Blow up oscillating solutions
to some nonlinear fourth order differential equations

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Abstract

We give strong theoretical and numerical evidence that solutions to some nonlinear fourth order ordinary differential equations blow up in finite time with infinitely many wild oscillations. We exhibit an explicit example where this phenomenon occurs. We discuss possible applications to biharmonic partial differential equations and to the suspension bridges model. In particular, we give a possible new explanation of the collapse of bridges.

Mathematics Subject Classification: 34C10, 34A12, 65L05.

1 Introduction

In this paper we study the differential equation

$$w''''(s)+kw''(s)+f(w(s))=0 \qquad (s\in\mathbb{R}) \tag{1}$$

where $k\in\mathbb{R}$ and f is a locally Lipschitz function.

This equation arises in several contexts. With no hope of being exhaustive, let us mention some models which lead to (1). When k is negative (1) is known as the extended Fisher-Kolmogorov equation, whereas when k is positive it is referred to as Swift-Hohenberg equation, see [19]. Equation (1) arises in the dynamic phase-space analogy of a nonlinearly supported elastic strut [12] and serves as a model of pattern formation in many physical, chemical or biological systems, see [4, 5] and references therein. It may also be used to investigate localization and spreading of deformation of a strut confined by an elastic foundation [20]. A particularly interesting model concerns traveling waves in a suspension bridge, see [8, 13, 17] and Section 3.1. Equation (1) also arises from a suitable transformation of some biharmonic pde's, see [2, 3] and Section 3.2. Last but not least, we mention the important book by Peletier-Troy [19] where one can find many other physical models, a survey of existing results, and further references.

The purpose of the present paper is to contribute to a better understanding of possible finite time blow up for solutions to (1) when the nonlinearity f satisfies

$$f\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\,,\quad f(t)\,t>0\quad \text{for every } t\in\mathbb{R}\setminus\{0\}. \tag{2}$$

Further assumptions on f will be needed in the sequel. We are interested in necessary and/or sufficient conditions for local solutions to be global.

The starting point of our study are the following results proved in [3]:

Proposition 1. *Let $k\in\mathbb{R}$ and assume that f satisfies* (2).

(i) *If a local solution w to* (1) *blows up at some finite $R\in\mathbb{R}$, then*

$$\liminf_{s\to R} w(s)=-\infty \qquad \textit{and} \qquad \limsup_{s\to R} w(s)=+\infty\,. \tag{3}$$

(ii) *If f also satisfies*

$$\limsup_{t\to+\infty}\frac{f(t)}{t}<+\infty \qquad \textit{or} \qquad \limsup_{t\to-\infty}\frac{f(t)}{t}<+\infty, \tag{4}$$

then any local solution to (1) *exists for all $s\in\mathbb{R}$.*

Proposition 1 (i) states that, under the sole assumption (2), the only way that finite time blow up can occur is with wide oscillations of the solution. Note also that if *both* the conditions in (4) are satisfied then global existence follows from classical theory of ode's. But (4) merely requires that f is "one-sided at most linear". Most of the nonlinearities f in the models mentioned before satisfy (4) which, in particular, includes the cases where f is either concave or convex. Several examples exhibited in [3] show that if (2) is violated, then local solutions to (1) may blow up in finite time. Motivated by [3, Problem 3] we are here interested in studying the case where (2) is satisfied but (4) fails.

In next section we bring strong evidence that if (4) fails then the solution to (1) blows up in finite time. Our first theoretical result (see Theorem 2) states that, under suitable assumptions on f and k, the solution to (1) exhibits "wide and thinning oscillations". By this we mean that the altitude of the oscillation increases and tends to infinity whereas its cycle (the distance between two consecutive zeros of the solution) decreases and tends to zero. Clearly, this *does not* prove that blow up occurs in finite time but, at least, it gives a strong hint in this direction. Our second theoretical result (see Theorem 3) gives an explicit example where the finite time blow up with wide and thinning oscillations indeed occurs. As far as we are aware, this is the first example which exhibits this phenomenon. As we shall see in Section 3.2, this statement has deep applications in some semilinear biharmonic problems. Moreover, in Section 3.3 we show that this kind of blow up phenomenon is typical of (at least) fourth order problems such as (1) since it does not occur in related lower order equations with f satisfying (4). Finite time blow up in more general situations is supported by our numerical results. In Section 4 we describe the numerical procedure used to obtain the plots and tables displayed in next section. We show there that the solution to (1) seems to blow up in finite time, both by plotting its wide oscillations and by approximating the sequence of its zeros.

We believe that this kind of finite time blow up can give an explanation to the celebrated collapse of the Tacoma Narrows Bridge [1]. We discuss in detail our point of view in Section 3.1.

This paper is organized as follows. In next section we state our main theoretical results and we describe our main numerical results. In Sections 3.1 and 3.2 we discuss the above mentioned suspension bridges model and some possible applications of (1) to biharmonic coercive elliptic partial differential equations; in particular, the latter enable us to prove Theorem 3. In Section 3.3 we show that the phenomena we find for the fourth order equation (1) do not hold for lower order equations. In Section 4 we explain how we obtained the numerical results. Sections 5 and 6 are devoted to the proof of Theorem 2. Finally, in Section 7 we give further numerical results which enable us to comment the assumptions on f and k.

2 Main results

Assume that f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) , \quad \text{there exists } \lambda, \delta, \gamma > 0 \text{ such that } f(t)t \geq \delta t^2 + \lambda |t|^{2+\gamma} \quad \text{for every } t \in \mathbb{R} \tag{5}$$

and notice that (5) strengthens (2). Consider (1) with $k = 0$:

$$w''''(s) + f(w(s)) = 0 \qquad (s \in \mathbb{R}). \tag{6}$$

We prove that if f is superlinear then any local solution w to (6) satisfying suitable initial conditions at $s = 0$ has infinitely many oscillations which tend to enlarge and to concentrate on small intervals:

Theorem 2. *Assume that f satisfies* (5)*. Let w be a local solution to* (6) *in a neighborhood of $s = 0$ such that*

$$w'(0)w''(0) - w(0)w'''(0) > 0 \, . \tag{7}$$

Let $R \in (0, +\infty]$ denote the supremum of the maximal interval of continuation of w. Then there exists an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ such that:

(i) $s_j \nearrow R$ as $j \to \infty$;

(ii) $w(s_j) = 0$ and w has constant sign in (s_j, s_{j+1}) for all $j \in \mathbb{N}$;

(iii) $\lim_{j \to \infty} (s_{j+1} - s_j) = 0$;

(iv) $\max_{s \in [s_j, s_{j+1}]} |w(s)| \to +\infty$ as $j \to \infty$.

Note that assumption (7) may possibly be relaxed but it cannot be completely removed since the trivial solution $w(s) \equiv 0$ clearly violates (ii). Note also that Theorem 2 is not completely satisfactory. First of all, the assumptions on f (see (5)) and on k ($k = 0$) are quite restrictive. Second thing, it does not state that the solution w blows up in finite time since it could be $R = +\infty$. However, Theorem 2 gives a strong hint in favor of a blow up in finite time. And its proof takes advantage of a quite simple behavior of the solution in the interval (s_j, s_{j+1}), see Lemma 10 below. Numerical plots, see Figures 8-11, show that if $k \neq 0$ then the oscillations of the solution can become very complicated. Finally, let us mention that thanks to the change of variables $s \mapsto -s$, Theorem 2 also applies to the infimum of the maximal interval of continuation.

