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LOWER-ORDER PERTURBATIONS OF CRITICAL GROWTH NONLINEARITIES IN SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract. The solvability of semilinear elliptic equations with nonlinearities in the critical growth range depends on the terms with lower-order growth. We generalize some known results to a wide class of lower-order terms and prove a multiplicity result in the left neighborhood of every eigenvalue of $-\Delta$ when the subcritical term is linear. The proofs are based on variational methods; to assure that the considered minimax levels lie in a suitable range, special classes of approximating functions having disjoint support with the Sobolev "concentrating" functions are constructed.

1. Introduction. In this paper we consider the semilinear problem

$$\begin{cases} -\Delta u = g(x,u) + |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, and $g(\cdot, s)$ has subcritical growth at infinity (i.e., $\lim_{|s|\to\infty} \frac{g(\cdot,s)}{|s|^{2^*-1}} = 0$). Consider the Hilbert space $H := H_0^1(\Omega)$ endowed with the Dirichlet scalar product. Let λ_k, $k \in \mathbb{N}$, be the eigenvalues of $-\Delta$ relative to the homogeneous Dirichlet problem in Ω; it is well-known that each eigenvalue has finite multiplicity μ_k, and that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \dots$ with $\lambda_k \to +\infty$ as $k \to \infty$. We denote by $\sigma(-\Delta) = \{\lambda_k\}$ the spectrum of $-\Delta$.

Define the functional $J : H \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx,$$

where $G(x,s) = \int_0^s g(x,t)dt$; if g is continuous we have $J \in C^1(H,\mathbb{R})$ and the critical points of the functional J correspond to (weak) solutions of equation (1).

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However, the standard variational arguments do not apply because the imbedding $H \subset L^{2^*}(\Omega)$ is not compact; i.e., the functional J does not satisfy the Palais-Smale condition ([15]) (PS condition). And indeed, for equations with critical growth, nontrivial solutions may not exist: if Ω is strictly star-shaped and $\lambda \leq 0$, then by a result of Pohožaev ([16]) the following equation has only the trivial solution $u \equiv 0$:

$$\begin{cases} -\Delta u = \lambda u + u|u|^{2^*-2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In recent years the situation of "lack of compactness" has been studied intensively, and has led to a good understanding of the underlying phenomena. Let S denote the best constant of the imbedding $H \subset L^{2^*}(\Omega)$ (see [20]); in a pioneering result Brezis-Nirenberg ([8]) showed that the functional J corresponding to (2), i.e., $J(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda}{2}\int u^2 - \frac{1}{2^*}\int |u|^{2^*}$, satisfies the PS condition in the interval $(0, \frac{S^{n/2}}{n})$, but fails to satisfy it at the level $\frac{S^{n/2}}{n}$ (see [12] for an alternative proof). The reason for the failure of the PS condition at level $\frac{S^{n/2}}{n}$ is explained by a profound result of Struwe ([18]): consider the "limiting problem"

$$\begin{cases} -\Delta u = u|u|^{2^*-2} & \text{in } \mathbb{R}^n \\ u(x) \to 0 & \text{as } |x| \to \infty; \end{cases} \tag{3}$$

the representation result of [18] states, roughly speaking, that if $\{u_m\} \subset H$ is a PS sequence for J, then it can be approximated (for $m \to \infty$) by a solution u^0 of (2) plus a certain number k of (suitably rescaled) solutions u^j $(j = 1, \dots, k)$ of (3). The crucial fact is that positive solutions of (3), which realize the constant S in the imbedding $H^1(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n)$, are unique up to rescaling, and they yield an increase of the level of the functional by $\frac{S^{n/2}}{n}$. In view of this, a possible strategy for proving existence results for equations of form (1) consists in constructing minimax levels for the functional J which lie in the interval $(0, \frac{S^{n/2}}{n})$; indeed, this is the method proposed in [8], and which we will use here. However, we note that with our assumptions on the lower-order term g the functional J may have negative critical levels, and that the above interval may not be a "range of compactness" for J. But we will prove that a PS sequence within the above energy range yields a nontrivial solution of (1) by means of its weak limit.

We recall the following known existence results for equations (1) and (2):

i) The first celebrated existence result for equation (2) is due to Brezis-Nirenberg ([8]) who proved that if $\lambda \in (\lambda^*, \lambda_1)$ (with $\lambda^* = 0$ if $n \geq 4$, and $0 < \lambda^*(\Omega) < \lambda_1$ if $n = 3$), then there exists a nontrivial positive solution of equation (2) (equations of more general form are also considered in [8]; see Section 2).

ii) Later, Cerami-Fortunato-Struwe ([12]) established the existence of a bifurcation from any eigenvalue λ_k of $-\Delta$ and estimated the left neighborhood $(\lambda_k - \nu, \lambda_k]$ of λ_k in which the bifurcation branch can be extended; more precisely, they showed that for $\nu = S|\Omega|^{-2/n}$, where S is the best Sobolev imbedding constant, the number of (pairs of) solutions of equation (2) is at least $m(\lambda) = \sum_{k \in K} \mu_k$ with $K := \{k \in \mathbb{N} : \lambda < \lambda_k < \lambda + \nu\}$.

iii) In a subsequent paper, Capozzi-Fortunato-Palmieri ([10]) proved that equation (2) admits nontrivial solutions for all $\lambda > 0$ if $n \geq 5$, and for all $\lambda > 0$ with $\lambda \notin \sigma(-\Delta)$ if $n = 4$ (see the remark after Corollary 1 below); see also [3] for an alternative proof in the case where $\lambda \notin \sigma(-\Delta)$. For the case $n = 3$ the existence of a nontrivial solution for all $\lambda > \frac{\lambda_1}{4}$ has been obtained by Comte ([14]) in the particular case that Ω is a ball.

