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On a Decomposition of the Hilbert Space L^2 and Its Applications to Stokes Problem.

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SUNTO - La scomposizione di Helmholtz-Weyl dello spazio L^2 è particolarmente adatta allo studio delle soluzioni del problema di Stokes. Si stabiliscono le principali proprietà della scomposizione e le loro applicazioni al problema di Stokes: opportune condizioni di bordo consentono di semplificare il problema; le tecniche utilizzate consentono anche lo studio di due problemi di tipo Stokes.

ABSTRACT - The Helmholtz-Weyl decomposition of the space L^2 is strictly related with the solutions of Stokes problem. We state the main properties of this decomposition and we apply them to Stokes problem: suitable boundary conditions reduce its solution to simpler problems; the techniques involved also allow to solve two Stokes-type problems.

1. - Introduction.

In this paper we show how the orthogonal decomposition of the Hilbert space L^2 introduced by Helmholtz and Weyl [9, 18] can be employed to solve Stokes and some related problems for a certain class of boundary conditions. If Ω is some region of the space and μ is a positive constant, Stokes problem

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consists in determining u and p satisfying

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega\,, \\ \nabla\cdot u = 0 & \text{in } \Omega\,, \end{cases}$$

for a given function f. In the sequel Ω represents an open connected bounded set of $\mathbb{R}^n$ with boundary $\partial\Omega \in C^3$; in some cases less regularity is sufficient but we will not go deep into this analysis. Bold capital letters ($\boldsymbol{L}^2$, $\boldsymbol{H}^1$, ...) represent functional spaces of vector functions and usual capital letters (L^2, H^1, ...) represent spaces of scalar functions: we set $L^2 := L^2(\Omega)$, ... and we specify the set only when it is not Ω. With H^m we denote the Hilbertian Sobolev spaces, D denotes the space of C^∞ functions with compact support in Ω and D' its dual space (space of distributions). We consider the spaces

$$\boldsymbol{G}_1 := \{f\in \boldsymbol{L}^2;\ \nabla\cdot f = 0,\ \gamma_n f = 0\},\quad \boldsymbol{G}_2 := \{f\in \boldsymbol{L}^2;\ \nabla\cdot f = 0,\ \exists g\in H^1,\ f=\nabla g\},$$
$$\boldsymbol{G}_3 := \{f\in \boldsymbol{L}^2;\ \exists g\in H_0^1,\ f=\nabla g\},\quad \boldsymbol{V} := \{f\in \boldsymbol{H}_0^1;\ \nabla\cdot f = 0\},$$
$$\boldsymbol{E} := \{f\in \boldsymbol{L}^2;\ \nabla\cdot f\in L^2\},\quad \boldsymbol{E}_0 := \{f\in \boldsymbol{E};\ \gamma_n f = 0\},\quad \boldsymbol{M} := \{f\in \boldsymbol{H}^1;\ \Delta f\in \boldsymbol{L}^2\},$$

where γ_n denotes the normal trace operator; γ_n is linear continuous and surjective from $\boldsymbol{E}$ onto $H^{-1/2}(\partial\Omega)$ and its kernel is $\boldsymbol{E}_0$. Obviously $\boldsymbol{V}\subset \boldsymbol{G}_1\subset \boldsymbol{E}_0$ and $\boldsymbol{G}_1\oplus \boldsymbol{G}_2\subset \boldsymbol{E}$; the spaces $\boldsymbol{V}$ and $\boldsymbol{E}$ are Hilbert spaces when endowed with the scalar products $(u, v)_{\boldsymbol{V}} := (\nabla u, \nabla v)_{\boldsymbol{L}^2}$ and $(u, v)_{\boldsymbol{E}} = (u, v)_{\boldsymbol{L}^2} + (\nabla\cdot u, \nabla\cdot v)_{L^2}$. It is well-known (see [9,15,18]) that $\boldsymbol{L}^2 = \boldsymbol{G}_1\oplus \boldsymbol{G}_2\oplus \boldsymbol{G}_3$ and that the spaces $\boldsymbol{G}_i$ ($i = 1, 2, 3$) are mutually orthogonal: we denote by P_i ($i = 1, 2, 3$) the orthogonal projectors of $\boldsymbol{L}^2$ onto $\boldsymbol{G}_i$; we also refer to [4,10] for a similar decomposition of $\boldsymbol{L}^p$ for all $p\in(1, \infty)$. The decomposition of a function $f\in \boldsymbol{L}^2$ following $\boldsymbol{G}_1\oplus \boldsymbol{G}_2\oplus \boldsymbol{G}_3$ is determined by solving the homogeneous Dirichlet problem for a Poisson equation and a Neumann problem for Laplace equation: let $f\in \boldsymbol{L}^2$ and denote $\varphi = \nabla\cdot f$ ($\varphi\in H^{-1}$), let ψ be the unique solution of

$$\text{(1)}\qquad \begin{cases} \Delta\psi = \varphi & \text{in } \Omega\,, \\ \psi\in H_0^1\,, \end{cases}$$

then $P_3 f = \nabla\psi$; let θ be the unique solution (up to the addition of constants) of the problem

$$\text{(2)}\qquad \begin{cases} \Delta\theta = 0 & \text{in } \Omega\,, \\ \dfrac{\partial\theta}{\partial n} = \gamma_n(f - P_3 f) & \text{on } \partial\Omega\,, \end{cases}$$

then $P_2 f = \nabla\theta$. Since $\nabla\cdot(f - P_3 f) = 0$, then $f - P_3 f\in \boldsymbol{E}$ and $\gamma_n(f - P_3 f)$

makes sense; using the generalized Stokes formula

$$\forall f \in \boldsymbol{E} \ \forall g \in H^1 \quad (f, \nabla g)_{L^2} + (\nabla \cdot f, g)_{L^2} = \langle \gamma_n f, \gamma g \rangle \tag{3}$$

(γ is the usual trace operator and $\langle \cdot, \cdot \rangle$ is the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$), one can verify that the compatibility condition for (2) is fulfilled; finally, we have $P_1 f = f - P_2 f - P_3 f$.

In this paper we establish some properties of the decomposition $\boldsymbol{L}^2 = \oplus_i \boldsymbol{G}_i$ and we apply them to solve Stokes problem with nonstandard boundary conditions; we point out that a somehow similar device has been used for Boussinesq equations in [13, 17] by considering a decomposition of the space of solenoidal periodic functions.

In Section 2 we study the behaviour of the spaces $\boldsymbol{G}_i$ when transformed by the Laplace operator and by the corresponding Green operator relative to the homogeneous Dirichlet problem. The proofs of the results in this section are quite simple but the statements are useful tools in the study of Stokes problem and are widely used in the sequel of the paper.

