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A note on the evolution Navier-Stokes equations with a pressure-dependent viscosity

Filippo Gazzola

Abstract. We consider Navier-Stokes equations with a pressure-dependent viscosity. Under suitable assumptions on the external force and on the initial data, we prove that the Cauchy-Dirichlet problem for the evolution equations admits a unique solution.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be open, bounded and with smooth boundary $\partial\Omega$; we consider the evolution Navier-Stokes equations with a pressure-dependent viscosity. Let f be the external force acting on the fluid, let u_0 be the initial velocity, then the Cauchy-Dirichlet problem reads

$$\begin{cases} \partial_t u - \nabla \cdot [\eta(p)(\nabla u + \nabla^T u)] + \nabla p + (u \cdot \nabla)u = f \qquad \text{in} \quad \Omega \times [0,T] \\ \nabla \cdot u = 0 \qquad \text{in} \quad \Omega \times [0,T] \\ u = 0 \qquad \text{on} \quad \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) \qquad \text{in} \quad \Omega \end{cases} \tag{1}$$

where u and p are respectively the (unknown) velocity and pressure of the fluid, while η is its viscosity; here and in the sequel, we use the notations $\partial_t = \frac{\partial}{\partial t}$ and $\nabla = \nabla_x$. It is well-known that viscosities of real fluids may depend on the pressure and physical experiments show that η has an exponential growth with respect to p, see [1, 11]. In a remarkable paper, Renardy [13] studies problem (1): under suitable assumptions on η he obtains a local existence and uniqueness result for (1) in the 3D case; also more general boundary conditions are considered in [13]. One of the basic remarks of Renardy is that the stationary problem may lose its ellipticity unless

$$\text{the eigenvalues of } (\nabla u + \nabla^T u) \text{ are strictly less than } \frac{1}{\sup \eta'} \ : \tag{2}$$

if the loss of ellipticity occurs we cannot expect well-posedness of (1). Due to the presence of the "multiplicative term" $\nabla \cdot [\eta(p)(\nabla u + \nabla^T u)]$, to solve (1) one needs to reason in functional spaces of smooth functions: a solution of (1) in such spaces (i.e. u being continuously differentiable with respect to t and taking values in a suitable space) does not exist if the compatibility condition between the initial and the boundary data is not satisfied. To guarantee the existence of data satisfying the equation at time $t = 0$, Renardy assumes that η is sublinear at infinity, that η' is bounded on $\mathbb{R}$ and that (2) holds; this is necessary in order to eliminate the pressure: indeed, instead of the Neumann boundary value problem for Laplace operator (as for the classical equations) one must here solve a similar problem for a nonlinear operator which is elliptic if (2) holds.

For the stationary problem, in the 3D case, the following results are known: given u, there exists a unique p satisfying the equation in a suitable weak sense (see [13]) and given p, there exists a unique u satisfying the equation in a complementary sense (see [5]). More precisely, in [5] the author studies the different behaviour of the stationary equation when projected onto the Helmholtz-Weyl spaces [15]: for further features of this decomposition we refer to [6, 14]. Independently of the space dimension, existence and uniqueness of (u, p) solving the stationary problem in presence of *almost conservative* forces f has been proved in [7]: such result is obtained by exploiting the projections of the equation onto these spaces.

In this paper we require η to satisfy the more realistic assumptions

$$\eta \in C^3(\mathbb{R}) \qquad \text{and} \qquad \inf_{x \in \mathbf{R}} \eta(x) = \eta_o > 0 \tag{3}$$

but to ensure the existence of data satisfying the equation at time $t = 0$ we take small initial velocities u_0. Then, independently of the space dimension, we prove an existence and uniqueness result for (1) when f is almost conservative and u_0 is sufficiently small; even if we pick some ideas from [13], the functional spaces considered and the techniques involved are different from those of Renardy. We first prove an existence and uniqueness result for a linear problem (see Proposition 2 below) and we apply a fixed point method to obtain a similar result for (1): in order to obtain a contractive map we do not require as usual T to be small enough but u_0 and f to satisfy the above restrictions; then, the contractive map is not defined on the whole functional space but only on a ball of it. In other words, in presence of conservative forces and zero initial velocity we prove in Proposition 3 below that there exists a unique equilibrium solution; in Section 4 we prove that for small initial velocities and almost conservative forces there exists a solution (u, p) of (1) which is unique in a sufficiently small neighborhood of equilibria. We point out that, due to the presence of the compatibility condition, it seems difficult to apply an inversion argument as in [7] in order to prove existence and uniqueness. We believe that the main interest of this paper is the unusual method employed to obtain the result: we refer to Section 5 for further comments on our result and on the techniques involved in the proof.

2. The existence and uniqueness result

We assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set satisfying

$$\partial\Omega \in C^2 \ . \tag{4}$$

Bold capital letters ($\mathbf{L}^s$, $\mathbf{W}^{m,s}$,...) represent functional spaces of vector functions and usual capital letters (L^s, $W^{m,s}$,...) represent spaces of scalar functions (to simplify notations we delete the domain of definition Ω); $|\cdot|_2$ denotes the L^2-norm. With $W^{m,s}$ we represent the Sobolev space of functions with generalized derivatives up to order m in L^s, with $\|\cdot\|_{m,s}$ we denote the corresponding norm and with $\mathbf{W}_0^{1,s}$ the $W^{1,s}$-closure of the space of smooth functions with compact support in Ω. Finally, L^∞ denotes the space of essentially bounded functions endowed with the sup norm which is denoted by $|\cdot|_\infty$.

