Nonlinear Analysis 70 (2009) 2965–2973

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Some new properties of biharmonic heat kernels

Filippo Gazzola [a,*], Hans-Christoph Grunau [b]

[a] Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32, 20133 Milano, Italy

[b] Fakultät für Mathematik, Otto-von-Guericke-Universität, Postfach 4120, 39016 Magdeburg, Germany

ARTICLE INFO

Keywords:
Biharmonic parabolic equations
Heat kernels

ABSTRACT

Contrary to the second-order case, biharmonic heat kernels are sign-changing. A deep knowledge of their behaviour may however allow us to prove positivity results for solutions of the Cauchy problem. We establish further properties of these kernels, we prove some Lorch–Szegö-type monotonicity results and we give some hints on how to obtain similar results for higher order polyharmonic parabolic problems.

1. Introduction

Consider the following Cauchy problem for the biharmonic heat equation

$$\begin{cases} u_t + \Delta^2 u = 0 & \text{in } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 1$ and $u_0 \in C^0 \cap L^\infty(\mathbb{R}^n)$. It is well known (see [1] and the references therein) that (1) admits a unique global in time solution explicitly given by

$$u(x,t) = \alpha_n t^{-n/4} \int_{\mathbb{R}^n} u_0(x-y) f_n\left(\frac{|y|}{t^{1/4}}\right) \mathrm{d}y, \quad (x,t) \in \mathbb{R}^{n+1}_+.$$

Here $\alpha_n > 0$ denotes a suitable normalization constant and

$$f_n(\eta) = \eta^{1-n} \int_0^\infty \mathrm{e}^{-s^4} (\eta s)^{n/2} J_{(n-2)/2}(\eta s) \mathrm{d}s, \tag{2}$$

where J_ν denotes the νth Bessel function of the first kind.

Contrary to the second-order heat equation, no general positivity preserving property holds for (1), namely the positivity of the initial datum u_0 *may not* imply positivity (in space and time) for the solution $u = u(x,t)$ of (1). However, a careful analysis of the kernels f_n in (2) enables us to obtain some restricted and somehow hidden positivity, see [1,2]. This property is called *eventual local positivity* and reflects the fact that, for suitable initial data, the solution of (1) becomes positive on compact domains of $\mathbb{R}^n$ for sufficiently large time t and the time depends on the compact set itself.

Let us also mention that positivity for (1) with a source term (namely $u_t + \Delta^2 u = f$) has been studied in [3] for linear problems when $f = f(x,t)$ and in [1] for nonlinear problems when $f = |u|^{p-1}u$ for $p > 1 + 4/n$ (the so-called super-Fujita case, see [4]). See also [5] for estimates, existence and decay of global solutions. We also refer [6,7] for related and blow-up results in the case $f = |u|^p$.

A better understanding of the behaviour of the kernels will certainly allow us to reach stronger results on positivity of solutions to (1). This is precisely the first goal of the present paper. After recalling in Section 2 some known results, in

* Corresponding author.
E-mail addresses: filippo.gazzola@polimi.it (F. Gazzola), hans-christoph.grunau@ovgu.de (H.-Ch. Grunau).

0362-546X/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.na.2008.12.039

Section 3 we establish some new features of the f_n-functions. These features enable us to reach the second purpose of this paper, namely Lorch–Szegö-type monotonicity results for the f-functions, see Section 4. This means that the sequence of moduli of certain weighted integrals of f_n between consecutive zeros of f_n is monotonically decreasing. Let us briefly explain how such monotonicity results may be used in order to obtain eventual local positivity. If the initial datum u_0 is positive and behaves like $|x|^{-\beta}$ $(0 < \beta < n)$ at ∞ (see (9)), it was proved in [1] that one should know that

$$\int_0^\infty \eta^{-\beta}\left[\eta^{n-1}f_n(\eta)\right]\mathrm{d}\eta = \int_0^\infty \eta^{n-\beta-2}\left[\eta f_n(\eta)\right]\mathrm{d}\eta > 0. \tag{3}$$

It is this constant times $t^{-\beta/4}$ which locally gives the asymptotic behaviour for $t \to \infty$ of the solution of the Cauchy problem. Below we will prove a Lorch–Szegö-type monotonicity result for $\eta \mapsto \eta f_n(\eta)$, $(n \geq 2)$, so that positivity of the above integral is immediate for $\beta \in [(n-2), n)$. In [1], inequality (3) (and more) was proved without referring explicitly to Lorch–Szegö-type results, but we think that our present approach gives a more natural interpretation of certain eventual local positivity features.

Finally, in Section 5 we give hints on how to extend some properties of the kernels and their consequences to *polyharmonic* heat equations.

