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ON THE MOMENTS OF SOLUTIONS TO LINEAR PARABOLIC EQUATIONS INVOLVING THE BIHARMONIC OPERATOR

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Abstract. We consider the solutions to Cauchy problems for the parabolic equation $u_\tau + \Delta^2 u = 0$ in $\mathbb{R}_+ \times \mathbb{R}^n$, with fast decay initial data. We study the behavior of their moments. This enables us to give a more precise description of the sign-changing behavior of solutions corresponding to positive initial data.

1. **Introduction.** The Cauchy problem

$$\begin{cases} u_\tau + \Delta^2 u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n\,, \\ u(0,y) = u_0(y) & \text{in } \mathbb{R}^n\,, \end{cases} \tag{1}$$

where $u_0 \in L^1(\mathbb{R}^n)$, and the behavior of its solutions $u = u(\tau, y)$, have been recently studied under several aspects. For general linear problems we refer to [2, 3, 4, 17, 20], for the semilinear equations $u_\tau + \Delta^2 u = |u|^{p-1}u$ and $u_\tau + \Delta^2 u = |u|^p$ we refer respectively to [13, 15, 18, 19] and [5, 9, 10, 11, 16], whereas for nonlinear problems with irregular initial data u_0 we refer to [6, 7, 8]. These papers show a quite strong interest about (1) developed in recent years.

It is well-known that the kernels f_n of the biharmonic heat operator change sign, see Section 4.1 where we recall their basic properties. Therefore, one expects the solution u to (1) to display at least one sign change even if the initial datum is nonnegative, namely $u_0 \geq 0$ in $\mathbb{R}^n$. This was explicitly shown in [13, 20] where a property named *eventual local positivity* was highlighted. Roughly speaking, for suitable initial data $u_0 \geq 0$ the solution u becomes positive on compact subsets of $\mathbb{R}^n$ for sufficiently large time τ although it is strictly negative in some other points. In other words, "negativity always exists but it goes to infinity in space as time goes to infinity".

The purpose of the present paper is to shed some further light on the long-time behavior of solutions to (1). Most of the classical methods usually exploited for the second order heat equation do not apply. For instance, any reasonable Lyapunov functional for (1) becomes very complicated due to the presence of fourth order derivatives, too many terms appear and the study of their signs is out of reach. Also standard entropy methods fail, due to the change of sign of the kernels and of the solution to (1); the usual entropy is $\int u \log u$ and cannot be considered. The sign change of the kernels f_n also forbids to analyze the behavior of suitable scaled ratios such as u/f_n (the solution u to (1) divided by the kernel f_n in (27)) in order to

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obtain Ornstein-Uhlenbeck-type equations. These difficulties force us use to develop alternative strategies.

We first introduce some energy-type functionals and describe their time evolution. Theorem 2.1 and Corollary 1 show the link between different energies and their decreasing properties with respect to τ. This enables us to deduce some striking properties of the positive and negative parts of the solutions to (1), see Remark 1.

Next, we study the behavior of the moments of the solutions to (1). To this end, we transform (1) into a Fokker-Planck-type equation, see (3). The first step then consists in determining the sign of the moments, with respect to general monomials, of the unique stationary solution, see Theorem 2.2; it turns out that these signs follow a quite elegant pattern. This result enables us to determine the behavior of the moments of general solutions to the Fokker-Planck equation, see Theorem 2.3; in particular, since these moments converge to the moments studied in Theorem 2.2, we may asymptotically determine their sign. Then we repeat the same study for the moments with respect to powers of $|x|$, that is, $|x|^b$ for all $b>-n$. The pattern for the stationary solution is even more elegant than the one for general monomials, see Theorem 2.4. We use this result in Theorem 2.5 in order to determine very precise properties of the corresponding moments of any solution to (1). Finally, with these results at hand we may find almost optimal thresholds between initial data u_0 for which the solution to (1) is always positive and the data for which it changes sign, see Corollaries 3 and 4.

This paper is organized as follows. In Section 2.1 we introduce and study some decreasing energies for solutions to (1). In Section 2.2 we study in full details the moments of solutions of the corresponding Fokker-Planck equation. Section 3 is devoted to the proofs of the main results. Finally, in the Appendix we recall the basic properties of the biharmonic heat kernels and of the spectrum of the biharmonic Fokker-Planck operator, and we conclude by suggesting an open problem.