We now give an explicit example where blow up occurs. Fix any integer $n \geq 5$, let $k = -\frac{n^2-4n+8}{2} < 0$, and let

$$f(t) = \left(\frac{n(n-4)}{4}\right)^2 t + |t|^{8/(n-4)}t$$

so that f satisfies (5). As a straightforward consequence of Theorem 6 below we obtain

Theorem 3. *Let $n \geq 5$ be an integer. There exists a solution $w = w(s)$ to the equation*

$$w''''(s) - \frac{n^2-4n+8}{2}w''(s) + \left(\frac{n(n-4)}{4}\right)^2 w(s) + |w(s)|^{8/(n-4)}w(s) = 0$$

which is defined in a neighborhood of $s = -\infty$ and such (3) *holds for some finite $R \in \mathbb{R}$.*

Theorem 3 proves that fourth order equations such as (1) may exhibit finite time blow up with wide oscillations. As we shall see in Section 3.3, this *is not* the case for lower order equations. Note that when $n = 8$, the equation in Theorem 3 simply becomes $w''''(s) - 20w''(s) + 64w(s) + w(s)^3 = 0$.

Let us now describe our numerical results. The first one concerns precisely the previous equation.

Numerical Result 4. *In the case $f(t) = 64t + t^3 = 0$, $k = -20$ the first* 18 *zeros of the solution w to* (1) *satisfying $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ are given by:*
$z_1 = 0.716$, $z_2 = 1.7977$, $z_3 = 2.13827$, $z_4 = 2.17358$, $z_5 = 2.18718$, $z_6 = 2.192412$, $z_7 = 2.194429$, $z_8 = 2.1952053$, $z_9 = 2.1955044$, $z_{10} = 2.1956196$, $z_{11} = 2.19566400$, $z_{12} = 2.19568109$, $z_{13} = 2.195687680$, $z_{14} = 2.195690216$, $z_{15} = 2.1956911931$, $z_{16} = 2.1956915694$, $z_{17} = 2.19569171433$, $z_{18} = 2.19569177015$.

Moreover the first 16 *critical levels (ordered on columns from left to right and then on consecutive lines) are given by*

1.00000e+000	*-7.28173e+001*	***5.54303e+002***	*-3.79831e+003*
2.56635e+004	*-1.73041e+005*	***2.56635e+004***	*-7.86188e+006*
1.16639e+006	*-3.57173e+008*	***5.29910e+007***	*-1.62267e+010*
2.40743e+009	*-7.37197e+011*	***1.09371e+011***	*-3.34914e+013*

where the levels of the relative maxima are highlighted in bold face.

Here and in what follows only the estimated correct digits are reported. We quote an increasing number of digits in the zeros z_k in order to emphasize the small differences which appear between two consecutive zeros. From the reported data, it appears that the amplitude of oscillations is increasing and that the length of cycles is decreasing with s, until a threshold value where numerical computation stops because of impossibility to reach the required accuracy by the variable step integrator in use. This may be a symptom of a vertical asymptote. We denote such threshold by TS. Here we have $TS = 2.1957$ (rounded to 5 significant digits). Figure 1 plots the computed solution until $s = 2.05281$, i.e. just a little after the second relative maximum point.

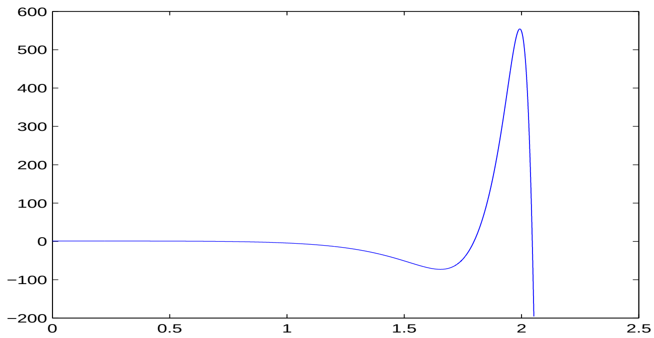

Figure 1: The solution to (1) with $f(t) = 64t + t^3 = 0$, $k = -20$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$.

Next, we slightly change the equation and we obtain

Numerical Result 5. *In the case* $f(t) = t + t^3$, $k = 0$ *the first* 20 *zeros of the solution* w *to* (1) *satisfying* $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ *are given by:*
$z_1 = 1.9$, $z_2 = 4.0$, $z_3 = 4.77$, $z_4 = 5.08$, $z_5 = 5.20$, $z_6 = 5.242$, $z_7 = 5.259$, $z_8 = 5.2661$, $z_9 = 5.2687$, $z_{10} = 5.26973$, $z_{11} = 5.27012$, $z_{12} = 5.27027$, $z_{13} = 5.270328$, $z_{14} = 5.270350$, $z_{15} = 5.2703590$, $z_{16} = 5.2703622$, $z_{17} = 5.27036356$, $z_{18} = 5.27036406$, $z_{19} = 5.27036424$, $z_{20} = 5.270364321$.

Moreover, the first 20 *critical levels (ordered on columns from left to right and then on consecutive lines) are given by*

1.0000e+000	*-7.3459e+000*	***4.9789e+001***	*- 3.3565e+002*
2.2622e+003	*- 1.5251e+004*	***1.0279e+005***	*-6.9287e+005*
4.6701e+006	*-3.1478e+007*	***2.1216e+008***	*-1.4299e+009*
9.6376e+009	*-6.4961e+010*	***4.3788e+011***	*-2.9514e+012*
1.9895e+013	*-1.3410e+014*	***9.0384e+014***	*-6.0917e+015*

where the levels of the relative maxima are highlighted in bold face.

In the case considered in the Numerical Result 5 it appears that $TS = 5.2704$ (rounded to 5 significant digits). In Figure 2 the second minimum point and the third maximum point can be easily recognized. It is worth noticing that the third maximum point is obtained at about $s = 5.17$ and then between this value and TS, further 7 relative maxima and 8 relative minima were computed.

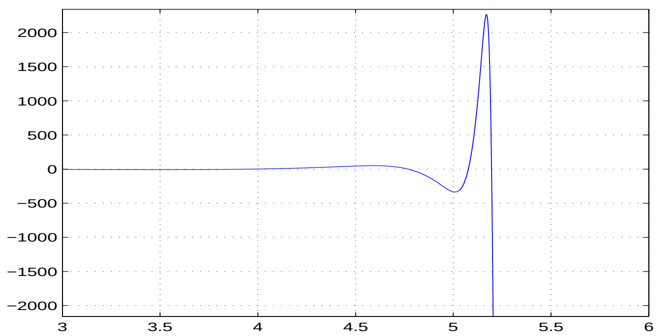

Figure 2: The solution to (1) with $f(t) = t + t^3$, $k = 0$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$.

It appears that the behavior of oscillations is similar to the Numerical Result 4. Analogous behaviors of solutions were found when computing the solution to (1) for many different values of k (including large negative values) and with different initial conditions.

In Figures 3 and 4 we display the plot of two solutions which have some small oscillations on a somehow large interval of time and then wide oscillations in a very small interval of time. Numerical results suggest that the blow up time for the solutions occur respectively for $TS = 96.5947$ and for $TS = 12.06618$.

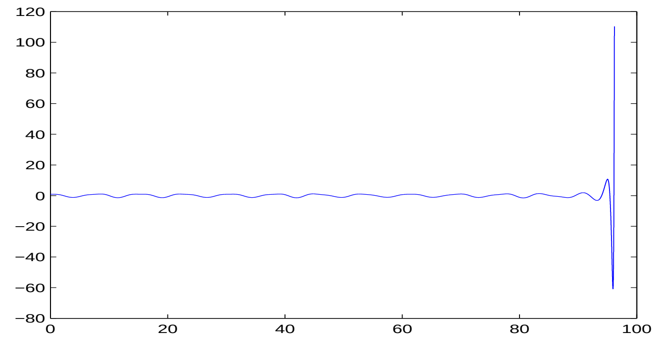

Figure 3: The solution to (1) with $f(t) = t + t^3$, $k = 3.6$, $[w(0), w'(0), w''(0), w'''(0)] = [0.9, 0, 0, 0]$.