In Section 3 we apply these results to Stokes equations for incompressible fluids: we define *independent motions* as the motions for which Stokes problem can be splitted into two different subproblems, the first one containing only the velocity u and the second one containing only the pressure p; this means that for independent motions we can modify independently the values of the velocity u and of the pressure p by changing in a suitable way the components of f. We study the problem with the three kinds of boundary conditions introduced by Girault [8] (see also [1]) and we show that with these boundary conditions the motion becomes independent: it is therefore possible to project the equation onto the spaces $\boldsymbol{G}_i$ and to decompose it in two subproblems; this leads to the exact value of p and to a very simple equation for u. Girault [8] shows that these boundary conditions are extremely useful for numerical approximations; we use her existence and uniqueness results in Theorems 3.1, 3.2 and 3.3 to obtain precise informations about the solution. Stokes problem becomes then easier to handle: it suffices to determine the components of f by solving (1) and (2) to obtain the exact value of the pressure p and to solve a system of four linear PDE's to determine the three components of the velocity u.

In Section 4, as further applications of Helmholtz-Weyl decomposition, we consider two modifications of the classical Stokes problem. These problems «approximate» in some sense the Stokes problem for compressible flow (see e.g. [16]) and the Stokes problem with a pressure-dependent viscosity (see [6,12]).

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2. – Some features of Helmholtz-Weyl decomposition.

2.1. *Basic properties of the spaces G_i.* In this section we state some properties of the functional spaces defined in the introduction; since the spaces G_i are mutually orthogonal and $G_1 \oplus G_2 \oplus G_3 = L^2$ we immediately infer that the spaces G_i are closed subspaces of L^2. As direct consequences of the definitions of G_2 and G_3 we obtain

PROPOSITION 2.1. *If $f \in G_2$ then f is harmonic in Ω; if $f \in G_3$ and $\nabla \cdot f = 0$ then $f \equiv 0$.*

Let us determine the intersection of G_i with the spaces E and E_0:

PROPOSITION 2.2. *There results:*

(i) $G_2 \cap E_0 = \{0\}$,

(ii) $G_3 \cap E = \{f \in L^2;\ \exists g \in H_0^1 \cap H^2,\ f = \nabla g\}$,

(iii) $G_3 \cap E_0 = \{f \in L^2;\ \exists g \in H_0^2,\ f = \nabla g\}$.

PROOF. (i) If $f \in G_2 \cap E_0$ then $\nabla \cdot f = 0$, $\gamma_n f = 0$ and there exists $g \in H^1$ such that $f = \nabla g$; therefore, $\Delta g = 0$ in Ω and $\partial g / \partial n = 0$ on $\partial \Omega$ which yields $g \equiv$ *constant* and $f \equiv 0$.

(ii) If $f \in G_3 \cap E$ then there exists $g \in H_0^1$ such that $f = \nabla g$ and $\Delta g \in L^2$; hence (see [7]), $g \in H_0^1 \cap H^2$. The converse inclusion $\{f \in L^2;\ \exists g \in H_0^1 \cap H^2,\ f = \nabla g\} \subseteq G_3 \cap E$ is trivial.

(iii) If $f \in G_3 \cap E_0$ then by (ii) we know that there exists $g \in H_0^1 \cap H^2$ such that $f = \nabla g$; moreover, $\partial g / \partial n = \gamma_n f = 0$ and the first inclusion follows. The converse inclusion is trivial. ∎

From the previous result we infer the following

PROPOSITION 2.3. *Let $H_0^{-1/2}(\partial\Omega) := \{\Phi \in H^{-1/2}(\partial\Omega);\ \langle \Phi, 1 \rangle = 0\}$; then γ_n is an isomorphism from G_2 onto $H_0^{-1/2}(\partial\Omega)$.*

PROOF. Since $G_2 \subset E$ we have $\gamma_n(G_2) \subseteq H^{-1/2}(\partial\Omega)$; hence, by (3) (with $f \in G_2$ and $g \equiv 1$) we obtain $\gamma_n(G_2) \subseteq H_0^{-1/2}(\partial\Omega)$. Next, we claim that $\forall \Phi \in H_0^{-1/2}(\partial\Omega)\ \exists! f \in G_2$ such that $\gamma_n f = \Phi$; let q be the unique (up to the addition of constants) solution of

$$\begin{cases} \Delta q = 0 & \text{in } \Omega , \\ \dfrac{\partial q}{\partial n} = \Phi & \text{on } \partial\Omega \ : \end{cases}$$

then $q \in H^1$. Set $f = \nabla q$, then $f \in \boldsymbol{G}_2$ and $\gamma_n f = \Phi$ and the existence of f is proved. Uniqueness follows from the fact that $\ker \gamma_n = \boldsymbol{E}_0$ and from Proposition 2.2(i). ∎

We now study the behaviour of the Laplacian over $\boldsymbol{G}_1$ and $\boldsymbol{G}_3$: the next proposition states that if $f \in \boldsymbol{M}$ has zero component on $\boldsymbol{G}_1$ (respectively $\boldsymbol{G}_3$) then also its Laplacian does.

PROPOSITION 2.4. *Let* $i \in \{1, 3\}$ *and let* $f \in \boldsymbol{M} \cap (\boldsymbol{G}_i \oplus \boldsymbol{G}_2)$; *then* $\Delta f \in \boldsymbol{G}_i \oplus \boldsymbol{G}_2$.

PROOF. If $f \in \boldsymbol{M} \cap \boldsymbol{G}_1$, then $\nabla \cdot (\Delta f) = \Delta(\nabla \cdot f) = 0$ and $\Delta f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$. If $f \in \boldsymbol{M} \cap \boldsymbol{G}_3$, then there exists $g \in H_0^1 \cap H^2$ such that $\Delta f = \Delta(\nabla g) = \nabla(\Delta g)$; hence, $\Delta g \in H^1$ and $\Delta f \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$. The results now follow directly from Proposition 2.1. ∎

We improve this result with a necessary and sufficient condition for a function $f \in \boldsymbol{M}$ to have its Laplacian in $\boldsymbol{G}_1 \oplus \boldsymbol{G}_2$: its divergence must be an harmonic function.

PROPOSITION 2.5. *Let* $f \in \boldsymbol{M}$ *and* $\varphi = \nabla \cdot f$; *then* $\Delta f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$ *if and only if* $\Delta \varphi = 0$.