We consider the spaces

$$\mathbf{G}_s := \{f \in \mathbf{L}^s;\ \nabla\cdot f = 0,\ \gamma_\nu f = 0\} \qquad \mathbf{G}_s^\perp := \{f \in \mathbf{L}^s;\ \exists g \in W^{1,s},\ f = \nabla g\}$$
$$\mathbf{V}_s := \{f \in \mathbf{W}_0^{1,s};\ \nabla\cdot f = 0\}$$

where γ_ν denotes the normal trace operator (in the sequel γ denotes the trace operator of order zero); it is well-known that $\mathbf{L}^s = \mathbf{G}_s \oplus \mathbf{G}_s^\perp$. We denote by $\mathcal{P}$ (resp. $\mathcal{Q}$) the projectors of $\mathbf{L}^s$ onto $\mathbf{G}_s$ (resp. $\mathbf{G}_s^\perp$): $\mathcal{P}$ and $\mathcal{Q}$ are linear continuous operators from $\mathbf{W}^{m,s}$ onto $\mathbf{W}^{m,s} \cap \mathbf{G}_s$ (resp. $\mathbf{W}^{m,s} \cap \mathbf{G}_s^\perp$), see [4]. Note that $\mathbf{V}_s$ is an interpolation space between $\mathbf{W}^{2,s} \cap \mathbf{V}_s$ and $\mathbf{G}_s$ (see [9] and the previous results in [2, 3]): by the results of Sections 1.2 and 3.2 in [12] we infer that there exists $\alpha \in (0,1)$ such that

$$\mathbf{V}_s \subseteq (\mathbf{W}^{2,s} \cap \mathbf{V}_s; \mathbf{G}_s)_{\alpha,\infty} \ . \tag{5}$$

For such α we introduce the Banach spaces $C^\alpha(0,T;\mathbf{W}^{m,s})$ and $C^{1,\alpha}(0,T;\mathbf{W}^{m,s})$ of Hölder continuous functions on $[0,T]$ with values in $\mathbf{W}^{m,s}$: we denote by $\|\cdot\|_{C^\alpha(\mathbf{W}^{m,s})}$ and $\|\cdot\|_{C^{1,\alpha}(\mathbf{W}^{m,s})}$ the corresponding norms.

As in [13] we assume that, for all $t \in [0,T]$, the mean value of $p(t)$ over Ω is given, say $\bar{p}(t)$: without loss of generality we take $\bar{p}(t) \equiv 0$. Assume that

$$s > n \ ; \tag{6}$$

for $m \in \{1,2\}$ and s satisfying (6) consider the space $\overline{W}^{m,s} := \{g \in W^{m,s};\ \int_\Omega g = 0\}$. We will prove:

Theorem 1. *Assume (3) (4) (6) and let $T > 0$, $\psi \in C^\alpha(0,T;\overline{W}^{1,s})$ be such that $\psi(0) \in \overline{W}^{2,s}$; there exist two constants $R_T = R_T(\psi) > 0$ and $U_T = U_T(\psi) > 0$ such that if*

(i) $\varphi \in C^\alpha(0,T;\mathbf{G}_s)$ *satisfies* $\varphi(0) \in \mathbf{W}^{1,s} \cap \mathbf{G}_s$ *and*
$\|\varphi\|_{C^\alpha(\mathbf{L}^s)} + \|\varphi(0)\|_{1,s} \le R_T$
(ii) $f = \varphi + \nabla\psi$
(iii) $u_0 \in \mathbf{W}^{3,s} \cap \mathbf{V}_s$ *and* $\|u_0\|_{3,s} \le U_T$
then there exists a unique $p_0 \in \overline{W}^{2,s}$ *such that*
$\mathcal{Q}(-\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla p_0 + (u_0 \cdot \nabla)u_0 - f(0)) = 0.$
Moreover, if f, u_0 *and* p_0 *satisfy*

$$-\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla p_0 + (u_0 \cdot \nabla)u_0 - f(0) = 0 \qquad \text{on} \quad \partial\Omega \tag{7}$$

then (1) admits a unique solution

$$(u,p) \in \Big(C^\alpha(0,T;\mathbf{W}^{2,s} \cap \mathbf{V}_s) \cap C^{1,\alpha}(0,T;\mathbf{G}_s)\Big) \times C^\alpha(0,T;\overline{W}^{1,s})$$

in a suitable neighborhood of $(0,\psi)$.