2. Preliminaries and notations

Let us first recall that thanks to Galaktionov–Pohožaev [8], we know that the f-functions have exponential decay at infinity. More precisely, for any integer $n \geq 1$ there exist $K = K_n > 0$, $\mu = \mu_n > 0$ such that

$$|f_n(\eta)| \leq K \exp\left(-\mu\eta^{4/3}\right) \quad \text{for all } \eta \geq 0. \tag{4}$$

Then, we recall some properties of the f-functions proved in [1]. Firstly, a recursion formula holds:

$$f_n'(\eta) = -\eta f_{n+2}(\eta) \quad \text{for all } n \geq 1. \tag{5}$$

Moreover, f_n satisfies the following third-order differential equation (see [1, Theorem 6])

$$f_n'''(\eta) + \frac{n-1}{\eta}f_n''(\eta) - \frac{n-1}{\eta^2}f_n'(\eta) - \frac{\eta}{4}f_n(\eta) = 0, \tag{6}$$

which we shall also exploit in the following equivalent form

$$(\Delta f_n)'(\eta) = \frac{\eta}{4}f_n(\eta). \tag{7}$$

Thanks to (5) and (6), in [1] the following result was proved

$$\int_0^\infty \eta^{n-1-\beta}f_n(\eta)\mathrm{d}\eta > 0 \quad \text{for all integer } n \geq 1 \text{ and all } \beta \in [0, n). \tag{8}$$

In turn, (8) was used to prove the *eventual local positivity* property for (1) with initial data of the kind

$$u_0(x) = \frac{1}{g(x) + |x|^\beta} \quad \text{where } g \in C^0(\mathbb{R}^n, \mathbb{R}_+) \text{ satisfies } \lim_{|x|\to\infty}\frac{g(x)}{|x|^\beta} = 0 \tag{9}$$

for some $\beta \in [0, n)$. By eventual local positivity we mean that the solution of (1) is (locally) positive on compact domains of $\mathbb{R}^n$ for (eventual) sufficiently large time t. The proof of (8) is quite lengthy and delicate.

It is also shown in [1, Theorem 7] that $f_n(\eta)$ changes sign infinitely many times as $\eta \to +\infty$. For a fixed n, we denote by $\{\zeta_j\}$ the sequence of all the zeroes of f_n:

$$f_n(\zeta_j) = 0 \quad 0 < \zeta_1 < \zeta_2 < \cdots.$$

In some situations, we need to distinguish between "ascending" zeroes (where $f_n' > 0$) and "descending" zeroes (where $f_n' < 0$). To this end, we denote by P_k (resp. N_k) the successive intervals, where f_n is positive (resp. negative) so that we have

$$[0, \infty) = \bigcup_{k=1}^\infty \left(\overline{P_k} \cup \overline{N_k}\right).$$

Moreover, we write

$$z_k^+ := \sup P_k = \inf N_k, \qquad z_k^- := \sup N_k = \inf P_{k+1} \quad (k \in \mathbb{N})$$

so that $\cup_k\{z_k^\pm\} \equiv \cup_j\{\zeta_j\}$ are the zeroes of f_n. Let

$$\mu_k \in P_k \quad \text{be such that} \quad f_n(\mu_k) = \max_{\eta\in P_k} f_n(\eta)$$

$$m_k \in N_k \quad \text{be such that} \quad f_n(m_k) = \min_{\eta\in N_k} f_n(\eta).$$

In particular, $f_n'(\mu_k) = f_n'(m_k) = 0$ and we know that

$$0 = \mu_1 < z_1^+ < m_1 < z_1^- < \mu_2 < z_2^+ < m_2 < z_2^- < \cdots.$$

3. Behaviour of the f-functions at some special points

3.1. Behaviour at zeroes

Proposition 1. *Assume that $n \geq 4$. Then, for all $j \geq 1$ we have $\zeta_j |f_n'(\zeta_j)| > \zeta_{j+1}|f_n'(\zeta_{j+1})|$.*

Proof. We use (7) and obtain, observing that $f_n(\zeta_j) = f_n(\zeta_{j+1}) = 0$:

$$\begin{aligned}
-\frac{1}{2}\left(\zeta_{j+1}^2 f_n'(\zeta_{j+1})^2 - \zeta_j^2 f_n'(\zeta_j)^2\right) &= -\frac{1}{2}\int_{\zeta_j}^{\zeta_{j+1}} \left(\eta^2 f_n'(\eta)^2\right)' \mathrm{d}\eta \\
&= -\int_{\zeta_j}^{\zeta_{j+1}} \left(\eta f_n'(\eta)^2 + \eta^2 f_n'(\eta) f_n''(\eta)\right) \mathrm{d}\eta \\
&= (n-2)\int_{\zeta_j}^{\zeta_{j+1}} \eta f_n'(\eta)^2 \mathrm{d}\eta - \int_{\zeta_j}^{\zeta_{j+1}} \eta^2 f_n'(\eta) \Delta f_n(\eta) \mathrm{d}\eta \\
&= (n-2)\int_{\zeta_j}^{\zeta_{j+1}} \eta f_n'(\eta)^2 \mathrm{d}\eta + 2\int_{\zeta_j}^{\zeta_{j+1}} \eta f_n(\eta)\Delta f_n(\eta)\mathrm{d}\eta + \int_{\zeta_j}^{\zeta_{j+1}} \eta^2 f_n(\eta)(\Delta f_n)'(\eta)\mathrm{d}\eta \\
&= (n-4)\int_{\zeta_j}^{\zeta_{j+1}} \eta f_n'(\eta)^2 \mathrm{d}\eta + 2(n-2)\int_{\zeta_j}^{\zeta_{j+1}} f_n(\eta) f_n'(\eta)\mathrm{d}\eta + \frac{1}{4}\int_{\zeta_j}^{\zeta_{j+1}} \eta^3 f_n(\eta)^2 \mathrm{d}\eta \\
&= (n-4)\int_{\zeta_j}^{\zeta_{j+1}} \eta f_n'(\eta)^2 \mathrm{d}\eta + \frac{1}{4}\int_{\zeta_j}^{\zeta_{j+1}} \eta^3 f_n(\eta)^2 \mathrm{d}\eta > 0
\end{aligned}$$

since we assume that $n \geq 4$. □

In lower dimensions one has a slightly weaker statement:

Proposition 2. *For any $n \geq 1$ and any $j \geq 1$ one has $|f_n'(\zeta_j)| > |f_n'(\zeta_{j+1})|$.*

Proof. The differential equation (7) directly shows:

$$\begin{aligned}
0 &< \int_{\zeta_j}^{\zeta_{j+1}} \frac{\eta}{4} f_n(\eta)^2 \mathrm{d}\eta = \int_{\zeta_j}^{\zeta_{j+1}} f_n(\eta)(\Delta f_n)'(\eta)\mathrm{d}\eta = -\int_{\zeta_j}^{\zeta_{j+1}} f_n'(\eta)\Delta f_n(\eta)\mathrm{d}\eta \\
&= -\int_{\zeta_j}^{\zeta_{j+1}} f_n'(\eta) f_n''(\eta)\mathrm{d}\eta - (n-1)\int_{\zeta_j}^{\zeta_{j+1}} \frac{f_n'(\eta)^2}{\eta}\mathrm{d}\eta \leq -\frac{1}{2} f_n'(\zeta_{j+1})^2 + \frac{1}{2} f_n'(\zeta_j)^2
\end{aligned}$$

and the statement follows. □

3.2. Behaviour at critical points

We observe first that in successive local maxima and minima, the f-functions are decreasing and increasing, respectively.

Proposition 3. *For any $n \geq 1$ and $k \geq 1$ we have*

$$f_n(\mu_k) > f_n(\mu_{k+1}), \qquad f_n(m_k) < f_n(m_{k+1}).$$

Proof. The statement follows directly from the Lorch–Szegö-type Theorem 1, which will be proved below, and integrating the recurrence relation (5). □

Proposition 4. *For all $n \geq 2$ and all $k \geq 1$ we have $|f_n''(\mu_k)| > |f_n''(m_k)| > |f_n''(\mu_{k+1})| > 0$. For all $k \geq 1$ we have $f_1''(\mu_k) < 0$ and $f_1''(m_k) > 0$.*

Proof. Assume first that $n \geq 2$. Since different f-functions are involved in the proof, we denote here μ_k^n, m_k^n and μ_{k+1}^n in order to emphasize their dependence on n. In view of the recursion formula (5), we know that

$$f_{n+2}(\mu_k^n) = f_{n+2}(m_k^n) = f_{n+2}(\mu_{k+1}^n) = 0. \tag{10}$$

Since $n+2 \geq 4$, by Proposition 1 we then obtain

$$\mu_k^n |f_{n+2}'(\mu_k^n)| > m_k^n |f_{n+2}'(m_k^n)| > \mu_{k+1}^n |f_{n+2}'(\mu_{k+1}^n)|. \tag{11}$$

By differentiating (5), we get $f_n''(\eta) = -f_{n+2}(\eta) - \eta f_{n+2}'(\eta)$. This, combined with (10) and (11) gives

$$|f_n''(\mu_k^n)| = \mu_k^n |f_{n+2}'(\mu_k^n)| > m_k^n |f_{n+2}'(m_k^n)| = |f_n''(m_k^n)| > \mu_{k+1}^n |f_{n+2}'(\mu_{k+1}^n)| = |f_n''(\mu_{k+1}^n)|.$$

The first two inequalities in the statement (for $n \geq 2$) are so proved. The last one holds since if we would have equality we would violate $|f_n''(\mu_{k+1})| > |f_n''(m_{k+1})|$.

Now assume that $n = 1$. Then (6) tells us that $f_1'''(\eta) < 0$ for $\eta \in (z_k^+, z_k^-)$ so that the map $\eta \mapsto f_1''(\eta)$ is strictly decreasing. Clearly, $f_1''(m_k) \geq 0$; if $f_1''(m_k) = 0$, then the just mentioned monotonicity would imply $f_1''(\eta) < 0$ for $\eta \in (m_k, z_k^-]$, contradicting the fact that m_k is a relative minimum for f_1. Similarly, one may proceed to show that $f_1''(\mu_k) < 0$. □

Proposition 5. *Let $n \geq 1$ and let $0 < \alpha < \beta$ be two critical points for f_n. Then the following two identities hold:*

$$\frac{1}{4}\int_\alpha^\beta \eta f_n(\eta)\mathrm{d}\eta = f_n''(\beta) - f_n''(\alpha),$$

$$\frac{1}{4}\int_\alpha^\beta \eta^{n-1} f_n(\eta)\mathrm{d}\eta = -2(n-2)\int_\alpha^\beta \eta^{n-4} f_n'(\eta)\mathrm{d}\eta + \beta^{n-2}f_n''(\beta) - \alpha^{n-2}f_n''(\alpha)\,;$$

when $n \geq 3$, the second identity also holds if $\alpha = 0$.