PROOF. We have $\Delta f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2 \Leftrightarrow \nabla \cdot (\Delta f) = 0 \Leftrightarrow \Delta(\nabla \cdot f) = 0 \Leftrightarrow \Delta \varphi = 0$. ∎

We wish to characterize the harmonic functions in $\boldsymbol{G}_1$ and $\boldsymbol{G}_3$; for $\boldsymbol{G}_1$ we refer to Proposition 3.6 while for $\boldsymbol{G}_3$ (see also Proposition 4.1) we prove

PROPOSITION. 2.6. *Let* $f \in \boldsymbol{M} \cap \boldsymbol{G}_3$; *then* $\Delta f = 0$ *if and only if* $\nabla \cdot f = k$ $(k \in \mathrm{R})$.

PROOF. If $f \in \boldsymbol{M} \cap \boldsymbol{G}_3$, then there exists $g \in H_0^1 \cap H^2$ such that $\nabla g = f$; hence, $\Delta f = 0 \Leftrightarrow \Delta(\nabla g) = 0 \Leftrightarrow \nabla(\Delta g) = 0 \Leftrightarrow \Delta g = k \Leftrightarrow \nabla \cdot f = k$. ∎

REMARKS. If the dimension is $n = 1$ we have $G_1 = \{0\}$, $G_2 \equiv \mathrm{R}$ and $G_3 = \{f \in L^2;\ \int_\Omega f = 0\}$. If $n = 3$ and if $\Omega \subset \mathrm{R}^3$ is connected but not simply connected the space G_1 can be decomposed as direct sum of $G_1' = \ker(\text{curl}) \cap G_1$ and its orthogonal complement G_1'' in G_1; if N denotes the order of connection of Ω it can be proved (see [3]) that $\dim(G_1') = N$: therefore, if Ω is simply connected then $G_1 = G_1''$, $\nabla \wedge (H^1) = \nabla \wedge (H^1 \cap G_1) = G_1 \oplus G_2$ and $[\nabla \wedge (H^1)]^\perp = G_3$.

2.2. *The homogeneous Dirichlet problem for Poisson equation.* In this section we consider the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \quad (f \in L^2), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

and we study its decomposition following $\oplus_i G_i$; consider Green's operator Γ relative to (4): Γ maps L^2 onto $H^2 \cap H_0^1$ and $\Gamma(f) = u =$ unique solution of (4). With our assumption on $\partial\Omega$, we have $P_i u \in H^2$ (see [15] p. 18); a first consequence of this fact is

PROPOSITION 2.7. *For all $u \in H_0^1$ we have* $\int_\Omega \nabla \cdot (P_3 u) = 0$.

PROOF. Since $P_i u \in H^1$ $(i = 1, 2, 3)$, from the divergence Theorem we get

$$\int_\Omega \nabla \cdot (P_1 u + P_2 u + P_3 u) = \int_{\partial\Omega} \gamma_n(u) = 0:$$

the result follows since the functions in $G_1 \oplus G_2$ are solenoidal. ∎

In problem (4) we have $u \in H^2 \cap H_0^1$ and therefore $\|u\|^2_{H_0^1} = -\int_\Omega u \cdot \Delta u$, that is, the only function u orthogonal (in L^2) to its Laplacian is the trivial function $u \equiv 0$; we have so proved

PROPOSITION 2.8. *Let $i \in \{1, 3\}$ and let $u \not\equiv 0$ satisfy $u \in H^2 \cap H_0^1 \cap G_i$; then $\Delta u \notin G_2$. Moreover, if $i \neq j$, $i \neq k$, $j \neq k$ then $\Gamma(G_i) \cap (G_j \oplus G_k) = \{0\}$ and $\Gamma(G_i \oplus G_j) \cap G_k = \{0\}$.*

Compare the first statement with Proposition 2.4: we already know that if $u \in H^2 \cap G_i$ $(i \in \{1, 3\})$ then $\Delta u \in G_2 \oplus G_i$; if in addition $u \in H_0^1$ then $P_i \Delta u$ cannot vanish unless $u \equiv 0$.

We now establish two properties of the space G_2:

PROPOSITION 2.9. (i) *There results $\Gamma(L^2) \cap G_2 = \{0\}$.*

(ii) *If $i \in \{1, 2, 3\}$, then $[P_i \Delta - \Delta P_i](H^2) \subseteq G_2$.*

PROOF. (i) If $u \in \Gamma(\boldsymbol{L}^2) \cap \boldsymbol{G}_2$, then $u \in \boldsymbol{H}_0^1 \cap \boldsymbol{G}_2$; hence, $u \equiv 0$ by Proposition 2.2(i).

(ii) Let $f \in \boldsymbol{H}^2$; if $i = 2$ the result follows directly from Proposition 2.1. If $i = 1$, by Proposition 2.4 we have $\Delta P_1 f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$, hence

$$[P_1 \Delta - \Delta P_1] f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2. \tag{5}$$

Since $P_i f \in \boldsymbol{H}^2$, $P_1 \Delta f = P_1(\Delta P_1 f)$ by Propositions 2.1 and 2.4. Furthermore, for all $v_1 \in \boldsymbol{G}_1$ we have $\int_\Omega \Delta P_1 f \cdot v_1 = \int_\Omega P_1(\Delta P_1 f)\cdot v_1$, hence $\int_\Omega [P_1 \Delta f - \Delta P_1 f]\cdot v_1 = 0$, which, together with (5), yields $[P_1 \Delta - \Delta P_1] f \in \boldsymbol{G}_2$. The result follows in the same fashion when $i = 3$. ∎

We now study the behaviour of the operator Γ on the spaces $\boldsymbol{G}_i$ with respect to vanishing components:

PROPOSITION 2.10. *Let $u \in \Gamma(\boldsymbol{G}_2)$ and $u \neq 0$; then $P_2 u \neq 0$ and $P_3 u \neq 0$.*

PROOF. Let $u \in \Gamma(\boldsymbol{G}_2)$: by Proposition 2.8 we have $P_2 u \neq 0$; by contradiction, if $P_3 u = 0$, we have $\gamma_n(u) = \gamma_n(P_1 u) + \gamma_n(P_2 u)$, hence, $\gamma_n(P_2 u) = 0$ which contradicts Proposition 2.2(i). ∎

PROPOSITION 2.11. *Let $f \in \boldsymbol{G}_1$ be such that $\Gamma(f) \notin \boldsymbol{G}_1$; then $P_i[\Gamma(f)] \neq 0$ $\forall i$.*