The aims of this paper are

a) to generalize the results i) and iii) to a broader class of subcritical perturbations of the critical growth term $|u|^{2^*-2}u$ in equation (2); in particular, we extend the results of [8] to the case where the functional J has a generalized mountain pass structure ([17]) (i.e., $\lambda_1 \leq \lim_{s\to 0} \frac{2G(x,s)}{s^2} < \infty$) instead of the classical mountain pass geometry ([2]) (i.e., $0 \leq \lim_{s \to 0} \frac{2G(x,s)}{s^2} < \lambda_1$), and the result of [10] to the case where a more general function $g(x,u)$ replaces the term λu in equation (2).

b) to show that the "bifurcation branch" starting in λ_{k+1} can be continued up to $\lambda_k - \delta_k$, for some $\delta_k > 0$. This yields a *multiplicity* result: for $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ there exist at least two (pairs of) solutions of equation (2); in particular, for $\lambda \in (\lambda_1 - \delta_1, \lambda_1)$, we find a "small" positive solution (the Brezis-Nirenberg solution) and a "large" solution which changes sign.

c) to extend the multiplicity result of Cerami-Fortunato-Struwe ([12]) for *odd* nonlinearities when $n \geq 5$: if the multiplicity of λ_k is μ_k, then there exists $\delta_k > 0$ such that if $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ then (2) has at least $2(\mu_k + 1)$ nontrivial solutions.

The basic idea of our approach is the following: since we are looking for solutions of equation (2) with $\lambda \geq \lambda_1$, the functional J has a generalized mountain-pass structure; this requires (in order to prove that the minimax level stays below $\frac{S^{n/2}}{n}$) an estimate of the maximum of the functional J over subsets of $V \oplus \mathbb{R}^+\{u_\varepsilon\}$, where V is some finite-dimensional subspace of H and u_ε is the concentrating (under rescaling) function mentioned above. Since V and u_ε are *not* orthogonal, this estimate involves "mixed terms" which are difficult to estimate (see [10]). However, an "orthogonalization" (in H as well as in L^p) of these functions can be obtained by replacing the space V by a space V_ε consisting of functions which approximate the functions in V, but are zero where u_ε is nonzero, that is, by disjoining their supports. Of course, now the approximation error of V_ε must be estimated, but this can be handled more easily than the mentioned mixed terms.

2. Statement of the results. We make the following assumption on the subcritical term g:

$$\begin{aligned} &g\colon \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function with} \\ &\forall \varepsilon > 0\ \exists a_\varepsilon \in L^{\frac{2n}{n+2}} \text{ such that } |g(x,s)| \leq a_\varepsilon(x) + \varepsilon |s|^{\frac{n+2}{n-2}} \\ &\text{for a.e. } x \in \Omega \text{ and } \forall s \in \mathbb{R}. \end{aligned} \tag{4}$$

The other assumptions are imposed on the primitive $G(x,s) = \int_0^s g(x,t)\,dt$: we first assume that

$$G(x,s) \geq 0 \quad \text{for a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}. \tag{5}$$

Note that the lower-order perturbation $g(x,u)$ is allowed to change sign.

For the further assumptions we distinguish two cases.

2.1. Nonresonance near the origin. Assume that there exist $k \in \mathbb{N}$, $\delta > 0$, $\sigma > 0$, and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

$$\frac{1}{2}(\lambda_k + \sigma)s^2 \leq G(x,s) \leq \frac{1}{2}\mu s^2 \quad \text{for a.e. } x \in \Omega, \quad \forall |s| \leq \delta; \tag{6}$$

furthermore, we assume that

$$G(x,s) \geq \frac{1}{2}(\lambda_k + \sigma)s^2 - \frac{1}{2^*}|s|^{2^*} \quad \text{for a.e. } x \in \Omega, \quad \forall s \neq 0. \tag{7}$$

As a consequence of (4), (6) we have $g(x,0) = 0$ for almost every $x \in \Omega$ and therefore $u \equiv 0$ solves (1).

In $\mathbb{R}^3$ we also need a growth condition at infinity:

there exists an open nonempty subset $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{G(x,s)}{s^4} = +\infty \quad \text{uniformly with respect to } x \in \Omega_0. \tag{8}$$

The reason for the additional assumption (8) in the case $n = 3$ will become clear in the proof of Lemma 5; a similar condition is considered in [8].

With the above assumptions we will prove

Theorem 1. *For $n \geq 4$ assume* (4)–(7), *for $n = 3$ assume* (4)–(8); *then equation* (1) *admits a nontrivial solution.*

Remarks. 1. When $n \geq 4$ this theorem generalizes the result of [10] (for $\lambda \notin \sigma(-\Delta)$) and of [3], where the particular case $g(x,s) = \lambda s$ is considered.

2. The case $\mu \in (0, \lambda_1)$ has been studied in [8], Corollaries 2.1 and 2.2. See also [11] for a different class of perturbations. In the forthcoming paper [4] the existence of solutions for $\mu \in (0, \lambda_1)$ is shown imposing only conditions (4), (5) and (6) (with $\lambda_0 = -\infty$). As already mentioned, these assumptions allow $g(x,s)$ to change sign. We refer also to [1] for the existence of *positive* solutions in the ball with g which changes sign.

3. Theorem 1 remains true replacing condition (5) by a weaker assumption which allows G also to change sign; see the Remark in Section 4.

4. When $n = 3$ our result extends Corollary 2.3 in [8] as we do not assume that $0 < \mu < \lambda_1$ nor $g(x,s) \geq 0$; of course we do not obtain a positive solution.

2.2. Resonance near the origin. Let us now consider the situation where $\frac{2G(x,s)}{s^2}$ is "possibly an eigenvalue" in a neighborhood of $s = 0$; i.e., assume that there exist $\delta > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

$$\frac{1}{2}\lambda_k s^2 \leq G(x,s) \leq \frac{1}{2}\mu s^2 \quad \text{for a.e. } x \in \Omega, \quad \forall |s| \leq \delta. \tag{9}$$

In this case we need to move σ from the quadratic part to the critical part in condition (6) and to impose a growth condition like (8) for all dimensions. More precisely, we assume that there exists $\sigma \in (0, 1/2^*)$ such that

$$G(x,s) \geq \frac{1}{2}\lambda_k s^2 - \left(\frac{1}{2^*} - \sigma\right)|s|^{2^*} \quad \text{for a.e. } x \in \Omega \quad \forall s \in \mathbb{R}; \tag{10}$$

moreover, we require that

there exists an open nonempty subset $\Omega_0 \subset \Omega \subset \mathbb{R}^n$, such that

$$\lim_{s \to +\infty} \frac{G(x,s)}{s^{8n/(n^2-4)}} = +\infty \quad \text{uniformly with respect to } x \in \Omega_0. \tag{11}$$

We will prove the following

Theorem 2. *Let $n \geq 3$ and assume* (4), (5), (9)–(11); *then equation* (1) *admits a nontrivial solution.*

Note that for $G(x,s) = \frac{1}{2}\lambda s^2$ condition (11) is satisfied for $n \geq 5$ (more precisely, for $n^2 - 4n - 4 > 0$ which may be of interest in view of radial solutions ([5])); therefore, from Theorems 1 and 2 we obtain

Corollary 1.