Proof. An integration by parts yields

$$\int_\alpha^\beta \frac{f_n''(\eta)}{\eta}\mathrm{d}\eta = \int_\alpha^\beta \frac{f_n'(\eta)}{\eta^2}\mathrm{d}\eta.$$

The first identity then follows by integrating (6) over $[\alpha, \beta]$.

Next, notice that further integrations by parts yield

$$\int_\alpha^\beta \eta^{n-2} f_n'''(\eta)\mathrm{d}\eta = (2-n)\int_\alpha^\beta \eta^{n-3} f_n''(\eta)\mathrm{d}\eta + \beta^{n-2}f_n''(\beta) - \alpha^{n-2}f_n''(\alpha),$$

$$\int_\alpha^\beta \eta^{n-3} f_n''(\eta)\mathrm{d}\eta = (3-n)\int_\alpha^\beta \eta^{n-4} f_n'(\eta)\mathrm{d}\eta.$$

The second identity then follows by multiplying (6) by η^{n-2} and integrating. □

By combining Propositions 4 and 5 we immediately obtain

Corollary 1. *Let $n \geq 2$. Then, for any $k \geq 1$, we have:*

$$\int_{\mu_k}^{\mu_{k+1}} \eta f_n(\eta)\mathrm{d}\eta > 0.$$

Another interesting property which holds on critical points is the following:

Proposition 6. *Let $n \geq 1$ and let $0 < \alpha < \beta$ be two critical points for f_n. Then*

$$4f_n''(\beta)^2 + (2-n)f_n(\beta)^2 < 4f_n''(\alpha)^2 + (2-n)f_n(\alpha)^2.$$

Proof. By using (7) and integrating by parts we find

$$\begin{aligned}
&\left[4f_n''(\beta)^2 + (2-n)f_n(\beta)^2\right] - \left[4f_n''(\alpha)^2 + (2-n)f_n(\alpha)^2\right] \\
&\quad = \left[4\Delta f_n(\beta)^2 + (2-n)f_n(\beta)^2\right] - \left[4\Delta f_n(\alpha)^2 + (2-n)f_n(\alpha)^2\right] \\
&\quad = \int_\alpha^\beta \left[4\Delta f_n(\eta)^2 + (2-n)f_n(\eta)^2\right]' \mathrm{d}\eta \\
&\quad = \int_\alpha^\beta \left[8\Delta f_n(\eta)(\Delta f_n)'(\eta) + 2(2-n)f_n(\eta)f_n'(\eta)\right] \mathrm{d}\eta \\
&\quad = 2\int_\alpha^\beta \left[\eta f_n(\eta)\Delta f_n(\eta) + (2-n)f_n(\eta)f_n'(\eta)\right] \mathrm{d}\eta \\
&\quad = 2\int_\alpha^\beta \left[\eta f_n(\eta)f_n''(\eta) + f_n(\eta)f_n'(\eta)\right] \mathrm{d}\eta \\
&\quad = 2\int_\alpha^\beta (\eta f_n'(\eta))' f_n(\eta)\mathrm{d}\eta \\
&\quad = -2\int_\alpha^\beta \eta f_n'(\eta)^2 \mathrm{d}\eta < 0.
\end{aligned}$$

This proves the statement. □

Proposition 7. *Let $n \geq 1$ and let $k \geq 1$. Then,*

$$\int_{m_k}^{\mu_{k+1}} \eta^{n-1} f_n(\eta) \mathrm{d}\eta < 0, \qquad \int_{\mu_{k+1}}^{m_{k+1}} \eta^{n-1} f_n(\eta) \mathrm{d}\eta > 0.$$

If $n \geq 3$, then the second inequality also holds when integrated over $[\mu_1, m_1] = [0, m_1]$.

Proof. Since the second inequality follows similarly, we only prove the first one.

Consider first the case $n = 1$. We have

$$\begin{aligned}\int_{m_k}^{\mu_{k+1}} f_1(\eta) \mathrm{d}\eta &= \frac{1}{z_k^-} \int_{m_k}^{z_k^-} z_k^- f_1(\eta) \mathrm{d}\eta + \frac{1}{z_k^-} \int_{z_k^-}^{\mu_{k+1}} z_k^- f_1(\eta) \mathrm{d}\eta \\ &< \frac{1}{z_k^-} \int_{m_k}^{z_k^-} \eta f_1(\eta) \mathrm{d}\eta + \frac{1}{z_k^-} \int_{z_k^-}^{\mu_{k+1}} \eta f_1(\eta)\, \mathrm{d}\eta = \frac{1}{z_k^-} \int_{m_k}^{\mu_{k+1}} \eta f_1(\eta) \mathrm{d}\eta \\ &= \frac{4}{z_k^-} \left(f_1''(\mu_{k+1}) - f_1''(m_k) \right) < 0,\end{aligned}$$

where we used Proposition 5 in the last equality and Proposition 4 for the last inequality.