2. **Asymptotic behavior of the solution.**

We first transform (1) into a Fokker-Planck-type equation. Let

$$R(\tau):=\sqrt[4]{4\,\tau+1}$$

so that $R(\tau)^3\,R'(\tau)\equiv 1$. Also put

$$u(\tau,y):=R(\tau)^{-n}\,v\left(\log R(\tau),\,\frac{y}{R(\tau)}\right). \tag{2}$$

Then take $t=\log R(\tau)$ and $x=y/R(\tau)$. Some lengthy but straightforward computations show that $v=v(t,x)$ solves

$$\left\{\begin{array}{ll} v_t+\mathcal{L}\,v=0 & \text{in } \mathbb{R}_+\times\mathbb{R}^n\,,\\ v(0,x)=u_0(x) & \text{in } \mathbb{R}^n\,, \end{array}\right. \tag{3}$$

where

$$\mathcal{L}\,v:=\Delta^2 v-\nabla\cdot(x\,v)\ . \tag{4}$$

In this paper, we will also consider the space $\mathcal{S}$ of smooth fast decaying functions

$$\mathcal{S}:=\{w\in C^\infty(\mathbb{R}^n)\,:\,|x|^a\,D^\alpha w(x)\to 0 \text{ as } |x|\to\infty \text{ for all } a\geq 0\ ,\ \alpha\in\mathbb{N}^n\}\ .$$

In order to describe the behavior of the solutions to (3), one needs first to characterize possible stationary solutions. From, for instance, [11, (1.7) and (1.8)], or [12], we recall

Proposition 1. *Up to a multiplication by a constant, there exists a unique nontrivial stationary solution to (3) which belongs to $\mathcal{S}$. This solution is radially symmetric and, if we further assume that $\int_{\mathbb{R}^n} v_\infty(x)\,dx = 1$, explicitly given by*

$$v_\infty(x) = 2^{(n+2)/4}\,\alpha_n\,|x|^{1-n/2}\int_0^\infty e^{-s^4}\,s^{n/2}\,J_{(n-2)/2}(\sqrt{2}\,|x|\,s)\,ds\ , \tag{5}$$

and satisfies

$$|v_\infty(x)| \le K\,e^{-\mu\,|x|^{4/3}} \qquad \textit{for all } x\in\mathbb{R}^n\,, \tag{6}$$

for some constants K, $\mu > 0$.

For the explicit value of μ in (6) see [22] and, for the case $n = 1$, see [14, (5.14)]. Notice that v_∞ can be expressed in terms of the kernel f_n: $v_\infty(x) = 2^{n/2}\,\alpha_n\,f_n(\sqrt{2}\,|x|)$, for any $x \in \mathbb{R}^n$, and (6) is therefore equivalent to

$$|f_n(x)| \le \frac{K}{2^{\frac{n}{2}}\,\alpha_n}\,e^{-\mu\,(\frac{1}{2}\,|x|^2)^{2/3}} \qquad \text{for all } x\in\mathbb{R}^n\,. \tag{7}$$

Although the functions v_∞ and f_n are strictly related we prefer to maintain the double notation since, in our setting, they play quite different roles; the former is a stationary solution to (3), the latter is the biharmonic heat kernel.

2.1. **Decreasing energies.** Assume that $u_0 \in L^1(\mathbb{R}^n)$ and let v be the solution to (3). To this solution we associate five different energy-type functionals

$$\mathcal{E}_0(t) := \int_{\mathbb{R}^n}|v(t,x)|^2dx\,,\quad \mathcal{E}_1(t) := \int_{\mathbb{R}^n}|\nabla v(t,x)|^2dx\,,\quad \mathcal{E}_2(t) := \int_{\mathbb{R}^n}|\Delta v(t,x)|^2dx\,,$$

$$\mathcal{E}_3(t) := \int_{\mathbb{R}^n}|\nabla\Delta v(t,x)|^2dx\,,\quad \mathcal{E}_4(t) := \int_{\mathbb{R}^n}|\Delta^2 v(t,x)|^2dx\,,$$

and we prove

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R}^n)$ and let v be the solution to (3). Then, for any $j \in \{0, 1, 2\}$, the energies of v satisfy the following ODE's:*

$$\mathcal{E}_j'(t) = -\,2\,\mathcal{E}_{j+2}(t) + (n+2j)\,\mathcal{E}_j(t)\ .$$

Undoing the change of variables (2) yields

Corollary 1. *Let $u_0 \in L^1(\mathbb{R}^n)$ and let u be the solution to (1). Then, for all $\tau > 0$ we have*