1. *If $n \geq 5$ then equation* (2) *admits a nontrivial solution $\forall\, \lambda > 0$.*
2. *If $n = 4$ then equation* (2) *admits a nontrivial solution $\forall\, \lambda > 0$ such that $\lambda \notin \sigma(-\Delta)$.*

Remark. This result is proved in [10]; it was pointed out to us by H. Brezis that the proof of Theorem 0.1 in [10] does not work if $n = 4$ and $\lambda \in \sigma(-\Delta)$: indeed the crucial estimate (2.10)a does not hold for $\tilde{u}^-$.

2.3. Multiplicity results in high dimensions. We also obtain a multiplicity result for equation (2): we prove that if $n \geq 5$ then the bifurcation branch starting from the eigenvalue $\lambda_{k+1} \in \sigma(-\Delta)$, $k \geq 1$, can be extended up to a left neighborhood of λ_k:

Theorem 3. *Let $n \geq 5$; then, $\forall\, \lambda_k \in \sigma(-\Delta)$, there exists $\delta_k > 0$ such that if $\lambda \in (\lambda_k - \delta_k, \lambda_k)$ equation* (2) *admits at least $\mu_k + 1$ (pairs of) nontrivial solutions.*

This result yields in particular a multiplicity result in the interval $(\lambda_1 - \delta_1, \lambda_1)$:

Corollary 2. *Let $n \geq 5$; then there exists a $\delta_1 \in (0, \lambda_1]$ such that in $(\lambda_1 - \delta_1, \lambda_1)$ there exist at least two solutions, the "small" positive solution of Brezis-Nirenberg, and a "large" solution which changes sign.*

The above corollary is already known if $n \geq 6$, for $\delta_1 = \lambda_1$, see [13, 21], but the result seems new in the case $n = 5$; furthermore, we point out that the solutions found here are below the energy level $\frac{1}{n}S^{n/2}$, while in [13] this may not be the case: there a "higher" energy estimate is obtained, and so our solutions (in the case $n \geq 6$) may be different from the ones found in [13].

Remark. Theorem 3 and Corollary 2 can be easily extended to equations of the more general form (1).

3. The variational characterization. The proofs of our results involve variational techniques as in [7, 8, 10, 12, 19]. As we deal with infinitesimal quantities we make use of the following Landau symbols: we denote

$$f(x) = o[g(x)] \quad \text{as } x \to x_0 \quad \text{if } \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$$

$$f(x) = O[g(x)] \quad \text{as } x \to x_0 \quad \text{if } \limsup_{x\to x_0} \frac{|f(x)|}{|g(x)|} < +\infty.$$

In this section we describe the variational characterization, which is based on a linking argument. In the standard variational procedure the PS condition is crucial; it can be stated as follows: A sequence $\{u_m\} \subset H$ is called a PS sequence for J at level c if $J(u_m) \to c$ and $J'(u_m) \to 0$ in H^{-1}. The functional J satisfies the PS condition at level c, if every PS sequence at level c has a convergent subsequence in H.

As already mentioned, the functional J does not satisfy the PS condition; however, the following result holds:

Lemma 1. *Assume* (4) *and let $\{u_m\} \subset H$ be a PS sequence for J; then there exists $u \in H$ such that $u_m \rightharpoonup u$ up to a subsequence and $J'(u) = 0$. Moreover, if $J(u_m) \to c$ with $c \in (0, \frac{S^{n/2}}{n})$ then $u \not\equiv 0$ and hence u is a nontrivial solution of* (1).

Proof. The proof is standard (see [8]); we briefly sketch it. Let

$$f(x,s) = g(x,s) + |s|^{2^*-2}s$$

and

$$F(x,s) = \int_0^s f(x,t)\,dt;$$

since (4) holds, we have

$$\exists \vartheta \in (0, \tfrac{1}{2})\ \exists \bar{s} > 0 \ \text{ such that } F(x,s) \leq \vartheta f(x,s)s \ \text{ for a.e. } x \in \Omega \quad \forall |s| \geq \bar{s}:$$

therefore $\{u_m\}$ is bounded (see [2]) and $\exists u$ such that $u_m \rightharpoonup u$ up to a subsequence. Furthermore, $J'(u) = 0$ by weak continuity of J'; see [18].

Assume $c \in (0, \frac{S^{n/2}}{n})$ and, for the sake of contradiction, $u \equiv 0$; as the term $g(x, u_m)u_m$ is subcritical, by Theorem 2.2.7 in [9] we infer from $J'(u_m)[u_m] = o(1)$ that

$$\|u_m\|^2 - \|u_m\|_{2^*}^{2^*} = o(1). \tag{12}$$

By the definition of S, $\|u\|^2 \geq S\,\|u\|_{2^*}^2$, for all $u \in H$, we obtain

$$o(1) \geq \|u_m\|^2(1 - S^{-2^*/2}\|u_m\|^{2^*-2}).$$

If now $\|u_m\| \to 0$ we contradict $c > 0$; therefore, $\|u_m\|^2 \geq S^{n/2} + o(1)$ and by (12) we get

$$\begin{aligned} J(u_m) &= \frac{1}{n}\|u_m\|^2 + \frac{n-2}{2n}(\|u_m\|^2 - \|u_m\|_{2^*}^{2^*}) + o(1) \\ &\geq \frac{1}{n}S^{n/2} + o(1) \end{aligned}$$

which contradicts $c < \frac{1}{n}S^{n/2}$.