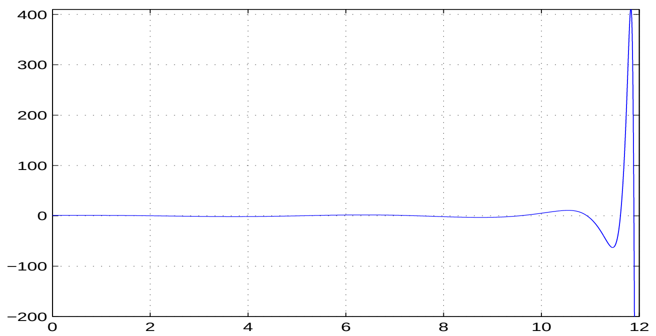

Figure 4: The solution to (1) with $f(t) = t + t^3$, $k = 3.5$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$.

Finally, for $f(t) = t + t^3$, the next table shows the numerically found blow up time $TS(k)$ (depending on k) of the solution w to (1) satisfying the initial conditions $[w(0), w'(0), w''(0), w'''(0)] = [\alpha, 0, 0, 0]$:

α	1	2	20	200	1000
$TS(-3.5)$	4.295821	3.526675	1.306347	0.4200780	0.1881457
$TS(-2)$	4.620923	3.714381	1.316391	0.4204113	0.1881759
$TS(0)$	5.270364	4.044220	1.330290	0.4208621	0.1882161
$TS(2.1)$	6.639968	4.568380	1.345553	0.4213355	0.1882583
$TS(2.9)$	7.897401	4.861199	1.351560	0.4215164	0.1882745
$TS(3.5)$	12.06618	5.145845	1.356139	0.4216523	0.1882865

This table suggests that TS is decreasing with respect to α (as expected) and increasing with respect to k. Other values of α and k display similar behaviors. However, for very large (positive) values of k and/or for very small (positive) values of α our numerical procedure does not show blow up but a somehow periodic behavior. In these cases we do not know if the solution is indeed periodic or if the blow up time TS is so large that the numerical procedure does not reach it with sufficient precision.

We also tried some asymmetric nonlinearity. In the case where

$$f(t) = \frac{t + t^3 + e^t - 1}{2},$$

$k = 2$, and $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$ the computed solution exhibits a threshold value $TS = 6.3009$. The relative maximum and minimum values of the solution were estimated numerically and are reported in the following tables. The maximum points and values are written in bold characters.

s	**0.0**	3.85325	**5.53421**	6.15714	**6.27537**	6.29695	**6.30086**	6.30091
$w(s)$	**1.0**	-3.3786	**11.055**	-184.06	**33.554**	-1.5026e+005	**73.377**	-2.3179e+010

The last two extremal values are obtained for values of s which differ less than 10^{-4}. Figure 5 displays the solution until the third maximum and shows that maxima values are much smaller than the absolute value of minima.

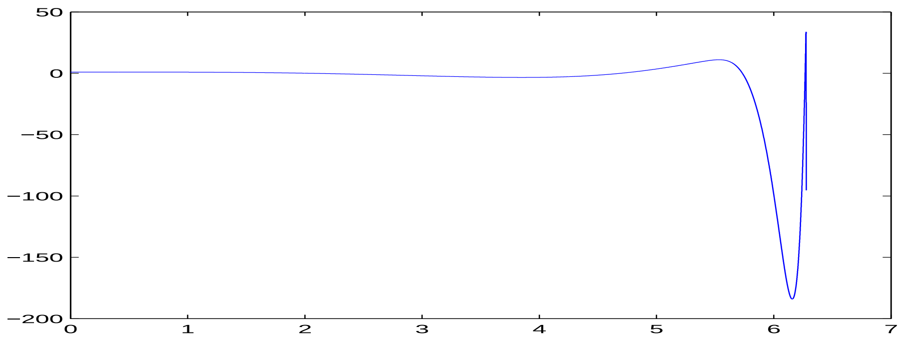

Figure 5: The solution to (1) with $f(t) = \frac{t+t^3+e^t-1}{2}$, $k = 2$, $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$.

3 Some related problems

3.1 Suspension bridges

Under suitable boundary and initial conditions, the following nonlinear beam equation was proposed by Lazer-McKenna [14] as a model for a suspension bridge

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x, t) \qquad x \in (0, L)\,, \quad t > 0\,, \tag{8}$$

where $L > 0$ denotes the length of the bridge, $u^+ = \max\{u, 0\}$, γu^+ represents the force due to the cables which are considered as a spring with a one-sided restoring force (equal to γu if u is downward positive and to 0 if u is upward negative), and W represents the forcing term acting on the bridge (including its own weight per unit length and the wind or other external sources). The solution u represents the vertical displacement when the beam is bending. Normalizing (8) by putting $\gamma = 1$ and $W \equiv 1$, and seeking traveling waves $u(x, t) = 1 + w(x - ct)$ to (8) leads to the equation

$$w''''(s) + kw''(s) + [w(s) + 1]^+ - 1 = 0 \qquad (s \in \mathbb{R},\ k = c^2)\,.$$

Finally, in order to maintain the same behavior but with a smooth nonlinearity, Chen-McKenna [8] suggest to consider the equation

$$w''''(s) + kw''(s) + e^{w(s)} - 1 = 0 \qquad (s \in \mathbb{R})\,, \tag{9}$$

which is exactly of the kind (1) with $f(t) = e^t - 1$ satisfying (2) and (4) but not (5). We wish to suggest here a variant of this model.

As pointed out by McKenna [16, Section 6], according to historical sources such as [1, 6], one of the most interesting behaviors for suspension bridges (including the Golden Gate and the Tacoma Narrows Bridge) is the following:

> large vertical oscillations can rapidly change, almost instantaneously, to a torsional oscillation.

A possible explanation to this fact is that *since the motion cannot be continued downwards due to the cables, when the bridge reaches its equilibrium position the existing energy generates a crossing wave, namely a torsional oscillation*. This is explained in Figure 6 where the dotted line displays the theoretical position of the bridge in absence of the action of the cables whereas the horizontal line denotes the real position of the bridge when stopped by the cables.

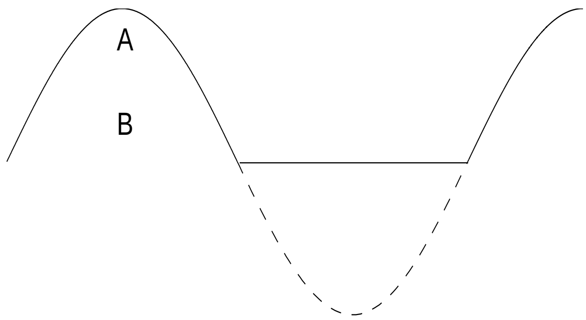

Figure 6: Action of the cables and of the wind on a suspension bridge.

Since the Tacoma Bridge collapse was due to a wide torsional motion of the bridge (see [23]), the bridge cannot be considered as a one dimensional beam. This problem was overcome in [10, Section 2.3] by introducing the *deflection from horizontal* as a second unknown function (besides the vertical displacement). Alternatively, we suggest to maintain the one dimensional model provided one also allows displacements *below the equilibrium position*; in other words, one can concentrate in the unknown function w both the vertical displacement and the deflection from horizontal. In this case the nonlinearity f in (1) should be unbounded also from below.

A further remark concerns the source f. It is clear that more the position of the bridge is far from the horizontal equilibrium position, more the action of the wind becomes relevant because the wind hits transversally the surface of the bridge. If ever the bridge would reach the limit vertical position, the wind would hit it orthogonally. This means that the forcing term f is superlinear, becoming more powerful for large displacements from the horizontal position, see again Figure 6 where in position A the impact of the wind is much more relevant than in position B.

As Theorems 2 and 3 and our numerical results seem to suggest, traveling waves with nonlinearities satisfying (5) blow up in finite time after wide oscillations. Is this the explanation of the Tacoma collapse?

