_{z_j}^{z_{j+1}} w(s)[kw''(s) + f(w(s))]\,\mathrm{d}s\\ &= -k\int_{z_j}^{z_{j+1}} w'(s)^2\,\mathrm{d}s + \int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s.\end{aligned}$$

Hence, if we integrate (28) over $[z_j, z_{j+1}]$ we obtain

$$\begin{aligned}&2\int_{z_j}^{z_{j+1}}[F(w(s)) + f(w(s))w(s)]\,\mathrm{d}s\\ &= \int_{z_j}^{z_{j+1}}[w''(s)^2 + kw'(s)^2]\,\mathrm{d}s + 2C(z_{j+1}-z_j),\end{aligned}$$

where C is as in (29). Using (11), (77), (78), the latter identity yields the estimate

$$\begin{aligned}&2\alpha\frac{q+2}{q+1}\int_{z_j}^{z_{j+1}}|w(s)|^{q+1}\,\mathrm{d}s + 2\beta\frac{p+2}{p+1}\int_{z_j}^{z_{j+1}}|w(s)|^{p+1}\,\mathrm{d}s\\ &\geqq (1+o(1))\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s + o(1)\\ &\geqq \frac{1+o(1)}{(z_{j+1}-z_j)^4}\int_{z_j}^{z_{j+1}} w(s)^2\,\mathrm{d}s + o(1),\end{aligned}$$

where $o(1)$ are infinitesimals due (76). We may then further estimate

$$[c(\beta,p)+o(1)]\,M_j^{p-1}\int_{z_j}^{z_{j+1}} w(s)^2\,\mathrm{d}s \geqq \frac{1+o(1)}{(z_{j+1}-z_j)^4}\int_{z_j}^{z_{j+1}} w(s)^2\,\mathrm{d}s$$

which finally gives

$$c(\beta,p)M_j^{p-1} \geqq \frac{1}{(z_{j+1}-z_j)^4}.$$

Step 8. Conclusion.

Since the above proof of Theorem 2 is quite lengthy and delicate, let us indicate the exact points where the statements were reached.

The fact that $R < +\infty$ is proved in Step 6. Statements (i) and (ii) are proved in Step 1. Statements (iii) and (iv) follow from Lemma 11. Statement (v) follows from (50) and (73). Statement (vi) is proved in (76) (first estimate) and in Step 7 (second estimate).

8. Proof of Theorem 3

By Proposition 1, we know that (8) holds. Denote by $[z_j, z_{j+1}]$ an interval of positivity (or negativity) for the solution w and note that two integrations by parts yield

$$\int_{z_j}^{z_{j+1}} w''''(s)w(s)\,\mathrm{d}s = \int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s - \Big[w''(s)w'(s)\Big]_{z_j}^{z_{j+1}}.$$

Hence, if we multiply (1) by $w(s)$ and integrate over (z_j, z_{j+1}), we obtain

$$\begin{aligned}\Big[w''(s)w'(s)\Big]_{z_j}^{z_{j+1}} &= \int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s + k\int_{z_j}^{z_{j+1}} w''(s)w(s)\,\mathrm{d}s\\ &\quad + \int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s. \end{aligned} \tag{79}$$

On the other hand, if we integrate the energy $\mathcal{E}$ in (28) over (z_j, z_{j+1}), by (29) we get

$$\begin{aligned}\Big[w''(s)w'(s)\Big]_{z_j}^{z_{j+1}} &= C(z_{j+1}-z_j) + \frac{3}{2}\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s + \frac{k}{2}\int_{z_j}^{z_{j+1}} w''(s)w(s)\,\mathrm{d}s\\ &\quad - \int_{z_j}^{z_{j+1}} F(w(s))\,\mathrm{d}s. \end{aligned} \tag{80}$$

By combining (79) with (80) we infer

$$\begin{aligned}\int_{z_j}^{z_{j+1}} \Big(f(w(s))w(s) + F(w(s))\Big)\,\mathrm{d}s &= C(z_{j+1}-z_j) + \frac{1}{2}\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s\\ &\quad - \frac{k}{2}\int_{z_j}^{z_{j+1}} w''(s)w(s)\,\mathrm{d}s. \end{aligned} \tag{81}$$

Next, we estimate

$$\begin{aligned} I(j) := \left|\int_{z_j}^{z_{j+1}} w''(s)w(s)\,\mathrm{d}s\right| &\leqq \int_{z_j}^{z_{j+1}} |w''(s)|\Big(f(w(s))w(s)\Big)^{\frac{1}{p+1}}\\ &\quad\times\left(\frac{|w(s)|^p}{|f(w(s))|}\right)^{\frac{1}{p+1}}\,\mathrm{d}s, \end{aligned}$$

so that, by Hölder's inequality,

$$I(j) \leqq \left(\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s\right)^{\frac{1}{p+1}}$$
$$\times\left(\int_{z_j}^{z_{j+1}} \left(\frac{|w(s)|^p}{|f(w(s))|}\right)^{\frac{2}{p-1}} \mathrm{d}s\right)^{\frac{p-1}{2(p+1)}}.$$

Next, we use (11) to estimate further

$$I(j) \leqq \frac{(z_{j+1}-z_j)^{\frac{p-1}{2(p+1)}}}{\rho^{\frac{1}{p+1}}} \left(\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s\right)^{\frac{1}{p+1}}.$$

In turn, by Young's inequality, we get

$$I(j) \leqq \frac{(z_{j+1}-z_j)^{\frac{p-1}{2(p+1)}}}{2\rho^{\frac{1}{p+1}}} \left(\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s + \left(\int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s\right)^{\frac{2}{p+1}}\right).$$

Recalling that $p > 1$, we have thus proved that

$$I(j) = o\left(\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s\right) + o\left(\int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s\right)$$

as $j \to \infty$. Inserting this estimate into (81) we get

$$\int_{z_j}^{z_{j+1}} \Big(f(w(s))w(s) + F(w(s))\Big)\,\mathrm{d}s \;+\; o\left(\int_{z_j}^{z_{j+1}} f(w(s))w(s)\,\mathrm{d}s\right)$$
$$= o(1) + \frac{1}{2}\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s \;+\; o\left(\int_{z_j}^{z_{j+1}} w''(s)^2\,\mathrm{d}s\right).$$

The result then follows by letting $j \to \infty$.

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