Remark. If in Lemma 1 we also assume that

$$\frac{1}{2}g(x,s)s - G(x,s) + \frac{1}{n}|s|^{2^*} \geq 0 \quad \forall s \in \mathbb{R} \quad \text{for a.e. } x \in \Omega$$

then every PS sequence $\{u_m\}$ is relatively compact. Indeed, if u is a critical point for J, then $J'(u)[u] = 0$, which substituted into $J(u)$ gives

$$J(u) = \frac{1}{2}\int_\Omega g(x,u)u - \int_\Omega G(x,u) + \frac{1}{n}\int_\Omega |u|^{2^*} \geq 0\,;$$

that is, all the critical levels of J are positive; then, by reasoning as in [13], we obtain that either $\{u_m\}$ is relatively compact or $c \geq \frac{1}{n}S^{n/2}$.

By Lemma 1, to prove Theorems 1–3 it suffices to build a PS sequence for J at a level strictly between 0 and $\frac{S^{n/2}}{n}$.

Denote by e_i an L^2 normalized eigenvector relative to $\lambda_i \in \sigma(-\Delta)$, let H^- denote the space spanned by the eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ and $H^+ := (H^-)^\perp$, and let $P_k : H \to H^-$ denote the orthogonal projection.

Without restrictions assume that $0 \in \Omega$, take m large enough so that $B_{2/m} \subset \Omega$ where $B_{2/m}$ denotes the ball of radius $2/m$ with center in 0; in the case of assumption (8) (respectively (11)) assume also that $0 \in \Omega_0$ and take m so large that $B_{2/m} \subset \Omega_0$. Consider the functions $\zeta_m : \Omega \to \mathbb{R}$ defined by

$$\zeta_m(x) := \begin{cases} 0 & \text{if } x \in B_{1/m} \\ m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m} \\ 1 & \text{if } x \in \Omega \setminus B_{2/m}. \end{cases}$$

Then, define the following "approximating eigenfunctions" $e_i^m := \zeta_m e_i$ and the space

$$H_m^- := \text{span}\{e_i^m;\ i = 1, \dots, k\}.$$

We first prove that the functions e_i^m converge to the eigenfunctions e_i and we estimate the approximation error:

Lemma 2. *As $m \to \infty$ we have*

$$e_i^m \to e_i \quad in\ H \quad and \qquad \max_{\{u \in H_m^- : \int u^2 = 1\}} \|u\|^2 \le \lambda_k + c_k m^{2-n}.$$

Proof. We have

$$\begin{aligned}\int_\Omega |\nabla(e_i^m - e_i)|^2 &= \int_\Omega |e_i \nabla \zeta_m + (\zeta_m - 1)\nabla e_i|^2 \\ &= \int_{A_m} |\nabla \zeta_m|^2 e_i^2 + 2\int_{A_m} (\nabla \zeta_m)(\zeta_m - 1) e_i \nabla e_i + \int_{B_{2/m}} (\zeta_m - 1)^2 |\nabla e_i|^2 \qquad (13) \\ &\le c\|e_i\|_\infty^2 m^{2-n} + c\|\nabla e_i\|_\infty \|e_i\|_\infty m^{1-n} + c\|\nabla e_i\|_\infty^2 m^{-n},\end{aligned}$$

where $c > 0$ is some positive constant independent of e_i.

Next, denote $\partial B = \{u \in H\ :\ \int_\Omega u^2 = 1\}$, and let $v \in H^- \cap \partial B$; i.e., $v = \sum_{j=1}^k \alpha_j e_j$ with $\sum_j \alpha_j^2 = 1$. Note that then $v_m := \zeta_m v = \sum_j \alpha_j \zeta_m e_j = \sum_j \alpha_j e_j^m$; i.e., $v_m \in H_m^-$. For each $v \in H^- \cap \partial B$ let $a_m(v) := \|v_m\|_2^{-1}$ (so that $a_m(v) v_m \in \partial B$); we first show

$$1 \le a_m(v) \le 1 + c\|v\|_\infty^2 m^{-n}\ :$$

this follows from

$$\begin{aligned} 1 &= \int_\Omega a_m^2(v) v_m^2 \\ &= a_m^2(v)\Big(1 - \int_{B_{2/m}} v^2 + \int_{A_m} \zeta_m^2 v^2\Big) \ \begin{cases} \le a_m^2(v) \\ \ge a_m^2(v)(1 - c\|v\|_\infty^2 m^{-n}). \end{cases}\end{aligned}$$

Let now $\overline{u}_m \in H_m^- \cap \partial B$ such that $\|\overline{u}_m\|^2 = \max_{u \in H_m^- \cap \partial B} \|u\|^2$. Then

$$\overline{u}_m = a_m(\overline{u}) \sum_j \overline{\alpha}_j e_j^m = a_m(\overline{u}) \zeta_m \sum_j \overline{\alpha}_j e_j = a_m(\overline{u}) \zeta_m \overline{u},$$

and hence, arguing as in (13),

$$\begin{aligned}\|\overline{u}_m\|^2 &= a_m^2(\overline{u}) \int_\Omega |\nabla(\zeta_m \overline{u})|^2 \\ &\le (1 + c\|\overline{u}\|_\infty^2 m^{-n}) \int_\Omega |\overline{u}\nabla\zeta_m + \zeta_m \nabla \overline{u}|^2 \\ &\le (1 + c\|\overline{u}\|_\infty^2 m^{-n})(c\|\overline{u}\|_\infty^2 m^{2-n} + c\|\nabla\overline{u}\|_\infty \|\overline{u}\|_\infty m^{1-n} + \|\overline{u}\|^2) \\ &\le c\|\overline{u}\|_\infty^2 m^{2-n} + c\|\nabla \overline{u}\|_\infty \|\overline{u}\|_\infty m^{1-n} + \lambda_k \\ &\le c_k m^{2-n} + \lambda_k.\end{aligned}$$