$$\int_{\mathbb{R}^n} u(\tau,y)\,dy = \int_{\mathbb{R}^n} u_0(y)\,dy\ , \tag{8}$$

$$\frac{d}{d\tau}\int_{\mathbb{R}^n} u(\tau,y)^2\,dy = -\,2\int_{\mathbb{R}^n}|\Delta u(\tau,y)|^2\,dy\ , \tag{9}$$

$$\frac{d}{d\tau}\int_{\mathbb{R}^n}|\nabla u(\tau,y)|^2\,dy = -\,2\int_{\mathbb{R}^n}|\nabla\Delta u(\tau,y)|^2\,dy\ ,$$

$$\frac{d}{d\tau}\int_{\mathbb{R}^n}|\Delta u(\tau,y)|^2\,dy = -\,2\int_{\mathbb{R}^n}|\Delta^2 u(\tau,y)|^2\,dy\ .$$

The proof of (8) follows by integrating (1) and by applying the divergence theorem. The other statements of Corollary 1 may be obtained either by direct computations using (1) and integrations by parts or as a straightforward consequence of Theorem 2.1, whence we omit their proof.

Remark 1. Denote by $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$ the positive and negative parts of a function u, so that $u = u^+ - u^-$. As already mentioned, the solution u to (1) may change sign, see [13, 20]. This happens, for instance, if $u_0 \in C^0_c(\mathbb{R}^n)$, $0 \not\equiv u_0 \geq 0$ in $\mathbb{R}^n$, see [20, Theorem 1]. In such case, (8) states that the map

$$\tau \mapsto \int_{\mathbb{R}^n} u(\tau, y)\, dy \qquad (\tau \geq 0)$$

is constant and equals a strictly positive number even if some pointwise negativity appears in $u(\tau, y)$. Hence,

$$\int_{\mathbb{R}^n} u^-(\tau, y)\, dy > \int_{\mathbb{R}^n} u^-(0, y)\, dy = 0 \quad \text{for all } \tau > 0\,,$$

$$\int_{\mathbb{R}^n} u^+(\tau, y)\, dy > \int_{\mathbb{R}^n} u^+(0, y)\, dy = \int_{\mathbb{R}^n} u_0(y)\, dy \quad \text{for all } \tau > 0\,;$$

here we use redundant notations ($u^+(0, y) = u_0^+(y) = u_0(y)$ and $u^-(0, y) = u_0^-(y) = 0$) in order to emphasize the *strict inequalities* between the mass of the positive (respectively, negative) part of the solution $u = u(\tau, y)$ and the the mass of the positive (respectively, negative) part of initial datum u_0.

On the other hand, (9) states that

$$\tau \mapsto \int_{\mathbb{R}^n} u(\tau, y)^2\, dy \qquad (\tau \geq 0)$$

decreases and, in particular, that

$$\int_{\mathbb{R}^n} u^+(\tau, y)^2\, dy < \int_{\mathbb{R}^n} u_0(y)^2\, dy = \int_{\mathbb{R}^n} u_0^+(y)^2\, dy \qquad (\tau > 0)\ .$$

Summarizing, the L^2-norm of the positive part of the solution u is **smaller** than the L^2-norm of the positive part of the initial datum u_0, whereas the L^1-norm of the positive part of the solution u is **larger** than the L^1-norm of the positive part of the initial datum u_0. □

2.2. **Behavior of the moments.** Here we are interested in the moments of the function v_∞ defined in (5) and to relate them to the moments of solutions v to (3). The prototype monomial in $\mathbb{R}^n$ is given by

$$P_m(x) = \prod_{i=1}^{n} x_i^{m_i} \qquad \text{for } m = (m_1, ..., m_n) \in \mathbb{N}^n$$

and its degree is $|m| = \sum_i m_i$. Then we define the P_m-moment of v_∞ by

$$\mathcal{M}_{P_m} := \int_{\mathbb{R}^n} P_m(x)\, v_\infty(x)\, dx \tag{10}$$

and we prove

Theorem 2.2. *For any $m = (m_1, ..., m_n) \in \mathbb{N}^n$ the following facts hold:*

1. $\mathcal{M}_{\Delta^2 P_m} = -|m|\, \mathcal{M}_{P_m}$,
2. *if $|m| \notin 4\mathbb{N}$ or if at least one of the m_i's is odd, then $\mathcal{M}_{P_m} = 0$,*
3. *if $|m| \in 8\mathbb{N}$ and all the m_i's are even, then $\mathcal{M}_{P_m} > 0$,*
4. *if $|m| \in 8\mathbb{N} + 4$ and all the m_i's are even, then $\mathcal{M}_{P_m} < 0$.*

Let $u_0 \in \mathcal{S}$ and consider the solution v to (3). Let P_m be as above and consider the (time-dependent) map

$$M_{P_m,u_0}(t) := \int_{\mathbb{R}^n} P_m(x)\, v(t,x)\, dx = \int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} x_i^{m_i}\right) v(t,x)\, dx\ .$$

The following result holds.