3.2 Biharmonic coercive equations

• Consider the critical growth biharmonic elliptic equation

$$\Delta^2 u + |u|^{8/(n-4)} u = 0 \qquad \text{in } \mathbb{R}^n, \tag{10}$$

where $n \geq 5$. The exponent $\frac{n+4}{n-4} = \frac{8}{n-4} + 1$ is critical in the sense of Sobolev embedding. The corresponding noncoercive equation $\Delta^2 u - |u|^{8/(n-4)} u = 0$ has been extensively studied, see [22] and subsequent work in [11, 15]. Here we are interested in equation (10) where the nonlinearity has the opposite sign. A problem raised by Lazzo-Schmidt [15] concerns the possibility that radial solutions to (10) have an oscillatory blow up behavior in finite or infinite time. By combining Proposition 1 with a recent result by D'Ambrosio-Mitidieri [9], we show that this is indeed the case.

Theorem 6. *Let $n \geq 5$ and let $u = u(r)$ be a nontrivial radially symmetric solution to the equation $\Delta^2 u + |u|^{8/(n-4)} u = 0$ in a neighborhood of the origin. Then there exists $\rho \in (0, \infty)$ such that*

$$\liminf_{r \nearrow \rho} u(r) = -\infty \qquad \textit{and} \qquad \limsup_{r \nearrow \rho} u(r) = +\infty\,.$$

Proof. In radial coordinates $r = |x|$, equation (10) reads

$$u''''(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r) - \frac{(n-1)(n-3)}{r^3} u'(r) + |u(r)|^{8/(n-4)} u(r) = 0 \quad (11)$$

for $r \in [0, \infty)$. The change of variables

$$u(r) \;=\; r^{-(n-4)/2}\, w(\log r) \quad (r > 0)\,, \qquad w(s) \;=\; e^{(n-4)s/2}\, u(e^s) \quad (s \in \mathbb{R}) \qquad (12)$$

was suggested in [11, (12)] in order to transform (11) into an autonomous equation. Indeed, with (12) we may rewrite (11) as

$$w''''(s) - \frac{n^2 - 4n + 8}{2} w''(s) + \left(\frac{n(n-4)}{4}\right)^2 w(s) + |w(s)|^{8/(n-4)} w(s) = 0 \qquad (s \in \mathbb{R})\,,$$

which is precisely of the form (1) with $k = -\frac{n^2-4n+8}{2} < 0$ and $f(w) = \left(\frac{n(n-4)}{4}\right)^2 w + |w|^{8/(n-4)} w$ so that f satisfies (5) and, in particular, f satisfies (2). By [9, Theorem 4.6], we know that the only globally defined (not necessarily radial) solution to (10) is $u \equiv 0$. This means that any nontrivial local solution u to (11) blows up in finite time, say for $r \nearrow \rho$. In view of (12), this proves that w blows up as $s \nearrow R = \log \rho$ and therefore Proposition 1 states that (3) holds. Going back to u by means of (12) proves the statement. $\Box$

• Let $k \in \mathbb{R}$ and consider now the equation

$$\Delta^2 u - 2(n-4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2} - (n-2)\left[(n-2)^2 + k\right] \frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4} \quad (13)$$

where $x \in \mathbb{R}^n \setminus \{0\}$ ($n \geq 2$). For $n = 4$ and $k = -4$, (13) simply becomes

$$\Delta^2 u + e^u = \frac{1}{|x|^4}\,, \qquad x \in \mathbb{R}^4 \setminus \{0\}.$$

As pointed out in [3], for any $k \in \mathbb{R}$, (13) admits an explicit global radial solution which is given by $\overline{u}(x) = -4 \log |x|$. To see this, one may write (13) in its radial form, that is

$$u''''(r) + 6\,\frac{u'''(r)}{r} + (7 + k)\,\frac{u''(r)}{r^2} + (1 + k)\,\frac{u'(r)}{r^3} + e^{u(r)} = \frac{1}{r^4}\,,$$

where $r = |x| \in (0, +\infty)$. Then, with the change of variables (see [2])

$$s = \log r \qquad w(s) := u(e^s) + 4s \qquad s \in \mathbb{R}\,,$$

one finds that $w = w(s)$ solves... (9)! It is quite surprising that (9) arises in so different contexts. Note that the singular solution $\overline{u}(x) = -4 \log |x|$ to (13) corresponds to the trivial solution $w \equiv 0$ to (9).

We refer to [3] for further interesting cases and some results related to (13).

3.3 Some lower order equations

• The function $f(t) = e^t - 1$ satisfies both (2) and (4). Therefore, Proposition 1 states that any local solution to the equation $w''''(s) + e^{w(s)} - 1 = 0$ is globally defined on $\mathbb{R}$. On the other hand, the corresponding *first order* equation

$$w'(s) + e^{w(s)} - 1 = 0 \qquad (14)$$

has the local solution $w(s) = -\log(1 - e^{-s})$ which is defined on $(0, +\infty)$ and blows up as $s \searrow 0$. Hence, (4) ensures global solvability to the fourth order equation (1) but not to the first order equation (14).

• Consider now the function $f(t) = t + t^3$ as in the Numerical Result 5. The *first order* equation

$$w'(s) + w(s) + w(s)^3 = 0$$

has the solutions

$$w(s) = \frac{\gamma}{\sqrt{e^{2s} - \gamma^2}} \,, \qquad \gamma \in \mathbb{R}$$

which, if $\gamma \neq 0$, blow up as $s \searrow \log|\gamma|$. The blow up occurs monotonically and therefore Theorem 2 does not apply to first order equations.

• Consider the *second order* equation

$$w''(s) + f(w(s)) = 0 \tag{15}$$

with f merely satisfying (2) so that the solution is concave (convex) whenever it is positive (negative). By multiplying (15) by $w'(s)$ and integrating we see that there exists $C \geq 0$ such that

$$w'(s)^2 + 2F(w(s)) = C \qquad \forall s \tag{16}$$

where $F(t) = \int_0^t f(\tau)d\tau$. Let us consider the Cauchy problem

$$w(0) = 0 \,, \qquad w'(0) = \gamma \tag{17}$$

and note that, by what follows, the case where $w(0) \neq 0$ can be deduced from this case. In view of (16) we have $\gamma^2 = C$. If $\gamma = 0$, then the unique solution is $w \equiv 0$ which is global. If $\gamma > 0$, then two cases have to be distinguished

$$(i) \ \lim_{t \to +\infty} F(t) \leq C \qquad (ii) \ \lim_{t \to +\infty} F(t) > C$$

where C is the constant in (16). Recall that this limit exists since $F' = f > 0$ on $\mathbb{R}_+$. If case (i) occurs then $F(w(s)) < C$ and, by (16) and the fact that $\gamma > 0$, $w'(s) > 0$ for all s so that the solution w is globally defined, increasing, and concave on $[0, +\infty)$. If case (ii) occurs, then w' vanishes and changes sign when $F(w(s))$ reaches C, say at $s = \bar{s}$. Indeed, (16) is not equivalent to (15) since, when multiplying by w', we have added the case where w' can vanish identically. A solution w to (15) cannot remain constant on an interval. Hence, w is strictly decreasing (and concave) in a right neighborhood of $\bar{s}$. When it reaches the value 0 a new Cauchy problem with $w' = -\sqrt{C} < 0$ appears. And then we have to distinguish again the two cases (i) and (ii) but with $+\infty$ replaced by $-\infty$. In case (i), the solution w is globally defined decreasing, and convex. In case (ii), the solution attains a global minimum where w' changes sign and goes back to 0 where the same Cauchy problem (17) appears because of (16); in this case, the solution is periodic. We have so shown that, in any case, local solutions to (15) are global.