PROOF. Let $u = \Gamma(f)$, then $P_1 u \neq 0$ by Proposition 2.8: we claim that $P_2 u = 0 \Leftrightarrow P_3 u = 0$. If $P_3 u = 0$ then $\gamma_n(P_2 u) = \gamma_n(u) - \gamma_n(P_1 u) = 0$; hence, $P_2 u = 0$ by Proposition 2.2(i). Conversely, let $P_2 u = 0$; we know that there exists $g \in H^3 \cap H_0^1$ such that $P_3 u = \nabla g$. As $f = \Delta(P_1 u + P_3 u)$, we have $\nabla \cdot (\Delta P_3 u) = \nabla \cdot (f - \Delta P_1 u) = 0$; therefore, $\Delta^2 g = \Delta(\nabla \cdot P_3 u) = 0$. Moreover, $\gamma_n(P_3 u) = \gamma_n(u) - \gamma_n(P_1 u) = 0$: hence, $\partial g/\partial n = 0$ on $\partial\Omega$. We then obtain the problem

$$\begin{cases} \Delta^2 g = 0 & \text{in } \Omega\,, \\ \dfrac{\partial g}{\partial n} = 0 & \text{on } \partial\Omega\,, \\ g = 0 & \text{on } \partial\Omega\,, \end{cases}$$

which yields $g = 0$, that is, $P_3 u = 0$. ∎

If $u \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ we know that $\Delta u \in \boldsymbol{L}^2$; we can thus decompose Δu:

PROPOSITION 2.12. *For all $u \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ there exist $v_1, v_2, v_3 \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ (uniquely determined) such that $\Delta v_i \in \boldsymbol{G}_i$ and $u = \sum_i v_i$.*

PROOF. We know that $P_i \Delta u \in \boldsymbol{L}^2$; consider the problems ($i = 1, 2, 3$)

$$\begin{cases} \Delta v_i = P_i \Delta u & \text{in } \Omega, \\ v_i \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1 : \end{cases}$$

these problems have a unique solution. ∎

REMARK. If $u \in \boldsymbol{H}^2 \cap \boldsymbol{V}$, then $\Delta u \in (\boldsymbol{G}_1 \oplus \boldsymbol{G}_2) \subset \boldsymbol{E}$ and we can define $\gamma_n(\Delta u) \in H^{-1/2}(\partial\Omega)$: with the notations of Proposition 2.12, we have $\gamma_n(\Delta u) = \gamma_n(\Delta v_2 + \Delta v_3)$ and $\gamma_n(\Delta v_1) = 0$.

The next result states that a function $f \in \boldsymbol{G}_1$ can be seen as the projection (over $\boldsymbol{G}_1$) of the Laplacian of a function in $\boldsymbol{H}^2 \cap \boldsymbol{G}_1$:

PROPOSITION. 2.13. *For all $f \in \boldsymbol{G}_1$ there exists $v \in \boldsymbol{H}^2 \cap \boldsymbol{G}_1$ such that $P_1 \Delta v = f$.*

PROOF. If $u = \Gamma(f)$, then $\sum_i \Delta P_i u = \Delta u = f$ with $P_i u \in \boldsymbol{H}^2 \cap \boldsymbol{G}_i$; as $\Delta P_2 u = 0$ by Proposition 2.1 and $\Delta P_3 u \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$ by Proposition 2.4, the result follows by setting $v = P_1 u$. ∎

In Section 3 we refine this result: Proposition 3.2 restricts the class of v to get uniqueness, Propositions 3.5 and 3.8 yield a function $v \in \boldsymbol{G}_1 \cap \boldsymbol{H}^2$ for which $\Delta v = f$ (without taking its projection).

3. – Independent motions for Stokes problem.

3.1. *The classical Stokes problem.* Stokes equations represent the stationary linearized form of Navier-Stokes equations that appear in fluid-mechanics problems for incompressible fluids, see [5, 10, 15]. Let $\Omega \subset \mathbb{R}^3$ be open and bounded with $\partial\Omega \in C^3$, let $f \in \boldsymbol{L}^2$ be a given vector function in Ω (the external force applied to the fluid): the unknowns are the vector function $u = (u_1, u_2, u_3)$ and the scalar function p representing respectively the velocity and the pressure of the fluid; u and p satisfy (in a suitable sense) the system

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \end{cases} \tag{6}$$

where $\mu > 0$ is the viscosity of the fluid. From the divergence Theorem and from the second of (6) we have a necessary condition for the solvability of (6): $\langle \gamma_n(u), 1\rangle = 0$.

In the sequel we will also deal with the problem ($\varphi \in H^1$)

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega\,, \\ \nabla\cdot u = \varphi & \text{in } \Omega\,; \end{cases} \tag{7}$$

in this case the compatibility condition becomes

$$\langle \gamma_n(u), 1\rangle = \int_\Omega \varphi(x)\,dx\,. \tag{8}$$

The aim of this section is to study under which conditions (6) can be splitted into two different problems for u and p; in these cases we say that the motion of the fluid is *independent*. The separation of the unknown functions u and p is useful for numerical methods: in [2] Stokes problem (with homogeneous Dirichlet condition for u) is divided into two different Dirichlet problems for u and p and this separation allows the introduction of a new numerical approach; similar results are also obtained in [1, 8] for more general boundary conditions.

By Propositions 2.1 and 2.4 we can decompose the first of (6) and obtain

$$-\mu P_1\Delta u = P_1 f\,, \qquad -\mu P_2\Delta u + P_2\nabla p = P_2 f\,, \qquad P_3\nabla p = P_3 f; \tag{9}$$

to get (6) we must add the condition $\nabla\cdot u = 0$. In particular, the previous decomposition states that if $P_3 f = 0$ (i.e. $\nabla\cdot f = 0$) then $\nabla p \in G_2$ and, by Proposition 2.1, we infer

PROPOSITION 3.1. *Let $f \in G_1 \oplus G_2$; if there exists (u, p) solving (6) then p is an harmonic function in Ω.*

Let us first consider the non-slip boundary condition

$$u = 0 \quad \text{on } \partial\Omega\,; \tag{10}$$

it is well-known that (6) (10) has a unique solution (uniqueness for p is to be intended up to the addition of a constant): if $f \in L^2$, there exists a unique $(u, p) \in (H^2 \cap V) \times (H^1/\mathbb{R})$ satisfying (6) almost everywhere in Ω and (10) in the trace Theorem sense. In particular, from the variational formulation of (6) (10) (see [11]) we infer

PROPOSITION 3.2. *For all $f \in G_1$ there exists a unique $u \in H^2 \cap V$ such that $P_1\Delta u = f$.*

3.2. *Independent and rotovorticous motions.* In this section we determine sufficient conditions that allow to split (6) in two different problems for u and p.

DEFINITION 3.1. *We say that the motion of a fluid is independent if it is possible to split the force f in two components, the first one acting only on u and the second one acting only on p.*

As already underlined in the introduction, the main feature of independent motions is that we can modify the velocity u and the pressure p of the fluid independently from each other by changing in a suitable way the components of the external force f. As particular components of the force f we consider its projections onto the spaces $\boldsymbol{G}_i$: in this case independent motions seem to be strictly related with the boundary conditions for problem (6); in Section 3.3 we will give three examples of boundary conditions which generate independent motions. A first kind of independent motions are the motions that we call *rotovorticous motions* because a condition on the curl of the vorticity is required. The vorticity vector ω is defined by

$$\omega = \nabla \wedge u\,;$$

if in some region of Ω we have $\omega = 0$ the motion is of pure deformation and is then called irrotational.