Theorem 2.3. *Assume that $u_0 \in \mathcal{S}$ is normalized in such a way that*

$$\int_{\mathbb{R}^n} u_0(x)\, dx = \int_{\mathbb{R}^n} v_\infty(x)\, dx = 1 \tag{11}$$

and let v denote the solution to (3). For any $t \geq 0$, the following facts hold:

1. $M'_{P_m,u_0}(t) = -\, M_{\Delta^2 P_m,u_0}(t) - |m|\, M_{P_m,u_0}(t)$ *for all* $m \in \mathbb{N}^n$,
2. $M_{P_m,u_0}(t) = e^{-|m|\,t} \int_{\mathbb{R}^n} P_m(x)\, u_0(x)\, dx$ *for all* $|m| \leq 3$,
3. $\lim_{t\to\infty} M_{P_m,u_0}(t) = \mathcal{M}_{P_m}$ *for all* $m \in \mathbb{N}^n$.

By combining Theorems 2.2 and 2.3, we infer

Corollary 2. *Assume that $u_0 \in \mathcal{S}$ is normalized in such a way that (11) holds and let v denote the solution to (3). Then*

$$\lim_{t\to\infty} M_{P_m,u_0}(t) \begin{cases} = 0 & \textit{if } |m| \notin 4\mathbb{N} \textit{ or if at least one of the } m_i\textit{'s is odd,} \\ > 0 & \textit{if } |m| \in 8\mathbb{N} \textit{ and all the } m_i\textit{'s are even,} \\ < 0 & \textit{if } |m| \in 8\mathbb{N}+4 \textit{ and all the } m_i\textit{'s are even.} \end{cases}$$

We have so far considered moments having polynomials of x as weights. In fact, also different kinds of moments are of interest. We now consider powers of $|x|$ which are polynomials only for even integer powers. For any $b > -n$ we define

$$\mathcal{M}_b := \int_{\mathbb{R}^n} |x|^b\, v_\infty(x)\, dx\ .$$

Note that for $b > -n$ the above integral is finite since $|x|^b\, v_\infty(x) \sim v_\infty(0)\, |x|^b$ as $x \to 0$ and v_∞ has exponential decay at infinity according to (6). If $P_m(x) = |x|^m$ for some $m \in 2\mathbb{N}$, then $\mathcal{M}_m$ coincides with $\mathcal{M}_{P_m}$ as defined in (10). We are again interested in the sign of these moments. By combining several arguments from [13] we prove

Theorem 2.4. *Assume that $n \geq 1$ and that $b > -n$. Then*

$$\begin{aligned} &\mathcal{M}_b > 0 \qquad \textit{for all} \qquad b \in (-n,2) \bigcup \Big(\bigcup_{k=0}^{\infty} (8k+6, 8k+10) \Big)\ , \\ &\mathcal{M}_b = 0 \qquad \textit{for all} \qquad b \in 4\mathbb{N}+2\ , \\ &\mathcal{M}_b < 0 \qquad \textit{for all} \qquad b \in \bigcup_{k=0}^{\infty} (8k+2, 8k+6)\ . \end{aligned}$$

Theorems 2.2 and 2.4 give further evidence to the sign-changing properties of the kernels f_n (recall that $v_\infty(x) = 2^{n/2}\, \alpha_n\, f_n(\sqrt{2}\,|x|))$, and they better describe how these infinitely many sign changes occur. They also show that the sign of the moments *do not* depend on n, see Figure 1; for instance, negativity of $\mathcal{M}_b$ occurs for $b \in (2,6) \cup (10,14) \cup ...$ regardless of the value of $n \geq 1$.