Of course, an alternative way to reach this conclusion, is to set $u := w$ and $v := w'$ and to consider the Hamiltonian system

$$\begin{cases} u' = v \\ v' = -f(u) \,. \end{cases}$$

The only stationary point is $(u, v) = (0, 0)$ corresponding to $w \equiv 0$ whereas the Hamiltonian is $H(u, v) = \frac{v^2}{2} + F(u)$ and is constant on solutions. A phase plane analysis then yields the same conclusions as above.

These arguments show that if f satisfies (2) then any local solution to (15) is globally defined and Theorem 2 *does not hold* for second order equations. The phenomenon in Theorem 2 is new and characterizes fourth order equations.

• One could wonder whether (15) is the correct *second order* equation corresponding to (1) since the first is a coercive problem and the latter is an indefinite problem. Roughly speaking, the energy for (1) is convex whereas the energy for (15) has a mountain-pass shape. However, if one considers instead the equation $-w'' + f(w) = 0$ with f satisfying (2), one readily sees that w is convex whenever it is positive and therefore that it blows up monotonically in finite time if f also satisfies (5), see e.g. the blow up technique developed by Mitidieri-Pohožaev [18]. In fact, the same occurs for (1) if we change sign to a nonlinearity f satisfying (5): with the same technique one sees that in finite time all derivatives of w tend to infinity.

• The *third order* equation $w'''(s)+w(s)^3=0$ admits the solutions $w(s)=\pm\sqrt{\frac{105}{8}}s^{-3/2}$ which are defined on $(0,+\infty)$ and blow up monotonically as $s\searrow 0$. Hence, also for third order equations, Proposition 1 (i) does not apply.

All the above examples show that fourth order equations such as (1) exhibit phenomena not visible for lower order equations.

4 The numerical procedure

In order to numerically evaluate zeros of the computed solution w to (1), we checked where the computed discrete function changed sign. Then, for each detected interval, we used two different methods:
a) one step of bisection method;
b) computation of exact zero of the linear interpolation polynomial.

From the known bisection error and the comparison between the two computed values of each zero, we obtain the estimated correct digits of values reported in Section 2.

Concerning the computation of the solution w, we chose to use standard numerical methods for stiff equations, i.e. we used the MATLAB solvers `ode15s`, `ode23s`, `ode23tb`, according to the required efficiency and accuracy. We remind that `ode15s` is a variable order solver with low/medium order of accuracy; `ode23s` is a one-step solver which can be in some case more effective than `ode15s`; `ode23tb` is an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two. Solutions computed by different methods were compared and then a reliable tolerance was chosen, in the sense that we used relative error threshold which revealed neither too tight nor too crude in order to guarantee the same results by different methods.

5 Preliminaries

Let f be as in (6) and put

$$F(t):=\int_0^t f(\tau)\,d\tau\,.$$

To equation (6) we associate the *energy function*

$$\mathcal{E}(s):=\frac{1}{2}\,w''(s)^2-w'(s)w'''(s)-F(w(s)). \tag{18}$$

Then, if w solves (6), there holds

$$\mathcal{E}'(s)=-\Big(w''''(s)+f(w(s))\Big)w'(s)=0 \quad\Longrightarrow\quad \mathcal{E}(s)=C\,, \tag{19}$$

for some $C\in\mathbb{R}$.

Moreover, we define

$$G(s):=w'(s)^2-w(s)w''(s). \tag{20}$$

Then

$$G'(s)=H(s):=w'(s)w''(s)-w(s)w'''(s) \tag{21}$$

and a short computation gives

$$G''(s)=H'(s)=w''(s)^2+w(s)f(w(s)). \tag{22}$$

If (2) holds, by (22) we infer that

$$G''(s)=H'(s)\geq 0 \qquad \text{so that} \qquad H \text{ is nondecreasing and } G \text{ is convex.} \tag{23}$$

Remark 7. In view of possible generalizations of our results, we point out that energy functionals may be associated to equation (1) also when $k \neq 0$. Instead of (18) we take

$$\mathcal{E}(s) = \frac{1}{2}\, w''(s)^2 - \frac{k}{2}\, w'(s)^2 - w'(s)w'''(s) - F(w(s))$$

which satisfies again (19) for some $C \in \mathbb{R}$. Moreover, instead of (20) and (21) one can define

$$G(s) = w'(s)^2 - w(s)w''(s) - \frac{k}{2}\, w(s)^2\,, \qquad G'(s) = H(s) = w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

and if $k \leq 0$ and (2) holds, then (23) still holds true.

In the sequel, we need of the following technical estimate:

Lemma 8. *For any $K > 0$ and any $t \geq 1$, the following inequality holds*

$$\frac{t^3}{K} - \frac{3t^2}{K+1} + \frac{3t}{K+2} - \frac{1}{K+3} \geq \frac{6t^3}{K(K+1)(K+2)(K+3)}.$$

Proof. Note first that

$$\frac{6}{K(K+1)(K+2)(K+3)} = \frac{1}{K} - \frac{3}{K+1} + \frac{3}{K+2} - \frac{1}{K+3}$$

so that the claim becomes

$$\frac{3(t^3-t^2)}{K+1} - \frac{3(t^3-t)}{K+2} + \frac{t^3-1}{K+3} \geq 0 \qquad \forall K > 0\, \forall t \geq 1\,.$$

After dividing by $(t-1)$, the latter inequality becomes

$$\frac{3t^2}{K+1} - \frac{3t(t+1)}{K+2} + \frac{t^2+t+1}{K+3} \geq 0 \qquad \forall K > 0\, \forall t \geq 1\,.$$

By eliminating the denominators and collecting terms, this is equivalent to

$$\phi(t) := (K^2+6K+11)t^2 - (2K^2+9K+7)t + (K^2+3K+2) \geq 0 \qquad \forall K > 0\, \forall t \geq 1\,.$$

Since $\phi(1) = 6$ and $\phi'(t) = 2(K^2+6K+11)t - (2K^2+9K+7) \geq 3K+15 > 0$ for $t \geq 1$, the claim follows. □

6 Proof of Theorem 2

If $R < +\infty$, Items (i)-(iii) follow from Proposition 1. In order to prove Item (iv), for all j let $m_j \in [s_j, s_{j+1}]$ be the point where $|w(s)|$ attains its maximum on $[s_j, s_{j+1}]$. By Proposition 1 we infer that

$$\limsup_{j\to\infty} |w(m_j)| = +\infty\,.$$

Therefore, there exists a subsequence $\{m_k\} \subset \{m_j\}$ such that

$$\lim_{k\to\infty} |w(m_k)| = +\infty\,.$$

In view of (19) we then infer

$$\lim_{k\to\infty} w''(m_k)^2 = 2 \lim_{k\to\infty} [C + F(w(m_k))] = +\infty\,.$$

In turn, (20) yields

$$\lim_{k\to\infty} G(m_k) = -\lim_{k\to\infty} w(m_k)w''(m_k) = +\infty\,.$$

By (23) we finally deduce that $\lim_{s\nearrow R} G(s) = +\infty$ without extracting subsequences. In particular, we get that

$$\lim_{j\to\infty} G(m_j) = -\lim_{j\to\infty} w(m_j)w''(m_j) = +\infty$$

on the whole sequence $\{m_j\}$ of maxima of $|w(s)|$. Using again (19) we obtain Item (iv). This completes the proof of Theorem 2 in the case $R < +\infty$.

For the case $R = +\infty$, in what follows we assume that the local solution $w = w(s)$ can be continued as $s \to +\infty$ so that w is defined (at least) on $[0, +\infty)$. For simplicity, with an abuse of language we call *global* any such solution. By [3, Theorem 4] we know that $w(s)$ changes sign infinitely many times as $s \to +\infty$; in particular, Items (i) and (ii) are straightforward and we need to prove Items (iii) and (iv). Before attacking directly these problems, we prove some qualitative properties of the solution.