Something should be said about the statement on the existence of p_0 and on assumption (7). Roughly speaking, since we deal with strict solutions (see, e.g. [12]) the equation (1) needs to make sense at time $t = 0$. Therefore, by projecting it onto the space $\mathbf{G}_s^\perp$ one has to prove the existence of a function p_0 such that $\mathcal{Q}(-\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla p_0 + (u_0 \cdot \nabla)u_0 - f(0)) = 0$; this will be proved in Proposition 1 below. Then, the equation is projected onto $\mathbf{G}_s$: in the interior of Ω the "free term" $\partial_t u|_{t=0}$ takes the values of $\mathcal{P}(\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] - (u_0 \cdot \nabla)u_0 + f(0))$; when the boundary $\partial\Omega$ is attained $\partial_t u|_{t=0}$ is no longer a free term because $\gamma(\partial_t u|_{t=0}) = 0$. This is the reason of the compatibility condition (7): as already mentioned in the introduction, a strict (Hölder-continuous) solution of (1) does not exist if (7) is not satisfied. In Lemmas 1 and 4 below we determine a sufficient condition for the existence of data satisfying (7).

3. Preliminary results

Define

$$\mathbf{X} := \Big(C^\alpha(0,T;\mathbf{W}^{2,s} \cap \mathbf{V}_s) \cap C^{1,\alpha}(0,T;\mathbf{G}_s)\Big) \times C^\alpha(0,T;\overline{W}^{1,s})$$

which is a Banach space when endowed with the norm

$$\|(u,p)\|_{\mathbf{X}} = \|u\|_{C^\alpha(\mathbf{W}^{2,s})} + \|u\|_{C^{1,\alpha}(\mathbf{L}^s)} + \|p\|_{C^\alpha(W^{1,s})}$$

for all $(u,p) \in \mathbf{X}$: consider also the Banach space $\mathbf{X}_0 = \{(u,p) \in \mathbf{X};\ u(0) \in \mathbf{W}^{3,s},\ p(0) \in W^{2,s}\}$ endowed with the norm $\|(u,p)\|_{\mathbf{X}_0} = \|(u,p)\|_{\mathbf{X}} + \|u(0)\|_{3,s} + \|p(0)\|_{2,s}$.

Let $\mathbf{F} := \{f \in C^\alpha(0,T;\mathbf{L}^s);\ f(0) \in \mathbf{W}^{1,s}\}$ and define $\|f\|_{\mathbf{F}} := \|f\|_{C^\alpha(\mathbf{L}^s)} + \|f(0)\|_{1,s}$; consider the space

$$\mathbf{Y} := \mathbf{F} \times \Big(\mathbf{W}^{3,s} \cap \mathbf{V}_s\Big)$$

and for all $(f, u_0) \in \mathbf{Y}$ define the norm

$$\|(f,u_0)\|_{\mathbf{Y}} = \|f\|_{\mathbf{F}} + \|u_0\|_{3,s} \ :$$

then, $\mathbf{Y}$ is a Banach space.

We first note that by (6) we infer

$$C^\alpha(0,T;\overline{W}^{1,s}) \subset L^\infty(0,T;L^\infty) \tag{8}$$

and that

$$\mathbf{W}^{2,s}\ ,\quad C^\alpha(0,T;W^{1,s}) \qquad \text{are Banach algebras.} \tag{9}$$

The first part of Theorem 1 is proved by the following

Proposition 1. *Assume (3) (4) (6): for all $g \in \mathbf{W}^{1,s}$ there exists a constant $\overline{U} = \overline{U}(\mathcal{Q}g) > 0$ such that if $u_0 \in \mathbf{W}^{3,s} \cap \mathbf{V}_s$ and $\|u_0\|_{3,s} \le \overline{U}$ then there exists a unique $p_0 \in \overline{W}^{2,s}$ satisfying*

$$\mathcal{Q}\Big(-\nabla\cdot[\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla p_0 + (u_0\cdot\nabla)u_0 - g\Big) = 0\ .$$

Proof. Define the operator

$$\begin{aligned}\Psi(u,p,g) = \Big(&\nabla\cdot(-\nabla\cdot[\eta(p)(\nabla u + \nabla^T u)] + \nabla p + (u\cdot\nabla)u - g),\\ &\gamma_\nu(-\nabla\cdot[\eta(p)(\nabla u + \nabla^T u)] + \nabla p + (u\cdot\nabla)u - g)\Big)\ ;\end{aligned}$$

by (3), (6), (9) and well-known continuity properties of the operator γ_ν, we have

$$\Psi \in C^1\Big((\mathbf{W}^{3,s}\cap\mathbf{V}_s) \times \overline{W}^{2,s} \times \mathbf{W}^{1,s};\ \mathbf{B}_s\Big)$$

where $\mathbf{B}_s = \{(\phi,\psi) \in L^s \times W^{1-1/s,s}(\partial\Omega);\ \exists h \in \mathbf{W}^{1,s},\ \phi = \nabla\cdot h,\ \psi = \gamma_\nu h\}$: then $\mathbf{B}_s$ is a Banach space. Next note that for all $g \in \mathbf{W}^{1,s}$ there exists a unique $p_g \in \overline{W}^{2,s}$ solving the problem

$$\begin{cases} \Delta p_g = \nabla\cdot g & \text{in}\quad \Omega\\ \dfrac{\partial p_g}{\partial\nu} = \gamma_\nu g & \text{on}\quad \partial\Omega\ : \end{cases}$$

therefore, $\Psi(0, p_g, g) = 0$.