Now consider the case $n \geq 2$. Because of Proposition 2 and (5) we have $f_n'(\eta) > 0$ for all $\eta \in (m_k, \mu_{k+1})$. Moreover, by Proposition 4 we know that $\mu_{k+1}^{n-2} f_n''(\mu_{k+1}) < 0$ and $m_k^{n-2} f_n''(m_k) > 0$. Therefore, by Proposition 5 we get

$$\frac{1}{4} \int_{m_k}^{\mu_{k+1}} \eta^{n-1} f_n(\eta) \mathrm{d}\eta = -2(n-2) \int_{m_k}^{\mu_{k+1}} \eta^{n-4} f_n'(\eta) \mathrm{d}\eta + \mu_{k+1}^{n-2} f_n''(\mu_{k+1}) - m_k^{n-2} f_n''(m_k) < 0.$$

The first inequality is so proved also for $n \geq 2$. □

3.3. Behaviour of the second derivative

Proposition 8. *For any $n \geq 1$ and any $k \geq 1$, the following two facts hold:*

$$f_n''(\eta) < 0 \quad \text{and} \quad f_n'''(\eta) > 0 \quad \text{for all } \eta \in (z_k^-, \mu_{k+1}],$$
$$f_n''(\eta) > 0 \quad \text{and} \quad f_n'''(\eta) < 0 \quad \text{for all } \eta \in (z_k^+, m_k].$$

Proof. Fix $k \geq 1$ and take $\eta \in (z_k^-, \mu_{k+1})$; then, $f_n(\eta) > 0$ and $f_n'(\eta) > 0$. Using (6) we then obtain

$$f_n'''(\eta) + \frac{n-1}{\eta} f_n''(\eta) > 0 \quad \text{for all } \eta \in (z_k^-, \mu_{k+1}). \tag{12}$$

Since μ_{k+1} is a maximum point for f_n, by Proposition 4 we infer that $f_n''(\mu_{k+1}) < 0$. By continuity, there exists $\overline{\eta} < \mu_{k+1}$ such that $f_n''(\eta) < 0$ for all $\eta \in (\overline{\eta}, \mu_{k+1}]$. The first statement follows if we show that $\overline{\eta} \leq z_k^-$. For contradiction, assume that $\overline{\eta} > z_k^-$. Then, we would have $f_n''(\overline{\eta}) = 0$ and $f_n'''(\overline{\eta}) \leq 0$ since f_n'' becomes negative for $\eta > \overline{\eta}$. This contradicts (12). Therefore, $f''(\eta) < 0$ for all $\eta \in (z_k^-, \mu_{k+1})$. By using (6) again, we then also infer that $f'''(\eta) > 0$ for all $\eta \in (z_k^-, \mu_{k+1}]$.

By reversing all the signs, we obtain similarly the statement in $(z_k^+, m_k]$. □

When $n = 1$ we have a more precise description of flex points:

Proposition 9. *For any $k \geq 1$ there exists a unique $\eta_k \in [\mu_k, m_k]$ and a unique $\eta^k \in [m_k, \mu_{k+1}]$ such that $f_1''(\eta_k) = f_1''(\eta^k) = 0$. Moreover, $\eta_k \in (\mu_k, z_k^+]$ and $\eta^k \in (m_k, z_k^-]$.*

Proof. Since the proofs of the two statements are similar, we only prove the first one. By Proposition 4 we have $f_1''(\mu_k) < 0$ and $f_1''(m_k) > 0$. Therefore, the equation $f_1''(\eta) = 0$ has an odd number of solutions in (μ_k, m_k). By Proposition 8 the equation $f_1''(\eta) = 0$ has no solutions in $(z_k^+, m_k]$. Assume for contradiction that $f_1''(\eta) = 0$ has at least three solutions in $[\mu_k, z_k^+]$. Then, in a descending flex point $\eta^* \in (\mu_k, z_k^+)$ we have $f_1''(\eta^*) = 0$ and $f_1'''(\eta^*) \leq 0$. This contradicts (6) since $f_1(\eta^*) > 0$. □

3.4. Integral properties on intervals containing 0

Proposition 10. *For all $n \geq 1$ and all $\gamma > -1$ the following implication holds:*

$$\int_0^a \eta^{\gamma+2} f_{n+2}(\eta) \mathrm{d}\eta \geq 0 \quad \text{for all } a > 0 \Longrightarrow \int_0^a \eta^{\gamma} f_n(\eta) \mathrm{d}\eta \geq 0 \quad \text{for all } a > 0.$$

Proof. In order to prove the statement it suffices to show that if the first inequality is true, then

$$\int_0^{z_k^-} \eta^\gamma f_n(\eta)\mathrm{d}\eta \geq 0 \quad \text{for all } k \geq 1.$$

By integrating by parts and (5) we have

$$\int_0^{z_k^-} \eta^\gamma f_n(\eta)\mathrm{d}\eta = -\frac{1}{\gamma+1}\int_0^{z_k^-} \eta^{\gamma+1} f_n'(\eta)\mathrm{d}\eta = \frac{1}{\gamma+1}\int_0^{z_k^-} \eta^{\gamma+2} f_{n+2}(\eta)\mathrm{d}\eta \geq 0$$

and the statement follows. □

Proposition 11. *For all $n \geq 1$, all $k \geq 1$ and all $\gamma > -1$ the following identity holds:*

$$\frac{1}{4}\int_0^{z_k^-} \eta^{\gamma+4} f_n(\eta)\mathrm{d}\eta = (\gamma+1)(\gamma+3)(n-\gamma-3)\int_0^{z_k^-} \eta^\gamma f_n(\eta)\mathrm{d}\eta + (z_k^-)^{\gamma+3}\Delta f_n(z_k^-) - (\gamma+3)(z_k^-)^{\gamma+2} f_n'(z_k^-).$$