Lemma 9. *Assume that f satisfies* (2). *Let w be a global solution to* (6). *Then, for the function H defined in* (21)*, the following alternative holds:*

(i) *If $H(s)$ is bounded as $s \to +\infty$, then $H(s) \le 0$ for all s and*

$$\lim_{s\to+\infty} H(s) = \lim_{s\to+\infty} w(s) = 0.$$

(ii) *If $H(0) > 0$, then*

$$\lim_{s\to+\infty} H(s) = +\infty\,, \qquad \lim_{s\to+\infty} G(s) = +\infty\,, \tag{24}$$

and $w(s)$ is unbounded as $s \to +\infty$.

Proof. By (23), $H(s)$ admits a limit as $s \to +\infty$, either finite or $+\infty$.

(i) Suppose that $H(s)$ is bounded as $s \to +\infty$, in which case (5) yields

$$\int_0^s [w''(t)^2 + \delta w(t)^2]\,dt \le \int_0^s [w''(t)^2 + w(t)f(w(t))]\,dt = \int_0^s H'(t)\,dt = H(s) - H(0) < \infty.$$

Letting $s \to +\infty$, we deduce that $w \in H^2(0,\infty)$ so that $w' \in H^1(0,\infty)$ and

$$\lim_{s\to+\infty} w'(s) = 0. \tag{25}$$

By [3, Theorem 4] we know that $w(s)$ changes sign infinitely many times as $s \to +\infty$. Then there exists a divergent sequence $\{s_j\}_{j\in\mathbb{N}}$ of zeros of $w(s)$ such that $w(s)$ has one sign on $[s_j, s_{j+1}]$ for each $j \in \mathbb{N}$. Since by (23) G admits a limit, by (25) we know that

$$\lim_{s\to+\infty} G(s) = \lim_{j\to+\infty} G(s_j) = \lim_{j\to+\infty} w'(s_j)^2 = 0.$$

This implies that $H(s) = G'(s) \to 0$ as $s \to +\infty$ and, again by (23), $H(s) \le 0$ for all s.

(ii) If $H(0) > 0$, then $H(s)$ is unbounded as $s \to +\infty$ by part (i). Suppose $w(s)$ is bounded as $s \to +\infty$ for the sake of contradiction and consider the divergent sequence $\{m_j\}_{j\in\mathbb{N}}$ of local extrema of w in the interval $[s_j, s_{j+1}]$. According to (19),

$$w''(m_j)^2 = C + 2F(w(m_j))$$

for all $j \in \mathbb{N}$. Since $w(s)$ is assumed to be bounded as $s \to +\infty$, the sequence $\{w''(m_j)\}_{j\in\mathbb{N}}$ is bounded. But from [3, Lemma 28] we know that

$$\max_{m_j\le s\le m_{j+1}} |w''(s)| \;=\; \max\{|w''(m_j)|, |w''(m_{j+1})|\}.$$

Hence, $w''(s)$ is uniformly bounded as $s \to +\infty$. Using the inequality

$$\sup_{s\ge0} w'(s)^2 \le 4\sup_{s\ge0}|w(s)| \cdot \sup_{s\ge0}|w''(s)|,$$

we conclude that $w'(s)$ is uniformly bounded as well. On the other hand, we have

$$\lim_{s\to+\infty} G'(s) = \lim_{s\to+\infty} H(s) = +\infty.$$

Hence,

$$\lim_{s\to+\infty} [w'(s)^2 - w(s)w''(s)] = \lim_{s\to+\infty} G(s) = +\infty.$$

This is absurd since w, w' and w'' are all uniformly bounded as $s \to +\infty$. $\square$

As already mentioned, by [3, Theorem 4] we know that $w(s)$ changes sign infinitely many times as $s \to +\infty$. Denote by $[s_j, t_j] \subset (0, +\infty)$ any interval such that

$$w(s_j) = w(t_j) = 0 \qquad \text{and} \qquad w(s) > 0 \quad \forall s \in (s_j, t_j) \tag{26}$$

and notice that $s_j \to +\infty$ as $j \to \infty$. All what follows can be extended to intervals where w is negative.

We prove some qualitative properties of w in these intervals:

Lemma 10. *Assume that f satisfies* (5)*. Let w be a global solution to* (6) *satisfying* (7)*. If j is sufficiently large, then on any interval $[s_j, t_j] \subset (0, +\infty)$ such that* (26) *holds, the following facts occur:*

(i) $0 < w'(s_j) < -w'(t_j)$ *and there exists a unique $m_j \in (s_j, t_j)$ such that $w'(m_j) = 0$;*

(ii) $w''(s_j) > 0$, $w''(t_j) < 0$, *and there exists a unique $r_j \in (s_j, t_j)$ such that $w''(r_j) = 0$, moreover $r_j < m_j$.*

Proof. Assumption (7) is equivalent to $H(0) > 0$, see (21). Then Lemma 9 (ii) states that there exists $\overline{s} \geq 0$ such that $G(\overline{s}) \geq 0$. Chose j sufficiently large so that $s_j > \overline{s}$. By (21) and (23), we know that $0 \leq G(\overline{s}) < w'(s_j)^2 = G(s_j) < G(t_j) = w'(t_j)^2$. Hence, $0 < w'(s_j) < -w'(t_j)$. Moreover, if w admits two critical points m_j and m_j', one of them (say m_j') would be a local minimum where $G(m_j') = -w(m_j')w''(m_j') \leq 0$ contradicting the monotonicity of G since $G(s_j) > 0$. This proves Item (i).

By (23) we infer that $0 < H(0) < w'(s_j)w''(s_j) = H(s_j) < H(t_j) = w'(t_j)w''(t_j)$ which, together with the just proved Item (i), shows that $w''(s_j) > 0$ and $w''(t_j) < 0$ and the existence of $r_j \in (s_j, t_j)$ such that $w''(r_j) = 0$ and w'' changes sign in r_j. Rewriting (6) as $[w''(s)]'' = -f(w(s)) < 0$ for $s \in (s_j, t_j)$, shows that $s \mapsto w''(s)$ is strictly concave and can therefore admit at most two zeros with sign changes. Since $w''(s_j) > 0$ and $w''(t_j) < 0$, we infer that the flex point $r_j \in (s_j, t_j)$ is unique. To complete the proof we still have to show that $r_j < m_j$. Since $w''(m_j) \leq 0$ we have that $r_j \in (s_j, m_j]$. But it cannot be $r_j = m_j$ since otherwise we would also have $w'''(m_j) = 0$ (recall that m_j is a local maximum for w) and, in turn, $H(m_j) = 0$, contradiction. $\square$

The qualitative properties of the solution found in Lemma 10 are displayed in Figure 7.

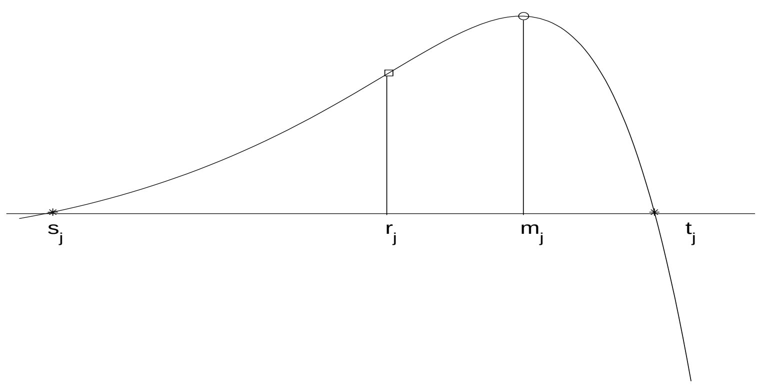

Figure 7: Qualitative behavior of the solution w in the interval $[s_j, t_j]$.

We point out that numerical results suggest that Lemma 10 *may not hold* whenever $k \neq 0$, see Figures 8-11 in next section.

We are now ready to prove Items (iii) and (iv) of Theorem 2. For Item (iv), see (33) below, whereas Item (iii) will follow once we are done with the next three steps, related to the points found in Lemma 10. In what follows, C denotes a positive constant which may vary from line to line.