Finally, note that $\Psi_p(0, p_g, g)$ is the linear (continuous) operator mapping $\overline{W}^{2,s}$ into $\mathbf{B}_s$ such that

$$\Psi_p(0, p_g, g)[q] = \left(\Delta q, \frac{\partial q}{\partial \nu}\right) \ :$$

therefore, $\Psi_p(0, p_g, g)$ is an isomorphism. The statement follows from the implicit function theorem [16] and from the fact that the neighborhood of u_0 to which it can be applied only depends on $\mathcal{Q}g$. □

Obviously, $f(0)$ is the particular function g which should be inserted in the statement of Proposition 1 to prove the first part of Theorem 1.

Next, we prove a crucial existence and uniqueness result for a linear problem which is of interest independently of our context:

Proposition 2. *Assume (3) (4) (6) and let $T > 0$, $\psi \in C^\alpha(0, T; \overline{W}^{1,s})$ be such that $\psi(0) \in \overline{W}^{2,s}$; then, for all $(g, v_0) \in \mathbf{Y}$ there exists a unique $q_0 \in \overline{W}^{2,s}$ solution of the problem*

$$\mathcal{Q}\Big(- \nabla \cdot [\eta(\psi(0))(\nabla v_0 + \nabla^T v_0)] + \nabla(q_0 - \psi(0)) - g(0)\Big) = 0 \ .$$

If the above function q_0 satisfies the compatibility condition

$$-\nabla \cdot [\eta(\psi(0))(\nabla v_0 + \nabla^T v_0)] + \nabla(q_0 - \psi(0)) = g(0) \qquad on \quad \partial\Omega \tag{10}$$

then there exists a unique solution $(v, q) \in \mathbf{X}_0$ of the linear problem

$$\begin{cases} \partial_t v - \nabla \cdot \big[\eta(\psi)(\nabla v + \nabla^T v)\big] + \nabla(q - \psi) = g & in \quad \Omega \times [0, T] \\ \nabla \cdot v = 0 \qquad in \quad \Omega \times [0, T] \\ v = 0 \qquad on \quad \partial\Omega \times [0, T] \\ v(0) = v_0 \qquad in \quad \Omega \ . \end{cases} \tag{11}$$

Moreover, there exists a constant $C_\psi > 0$ (depending on $\|\psi\|_{C^\alpha(W^{1,s})}$) such that

$$\|(v, q - \psi)\|_{\mathbf{X}} \le C_\psi \|(g, v_0)\|_{\mathbf{Y}} \ .$$

Proof. To prove the first statement take $q_0 \in \overline{W}^{2,s}$ such that

$$\nabla q_0 = \mathcal{Q}\Big(\nabla \cdot [\eta(\psi(0))(\nabla v_0 + \nabla^T v_0)] + g(0)\Big) + \nabla\psi(0) \ .$$

For all $t \in [0,T]$ define the operator $A_\psi(t) : \mathbf{W}^{2,s} \cap \mathbf{V}_s \to \mathbf{G}_s$ by

$$A_\psi(t)v = -\mathcal{P}\Big(\nabla \cdot [\eta(\psi(t))(\nabla v + \nabla^T v)]\Big) \ :$$

by (8) and by the arguments in the proof of Lemma 2 in [7] we can apply Theorem 5.2 in [8] (see also [10]) and therefore the operator $A_\psi(t)$ is sectorial for all t; this can also be obtained by reasoning as in Section 3.1.1 in [12]. Hence, $A_\psi(t)$ generates an analytic semigroup in $\mathbf{G}_s$. Note that the domain of $A_\psi(t)$ does not depend on t (it is $\mathbf{W}^{2,s} \cap \mathbf{V}_s$ for all t) and that the map $t \mapsto A_\psi(t)$ is Hölder continuous with values in $L(\mathbf{W}^{2,s} \cap \mathbf{V}_s, \mathbf{G}_s)$; then, by (5) and (10), all the assumptions of Proposition 6.1.3 in [12] are fulfilled: therefore, there exists a unique $v \in C^\alpha(0,T;\mathbf{W}^{2,s} \cap \mathbf{V}_s) \cap C^{1,\alpha}(0,T;\mathbf{G}_s)$ solving the first of (11) projected onto $\mathbf{G}_s$ and satisfying

$$\|v\|_{C^\alpha(\mathbf{W}^{2,s})} + \|v\|_{C^{1,\alpha}(\mathbf{L}^s)} \le C'_\psi(\|\mathcal{P}g\|_{\mathbf{F}} + \|v_0\|_{3,s}) \ .$$

Then, one takes $q \in C^\alpha(0,T;\overline{W}^{1,s})$ such that

$$\nabla q = \mathcal{Q}\Big(\nabla \cdot \big[\eta(\psi)(\nabla v + \nabla^T v)\big] + g\Big) + \nabla\psi \ :$$

the estimate of $\|q-\psi\|_{C^\alpha(W^{1,s})}$ is obtained by means of that relative to v and the proof is complete. □