DEFINITION 3.2. *We say that a motion is rotovorticous if* $\nabla \wedge \omega \in \boldsymbol{E}_0$.

Making use of Definitions 3.1 and 3.2, when we deal with the more general problem (7) we will speak about independent and rotovorticous solutions (instead of motions). As particular rotovorticous motions (solutions) we have irrotational motions, motions with constant vorticity vector and motions for which $\omega \in \boldsymbol{D}$. According to the definition of $\boldsymbol{E}_0$, rotovorticous solutions satisfy $\nabla \wedge \omega \in \boldsymbol{L}^2$, $\nabla \cdot (\nabla \wedge \omega) \in L^2$ and $\exists \{\psi_n\} \subset \boldsymbol{D}$ such that $\psi_n \to \nabla \wedge \omega$ in $\boldsymbol{E}$. The first condition is fulfilled when $\Delta u \in \boldsymbol{L}^2$ and $\varphi \in H^1$ since $\nabla \wedge \omega = \nabla\varphi + \Delta u$, the second condition is always fulfilled since $\nabla \cdot (\nabla \wedge \omega) = 0$; for the third condition we know that there exists $\{\psi_n\} \subset \boldsymbol{D}$ such that $\psi_n \to \nabla \wedge \omega$ in $\boldsymbol{L}^2$ ($\boldsymbol{D}$ is dense in $\boldsymbol{L}^2$) and since the derivation operator is continuous in D' we also have $\nabla \cdot \psi_n \to \nabla \cdot (\nabla \wedge \omega) = 0$ in D' but we do not know if this convergence is also in the strong topology of L^2.

Note also that from (3) (with $g \equiv 1$, $f = \nabla \wedge \omega$) we have

$$u \in \boldsymbol{H}^2 \Rightarrow \langle \gamma_n(\nabla \wedge \omega), 1 \rangle = 0\,; \tag{11}$$

thus, if $f \in \boldsymbol{L}^2$ and $\varphi \in H^1$, we have a motion for which (11) holds, while only for rotovorticous motions the stronger condition $\gamma_n(\nabla \wedge \omega) = 0$ is true. Since

$\nabla \cdot (\nabla \wedge \omega) = 0$, rotovorticous solutions satisfy $\nabla \wedge \omega \in \boldsymbol{G}_1$ while the other solutions satisfy $\nabla \wedge \omega \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$. The main simplification due to the rotovorticous property of the solution of problem (7) (and (6)) is illustrated by the following

PROPOSITION 3.3. *If $\varphi \in M$ and $\Delta u \in \boldsymbol{L}^2$, a solution of* (7) *is rotovorticous if and only if $\gamma_n(\Delta u) = \partial\varphi/\partial n$.*

PROOF. We have $\varphi \in M \Leftrightarrow \nabla\varphi \in \boldsymbol{E}$ and $\gamma_n(\Delta u) = \gamma_n(\nabla \wedge \omega) + \gamma_n(\nabla\varphi) = \partial\varphi/\partial n \in H^{-1/2}(\partial\Omega)$. ■

3.3. *Boundary conditions generating independent motions.* In this section we deal with problem (6) while problem (7) will be discussed in Section 4.1; we assume that $f \in \boldsymbol{L}^2$. The boundary conditions that can be added to (6) are of different kind according to the physical problem considered. For problem (6) we always have $\Delta u \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$ and $\nabla p \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$. A possible kind of independent motion is then found when $P_2 \Delta u$ or $P_2 \nabla p$ vanish; the next result relates this remark with rotovorticous motions:

PROPOSITION 3.4. *The motion of an incompressible fluid* (*as described by* (6)) *is rotovorticous if and only if $P_2 \Delta u = 0$* (*i.e. $P_2 \nabla p = P_2 f$*).

PROOF. Since $\nabla \cdot u = 0$ we have $\nabla \wedge \omega = -\Delta u$ and the result follows from Proposition 3.3. ■

We know that there exists $g_2 \in H^1$ and $g_3 \in H_0^1$ such that $\nabla g_i = P_i f$ ($i = 2, 3$); let

(12) $$g = g_2 + g_3,$$

in this section g_i and g will always represent these functions.

A first example where the motion becomes rotovorticous is when to equations (6) we associate the boundary conditions

(13) $$\gamma_n(u) = \gamma_n(\omega) = \gamma_n(\nabla \wedge \omega) = 0\,;$$

with these conditions we obtain an improvement of a result in [8]: we characterize the solution by means of the components $P_i f$ of f.

THEOREM 3.1. *There exists a unique solution $(u, p) \in (\boldsymbol{H}^2 \cap \boldsymbol{G}_1) \times (H^1/\mathrm{R})$ of* (6) *with boundary conditions* (13); *this unique solution is given by* (*g as in* (12), $k \in \mathrm{R}$)

(14) $$p(x) = g(x) + k$$

and by the unique solution u *of the problem*

$$\begin{cases} \Delta u = -\dfrac{1}{\mu}\cdot P_1 f & in\ \Omega\,, \\ \nabla\cdot u = 0 & in\ \Omega\,, \\ (13)\,. \end{cases}$$

PROOF. Existence and uniqueness of a solution $(u, p) \in \boldsymbol{G}_1 \times (H^1/\mathrm{R})$ such that $\omega \in \boldsymbol{E}$ is proved in [8]. Now, use twice Proposition 1.4 in [3] to obtain $u \in \boldsymbol{H}^2$: thus $\Delta u \in \boldsymbol{L}^2$ and (6) holds almost everywhere; moreover, $\nabla \wedge \omega \in \boldsymbol{E}$ and (13) is defined in the trace Theorem sense. By (13) and Proposition 3.4, $P_2 \Delta u = 0$ and u is solution of the above problem; by (9) we have $\nabla p = P_2 f + P_3 f$ and the result follows. ■

REMARK. If we take $f \in \boldsymbol{E}$ we have $P_3 f = f - P_1 f - P_2 f \in \boldsymbol{E}$ and therefore $p \in M$.

From Theorem 3.1 we infer directly

PROPOSITION 3.5. *For all* $f \in \boldsymbol{G}_1$ *there exists a unique* $u \in \boldsymbol{H}^2 \cap \boldsymbol{G}_1$ *with* $\nabla \wedge u \in \boldsymbol{G}_1$ *and* $\nabla \wedge (\nabla \wedge u) \in \boldsymbol{G}_1$ *such that* $f = \Delta u$.