Step 1. We prove that $\lim_{j\to\infty}(r_j - s_j) = 0$.

Assume for contradiction that the claim is false so that $\limsup_{j\to\infty}(r_j - s_j) > 0$. Then there exists $a > 0$ and a subsequence (still denoted in the same way) such that $(r_j - s_j) \geq a$ for all j. By Lemma 9 we know that $w'(s_j)^2 = H(s_j) \to +\infty$ as $j \to \infty$. In turn, by Lemma 10, we know that $w'(s) \to +\infty$ for all $s \in [s_j, r_j]$ as $j \to \infty$. Finally, this proves that

$$w(s_j + \sigma) \to +\infty \qquad \forall \sigma \in (0, r_j - s_j] \quad \text{as } j \to \infty\,. \tag{27}$$

Let $h(s) := (s - s_j)^3(r_j - s)^4$. By (27) and assumption (5), we infer that

$$h(s_j + \sigma)f(w(s_j + \sigma)) + h''''(s_j + \sigma)w(s_j + \sigma) \to +\infty \qquad \forall \sigma \in (0, r_j - s_j) \quad \text{as } j \to \infty\,. \tag{28}$$

Multiply (6) by $h(s)$ and integrate over $[s_j, r_j]$. Since h, h', h'' vanish in $\{s_j, r_j\}$ and $h'''(r_j) = 0$, four integration by parts yield

$$\int_{s_j}^{r_j} [h(s)f(w(s)) + h''''(s)w(s)]\, ds = 0\,. \tag{29}$$

This contradicts (28) unless

$$\lim_{j\to\infty}(r_j - s_j) = 0\,. \tag{30}$$

This proves the claim of Step 1.

Step 2. We prove that $\lim_{j\to\infty}(t_j - m_j) = 0$.

Let $h(s) := (s - s_j)^3(t_j - s)^3$, multiply (6) by $h(s)$, and integrate over $[s_j, t_j]$. Since h, h', h'' vanish in $\{s_j, t_j\}$, four integration by parts yield

$$72\int_{s_j}^{t_j} [(t_j - s)^2 - 3(t_j - s)(s - s_j) + (s - s_j)^2]w(s)\, ds = \int_{s_j}^{t_j} (s - s_j)^3(t_j - s)^3 f(w(s))\, ds\,. \tag{31}$$

Since

$$(t_j - s)^2 - 3(t_j - s)(s - s_j) + (s - s_j)^2 < (t_j - s)^2 + 2(t_j - s)(s - s_j) + (s - s_j)^2 = (t_j - s_j)^2,$$

from (31) we infer

$$72(t_j - s_j)^2 \int_{s_j}^{t_j} w(s)\, ds > \int_{s_j}^{t_j} (s - s_j)^3(t_j - s)^3 f(w(s))\, ds\,.$$

If we let $M_j = w(m_j)$ denote the maximum of w over $[s_j, t_j]$, the last inequality yields

$$72M_j(t_j - s_j)^3 > \int_{s_j}^{t_j} (s - s_j)^3(t_j - s)^3 f(w(s))\, ds\,. \tag{32}$$

Note that by (20) and (24), we have

$$-M_j w''(m_j) = G(m_j) \to +\infty \qquad \text{as } j \to +\infty$$

whereas, by (19),

$$\frac{w''(m_j)^2}{2} - F(M_j) = C \qquad \forall j\,.$$

By combining these two facts, we readily obtain

$$\lim_{j\to+\infty} M_j = +\infty\,. \tag{33}$$

Our purpose here is to estimate

$$I_1 := \int_{m_j}^{t_j} (s-s_j)^3 (t_j-s)^3 f(w(s))\, ds\,.$$

In view of Lemma 10 we know that w is concave over $[m_j, t_j]$ so that

$$w(s) \geq \frac{M_j(t_j-s)}{t_j-m_j} \qquad \forall s\in[m_j,t_j]\,.$$

Then, by assumption (5), we infer that

$$f(w(s)) \geq \lambda w(s)^{1+\gamma} \geq \lambda M_j^{1+\gamma}\frac{(t_j-s)^{1+\gamma}}{(t_j-m_j)^{1+\gamma}} \qquad \forall s\in[m_j,t_j]\,.$$

Hence,

$$I_1 \geq \frac{\lambda\, M_j^{1+\gamma}}{(t_j-m_j)^{1+\gamma}} \int_{m_j}^{t_j} (s-s_j)^3 (t_j-s)^{4+\gamma}\, ds\,.$$

With the change of variables $t = t_j - s$, the previous estimate becomes

$$\begin{aligned} I_1 &\geq \frac{\lambda\, M_j^{1+\gamma}}{(t_j-m_j)^{1+\gamma}} \int_0^{t_j-m_j} (t_j-s_j-t)^3\, t^{4+\gamma}\, dt \\ &= \lambda\, M_j^{1+\gamma}(t_j-m_j)^4 \left[\tfrac{(t_j-s_j)^3}{5+\gamma} - \tfrac{3(t_j-s_j)^2(t_j-m_j)}{6+\gamma} + \tfrac{3(t_j-s_j)(t_j-m_j)^2}{7+\gamma} - \tfrac{(t_j-m_j)^3}{8+\gamma}\right] \\ &\geq \tfrac{6\lambda\, M_j^{1+\gamma}}{(5+\gamma)(6+\gamma)(7+\gamma)(8+\gamma)}(t_j-s_j)^3(t_j-m_j)^4 = C\, M_j^{1+\gamma}(t_j-s_j)^3(t_j-m_j)^4 \end{aligned}$$

where, for the last inequality, we used Lemma 8 with $K = 5+\gamma$ and $t = \frac{t_j-s_j}{t_j-m_j}$. The just obtained lower bound for I_1, combined with (32), yields

$$t_j - m_j \leq \frac{C}{M_j^{\gamma/4}}\,. \tag{34}$$

Together with (33), this proves the claim of Step 2.

Step 3. We prove that $\lim_{j\to\infty}(m_j - r_j) = 0$.
Here, we estimate

$$I_2 := \int_{r_j}^{m_j} (s-s_j)^3 (t_j-s)^3 f(w(s))\, ds\,.$$

By Lemma 10 we know that w is concave over $[r_j, m_j]$ so that (this inequality is far from being optimal!)

$$w(s) \geq \frac{M_j(s-r_j)}{m_j-r_j} \qquad \forall s\in[r_j,m_j]\,.$$

Then, by assumption (5), we infer that

$$f(w(s)) \geq \lambda w(s)^{1+\gamma} \geq \lambda M_j^{1+\gamma}\frac{(s-r_j)^{1+\gamma}}{(m_j-r_j)^{1+\gamma}} \qquad \forall s\in[r_j,m_j]\,.$$

Hence, since $s_j < r_j$,

$$I_2 \geq \frac{\lambda\, M_j^{1+\gamma}}{(m_j-r_j)^{1+\gamma}} \int_{r_j}^{m_j} (t_j-s)^3 (s-r_j)^{4+\gamma}\, ds\,.$$

With the change of variables $t = s - r_j$, the previous estimate becomes

$$\begin{aligned}
I_2 &\geq \frac{\lambda\, M_j^{1+\gamma}}{(m_j - r_j)^{1+\gamma}} \int_0^{m_j - r_j} (t_j - r_j - t)^3 t^{4+\gamma}\, dt \\
&= \lambda\, M_j^{1+\gamma} (m_j - r_j)^4 \left[\tfrac{(t_j - r_j)^3}{5+\gamma} - \tfrac{3(t_j-r_j)^2(m_j-r_j)}{6+\gamma} + \tfrac{3(t_j-r_j)(m_j-r_j)^2}{7+\gamma} - \tfrac{(m_j-r_j)^3}{8+\gamma} \right] \\
&\geq C\, M_j^{1+\gamma} (t_j - r_j)^3 (m_j - r_j)^4,
\end{aligned}$$

where, for the last inequality, we used Lemma 8 with $K = 5 + \gamma$ and $t = \frac{t_j - r_j}{m_j - r_j}$. The just obtained lower bound for I_2, combined with (32), yields

$$M_j^{\gamma} (t_j - r_j)^3 (m_j - r_j)^4 \leq C (t_j - s_j)^3 \,. \tag{35}$$

For any integer j, two cases may occur:

$$(a)\; t_j - r_j \geq \frac{t_j - s_j}{M_j^{\gamma/14}} \qquad (b)\; t_j - r_j < \frac{t_j - s_j}{M_j^{\gamma/14}} \,.$$