4. Proof of Theorem 1

Take $f \in \mathbf{F}$, define $\psi \in C^\alpha(0,T;\overline{W}^{1,s})$ by

$$\nabla\psi(t) = \mathcal{Q}f(t) \qquad \forall t \in [0,T] \tag{12}$$

and let

$$\overline{U} = \overline{U}(\nabla\psi(0)) \tag{13}$$

be the constant found in Proposition 1; clearly, $\mathcal{P}f(0) \in \mathbf{W}^{1,s} \cap \mathbf{G}_s$ and we consider

$$\begin{aligned}
&\mathbf{X}_f := \Big\{(v,q) \in \mathbf{X}_0;\\
&\mathcal{Q}\Big(-\nabla \cdot [\eta(q(0))(\nabla v(0) + \nabla^T v(0))] + \nabla[q(0)-\psi(0)] + [v(0)\cdot\nabla]v(0)\Big) = 0;\\
&\gamma\Big(\nabla[q(0)-\psi(0)] - \mathcal{P}f(0) - \nabla \cdot [\eta(q(0))(\nabla v(0) + \nabla^T v(0))]\Big) = 0\Big\}.
\end{aligned}$$

To satisfy (7) a suitable relation between f and u_0 is needed: therefore, we define

$$\mathcal{U}_f^0 := \{u_0 \in \mathbf{W}^{3,s} \cap \mathbf{V}_s;\ \exists (u,p) \in \mathbf{X}_f,\ u(0) = u_0\}\ .$$

We first prove

Lemma 1. *For all $f \in \mathbf{F}$ the set $\mathbf{X}_f$ is closed in $\mathbf{X}_0$.*
Moreover, there exists $K > 0$ such that if $\|\mathcal{P}f\|_{\mathbf{F}} \le K$ then there exists $v_0 \in \mathcal{U}_f^0$ such that $\|v_0\|_{3,s} \le \overline{U}$ ($\overline{U}$ as in (13)); in particular, $\mathbf{X}_f \neq \emptyset$.

Proof. Since the operators $\mathcal{Q}$ and γ are continuous in $\mathbf{W}^{1,s}$ we have that $\mathbf{X}_f$ is closed.

If $\|\mathcal{P}f(0)\|_{1,s}$ is small enough (and this is certainly the case if $\|\mathcal{P}f\|_{\mathbf{F}} \le K$ for a suitable K) then, by the result in [7] we know that there exists a unique $(v_0, q_0) \in (\mathbf{W}^{3,s} \cap \mathbf{V}_s) \times \overline{W}^{2,s}$ such that

$$-\nabla \cdot [\eta(q_0)(\nabla v_0 + \nabla^T v_0)] + \nabla q_0 + (v_0 \cdot \nabla) v_0 = f(0) \qquad \text{in} \quad \Omega\ :$$

hence $v_0 \in \mathcal{U}_f^0$ and $(v_0, q_0) \in \mathbf{X}_f$. Moreover, it is proved in [7] that the operator corresponding to the stationary problem is a local homeomorphism in a neighborhood of $(0, \psi(0))$; therefore,

$$\|v_0\|_{3,s} + \|q_0 - \psi(0)\|_{2,s} \le \Gamma_f$$

where $\Gamma_f \to 0$ as $\|\mathcal{P}f(0)\|_{1,s} \to 0$. Then, if K is small enough we have $\|v_0\|_{3,s} \le \overline{U}$. □

Let $L_\psi : \mathbf{X}_0 \to \mathbf{Y}$ be the linear operator defined by (11), that is

$$L_\psi(u,p) = \Big(\partial_t u - \nabla \cdot \big[\eta(\psi)(\nabla u + \nabla^T u)\big] + \nabla(p - \psi);\ u(0)\Big)$$

and let $R_\psi : \mathbf{X} \to C^\alpha(0,T;\mathbf{L}^s)$ be the "residual" operator defined by

$$R_\psi(u,p) := \nabla \cdot [(\eta(p) - \eta(\psi))(\nabla u + \nabla^T u)] - (u \cdot \nabla) u\ ;$$

in the sequel, we omit the subscript on L and R.

Lemma 2. *Let f, ψ be as in (12), assume that $\|\mathcal{P}f\|_{\mathbf{F}} \le K$ (K as in Lemma 1) and that $u_0 \in \mathcal{U}_f^0$ satisfies $\|u_0\|_{3,s} \le \overline{U}$ ($\overline{U}$ as in (13)). Then, for all $(u',p') \in \mathbf{X}_f$ satisfying $u'(0) = u_0$ there exists a unique $(u,p) \in \mathbf{X}_f$ such that*

$$L(u,p) = \Big(\mathcal{P}f + R(u',p');\ u_0\Big)\ . \tag{14}$$

Proof. Lemma 1 ensures that the set specified by the assumptions is nonempty.

Existence and uniqueness of $(u,p) \in \mathbf{X}_0$ follow by Proposition 2 if the corresponding compatibility condition 10 holds. By Proposition 1, there exists a unique $p_0 \in \overline{W}^{2,s}$ satisfying

$$\mathcal{Q}\Big(-\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla p_0 - \nabla\psi(0) + (u_0 \cdot \nabla)u_0 \Big) = 0 \ ;$$

by Proposition 2, there exists a unique $q_0 \in \overline{W}^{2,s}$ such that

$$\mathcal{Q}\Big(-\nabla \cdot [\eta(p_0)(\nabla u_0 + \nabla^T u_0)] + \nabla q_0 - \nabla\psi(0) + (u_0 \cdot \nabla)u_0 \Big) = 0 \ :$$

hence, $q_0 = p_0$ and (10) is fulfilled because $(u', p') \in \mathbf{X}_f$. This also proves that $(u,p) \in \mathbf{X}_f$. □

Lemma 2 defines a map $\Lambda : \mathbf{X}_f \to \mathbf{X}_f$ such that $\Lambda(u', p') = (u,p)$; (1) admits a solution if Λ admits a fixed point.