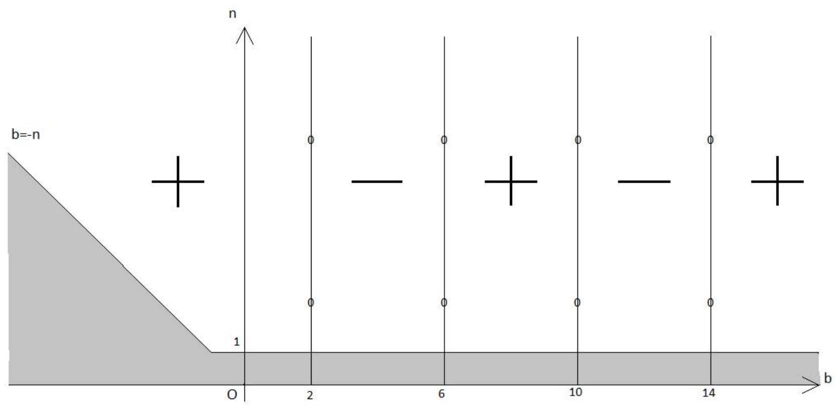

FIGURE 1. Sign of $\mathcal{M}_b$.

In the particular case where $|m| = 2k$ and $P_m(x) = |x|^{2k}$ we may give a simple characterization of the moments of a solution to (3). Consider a solution v to (3) with initial data u_0. For all $b \geq 0$ we put

$$M_{b,u_0}(t) := \int_{\mathbb{R}^n} |x|^b \, v(t,x) \, dx$$

and we prove

Theorem 2.5. *Assume that $u_0 \in \mathcal{S}$ is normalized in such a way that (11) holds and let v denote the solution to (3). Then for any $k \in \mathbb{N}$, $k \geq 2$, the above defined functions satisfy the following ODE*

$$M'_{2k,u_0}(t) + 2k\, M_{2k,u_0}(t) = -\, 2k\,(2k-2)\,(2k+n-2)\,(2k+n-4)\, M_{2k-4,u_0}(t) \; . \tag{12}$$

Moreover, for any $k \in \mathbb{N}$, we have

$$\lim_{t \to +\infty} M_{2k,u_0}(t) = \mathcal{M}_{2k} \tag{13}$$

and the following explicit representation

$$M_{2k,u_0}(t) = \sum_{j=0}^{k} a_j^k \, e^{-2jt} \, , \tag{14}$$

where $a_0^k = \mathcal{M}_{2k}$ and

$$(i) \qquad a_k^k = M_{2k,u_0}(0) + 2k\,(k-1)\,(2k+n-2)\,(2k+n-4) \sum_{j=0}^{k-2} \frac{a_j^{k-2}}{k-j} \; ,$$

$$(ii) \qquad a_{k-1}^k = 0 \quad \text{if } k \geq 1 \; ,$$

$$(iii) \qquad a_j^k = -\tfrac{2k\,(k-1)\,(2k+n-2)\,(2k+n-4)}{k-j} \, a_j^{k-2} \quad \text{if } k \geq 2 \text{ and } j = 0, ..., k-2 \; .$$

In (i) we use the convention that $\sum_{j=0}^{k-2} = 0$ if $k \leq 1$.

Formula (14) shows, for instance, that

$$M_{0,u_0}(t) \equiv \int_{\mathbb{R}^n} u_0(x) \, dx \; , \qquad M_{2,u_0}(t) = e^{-2t} \int_{\mathbb{R}^n} |x|^2 \, u_0(x) \, dx \; ,$$

$$M_{4,u_0}(t) = -\,2n\,(n+2)\int_{\mathbb{R}^n} u_0(x)\,dx + \int_{\mathbb{R}^n}\Big[|x|^4 + 2n\,(n+2)\Big]u_0(x)\,dx\; e^{-4t}\ ,$$

$$\begin{aligned} M_{6,u_0}(t) &= -\,6\,(n+4)\,(n+2)\int_{\mathbb{R}^n} |x|^2\,u_0(x)\,dx\; e^{-2t} \\ &\quad + \left(\int_{\mathbb{R}^n} |x|^6\,u_0(x)\,dx + 6\,(n+4)\,(n+2)\int_{\mathbb{R}^n} |x|^2\,u_0(x)\,dx\right) e^{-6t}\ . \end{aligned}$$

Remark 2. Even if $b \notin 2\mathbb{N}$ (so that $|x|^b$ is not a polynomial) we may still define the map M_{b,u_0} and, arguing as for (12), for all $b \in [4,\infty)$ we obtain

$$M'_{b,u_0}(t) + b\,M_{b,u_0}(t) = -\,b\,(b-2)\,(b+n-2)\,(b+n-4)\,M_{b-4,u_0}(t)\ .$$