As in [8], we consider the family of functions

$$u_\varepsilon^*(x) := \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{[\varepsilon^2 + |x|^2]^{\frac{n-2}{2}}} \quad (\varepsilon > 0), \tag{14}$$

which solve (3) and satisfy $\|u_\varepsilon^*\|^2 = \|u_\varepsilon^*\|_{2^*}^{2^*} = S^{n/2}$ for all $\varepsilon > 0$. Define a positive cut-off function $\eta \in C_c^\infty(B_{1/m})$ such that $\eta \equiv 1$ in $B_{1/2m}$, $\eta \le 1$ in $B_{1/m}$ and $\|\nabla \eta\|_\infty \le 4m$; consider the sequence of functions $u_\varepsilon(x) := \eta(x) \cdot u_\varepsilon^*(x)$. As $\varepsilon \to 0$ we have the following estimates which are due to Brezis-Nirenberg ([8]) (see also Lemma 2.1 in [10] and page 164 in [19]):

$$\|u_\varepsilon\|^2 = S^{n/2} + O(\varepsilon^{n-2}) \quad \|u_\varepsilon\|_{2^*}^{2^*} = S^{n/2} + O(\varepsilon^n). \tag{15}$$

For $v \in H_m^- \oplus \mathbb{R}^+\{u_\varepsilon\}$ we now write $v = w + \alpha u_\varepsilon$, where by definition

$$\text{supp}(u_\varepsilon) \cap \text{supp}(w) = \emptyset. \tag{16}$$

Several remarks are now in order:

(R_1) If (4) and either (6) or (9) hold then there exist $\alpha, \rho > 0$ such that

$$J(v) \ge \alpha \quad \forall v \in \partial B_\rho \cap H^+.$$

Indeed, by the assumptions we infer that there exists $C > 0$ such that $G(x,s) \le \frac{1}{2}\mu s^2 + C|s|^{2^*}$ and therefore $J(v) \ge c_1\|v\|^2 - c_2\|v\|^{2^*}$ for all $v \in H^+$.

(R_2) Define

$$Q_m^\varepsilon := [(B_R \cap H_m^-) \oplus [0, R]\{u_\varepsilon\}];$$

then $\exists R > \rho$ such that $\max_{v \in \partial Q_m^\varepsilon} J(v) \le \omega_m$ with $\omega_m \to 0$ as $m \to \infty$.

Indeed, by (7) we clearly have

$$\lim_{m\to\infty} \max_{v \in H_m^-} J(v) = 0$$

and by (5)

$$J(ru_\varepsilon) \le \frac{1}{2} r^2 \|u_\varepsilon\|^2 - \frac{1}{2^*} r^{2^*} \|u_\varepsilon\|_{2^*}^{2^*}$$

which, by (15), becomes negative if $r = R$ and R is large enough; therefore, $J(v) \le \omega_m$ for all $v \in (H_m^-) \cup (H_m^- \oplus R\{u_\varepsilon\})$; finally, since $\max_{0 \le r \le R} J(ru_\varepsilon) < +\infty$, if $v \in [(\partial B_R \cap H_m^-) \oplus [0, R]\{u_\varepsilon\}]$, by (16) we obtain $J(v) \le 0$ for large enough R.

(R_3) The functional J satisfies all the assumptions of the linking theorem [17] except for the PS condition.

Indeed, in view of Lemma 2 we have: if m is large enough, then

$$P_k H_m^- = H^- \quad \text{and} \quad H_m^- \oplus H^+ = H,$$

where $P_k : H \to H^-$ is the projection introduced above; therefore, $\partial B_\rho \cap H^+$ and ∂Q_m^ε link (cf. [17]).

The next sections are devoted to the construction of PS sequences at level $c < \frac{S^{n/2}}{n}$ which, by Lemma 1, yield a nontrivial solution of (1).

We will proceed as follows: Let

$$\Gamma := \{h \in C(\bar{Q}_m^\varepsilon, H) : h(v) = v, \ \forall v \in \partial Q_m^\varepsilon\};$$

then by standard methods we obtain a PS sequence for J at level

$$c = \inf_{h\in\Gamma} \max_{v\in Q_m^\varepsilon} J(h(v)).$$

Moreover, since the identity $Id \in \Gamma$, we have

$$c \le \max_{v\in Q_m^\varepsilon} J(v);$$

Theorems 1–3 follow if we can prove that for ε small enough

$$\sup_{v\in Q_m^\varepsilon} J(v) < \frac{1}{n} S^{n/2}. \tag{17}$$

4. Proof of Theorem 1. In this section we prove Theorem 1; so, assume hypotheses (4)–(7) if $n \ge 4$, (4)–(8) if $n = 3$ and choose m large enough so that

$$c_k m^{2-n} < \sigma \tag{18}$$

where c_k is as in Lemma 2 and σ is as in (6).

We claim that (17) holds; for the sake of contradiction assume that

$$\forall \varepsilon > 0 \quad \sup_{v\in Q_m^\varepsilon} J(v) \ge \frac{1}{n} S^{n/2}. \tag{19}$$

As the set $\{v \in Q_m^\varepsilon : J(v) \ge 0\}$ is compact, the supremum in (19) is attained. Therefore, for all $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \ge 0$ such that, for $v_\varepsilon := w_\varepsilon + t_\varepsilon u_\varepsilon$, we have

$$J(v_\varepsilon) = \max_{v\in Q_m^\varepsilon} J(v) \ge \frac{1}{n} S^{n/2};$$

that is,

$$\frac{1}{2}\|v_\varepsilon\|^2 - \int_\Omega G(x, v_\varepsilon) - \frac{1}{2^*}\|v_\varepsilon\|_{2^*}^{2^*} \ge \frac{1}{n} S^{n/2}, \quad \forall\, \varepsilon > 0. \tag{20}$$

Let us establish the main properties of the sequences $\{t_\varepsilon\}$ and $\{w_\varepsilon\}$: by (R_2) in the previous section we immediately obtain that the sequences $\{t_\varepsilon\} \subset \mathbb{R}^+$ and $\{w_\varepsilon\} \subset H_m^-$ are bounded. Hence, up to subsequences we may assume that

$$t_\varepsilon \to t_0 \ge 0 \qquad w_\varepsilon \to w_0 \in H_m^-,$$

where the convergence of $\{w_\varepsilon\}$ can be viewed in any norm since the space H_m^- is finite dimensional. As $w_\varepsilon \in H_m^-$, by using Lemma 2, (7) and (18) we have

$$\begin{aligned} J(w_\varepsilon) &\le \frac{\lambda_k + c_k m^{2-n}}{2}\|w_\varepsilon\|_2^2 - \int_\Omega G(x, w_\varepsilon) - \frac{1}{2^*}\|w_\varepsilon\|_{2^*}^{2^*} \\ &\le \frac{c_k m^{2-n} - \sigma}{2}\|w_\varepsilon\|_2^2 \le 0. \end{aligned} \tag{21}$$

Furthermore, we have

Lemma 3. *If* (20) *holds, then* $t_0 = 1$.