Let B_r denote the $\mathbf{X}$-ball of radius r centered at $(0,\psi)$, that is

$$B_r := \{(u,p) \in \mathbf{X};\ \|(u, p-\psi)\|_{\mathbf{X}} \leq r\} \ ;$$

we need a technical result:

Lemma 3. *Assume (3), (6); then for all $r > 0$ there exists $\eta_r > 0$ such that if $\|p_i - \psi\|_{C^\alpha(W^{1,s})} \leq r$ $(i = 1, 2)$ then*

$$\|\eta(p_1) - \eta(p_2)\|_{C^\alpha(W^{1,s})} \leq \eta_r \|p_1 - p_2\|_{C^\alpha(W^{1,s})} \ .$$

Moreover, the map $r \mapsto \eta_r$ is monotone non-decreasing.

Proof. For all $k = 0, ..., 3$, from (3) we infer that $\eta^{(k)} \in L^\infty_{loc}(\mathbb{R})$ and, by (8), we get

$$\eta_r^k := \sup_{(u,p) \in B_r} |\eta^{(k)}(p)|_{L^\infty(L^\infty)} < \infty \ ;$$

hence, by Lagrange theorem and again by (8), for all $k = 0, ..., 2$ we obtain

$$|\eta^{(k)}(p_1) - \eta^{(k)}(p_2)|_{L^\infty(L^\infty)} \leq c\eta_r^{k+1} \|p_1 - p_2\|_{C^\alpha(W^{1,s})}$$

and the results follow. □

Define $B_r^f = B_r \cap \mathbf{X}_f$ and, for all $u_0 \in \mathcal{U}_f^0$, $B_r^f(u_0) = \{(u,p) \in B_r^f,\ u(0) = u_0\}$: obviously, $B_r^f(u_0)$ is closed in $\mathbf{X}_0$. Under suitable conditions, Λ maps the "small" balls into themselves:

Lemma 4. *Let $f \in \mathbf{F}$ and ψ satisfy (12). There exists a constant $\rho = \rho(\psi) > 0$ such that for all $r \in (0, \rho]$*

(i) there exists $M_r = M(r, \psi) > 0$ such that if $\|\mathcal{P}f\|_{\mathbf{F}} \leq M_r$ then $B_r^f \neq \emptyset$

(ii) there exists $K_r \leq M_r$ such that if $\|\mathcal{P}f\|_{\mathbf{F}} \leq K_r$, if $u_0 \in \mathcal{U}_f^0$ satisfies $\|(\mathcal{P}f, u_0)\|_{\mathbf{Y}} \leq K_r$ and $B_r^f(u_0) \neq \emptyset$, then $\Lambda(B_r^f(u_0)) \subseteq B_r^f(u_0)$.

Proof. (i) For all r, take $M_r \leq K$ (K as in Lemma 1) and let (v_0, q_0) denote the "stationary solution" as in the proof of Lemma 1: then

$$\|v_0\|_{3,s} + \|q_0 - \psi(0)\|_{2,s} \leq \Gamma(M_r)$$

with $\lim_{x \to 0} \Gamma(x) = 0$. Take $(v, q) = (v_0, \psi - \psi(0) + q_0)$; if for a given $r > 0$ we choose M_r small enough (say $M_r \leq N_r$) we have $(v, q) \in B_r^f$: indeed,

$$\|(v, q - \psi)\|_{\mathbf{X}} \leq C_0(\|v_0\|_{3,s} + \|q_0 - \psi(0)\|_{2,s}) \leq C_0 \Gamma(M_r)$$

and it suffices to take N_r such that $\Gamma(N_r) = \frac{r}{C_0}$. This proves that for all $r > 0$, and all $f \in \mathbf{Y}$ satisfying $\|\mathcal{P}f\|_{\mathbf{F}} \leq M_r \leq N_r$ we have $B_r^f \neq \emptyset$.

(ii) Let $r > 0$ and take $K_r \in (0, M_r)$. Take $f \in \mathbf{Y}$ such that $\|\mathcal{P}f\|_{\mathbf{Y}} + \Gamma(\|\mathcal{P}f\|_{\mathbf{Y}}) \leq K_r$; then, by reasoning as in (i), the assumptions make sense: in particular, there exists u_0 such that $B_r^f(u_0) \neq \emptyset$. For all $(u', p') \in B_r^f(u_0)$, by (6) and (9), we obtain

$$\begin{aligned}\|R(u', p')\|_{C^\alpha(\mathbf{L}^s)} &\leq c\|[\eta(p') - \eta(\psi)](\nabla u' + \nabla^T u')\|_{C^\alpha(\mathbf{W}^{1,s})} + \|(u' \cdot \nabla)u'\|_{C^\alpha(\mathbf{L}^s)} \\ &\leq c\|\eta(p') - \eta(\psi)\|_{C^\alpha(W^{1,s})}\|u'\|_{C^\alpha(\mathbf{W}^{2,s})} + c\|u'\|^2_{C^\alpha(\mathbf{W}^{2,s})} \\ &\leq c\eta_r\|p' - \psi\|_{C^\alpha(W^{1,s})}\|u'\|_{C^\alpha(\mathbf{W}^{2,s})} + c\|u'\|^2_{C^\alpha(\mathbf{W}^{2,s})} \leq C_1 r^2\end{aligned}$$

the third inequality being a direct consequence of Lemma 3; hence, by Proposition 2 and (14)

$$\|(u, p - \psi)\|_{\mathbf{X}} \leq C_\psi(\|(\mathcal{P}f, u_0)\|_{\mathbf{Y}} + C_1 r^2)\ .$$