Another consequence of Theorem 3.1 is a characterization of the nontrivial harmonic functions of $\boldsymbol{M} \cap \boldsymbol{G}_1$: at least one between $P_2(\nabla \wedge f)$ and $P_2[\nabla \wedge (\nabla \wedge f)]$ must be different from 0.

PROPOSITION 3.6. *Let* $f \in \boldsymbol{M} \cap \boldsymbol{G}_1$ *be such that* $\Delta f = 0$; *if* $\nabla \wedge f \in \boldsymbol{G}_1$ *and* $\nabla \wedge (\nabla \wedge f) \in \boldsymbol{G}_1$, *then* $f \equiv 0$.

PROOF. By the assumptions we obtain $\gamma_n(\nabla \wedge f) = \gamma_n[\nabla \wedge (\nabla \wedge f)] = 0$; hence, $f \equiv 0$ by the uniqueness result of Theorem 3.1. ■

A second example of boundary conditions generating rotovorticous motions is

$$\omega \wedge n = 0 \ \text{ on } \partial\Omega\,, \qquad \gamma_n(u) = 0\,. \tag{15}$$

Conditions (15) approximate in some sense the classical free-slip (stress-free) boundary conditions: in particular, these conditions coincide in the regions of the boundary where the curvature is zero. Note also that (13) and (15) are geometrically complementary: the former says that the vorticity vector ω lies on the tangent plane to $\partial\Omega$ while the latter says that ω is orthogonal to $\partial\Omega$.

The following existence and uniqueness result holds:

THEOREM 3.2. *There exists a unique solution* $(u, p) \in (\boldsymbol{M} \cap \boldsymbol{G}_1) \times (H^1/\mathbb{R})$ *of* (6) (15) *given by* (14) *and by the unique solution* u (*almost everywhere*) *of the problem*

$$\begin{cases} \Delta u = -\dfrac{1}{\mu} \cdot P_1 f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ (15). \end{cases}$$

PROOF. Let $\boldsymbol{U} := \{v \in \boldsymbol{G}_1;\ \nabla \wedge v \in \boldsymbol{L}^2\}$; it is proved in [8] that there exists a unique solution $(u, p) \in \boldsymbol{U} \times H^1$ such that

$$\begin{cases} \mu \cdot \displaystyle\int_\Omega (\nabla \wedge u) \cdot (\nabla \wedge v) = \int_\Omega f \cdot v & \forall v \in \boldsymbol{U}, \\ \displaystyle\int_\Omega \nabla p \cdot \nabla q = \int_\Omega f \cdot \nabla q & \forall q \in H^1; \end{cases}$$

h e nce , ∇p and f have the same component over $\boldsymbol{G}_2 \oplus \boldsymbol{G}_3$. Applying Proposition 1.4 in [3] we have $u \in \boldsymbol{H}^1$; note that $\int_\Omega (\nabla \wedge u) \cdot (\nabla \wedge v) = -\langle \Delta u, v \rangle$ for all $v \in \boldsymbol{V}$ so that $u \in \boldsymbol{M}$ by a density argument and the result follows. ∎

A direct consequence of these results is a refinement of Proposition 3.1:

PROPOSITION 3.7. *Assume either* (13) *or* (15); *then*

(i) *if* $f \in \boldsymbol{G}_1$ *the function* p *solving* (6) *is a constant function in* Ω,

(ii) *if* $f \in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2$ *the solution* p *of* (6) *is an harmonic function in* Ω.

Another corollary of Theorem 3.2 is

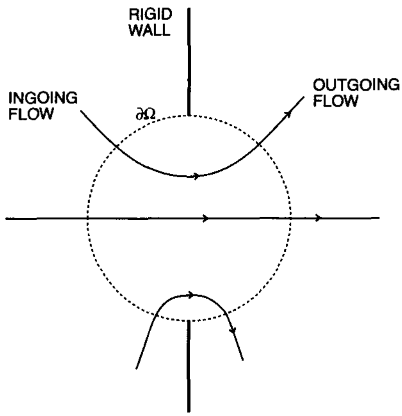
PROPOSITION 3.8. *For all* $f \in \boldsymbol{G}_1$ *there exists a unique* $u \in \boldsymbol{M} \cap \boldsymbol{G}_1$ *with* $(\nabla \wedge u) \wedge n = 0$ *on* $\partial\Omega$ *such that* $f = \Delta u$.

REMARK. For all the boundary conditions so far considered (i.e. (10), (13) and (15)) the following equivalence holds: $P_1 f = 0 \Leftrightarrow u \equiv 0$.

Next we study another kind of independent motion; we deal with the case where $P_2 \nabla p = 0$: to this aim we consider the boundary conditions

$$u \wedge n = 0 \ \text{ on } \partial\Omega, \qquad \gamma p = \lambda \ (\lambda \in \mathbb{R}). \tag{16}$$

Conditions (16) are available, for instance, in the following inflow-outflow problem: consider a rigid wall separating two basins and imagine that a fluid goes from one basin to the other through a circular hole; making a section in the direction of the wall, the situation can be described by the following picture ($\partial\Omega$ is the dotted line).



We also refer to [1] where conditions (16) are taken to describe flows in a network of pipes. Note also that (16) requires u to be normal to $\partial\Omega$ while (13) and (15) require u to be tangent to $\partial\Omega$: for problem (6) (16) we obviously do not have $u \in \boldsymbol{G}_1$ unless $\gamma u = 0$. The following result holds:

THEOREM 3.3. *There exists a unique solution* $(u, p) \in \boldsymbol{M} \times H^1$ *of problem* (6) (16); *if* g_3 *is as in* (12) *and* λ *is as in* (16) *this unique solution is given by*

$$p(x) = g_3(x) + \lambda$$

and by the unique solution u *(almost everywhere) of the problem*

$$\begin{cases} \Delta u = -\dfrac{1}{\mu}\cdot(P_1 f + P_2 f) & \text{in } \Omega\,, \\ \nabla\cdot u = 0 \quad \text{in } \Omega\,, \\ u \wedge n = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

PROOF. Existence, uniqueness and regularity are proved in [1,8]; we have $\nabla p \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$, but $\nabla p = \nabla(p - \lambda)$ and $(p - \lambda) \in H_0^1$, thus $\nabla p \in \boldsymbol{G}_3$ ($P_2 \nabla p = 0$) which yields $\nabla p = P_3 f$; from (9) we have $\Delta u = -1/\mu\cdot(P_1 f + P_2 f)$ since $-\mu P_2 \Delta u = P_2 f$. ■

To conclude this section we remark that Theorems 3.1, 3.2 and 3.3 allow to determine the exact value of the pressure p when (1) and (2) are solved. Moreover, for u we have a linear differential system of four equations to determine its three components: for this system we do not wonder about its solvability because of the above theorems. The solution of Stokes problem for independent motions is then reduced to the solution of three simpler problems; it would be interesting to obtain similar results for the non-slip boundary condition (10).