Proof. As G has subcritical growth at infinity and as $\{t_\varepsilon\}$ is bounded, we get

$$\lim_{\varepsilon\to 0} \int_\Omega G(x, t_\varepsilon u_\varepsilon) = 0;$$

hence, by (15) we obtain

$$J(t_\varepsilon u_\varepsilon) \le S^{n/2}\Big(\frac{t_0^2}{2} - \frac{t_0^{2^*}}{2^*}\Big) + o(1),$$

which combined with (16) and (21) gives

$$J(v_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon) \le \phi(t_0) \cdot S^{n/2} + o(1),$$

where $\phi(x) = \frac{x^2}{2} - \frac{x^{2^*}}{2^*}$. To conclude note that

$$\max_{x\ge 0} \phi(x) = \phi(1) = \frac{1}{n} \quad \text{and} \quad \phi(x) < \frac{1}{n} \quad \forall x \ge 0, \ x \ne 1;$$

thus, if $t_0 \ne 1$ then we contradict (20) for ε small enough. □

Next, we estimate the main order terms in $J(t_\varepsilon u_\varepsilon)$:

Lemma 4. *As $\varepsilon \to 0$ we have*

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon\|^2 - \frac{1}{2^*}\|t_\varepsilon u_\varepsilon\|_{2^*}^{2^*} \le \frac{1}{n}S^{n/2} + O(\varepsilon^{n-2}).$$

Proof. Using (15) we get (as $\varepsilon \to 0$)

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon\|^2 \le \frac{1}{2}S^{n/2} + \frac{t_\varepsilon^2 - 1}{2}S^{n/2} + O(\varepsilon^{n-2})$$
$$\frac{1}{2^*}\|t_\varepsilon u_\varepsilon\|_{2^*}^{2^*} \ge \frac{n-2}{2n}S^{n/2} + \frac{n-2}{2n}(t_\varepsilon^{2^*} - 1)S^{n/2} + O(\varepsilon^n);$$

hence,

$$\frac{1}{2}\|t_\varepsilon u_\varepsilon\|^2 - \frac{1}{2^*}\|t_\varepsilon u_\varepsilon\|_{2^*}^{2^*} \le \frac{1}{n}S^{n/2} + \frac{1}{2}\big(t_\varepsilon^2 - 1 - \frac{n-2}{n}(t_\varepsilon^{2^*} - 1)\big)S^{n/2} + O(\varepsilon^{n-2}).$$

To conclude, note that $\max_{x\ge 0}\{x^2 - 1 - \frac{n-2}{n}(x^{2^*} - 1)\} = 0$. □

Finally, we estimate the lower-order term $\int_\Omega G(x, t_\varepsilon u_\varepsilon)$:

Lemma 5. *There exists a function $\tau = \tau(\varepsilon)$ such that $\lim_{\varepsilon\to 0}\tau(\varepsilon) = +\infty$ and such that for ε small enough we have*

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge \tau(\varepsilon)\cdot\varepsilon^{n-2}.$$

Proof. Let us first consider the case $n = 3$: for ε small enough we have $B_\varepsilon \subset B_{1/2m} \subset \Omega_0$ and by (14)

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge \int_{B_\varepsilon} G\big(x, t_\varepsilon \frac{[3\varepsilon^2]^{1/4}}{[\varepsilon^2 + |x|^2]^{1/2}}\big).$$

By (8) we infer that $\exists\, \bar{s} > 0$ and an increasing function $\varphi = \varphi(s)$ with

$$\lim_{s\to+\infty}\varphi(s) = +\infty$$

such that if $s \ge \bar{s}$ then

$$G(x, s) \ge \varphi(s)s^4 \quad \text{for a.e. } x \in \Omega_0. \tag{22}$$

Next, note that if ε is small enough we have (recall that $t_\varepsilon \to 1$ by Lemma 3)

$$t_\varepsilon\frac{[3\varepsilon^2]^{1/4}}{[\varepsilon^2 + |x|^2]^{1/2}} > \bar{s}, \quad \forall\, x \in B_\varepsilon;$$

hence, by (22),

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge c\int_{B_\varepsilon}\varphi(\frac{1}{2}\varepsilon^{-1/2})\cdot(\varepsilon^{-1/2})^4 \ge c\varphi(\frac{1}{2}\varepsilon^{-1/2})\cdot\varepsilon$$

and the assertion follows by taking $\tau(\varepsilon) = c\varphi(\frac{1}{2}\varepsilon^{-1/2})$.

Let us now prove the result in the case $n \ge 4$. A straightforward calculation yields

$$t_\varepsilon u_\varepsilon^*(x) = \gamma \quad \Longleftrightarrow \quad |x| = \Phi(\gamma) := \big((t_\varepsilon/\gamma)^{2/(n-2)}\sqrt{n(n-2)}\cdot\varepsilon - \varepsilon^2\big)^{1/2};$$

then, there exists $c_1 > 0$ such that, for ε small enough, we have $\Phi(\delta) < c_1\sqrt{\varepsilon} < (2m)^{-1}$ where δ is as in (6); hence, by (5) and (6) we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge c\int_{\Phi(\delta)}^{1/2m} u_\varepsilon^2(r)\, r^{n-1}\, dr \ \ge c\int_{c_1\sqrt{\varepsilon}}^{1/2m}\frac{\varepsilon^{n-2}\cdot r^{n-1}}{(\varepsilon^2 + r^2)^{n-2}}\, dr \tag{23}$$

and therefore

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge c\varepsilon^{n-2}\int_{c_1\sqrt{\varepsilon}}^{1/2m} r^{3-n}\, dr$$
$$\ge \begin{cases} c\varepsilon^{n-2}\cdot\varepsilon^{2-n/2} & \text{if } n \ge 5 \\ c\varepsilon^2\cdot|\log\varepsilon| & \text{if } n = 4. \end{cases} \tag{24}$$

Thus, the result is proved for all $n \ge 3$. □

The *proof of Theorem* 1 is now easily completed: by (16), (21) and Lemmas 4 and 5 (which hold because we assumed (20)) we have

$$J(v_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon) \le \frac{1}{n}S^{n/2} + (c - \tau(\varepsilon))\cdot\varepsilon^{n-2}, \tag{25}$$

which contradicts (20) for ε small enough; thus (17) holds.