Let $\rho := \frac{1}{2C_\psi C_1}$ (so that $C_\psi C_1 \rho^2 = \frac{\rho}{2}$) and for all $r \in (0, \rho]$ define $K_r := \min\{\overline{U}, M_r, \frac{r}{2C_\psi}\}$ where $\overline{U}$ is as in (13); then, if $(u', p') \in B_r^f$, we have

$$\|\Lambda(u', p') - (0, \psi)\|_{\mathbf{X}} = \|(u, p - \psi)\|_{\mathbf{X}} \leq r\ .$$

□

To prove that Λ is a contractive map, we prove that the operator R is Lipschitz-continuous in B_ρ^f:

Lemma 5. *Let ρ be as in Lemma 4 and let $r \in (0,\rho]$; there exists a constant $C_r > 0$ such that if $(u_i', p_i') \in B_r^f$ (i=1,2) then*

$$\|R(u_1',p_1') - R(u_2',p_2')\|_{C^\alpha(\mathbf{L}^s)} \le C_r \|(u_1'-u_2', p_1'-p_2')\|_{\mathbf{X}} \ .$$

Moreover, the map $r \mapsto C_r$ is monotone non-decreasing and $\lim_{r\to 0} C_r = 0$.

Proof. Take $(u_i', p_i') \in B_r^f$ and set $R_i := R(u_i', p_i')$ $(i = 1,2)$; we have

$$\begin{gathered} R_1 - R_2 = \nabla \cdot \left[(\eta(p_1') - \eta(p_2'))(\nabla u_1' + \nabla^T u_1')\right] \\ +\nabla \cdot \left[(\eta(p_2') - \eta(\psi))(\nabla(u_1'-u_2') + \nabla^T(u_1'-u_2'))\right] \\ +(u_1'\cdot\nabla)(u_2'-u_1') + \left[(u_2'-u_1')\cdot\nabla\right] u_2' \end{gathered}$$

and therefore, by (9) and Lemma 3 we obtain

$$\begin{aligned} \|R_1 - R_2\|_{C^\alpha(\mathbf{L}^s)} \le c\eta_r \Big(& \|p_1'-p_2'\|_{C^\alpha(W^{1,s})}\|u_1'\|_{C^\alpha(\mathbf{W}^{2,s})} + \\ & \|p_1'-\psi\|_{C^\alpha(W^{1,s})}\|u_1'-u_2'\|_{C^\alpha(\mathbf{W}^{2,s})}\Big) \\ & + c(\|u_1'\|_{C^\alpha(\mathbf{W}^{2,s})} + \|u_2'\|_{C^\alpha(\mathbf{W}^{2,s})})\|u_1'-u_2'\|_{C^\alpha(\mathbf{W}^{2,s})} \\ \le C_r (& \|p_1'-p_2'\|_{C^\alpha(W^{1,s})} + \|u_1'-u_2'\|_{C^\alpha(\mathbf{W}^{2,s})}) \end{aligned}$$

with $r \mapsto C_r$ being as in the statement. □

Making suitable assumptions on f, u_0 and r we prove that $\Lambda : B_r^f(u_0) \to B_r^f(u_0)$ is a contractive map:

Lemma 6. *Let $f \in \mathbf{F}$ and ψ satisfy (12); there exists $\bar r = \bar r(\psi) > 0$ and a constant $\delta < 1$ such that if $u_0 \in \mathcal{U}_f^0$ satisfies $\|(\mathcal{P}f, u_0)\|_{\mathbf{Y}} \le K_{\bar r}$ ($K_{\bar r}$ as in Lemma 4) then for all $(u_i', p_i') \in B_{\bar r}^f(u_0)$ (i=1,2) we have*

$$\|\Lambda(u_2',p_2') - \Lambda(u_1',p_1')\|_{\mathbf{X}} \le \delta \|(u_2',p_2') - (u_1',p_1')\|_{\mathbf{X}} \ .$$

Proof. Take $\bar r \le \rho$: by Lemma 4 the set $B_{\bar r}^f(u_0)$ is nonempty. Let $\Lambda(u_i',p_i') = (u_i, p_i)$ $(i = 1,2)$, then by (14), $L(u_i,p_i) = \Big(\mathcal{P}f + R(u_i',p_i'); u_i'(0)\Big)$ and therefore

$$L(u_2-u_1, p_2-p_1) = \Big(R(u_2',p_2') - R(u_1',p_1'); u_2'(0) - u_1'(0)\Big) \ ;$$

since $u_1'(0) = u_2'(0) = u_0$, by Proposition 2 and Lemma 5 we get

$$\|\Lambda(u_2',p_2') - \Lambda(u_1',p_1')\|_{\mathbf{X}} = \|(u_2-u_1, p_2-p_1)\|_{\mathbf{X}} \le C_\psi C_r \|(u_2'-u_1', p_2'-p_1')\|_{\mathbf{X}} \ .$$