Proof. Using (7) and several integrations by parts yield

$$\begin{aligned}\frac{1}{4}\int_0^{z_k^-} \eta^{\gamma+4} f_n(\eta)\mathrm{d}\eta &= \int_0^{z_k^-} \eta^{\gamma+3}(\Delta f_n)'(\eta)\mathrm{d}\eta \\ &= -(\gamma+3)\int_0^{z_k^-} [\eta^{\gamma+2} f_n''(\eta) + (n-1)\eta^{\gamma+1} f_n'(\eta)]\mathrm{d}\eta + (z_k^-)^{\gamma+3}\Delta f_n(z_k^-) \\ &= (\gamma+1)(\gamma+3)(n-\gamma-3)\int_0^{z_k^-} \eta^\gamma f_n(\eta)\mathrm{d}\eta + (z_k^-)^{\gamma+3}\Delta f_n(z_k^-) - (\gamma+3)(z_k^-)^{\gamma+2} f_n'(z_k^-).\end{aligned}$$

This proves the statement. □

4. Lorch–Szegö-type monotonicity results for the *f*-functions

Theorem 1. *For any $n \geq 1$ and $k \geq 1$ we have that*

$$\int_{z_k^-}^{z_{k+1}^-} \eta f_n(\eta)\mathrm{d}\eta > 0 \quad \textit{and} \quad \int_{z_k^+}^{z_{k+1}^+} \eta f_n(\eta)\mathrm{d}\eta < 0.$$

If additionally $n \geq 2$ then we also have that

$$\int_0^{z_1^-} \eta f_n(\eta)\mathrm{d}\eta > 0.$$

Proof. We first claim that for all $k \geq 1$ we have

$$\Delta f_n(z_k^-) < 0, \qquad \Delta f_n(z_k^+) > 0. \tag{13}$$

To this end, we remark that (7) shows that Δf_n is strictly increasing on $\bigcup_{k=1}^\infty P_k$ and strictly decreasing on $\bigcup_{k=1}^\infty N_k$. In the local minima $m_k \in N_k$ we have that $\Delta f_n(m_k) = f_n''(m_k) \geq 0$ and so for $z_k^+ = \inf N_k$ we conclude that $\Delta f_n(z_k^+) > 0$. Analogously, $\mu_{k+1} \in P_{k+1}$, $\Delta f_n(\mu_{k+1}) = f_n''(\mu_{k+1}) \leq 0$, $z_k^- = \inf P_{k+1}$, $\Delta f_n(z_k^-) < 0$. Hence, (13) is proved.

Next, we show that

$$\Delta f_n(z_k^-) < \Delta f_n(z_{k+1}^-) < 0. \tag{14}$$

In order to prove (14), we multiply (7) by Δf_n. Then, by integrating over (z_k^-, z_{k+1}^-) we obtain:

$$\begin{aligned}\left[(\Delta f_n(\eta))^2\right]_{z_k^-}^{z_{k+1}^-} &= 2\int_{z_k^-}^{z_{k+1}^-} \Delta f_n(\eta)(\Delta f_n)'(\eta)\mathrm{d}\eta \\ &= \frac{1}{2}\int_{z_k^-}^{z_{k+1}^-} \eta f_n''(\eta) f_n(\eta)\mathrm{d}\eta + \frac{n-1}{2}\int_{z_k^-}^{z_{k+1}^-} f_n'(\eta) f_n(\eta)\mathrm{d}\eta \\ &= -\frac{1}{2}\int_{z_k^-}^{z_{k+1}^-} \eta f_n'(\eta)^2\mathrm{d}\eta < 0.\end{aligned}$$

So, $\Delta f_n(z_k^-)^2 > \Delta f_n(z_{k+1}^-)^2$. Together with (13), this proves (14).

We may now obtain the first statement of Theorem 1 from the differential equation (7) by integration:

$$\frac{1}{4}\int_{z_k^-}^{z_{k+1}^-} \eta f_n(\eta)\mathrm{d}\eta = \int_{z_k^-}^{z_{k+1}^-} (\Delta f_n)'(\eta)\mathrm{d}\eta = (\Delta f_n)(z_{k+1}^-) - (\Delta f_n)(z_k^-) > 0.$$

In the last step we made use of (14). The inequality $\int_{z_k^+}^{z_{k+1}^+} \eta f_n(\eta)\mathrm{d}\eta < 0$ may be proved similarly.