Denote by A the set of integers j for which case (a) occurs. Then (35) yields

$$m_j - r_j \leq \frac{C}{M_j^{11\gamma/56}} \qquad \forall j \in A$$

and, if A contains infinitely many indices j, by (34) we infer that

$$t_j - r_j = (t_j - m_j) + (m_j - r_j) \leq \frac{C}{M_j^{\gamma/4}} + \frac{C}{M_j^{11\gamma/56}} \leq \frac{C}{M_j^{11\gamma/56}}$$

for sufficiently large $j \in A$. Inserting this into the inequality for case (a) gives $t_j - s_j \leq \frac{C}{M_j^{\gamma/8}}$ and, by (33),

$$\lim_{j\to\infty,\, j\in A} (t_j - s_j) = 0 \,. \tag{36}$$

Denote by B the set of integers j for which case (b) occurs. Then

$$t_j - r_j < \frac{t_j - r_j + r_j - s_j}{M_j^{\gamma/14}} \qquad \forall j \in B$$

and therefore

$$\left(1 - \frac{1}{M_j^{\gamma/14}}\right)(t_j - r_j) \leq \frac{r_j - s_j}{M_j^{\gamma/14}} \qquad \forall j \in B \,.$$

Hence, if B contains infinitely many indices j, by (30) and (33) we infer that

$$\lim_{j\to\infty,\, j\in B} (t_j - r_j) = 0 \,. \tag{37}$$

The combination of (36) and (37) proves the claim of Step 3. □

Remark 11. If one could obtain *quantitative* information such as (34) also for $(r_j - s_j)$ and $(m_j - r_j)$, one would have a quantitative inequality of the kind

$$t_j - s_j \leq \Phi(M_j) = o(1) \qquad \text{as } j \to +\infty \tag{38}$$

for some function Φ. Assuming without loss of generality that $s_0 = 0$, using (38) one could try to prove that

$$R = \sum_{j=0}^{+\infty} (t_j - s_j) \leq \sum_{j=0}^{+\infty} \Phi(M_j) < +\infty \,.$$

This would prove that w blows up in finite time R. But a quantitative property such as (38) seems quite challenging.

7 Further comments and plots

Remark 12. In the proof of Theorem 2 we used [3, Theorem 4] to find a sequence of intervals $[s_j, t_j] \subset (0, +\infty)$ such that (26) holds. This result can be applied to the case where in (1) we have $k \geq 0$ but *not* in the case $k < 0$. To see this, consider the linear equation

$$w''''(s) - 2w''(s) + w(s) = 0 \qquad \text{in } \mathbb{R}\,.$$

Here, $f(t) = t$ so that (2) is fulfilled. However, the solutions to this equation are linear combinations of the four functions $\{e^s, e^{-s}, se^s, se^{-s}\}$ and therefore do not exhibit infinitely many sign changes. This shows that the extension of the proof of Theorem 2 is not possible in the case $k < 0$.

Remark 13. The function $w(s) = e^s \sin s$ satisfies the equation $w'''' + 4w = 0$, namely (6) with $f(t) = 4t$. This function f satisfies (2) (and (4)) but *not* (5). In this case, Theorem 2 does not hold since the intervals $[s_j, t_j]$ have constant width π.

We now show some plots for equation (9), that is, when

$$f(t) = e^t - 1\,.$$

Once more we underline that such f satisfies both (2) and (4) (so that the solution is global by Proposition 1) but *not* (5). For different values of k and of initial values of w, the pictures below show that Lemma 10 does not hold if $k > 0$, although [3, Theorem 4] ensures the existence of a sequence of intervals $[s_j, t_j] \subset (0, +\infty)$ such that (26) holds.

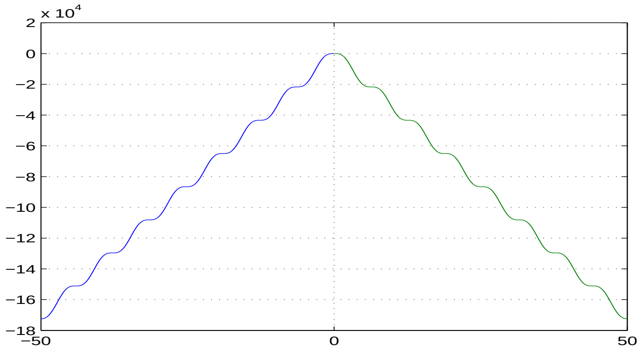

Figure 8: The solution to (9) with $k = 1$, $[w(0), w'(0), w''(0), w'''(0)] = [10, 0, -10, 0]$.

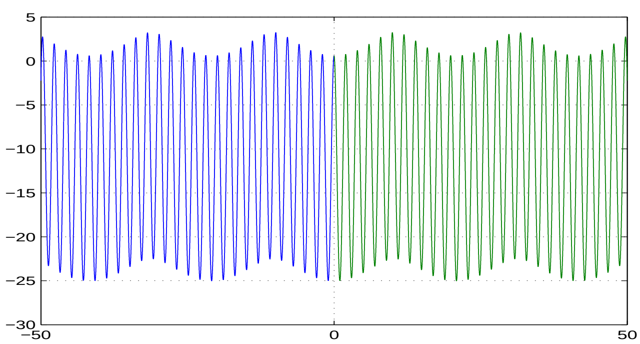

Figure 9: The solution to (9) with $k = 10$, $[w(0), w'(0), w''(0), w'''(0)] = [0.6, 0, -128, 0]$.

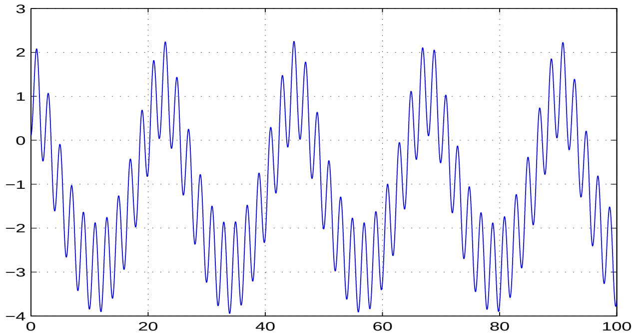

Figure 10: The solution to (9) with $k = 10$, $[w(0), w'(0), w''(0), w'''(0)] = [0.1, 0, 10, 0]$.

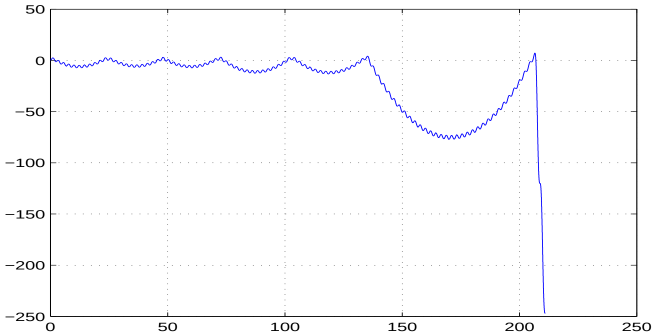

Figure 11: The solution to (9) with $k = 8$, $[w(0), w'(0), w''(0), w'''(0)] = [0.1, 0, 10, 0]$.

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