□

Note that Theorems 2.3 and 2.5 also hold in a weaker form if $u_0 \in L^1(\mathbb{R}^n)$ and $|x|^a\,u_0 \in L^1(\mathbb{R}^n)$ for some $a \geq 4$. In this case, the statements hold true under the additional restriction that $|m| \leq a$. In particular, we have the following

Corollary 3. *Assume that $(1+|x|^4)\,u_0 \in L^1(\mathbb{R}^n)$ and that (11) holds. If v denotes the solution to (3), then*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^n} |x|^4\,v(t,x)\,dx = \mathcal{M}_4 < 0\ .$$

By combining (8) with Corollary 3 and with [13, Proposition A.6], we obtain

Corollary 4. *Assume that $u_0 > 0$ a.e. in $\mathbb{R}^n$.*
(i) If $(1+|x|^4)\,u_0 \in L^1(\mathbb{R}^n)$, then the solution u to (1) changes sign.
(ii) If $n = 1$, there exists $\beta_0 > 0$ such that if $\beta \in (0,\beta_0)$ and $u_0(x) = |x|^{-\beta}$, then the solution u to (1) is a.e. positive in $\mathbb{R}_+ \times \mathbb{R}$.

Corollary 4 can be interpreted as follows. From [20, Theorem 1] we know that solutions u to (1) with compactly supported nonnegative initial data u_0 display the eventual local positivity property, that is, $u(\tau,y)$ becomes eventually positive on any compact subset of $\mathbb{R}^n$ but it is always strictly negative somewhere in a neighborhood of $|y| = \infty$. This happens because the biharmonic heat kernels exhibit oscillations and, outside the support of u_0, they "push below zero" the initial datum. The same happens if $u_0 > 0$ but u_0 is "very close to zero", see statement (i). On the other hand, if $u_0 > 0$ and u_0 is "far away from zero" then the kernels do not have enough negative strength to push the solution below zero, see statement (ii). The trivial case $u_0 \equiv 1$ (which is a stationary solution to (1)!) well explains this situation.

3. Proofs.

3.1. Proof of Theorem 2.1.

Let f_n be the biharmonic heat kernels, see (27), and let $\alpha_n > 0$ be a normalization constant, see (28). Then the solution u to (1) is explicitly given in terms of a convolution with the initial datum

$$\begin{aligned} u(\tau,y) &= \alpha_n\,\tau^{-n/4}\int_{\mathbb{R}^n} u_0(y-z)\,f_n\left(\frac{|z|}{\tau^{1/4}}\right)dz \qquad (15) \\ &= \alpha_n\int_{\mathbb{R}^n} u_0(y-\tau^{1/4}z)\,f_n(|z|)\,dz\ , \qquad (\tau,y) \in \mathbb{R}_+ \times \mathbb{R}^n\,. \end{aligned}$$

A well-known fact is that $u(\tau,y)$ decays to 0 as $\tau \to +\infty$, see e.g. [13, Theorem 1.1]. Moreover, by differentiating under integral sign (15) and by applying

Lebesgue's dominated convergence theorem (this can be done in view of (7)), one obtains

Proposition 2. *Let $u_0 \in L^1(\mathbb{R}^n)$ and let u be given by (15). Then*

$$\lim_{|y|\to\infty} D^\alpha u(\tau, y) = 0$$

for any $\tau > 0$ and any multiindex α, where D^α denotes the corresponding differentiation with respect to the y-variables. Therefore, if v denotes the corresponding solution to (3), we have

$$\lim_{|x|\to\infty} D^\alpha v(t, x) = 0$$

for any $t > 0$ and any multiindex α, where D^α now denotes differentiation with respect to the x-variables.

Proposition 2 allows integrations by parts with no terms at infinity. If $j = 0$ we have

$$\mathcal{E}_0'(t) = 2\int_{\mathbb{R}^n} v\, v_t\, dx = -2\int_{\mathbb{R}^n} v\,\mathcal{L}\, v\, dx$$

so that two integrations by parts yield

$$\mathcal{E}_0'(t) = -\,2\int_{\mathbb{R}^n} |\Delta v|^2\, dx + 2\int_{\mathbb{R}^n} v\,\nabla\cdot(x\, v)\, dx\ .$$

By computing $\nabla\cdot(x\, v) = n\, v + x\cdot\nabla v$ and with a further integration by parts we obtain the statement for $j = 0$.