Remark. If $n \ge 4$ assumption (5) can be weakened as follows:

$$\exists\gamma > 0, \exists\alpha < \frac{n}{n-2} \text{ such that } G(x, s) \ge -\gamma|s|^\alpha, \text{ for a.e. } x \in \Omega,\ \forall s \in \mathbb{R}. \tag{5'}$$

Then one has

Theorem 1′. *Assume* (4), (5′), (6), (7). *If $n = 4$, then equation* (1) *has a nontrivial solution. If $n \ge 5$,* (1) *has a nontrivial solution provided that $\gamma = \gamma(\alpha)$ in* (5′) *is sufficiently small.*

Proof. The only change in the proof concerns the estimates (23) and (24): by (5′) and (6) we obtain

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \ge c\int_{c_1\sqrt{\varepsilon}}^{1/2m}\frac{\varepsilon^{n-2}\cdot r^{n-1}}{(\varepsilon^2 + r^2)^{n-2}}\, dr - c\gamma\int_0^{c_1\sqrt{\varepsilon}}\big(\frac{\varepsilon^{\frac{n-2}{2}}}{(\varepsilon^2 + r^2)^{\frac{n-2}{2}}}\big)^\alpha r^{n-1}\, dr,$$

and therefore

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq c\varepsilon^{n-2} \int_{c_1\sqrt{\varepsilon}}^{1/2m} r^{3-n}\, dr \; - c\gamma \varepsilon^{\alpha\frac{n-2}{2}} \int_0^{c_1\sqrt{\varepsilon}} \frac{r^{n-1}}{r^{\alpha(n-2)}} dr,$$

and finally

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon) \geq \begin{cases} c\varepsilon^{n-2} \cdot \varepsilon^{2-n/2} - c\gamma\varepsilon^{n/2} & \text{if } n \geq 5 \\ c\varepsilon^2 \cdot |\log\varepsilon| - c\gamma\varepsilon^2 & \text{if } n = 4. \end{cases}$$

With these estimates we again obtain (25), which completes the proof.

5. Proof of Theorem 2. The proof of Theorem 2 follows the same lines as that of Theorem 1; however, some refinements of the estimates are required. In this section we assume hypotheses (4), (5), (9)–(11). In order to emphasize the dependence on m we denote v_ε^m, u_ε^m, w_ε^m instead of u_ε, w_ε, v_ε (this dependence is "hidden" in the cut-off function η).

As in the previous section we want to show (17) and we reason by contradiction assuming that (19) holds: for all m large enough (say $m \geq \bar{m}$) and all $\varepsilon > 0$ there exist $v_\varepsilon^m \in Q_m^\varepsilon$ and $t_\varepsilon \geq 0$ such that

$$\frac{1}{2}\|v_\varepsilon^m\|^2 - \int_\Omega G(x, v_\varepsilon^m) - \frac{1}{2^*}\|v_\varepsilon^m\|_{2^*}^{2^*} \geq \frac{1}{n}S^{n/2} \quad \forall\, \varepsilon > 0, \quad \forall\, m \geq \bar{m}. \tag{26}$$

If (26) holds, then the sequences $\{t_\varepsilon\}$ and $\{w_\varepsilon^m\}$ satisfy again

$$t_\varepsilon \geq c > 0 \quad \text{and} \quad \|w_\varepsilon^m\| \leq c. \tag{27}$$

The following lemma contains a refinement of the estimates (15) which highlights the dependence on m:

Lemma 6. *Let $m \to \infty$ and assume that $\varepsilon = \varepsilon(m) = o(1/m)$; then*

$$\|u_\varepsilon^m\|^2 = S^{n/2} + O[(\varepsilon m)^{n-2}], \quad \|u_\varepsilon^m\|_{2^*}^{2^*} = S^{n/2} + O[(\varepsilon m)^n].$$

Moreover, if (11) *holds, then there exists a function ϕ such that* $\lim_{x\to+\infty}\phi(x) = +\infty$ *and*

$$\int_\Omega G(x, u_\varepsilon^m) \geq \varepsilon^{n(n-2)/(n+2)}\phi(\varepsilon^{-1}).$$

Proof. As $\|\nabla\eta\|_\infty \leq 4m$, the proof of the estimates $\|u_\varepsilon^m\|^2 = S^{n/2} + O[(\varepsilon m)^{n-2}]$ and $\|u_\varepsilon^m\|_{2^*}^{2^*} = S^{n/2} + O[(\varepsilon m)^n]$ can be obtained as for (15); see [8].