4. - Two Stokes-type problems

4.1. *The problem with $\nabla \cdot u = \varphi$.* In this section we consider problem (7) with $f \in \boldsymbol{L}^2$, $\varphi \in M$; this problem is not Stokes problem for compressible flows but it can be used to prove some related results (see e.g. [16]). We first prove that the divergence operator is an isomorphism from $\boldsymbol{H}^2 \cap \boldsymbol{G}_3$ onto H^1:

PROPOSITION 4.1. *Let $\varphi \in H^1$, then there exists a unique $v \in (\boldsymbol{H}^2 \cap \boldsymbol{G}_3)$ such that*

$$\nabla \cdot v = \varphi . \tag{17}$$

PROOF. Let ψ be the unique solution of the problem

$$\begin{cases} \Delta\psi = \varphi & \text{in } \Omega , \\ \psi = 0 & \text{on } \partial\Omega : \end{cases}$$

then $\psi \in H_0^1 \cap H^3$ (see [7]). Now let $v = \nabla\psi$, then $v \in \boldsymbol{H}^2 \cap \boldsymbol{G}_3$ and $\nabla \cdot v = \Delta\psi = \varphi$ and existence is proved. To prove uniqueness, let $v' \in \boldsymbol{G}_3$ be another solution of (17), then $v - v' \in \boldsymbol{H}^2 \cap \boldsymbol{G}_3$; moreover, $\nabla \cdot (v - v') = 0$: by Proposition 2.1 we obtain $v = v'$. ■

From now on, v denotes the unique function defined by (17): note that $\Delta v = \nabla\varphi$; hence, if we set $w = u - v$ and if u satisfies (7) then w satisfies

$$\begin{cases} -\mu\Delta w + \nabla(p - \mu\varphi) = f & \text{in } \Omega , \\ \nabla \cdot w = 0 & \text{in } \Omega . \end{cases} \tag{18}$$

Using the results of Section 3.3 we can determine boundary conditions generating independent solutions for (7): if $\varphi \in M$, we consider v given by (17) and the boundary conditions

$$\gamma_n(u) = \gamma_n(v), \qquad \gamma_n(\omega) = \gamma_n(\nabla \wedge \omega) = 0 . \tag{19}$$

We can prove

THEOREM 4.1. *Let $\varphi \in M$, $f \in \boldsymbol{L}^2$ and $g \in H^1$ be such that $\nabla g = P_2 f + P_3 f$. Then there exists a unique solution $(u, p) \in \boldsymbol{H}^2 \times (H^1/\mathbb{R})$ of (7), (19) given by*

$$p(x) = \mu\varphi(x) + g(x) + k \qquad (k \in \mathbb{R})$$

and by the unique solution u of the problem

$$\begin{cases} \Delta u = -\dfrac{1}{\mu} \cdot P_1 f + \nabla\varphi & \text{in } \Omega , \\ \nabla \cdot u = \varphi & \text{in } \Omega \\ (19) , \end{cases}$$

the equations being satisfied almost everywhere in Ω.

PROOF. As $v \in \boldsymbol{H}^2 \cap \boldsymbol{G}_3$ we have $\nabla \wedge v = \nabla \wedge (\nabla \wedge v) = 0$; let $w = u - v$, then to (18) we must add the boundary conditions

$$\gamma_n(w) = \gamma_n(\nabla \wedge w) = \gamma_n[\nabla \wedge (\nabla \wedge w)] = 0$$

which, by Theorem 3.1 yield a unique solution (w, p) given by $p(x) = \mu\varphi(x) + g(x) + k$ and

$$\begin{cases} \Delta w = -\dfrac{1}{\mu} \cdot P_1 f & \text{in } \Omega , \\ \nabla \cdot u = 0 & \text{in } \Omega , \\ \gamma_n(w) = \gamma_n(\nabla \wedge w) = \gamma_n[\nabla \wedge (\nabla \wedge w)] = 0 , \end{cases}$$

and the result is proved. ■

Reasoning as for Theorem 4.1 we can prove

THEOREM 4.2. *Let $\varphi \in M$, let g and v have the same meaning as in Theorem 4.1; then there exists a unique solution $(u, p) \in \boldsymbol{M} \times (H^1/\mathbb{R})$ of (7) with boundary conditions*

$$\gamma_n(u) = \gamma_n(v), \qquad \omega \wedge n = 0 \ \text{ on } \partial\Omega .$$

This unique solution is given by ($k \in \mathbb{R}$)

$$p(x) = \mu\varphi(x) + g(x) + k$$

and by the unique solution u *(almost everywhere) of the problem*

$$\begin{cases} \Delta u = -\dfrac{1}{\mu}\cdot P_1 f + \nabla\varphi & \text{in } \Omega\,, \\ \nabla\cdot u = \varphi \quad \text{in } \Omega\,, \\ \gamma_n(u) = \gamma_n(v)\,, \quad \omega \wedge n = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Note that by (17) the condition (8) is satisfied for both Theorems 4.1 and 4.2.

In a completely similar way we can also prove the analogue of Theorem 3.3 and find independent solutions of (7) which are not rotovorticous:

THEOREM 4.3. *Let* $\varphi \in M$, *let* $g_3 \in H_0^1$ *be such that* $P_3 f = \nabla g_3$ *and let* v *be as in* (17); *then there exists a unique solution* $(u, p) \in \boldsymbol{M} \times H^1$ *of* (7) *with boundary conditions*

$$\gamma p = \lambda + \mu\cdot\gamma\varphi \;\; (\lambda \in \mathbb{R})\,, \qquad u \wedge n = v \wedge n \;\; \text{on } \partial\Omega\,.$$

This unique solution is given by

$$p(x) = \mu\varphi(x) + g_3(x) + \lambda$$

and by the unique solution u *(almost everywhere) of*

$$\begin{cases} \Delta u = -\dfrac{1}{\mu}\cdot(P_1 f + P_2 f) + \nabla\varphi & \text{in } \Omega\,, \\ \nabla\cdot u = \varphi \quad \text{in } \Omega\,, \\ u \wedge n = v \wedge n & \text{on } \partial\Omega\,. \end{cases}$$

4.2. *The problem with* $\mu = \mu(p)$. In this section we modify slightly problem (6) (10) and we solve the nonlinear problem

$$(20) \qquad \begin{cases} -\mu(p)\Delta u + \nabla p = f & \text{in } \Omega\,, \\ \nabla\cdot u = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \\ \dfrac{\partial p}{\partial n} = f\cdot n & \text{on } \partial\Omega\,, \end{cases}$$

where n represents the unit outer normal on $\partial\Omega$: μ is no longer a constant but it is now supposed to depend on p. The Neumann condition states that the external force f applied at the boundary determines the variation of the pressure p; in fact, by applying the normal trace operator γ_n to the first of (20) as in [1], we see that this condition «forces» problem (20) for which we cannot expect in general the existence of a solution: the solution needs also to satisfy $\gamma_n(\Delta u) = 0$. The first overdetermined problem was solved by Serrin [14]: he proved that if certain problems in potential theory admit a solution u, then the domain must be a ball and u is radially symmetric. Here we do not wonder about the properties of Ω but we give a necessary condition on the data f for problem (20) to have a solution; a sufficient condition to obtain a (trivial) solution of (20) will also be given in this section.