If $j = 1$ we have

$$\mathcal{E}_1'(t) = 2\int_{\mathbb{R}^n} \nabla v\,\nabla v_t\, dx = -\,2\int_{\mathbb{R}^n} \Delta v\, v_t\, dx = 2\int_{\mathbb{R}^n} \Delta v\,\mathcal{L}\, v\, dx\,.$$

Since there are no boundary terms, an integration by parts shows that

$$\int_{\mathbb{R}^n} \Delta v\,(x\cdot\nabla v)\, dx = \frac{n-2}{2}\int_{\mathbb{R}^n} |\nabla v|^2\, dx\,.$$

Hence, proceeding as above we obtain the statement for $j = 1$.

If $j = 2$ we have

$$\mathcal{E}_2'(t) = 2\int_{\mathbb{R}^n} \Delta v\,\Delta v_t\, dx = 2\int_{\mathbb{R}^n} \Delta^2 v\, v_t\, dx = -2\int_{\mathbb{R}^n} \Delta^2 v\,\mathcal{L}\, v\, dx\,.$$

In view of the Rellich-type identity due to Mitidieri [23, (2.6)] we have

$$\int_{\mathbb{R}^n} \Delta^2 v\,(x\cdot\nabla v)\, dx = \frac{4-n}{2}\int_{\mathbb{R}^n} |\Delta v|^2\, dx$$

and, once again, proceeding as above proves the statement also for $j = 2$.

3.2. **Proof of Theorem 2.2.** Recalling that v_∞ is a stationary solution to (3), several integrations by parts, which are allowed by Proposition 2, show that

$$\begin{aligned}\mathcal{M}_{\Delta^2 P_m} &= \int_{\mathbb{R}^n} (\Delta^2 P_m)\, v_\infty\, dx = \int_{\mathbb{R}^n} P_m\,\Delta^2 v_\infty\, dx \\ &= \int_{\mathbb{R}^n} P_m\,\nabla\cdot(x\, v_\infty)\, dx = -\,|m|\int_{\mathbb{R}^n} P_m\, v_\infty\, dx = -\,|m|\,\mathcal{M}_{P_m}\end{aligned}$$

which proves Item 1, while it is straightforward to check that

$$\mathcal{M}_{P_m} = 0 \qquad \text{for all } 1 \le |m| \le 3\ . \tag{16}$$

The first part of Item 2 follows by combining Item 1 and (16). The second part of Item 2 follows by recalling that v_∞ is even with respect to all the variables x_i and, therefore, if P_m is odd with respect to some x_i so is $P_m\, v_\infty$.

We prove Items 3 and 4 at the same time. We know that Item 3 is true if $|m| = 0$ (see Theorem 2.4 with $b = 0$ for further details). If $|m| = 4$ and all the m_i's are even, then either $\Delta^2 P_m = 24$ (if $P_m(x) = x_i^4$ for some i) or $\Delta^2 P_m = 8$ (if $P_m(x) = x_i^2\, x_j^2$ for some $i \neq j$). In any case, ΔP_m is a positive constant $\gamma > 0$ so that Item 1 shows $\gamma \int_{\mathbb{R}^n} v_\infty\, dx = -\,4\,\mathcal{M}_{P_m}$. This shows that $\mathcal{M}_{P_m} < 0$. By repeating the use of Item 1, we see that both Items 3 and 4 hold.

3.3. **Proof of Theorem 2.3.** Note first that, by Proposition 2, the moments of v are well defined. We have

$$M'_{P_m,u_0}(t) = \int_{\mathbb{R}^n} P_m\, v_t\, dx = -\int_{\mathbb{R}^n} P_m\, \mathcal{L}\, v\, dx\ . \tag{17}$$

Once more, Proposition 2 allows several integrations by parts which show that

$$-\int_{\mathbb{R}^n} P_m\, \Delta^2 v\, dx = -\int_{\mathbb{R}^n} v\, \Delta^2 P_m\, dx \text{ and } \int_{\mathbb{R}^n} P_m\, \nabla\cdot(x\, v)\, dx = -\,|m| \int_{\mathbb{R}^n} P_m\, v\, dx\, . \tag{18}$$

This proves Item 1.

If $|m| \leq 3$, then $\Delta^2 P_m = 0$. In view of (17) and (18) this tells us that

$$M'_{P_m,u_0}(t) = -\,|m|\, M_{P_m,u_0}(t)$$

which proves Item 2 upon integration of the ODE.