By (11) there exists an increasing function τ such that $\lim_{x\to+\infty}\tau(x) = +\infty$ satisfying $G(x, s) \geq \tau(s) \cdot s^{8n/(n^2-4)}$ for almost every $x \in \Omega_0$ and for all $s \geq \bar{s}$ (for a suitable $\bar{s} > 0$); therefore, reasoning as in the proof of Lemma 5 (in the case $n = 3$) we obtain

$$\begin{aligned}\int_\Omega G(x, t_\varepsilon u_\varepsilon^m) &\geq c\int_0^\varepsilon \Big(\frac{\varepsilon^{(n-2)/2}}{(\varepsilon^2+r^2)^{(n-2)/2}}\Big)^{8n/(n^2-4)} \cdot \tau\Big(\frac{\varepsilon^{(n-2)/2}}{(\varepsilon^2+r^2)^{(n-2)/2}}\Big) \cdot r^{n-1}\, dr \\ &\geq c(\varepsilon^{(2-n)/2})^{8n/(n^2-4)} \cdot \tau(c\varepsilon^{(2-n)/2}) \cdot [r^n]_0^\varepsilon \\ &\geq c\varepsilon^{n(n-2)/(n+2)} \cdot \tau(c\varepsilon^{(2-n)/2})\end{aligned}$$

and the result follows by setting $\phi(x) = c\tau(cx^{(n-2)/2})$. □

We now choose a suitable $\varepsilon = \varepsilon(m)$ in order to deal only with the parameter m, and to arrive at a contradiction to (26) for m large enough: we take

$$\varepsilon(m) = m^{-(n+2)/2}; \tag{28}$$

therefore, as $m \to \infty$, $\varepsilon(m) = o(1/m)$ and Lemma 6 applies. From now on, we denote by v^m, u^m, w^m the functions v_ε^m, u_ε^m, w_ε^m with the above choice of ε and with t_m the corresponding t_ε.

We first estimate $J(t_m u^m)$: by Lemma 6 we infer

Lemma 7. *If m is large enough we have*

$$J(t_m u^m) \leq \frac{1}{n}S^{n/2} - cm^{n(2-n)/2}\phi(cm^{(n+2)/2}),$$

where ϕ is the function defined in Lemma 6.

Proof. For the moment assume just that $\varepsilon = \varepsilon(m) = o(1/m)$ as $m \to \infty$; then by Lemma 6, (27) and by reasoning as in the proof of Lemma 4 we obtain

$$J(t_\varepsilon u_\varepsilon^m) \leq \frac{1}{n}S^{n/2} + O[(\varepsilon m)^{n-2}] - \varepsilon^{n(n-2)/(n+2)}\phi(\varepsilon^{-1}).$$

With the choice of ε as in (28) we get

$$J(t_\varepsilon u_\varepsilon^m) \leq \frac{1}{n}S^{n/2} + O[m^{n(2-n)/2}] - cm^{n(2-n)/2}\phi(m^{(n+2)/2}),$$

and hence

$$J(t_m u^m) = J(t_\varepsilon u_\varepsilon^m) \leq \frac{1}{n}S^{n/2} - cm^{n(2-n)/2}\phi(m^{(n+2)/2}).$$

Finally, we estimate the part of the functional relative to w^m:

Lemma 8. *If m is large enough we have*

$$J(w^m) \le cm^{n(2-n)/2}.$$

Proof. By (10) and Lemma 2 we get (for large m)

$$J(w^m) \le \frac{1}{2}\|w^m\|^2 - \frac{\lambda_k}{2}\|w^m\|_2^2 - \sigma\|w^m\|_{2^*}^{2^*} \le c_1\|w^m\|_2^2 \cdot m^{2-n} - c_2\|w^m\|_2^{2^*}.$$

Consider the function $h(x) = c_1 m^{2-n} \cdot x^2 - c_2 \cdot x^{2n/(n-2)}$: its derivative vanishes for $x = cm^{-(n-2)^2/4}$, and therefore, using (27), we have $h(\|w^m\|_2) \le cm^{n(2-n)/2}$; with this and (29) we obtain the result.

The proof of Theorem 2 is now obtained by (16) and Lemmas 7 and 8:

$$\begin{aligned} J(v^m) &= J(t_m u^m) + J(w^m) \\ &\le \frac{1}{n}S^{n/2} - cm^{n(2-n)/2}\big(\phi(m^{(n+2)/2}) - 1\big) < \frac{1}{n}S^{n/2}, \end{aligned}$$

for m sufficiently large. This contradicts (26), and the proof of Theorem 2 is complete.

6. Proof of Theorem 3. For all $\lambda > 0$ define

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*}.$$

Let $k \in \mathbb{N}$ and choose $\varepsilon(m)$ as in (28): by Lemmas 7 and 8 we infer that there exists a function $\delta(m)$ with $\lim_{m\to\infty}\delta(m) = 0$ and $\bar{m} \in \mathbb{N}$ such that

$$\delta(m) > 0 \quad \forall m \ge \bar{m} \quad \text{and} \quad \sup_{v\in Q_m} J_{\lambda_k}(v) < \frac{1}{n}S^{n/2} - \delta(m);$$

here we write Q_m instead of Q_m^ε since $\varepsilon = \varepsilon(m)$ is given. Therefore, there exists $\tilde{m} \ge \bar{m}$ such that

$$\sup_{m\ge\bar{m}} \delta(m) = \delta(\tilde{m}) =: \tilde{\delta} > 0$$

and

$$\sup_{v\in Q_{\tilde{m}}} J_{\lambda_k}(v) \le \frac{1}{n}S^{n/2} - \tilde{\delta}. \tag{30}$$

As the set $Q_{\tilde{m}}$ is compact, we have

$$\sup_{v\in Q_{\tilde{m}}} \frac{1}{2}\|v\|_2^2 = \gamma < \infty. \tag{31}$$

Let $\delta_k := \tilde{\delta}/\gamma$ and let $\lambda \in (\lambda_k - \delta_k, \lambda_k)$; consider the spaces

$$V = \text{span}\{e_i^{\tilde{m}} : i = 1, \dots, k\} \oplus \mathbb{R}\{u_\varepsilon\} \quad W = \overline{\text{span}\{e_i : i \ge k\}}.$$

As $\lambda < \lambda_k$ we obviously have

$$\exists\, \alpha, \rho > 0 \text{ such that } J(v) \ge \alpha\,, \quad \forall\, v \in \partial B_\rho \cap W;$$

moreover, by (30) and (31) we have

$$\sup_{v\in Q_{\tilde{m}}} J_\lambda(v) \le \frac{1}{n}S^{n/2} - \tilde{\delta} + (\lambda_k - \lambda)\gamma < \frac{1}{n}S^{n/2}.$$

To complete the proof of Theorem 3 it now suffices to apply Theorem 2.4 in [6] (see also the restatement in [12]) with $b = \frac{1}{n}S^{n/2}$ and using that $\dim V - \text{codim} W = \mu_k + 1$.

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