Note also that system (20) does not represent Stokes equations with a pressure-dependent viscosity: indeed, the first equation should be derived from $-\nabla\cdot[\mu(p)(\nabla u + \nabla^T u)] + \nabla p = f$ (see [12]), where $\nabla^T u$ denotes the transposed tensor of ∇u. However, if we suppose that the dependence $\mu = \mu(p)$ is not very important (that is, if $\mu'(p)$ is «small») we can neglect the term $-\mu'(p)(\nabla u + \nabla^T u)\nabla p$ and obtain the first of (20).

To solve problem (20) we assume that $\Omega \subseteq \mathbf{R}^n$ is open and bounded with boundary $\partial\Omega \in C^3$ and that

$$\begin{cases} \mu \text{ is absolutely continuous}, \\ \inf_{\mathbf{R}} |\mu(x)| > 0 \\ \mu'(x) \text{ keeps constant sign in } \mathbf{R}, \\ \mu \in W^{1,\infty}(\mathbf{R}), \end{cases} \tag{21}$$

$$f \in \boldsymbol{E}, \tag{22}$$

$$\text{the mean value } \bar{p} \text{ of } p \text{ over } \Omega \text{ is known}. \tag{23}$$

In the sequel we denote $f_1 = P_1 f$, $f' = P_2 f + P_3 f$ and with h a function such that $\nabla h = f'$; from (22) we infer that $f' \in \boldsymbol{E}$ and by Proposition 2.2(ii), $h \in M$.

If we set $q = p - h$, $\theta(q) = \mu(q + h)$ system (20) becomes

$$\begin{cases} -\theta(q)\Delta u + \nabla q = f_1 \quad \text{in } \Omega, \\ \nabla\cdot u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \dfrac{\partial q}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{24}$$

and (23) yields $\bar{q} = (1/|\Omega|)\int_\Omega q = \bar{p} - \bar{h}$ where $\bar{h} = (1/|\Omega|)\int_\Omega h$; moreover, assumptions (21) on μ can be «translated» into similar assumptions on θ. After solving system (24) we shall immediately infer a solution of (20); indeed if (u, q) solves the first, $(u, q+h)$ will solve the latter. So, take the divergence of the vector equation in (24) to obtain $\Delta q - \theta'(q)\Delta u \cdot \nabla q = 0$; from the vector equation itself we get $\Delta u = (1/\theta(q)) \cdot (\nabla q - f_1)$ which, inserted in the previous one and in (24), yields the following problem

$$\begin{cases} \theta(q)\Delta q - \theta'(q)|\nabla q|^2 + \theta'(q) f_1 \cdot \nabla q = 0 & \text{in } \Omega\,, \\ \dfrac{\partial q}{\partial n} = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{25}$$

We can prove

PROPOSITION 4.2. *Let* (21)-(23) *hold; then problem* (25) *has a unique solution* $q \in M$ *which is constant in* Ω *and is given by* $q(x) \equiv \bar{q}$.

PROOF. Let m be any positive integer, multiply (25) by $\theta^m(q)$ and integrate over Ω:

$$\int_\Omega [\theta^{m+1}(q)\Delta q - \theta'(q)\theta^m(q)|\nabla q|^2 + \theta'(q)\theta^m(q) f_1 \cdot \nabla q] = 0\,.$$

Bearing in mind the boundary condition of (25) and integrating by parts we get

$$\int_\Omega \theta'(q)\theta^m(q)[(m+2)|\nabla q|^2 - f_1 \cdot \nabla q] = 0\,;$$

if $\nabla q \not\equiv 0$, we get a contradiction from the arbitrariness of m and from (21). ■

As a consequence, we obtain

PROPOSITION 4.3. *Let* (21)-(23) *hold, and assume that* (20) *has a solution* (u, p) *with* $p \in M$; *then* $p(x) = \bar{q} + h(x) = h(x) + \bar{p} - \bar{h}$.

We can now go back to problem (20); from Proposition 4.3 we infer that u must be the unique solution of the following Dirichlet problem for Poisson's equation:

$$\begin{cases} \Delta u = -\dfrac{1}{\mu(h + \bar{p} - \bar{h})} \cdot f_1 & \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{26}$$

The problem now is that the unique solution u of (26) may not be solenoidal; hence, we have

THEOREM 4.4. *Assume* (21)-(23), *then problem* (20) *has a solution* $(u, p) \in (\boldsymbol{H}^2 \cap \boldsymbol{V}) \times M$ *if and only if*

$$\Gamma\left(\frac{1}{\mu(h + \overline{p} - \overline{h})} \cdot f_1\right) \in \boldsymbol{G}_1 .$$

In this case the solution is unique and p *is given by Proposition* 4.3 *while* u *is given by*

$$u = \Gamma\left(\frac{1}{\mu(h + \overline{p} - \overline{h})} \cdot f_1\right).$$

From Theorem 4.4 we obtain

PROPOSITION 4.4. *Assume* (21)-(23) *and suppose that* $f_1 = 0$; *then problem* (20) *has a unique solution* (u, p) *given by* $u \equiv 0$ *and* $p(x) = h(x) + \overline{p} - \overline{h}$.

Our last result is a necessary condition (on the data f) for the existence of a solution of problem (20); we know that $f_1 \perp f'$ in the sense of $\boldsymbol{L}^2$, the necessary condition is the much stronger $f_1 \perp f'$ in almost every point of Ω:

THEOREM 4.5. *Assume* (21)-(23). *If problem* (20) *has a (unique) solution* $(u, p) \in (\boldsymbol{H}^2 \cap \boldsymbol{V}) \times M$ *then* $f_1(x) \cdot f'(x) = 0$ *almost everywhere in* Ω.

PROOF. Taking the divergence of the first of (26) yields

$$\frac{\mu'(h + \overline{p} - \overline{h})}{\mu^2(h + \overline{p} - \overline{h})} f_1 \cdot \nabla h = 0$$

and with assumptions (21) we get the result. ■

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