By Lemma 5 we can choose $\bar{r} = \bar{r}(\psi)$ so that $\delta := C_\psi C_{\bar{r}} < 1$. □

Proof of Theorem 1. Let $\bar{r}$ be as in Lemma 6; if $u_0 \in \mathcal{U}_f^0$ and $\|(\mathcal{P}f, u_0)\|_{\mathbf{Y}} \leq K_{\bar{r}}$, then Lemmas 1 and 4 imply that $B_{\bar{r}}^f(u_0)$ is closed and nonempty; Lemma 6 implies that Λ has a unique fixed point $(u,p) \in B_{\bar{r}}^f(u_0)$. By the definition of Λ given in Lemma 2, (u,p) satisfies (7) and solves (1). □

5. Concluding remarks

Let us first notice that, as for the classical equations (i.e. for $\eta(p) \equiv \eta$), in presence of conservative forces and zero initial velocity one can prove that the solution of (1) describes the static of the fluid: in this case the existence of a suitable p_0 satisfying the equation at time $t = 0$ and the corresponding compatibility condition is straightforward. This result can be obtained as a trivial consequence of Theorem 1 or with a direct proof as follows:

Proposition 3. *Assume (3) (4) (6); take $T > 0$, $u_0 \equiv 0$ and $f \in C^\alpha(0,T;\mathbf{G}_s^\perp)$. Then (1) admits a unique solution $(u,p) \in \mathbf{X}$ given by*

$$u \equiv 0 \qquad p(x,t) = \psi(x,t)$$

where ψ is the zero mean value potential of f with respect to x (i.e. $\nabla\psi = f$ and $\int_\Omega \psi = 0$).

Proof. Let $(u,p) \in \mathbf{X}$ be a solution of (1): multiply the first of (1) by $u(t)$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|_2^2 + \int_\Omega \eta(p(t))(\nabla u(t) + \nabla^T u(t)) : \nabla u(t) = 0 \qquad \text{for all } t \in [0,T] \ ;$$

by (3) we get $\frac{d}{dt}|u(t)|_2^2 \leq 0$ which, together with $u_0 \equiv 0$ yields $u(x,t) \equiv 0$ and the result follows. □

• In the proof of Theorem 1 a major role is played by Proposition 2: different regularity results can be obtained by applying the same arguments, provided one deals with a pair of optimal regularity (see [12]) for (11). □

• In Lemma 4, to prove that $B_r^f \neq \emptyset$, we chose as initial velocity u_0 the solution of the stationary problem: the corresponding solution $(u,p) \in \mathbf{X}_0$ of (1) then satisfies $\partial_t u|_{t=0} = 0$. □

• Define the operator

$$\Phi(u,p) := \partial_t u - \nabla \cdot [\eta(p)(\nabla u + \nabla^T u)] + \nabla p + (u \cdot \nabla)u \ :$$

by standard methods one can prove that $\Phi \in C^1(\mathbf{X},\mathbf{Y})$; note that $\Phi'(0,0)$ is the linear operator corresponding to the classical Stokes problem. Moreover, by Proposition 2 we infer that for all $p \in C^\alpha(0,T;\overline{W}^{1,s})$ the linear operator $\Phi'(0,p)$ is injective and

$$\gamma_p := \inf_{\|(v,q)\|_{\mathbf{x}}=1} \|\Phi'(0,p)[v,q]\|_{\mathbf{Y}} > 0 \ .$$

□

• As we already pointed out in the introduction, our result is somehow unusual as we get a solution of (1) for $T < +\infty$ and for small data $(\mathcal{P}f, u_0) \in \mathbf{Y}$; one should expect either local existence for arbitrary data or global existence for small data. In particular, for $T = +\infty$ our result yields a solution only if $(\mathcal{P}f, u_0) = (0,0)$: in this case we obtain the trivial solution. On the other hand, we can say that the equation in (1) is not really parabolic since the stationary equation may not be elliptic: the restrictions on the data and on T are needed to ensure the well-posedness of (1); in some sense, this is the price we must pay to weaken Renardy's assumptions on η. Moreover, it seems difficult to find analogies with the classical Navier-Stokes equations: we think that (1) are not only more complicated but also of completely different nature. Indeed, when we assign the initial data u_0, the pressure p is introduced constrained by the compatibility condition which is a nonlinear problem not necessarily well-posed, instead of the very simple classical linear elliptic Neumann problem (corresponding to the Hodge projection): this means that p and u are not so "independent" as for the classical equations. The classical equations are quasilinear: in this case, as u and p depend on each other, we feel that the equations are more likely to be fully nonlinear. These are the reasons of our somehow unusual method: the possible loss of ellipticity of the stationary problem and the more strong interference of p and u yield some theoretical and formal complications with respect to the standard parabolic theory and to the classical Navier-Stokes equations.

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Filippo Gazzola
Dipartimento di Scienze T.A.
via Cavour 84
15100 Alessandria (Italy)
(e-mail: Gazzola@venere.unial.it

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