Now assuming $n \geq 2$ and integrating over $(0, z_1^-)$ we obtain:

$$\begin{aligned}
[\Delta f_n(\eta)]_0^{z_1^-} &= 2\int_0^{z_1^-} \Delta f_n(\eta)(\Delta f_n)'(\eta)\mathrm{d}\eta \\
&= \frac{1}{2}\int_0^{z_1^-} \eta f_n''(\eta)f_n(\eta)\mathrm{d}\eta + \frac{n-1}{2}\int_0^{z_1^-} f_n'(\eta)f_n(\eta)\mathrm{d}\eta \\
&= -\frac{1}{2}\int_0^{z_1^-} \eta f_n'(\eta)^2\mathrm{d}\eta + \frac{n-2}{2}\int_0^{z_1^-} f_n'(\eta)f_n(\eta)\mathrm{d}\eta \\
&= -\frac{1}{2}\int_0^{z_1^-} \eta f_n'(\eta)^2\mathrm{d}\eta - \frac{n-2}{4}f_n(0)^2 < 0,
\end{aligned}$$

so that

$$\Delta f_n(0) < \Delta f_n(z_1^-) < 0.$$

The last statement of Theorem 1 now follows with the same argument as above. □

When starting integration from $\eta = 0$, the following result holds:

Theorem 2. *Let $n \geq 3$ and assume that $h \in C^1(0, +\infty)$ satisfies $h(\eta) > 0$, $h'(\eta) \leq 0$ for all $\eta > 0$ and*

$$\int_0 h(\eta)\eta^{(n-1)/2}\mathrm{d}\eta < \infty.$$

Then, for any $a > 0$ we have

$$\int_0^a h(\eta)\eta^{(n-1)/2} f_n(\eta)\mathrm{d}\eta > 0.$$

Proof. By making use of the definition of f_n we see that

$$I_a := \int_0^a h(\eta)\eta^{(n-1)/2} f_n(\eta)\mathrm{d}\eta = \int_0^a h(\eta)\eta^{(1-n)/2}\int_0^\infty \mathrm{e}^{-s^4}(\eta s)^{n/2}J_{(n-2)/2}(\eta s)\mathrm{d}s\mathrm{d}\eta.$$

Therefore, the change of variables $z = \eta s$ yields

$$I_a = \int_0^a h(\eta)\eta^{-(n+1)/2}\int_0^\infty \mathrm{e}^{-(z/\eta)^4}z^{n/2}J_{(n-2)/2}(z)\mathrm{d}z\mathrm{d}\eta.$$

In view of the integrability condition on $\eta \mapsto h(\eta)\eta^{(n-1)/2}$, Fubini's Theorem implies that

$$I_a = \int_0^\infty \left(\sqrt{z}J_{(n-2)/2}(z)\right) z^{(n-1)/2}\int_0^a h(\eta)\eta^{-(n+1)/2}\mathrm{e}^{-(z/\eta)^4}\mathrm{d}\eta\mathrm{d}z.$$

For all $z > 0$, let

$$g(z) := z^{(n-1)/2}\int_0^a h(\eta)\eta^{-(n+1)/2}\mathrm{e}^{-(z/\eta)^4}\mathrm{d}\eta.$$

Then, by differentiating and integrating by parts

$$\begin{aligned}
g'(z) &= \frac{n-1}{2}z^{(n-3)/2}\int_0^a h(\eta)\eta^{-(n+1)/2}\mathrm{e}^{-(z/\eta)^4}\mathrm{d}\eta - z^{(n-1)/2}\int_0^a h(\eta)\eta^{-(n+1)/2}\frac{4z^3}{\eta^4}\mathrm{e}^{-(z/\eta)^4}\mathrm{d}\eta \\
&= z^{(n-3)/2}\left(\int_0^a h'(\eta)\eta^{(1-n)/2}\mathrm{e}^{-(z/\eta)^4}\mathrm{d}\eta - h(a)\,a^{(1-n)/2}\mathrm{e}^{-(z/a)^4}\right).
\end{aligned}$$

Since $h' \leq 0$ and $h > 0$, this shows that g is strictly decreasing. Since we assumed $n \geq 3$, the Lorch–Szegö Theorem [9] (see also [10, Corollary 4.15.2]) applies and the proof is complete. □

In the particular case of powers, we have the following statement

Theorem 3. *Let $n \geq 3$. Then for any $\gamma \in (-1, \frac{n-1}{2}]$ and any $a > 0$ we have*

$$\int_0^a \eta^\gamma f_n(\eta)\mathrm{d}\eta > 0.$$

Let $n \geq 1$. Then, for any $k \geq 1$ we have

$$\int_0^{z_k^-} \eta^{n+1} f_n(\eta)\mathrm{d}\eta < 0.$$

Proof. The first statement is a direct consequence of Theorem 2.

By taking $\gamma = n - 3$ in Proposition 11, we obtain

$$\frac{1}{4}\int_0^{z_k^-} \eta^{n+1} f_n(\eta)\mathrm{d}\eta = (z_k^-)^n \Delta f_n(z_k^-) - n(z_k^-)^{n-1} f_n'(z_k^-) < 0,$$

where the inequality follows from (13) and from the fact that $f_n'(z_k^-) > 0$. □

Remark 1. In view of the discussion in the introduction, see (3), it would be interesting to prove Theorem 1 not only for $\eta f_n(\eta)$ but also for $\eta^\gamma f_n(\eta)$ with $\gamma \geq 1$ as large as possible. This is related to finding the *largest* value γ_n of γ for which

$$\int_0^a \eta^\gamma f_n(\eta)\mathrm{d}\eta \geq 0 \quad \text{for all } a > 0.$$

Note that for any $\gamma \in (-1, \gamma_n)$ the above integral is finite and strictly positive. By Theorem 3 we know that for any $n \geq 1$ we have $\gamma_n < n + 1$. Moreover, if $n \geq 3$ then $\gamma_n \geq \frac{n-1}{2}$, see again Theorem 3. If $n = 2$, then $\gamma_2 \geq 1$, see Theorem 1. If $n = 1$, by repeating the arguments in the proof of [1, Lemma A.4], we have that

$$\int_0^{z_1^-} f_1(\eta)\mathrm{d}\eta > 0$$

which, combined with Theorem 1, yields $\gamma_1 \geq 0$. By Proposition 10 we know that $\gamma_{n+2} \leq \gamma_n + 2$. Although (8) holds for $\beta = 0$, numerical experiments suggest that $\gamma_n < n - 1$, at least in high dimensions n. For instance, if $n = 20$, it seems that $16 < \gamma_{20} < 17$.

5. Extensions to polyharmonic heat kernels

In this section we briefly sketch some properties of higher order *polyharmonic* heat kernels. The proofs can be obtained by slightly modifying the arguments in [1]. So, for $m \geq 2$ consider the following Cauchy problem for the polyharmonic heat equation

$$\begin{cases} u_t + (-\Delta)^m u = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u\,(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{15}$$

where $n \geq 1$ and $u_0 \in C^0 \cap L^\infty(\mathbb{R}^n)$. Then, (1) admits a unique global in time solution explicitly given by

$$u(x, t) = \alpha t^{-n/2m} \int_{\mathbb{R}^n} u_0(x - y) f_{m,n}\left(\frac{|y|}{t^{1/2m}}\right)\mathrm{d}y \quad (x, t) \in \mathbb{R}^{n+1}_+,$$

where $\alpha = \alpha_{m,n} > 0$ is a suitable normalization constant and

$$f_{m,n}(\eta) = \eta^{1-n} \int_0^\infty \mathrm{e}^{-s^{2m}} (\eta s)^{n/2} J_{(n-2)/2}(\eta s)\mathrm{d}s.$$

By arguing as in [1] and using [10, Section 4.62], we find that (5) still holds, independently of m. Moreover, the following $(2m - 1)$th-order differential equation generalizes (7):

$$[\Delta^{m-1} f_{m,n}]'(\eta) = \frac{(-1)^m}{2m} \eta f_{m,n}(\eta) \quad \text{for all } n \geq 1. \tag{16}$$

It is straightforward that (16) coincides with (7) if $m = 2$, whereas it reduces to $f'(\eta) = -\frac{1}{2}\eta f(\eta)$ whenever $m = 1$ (recall that in the latter case, the kernel f is *independent* of n).

With these two identities, we can prove results similar to (8). We can show that

$$C_{m,n,\beta} := \int_0^\infty \eta^{n-1-\beta} f_{m,n}(\eta)\mathrm{d}\eta > 0 \quad \text{for all integer } n \geq 1 \text{ and all } \beta \in [0, n). \tag{17}$$

The proof of (17) follows the lines of [1], that is, we combine the use of Fubini's Theorem, Lorch–Szegö's Theorem, several integration by parts and two crucial recurrence formulas based on (5) and (16), namely

$$C_{m,n+2,\beta} = (n-\beta)C_{m,n,\beta} \quad \text{for all } \beta \in (0,n)$$

$$C_{m,n,\beta} = 2m \prod_{j=1}^{m-1} \left[(2m+\beta-2j)(n+2j-2m-\beta)\right] C_{m,n+2,\beta+2m} \quad \text{for all } \beta \in (0, n+2-2m).$$

Clearly, we expect that (17) may enable one to prove eventual local positivity results for (15) when u_0 is as in (9).

References

[1] A. Ferrero, F. Gazzola, H.-Ch. Grunau, Decay and eventual local positivity for biharmonic parabolic equations, Discrete Contin. Dyn. Syst. 21 (2008) 1129–1157.
[2] F. Gazzola, H.-Ch. Grunau, Eventual local positivity for a biharmonic heat equation in $\mathbb{R}^n$, Discrete Contin. Dyn. Syst. S 1 (2008) 83–87.
[3] E. Berchio, On the sign of solutions to some linear parabolic biharmonic equations, Adv. Differential Equations 13 (2008) 956–976.
[4] H. Fujita, On the blowing up of solutions of the Cauchy problem $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect. I Math. 13 (1996) 109–124.
[5] F. Gazzola, H.-Ch. Grunau, Global solutions for superlinear parabolic equations involving the biharmonic operator for initial data with optimal slow decay, Calcu. Variations 30 (2007) 389–415.
[6] G. Caristi, E. Mitidieri, Existence and nonexistence of global solutions of higher-order parabolic problems with slow decay initial data, J. Math. Anal. Appl. 279 (2003) 710–722.
[7] Yu.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, S.I. Pohožaev, Global solutions of higher-order semilinear parabolic equations in the supercritical range, Adv. Differential Equations 9 (2004) 1009–1038.
[8] V.A. Galaktionov, S.I. Pohožaev, Existence and blow-up for higher-order semilinear parabolic equations: Majorizing order-preserving operators, Indiana Univ. Math. J. 51 (2002) 1321–1338.
[9] L. Lorch, P. Szegö, Higher monotonicity properties of certain Sturm–Liouville functions, Acta Math. 109 (1963) 55–73.
[10] G.E. Andrews, R. Askey, R. Roy, Special functions, in: Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.