Item 3 is proved by induction as follows. We first claim that if $4k < |m| < 4(k{+}1)$ for some $k \in \mathbb{N}$ then $\lim_{t\to\infty} M_{P_m,u_0}(t) = 0$. For $k = 0$ this is a straightforward consequence of Item 2. Assume that the above statement has been proved for some $k \in \mathbb{N}$. Then we take $4(k+1) < |m| < 4(k+2)$ and we rewrite Item 1 as

$$\frac{d}{dt}\Big(e^{|m|\,t} M_{P_m,u_0}(t)\Big) = -\, e^{|m|\,t} M_{\Delta^2 P_m,u_0}(t)\ . \tag{19}$$

Since $\Delta^2 P_m$ is a polynomial of degree $|m|{-}4 \in (4k, 4(k{+}1))$ we know by assumption that $M_{\Delta^2 P_m,u_0}(t) = o(1)$ as $t \to \infty$. Hence, integrating (19) gives

$$e^{|m|\,t} M_{P_m,u_0}(t) = M_{P_m,u_0}(0) - \int_0^t e^{|m|\tau} M_{\Delta^2 P_m,u_0}(\tau)\, d\tau \tag{20}$$

and proves the claim also for $4(k+1) < |m| < 4(k+2)$.

Similarly, we prove Item 3 when $|m| = 4k$ for some $k \in \mathbb{N}$. When $k = 0$, Item 2 yields $M_{0,u_0}(t) \equiv \int_{\mathbb{R}^n} v_\infty\, dx$ so that Item 3 follows by (11). When $k = 1$, Item 3 follows by Item 1 and by taking into account that $M_{\Delta^2 P_m,u_0}(t)$ equals a constant times $\int_{\mathbb{R}^n} v_\infty\, dx$. Assume that Item 3 has been proved for some $k \in \mathbb{N}$, $k \geq 1$. By Item 1 we still know that (19) holds. Since $\Delta^2 P_m$ is a polynomial of degree $|m| - 4 = 4k$ we know by assumption that $M_{\Delta^2 P_m,u_0}(t) = \mathcal{M}_{P_m} + o(1)$ as $t \to \infty$. Hence, by (20) we infer that

$$\begin{aligned}\lim_{t\to\infty} M_{P_m,u_0}(t) &= \lim_{t\to\infty} \left(M_{P_m,u_0}(0) e^{-|m|\,t} - e^{-|m|\,t} \int_0^t e^{|m|\tau} M_{\Delta^2 P_m,u_0}(\tau)\, d\tau \right) \\ &= -\lim_{t\to\infty} \frac{M_{\Delta^2 P_m,u_0}(t)}{|m|} = -\,\frac{\mathcal{M}_{\Delta^2 P_m}}{|m|} = \mathcal{M}_{P_m}\end{aligned}$$

where we used de l'Hopital's rule, the assumption, and Item 1 in Theorem 2.2. This completes the proof of Item 3.

3.4. **Proof of Theorem 2.4.** Up to the constant ω_n (that is, surface measure of the unit ball in $\mathbb{R}^n$) the $\mathcal{M}_b$'s coincide with

$$\int_0^\infty r^{n+b-1}\, v_\infty(r)\, dr$$

if we abusively write $v_\infty(x) = v_\infty(r)$, $r = |x|$. In view of [13, Proposition 3.2] we know that

$$\mathcal{M}_b > 0 \qquad \text{for all } b \in (-n, 0]\ . \tag{21}$$

Note that $\mathcal{M}_b$ differs from $C_{n,\beta}$ in [13] by a positive factor.

For the next steps we need to emphasize the dependence of $\mathcal{M}_b$ on n and we denote it by $\mathcal{M}_b\,(n)$. Then we remark that the recurrence formula (29) in [13, Lemma 4.2] can be extended to all $b > -n$. More precisely, an integration by parts combined with (29) gives

$$\int_0^\infty \eta^{n+b+1} f_{n+2}(\eta)\, d\eta = (n+b) \int_0^\infty \eta^{n+b-1} f_n(\eta)\, d\eta \qquad \text{for all } b > -n\ .$$

In terms of $\mathcal{M}_b$ this proves that

$$\mathcal{M}_b\,(n+2) = \frac{n+b}{2}\, \mathcal{M}_b\,(n) \qquad \text{for all } b > -n \tag{22}$$

and shows that the sign of $\mathcal{M}_b\,(n)$ does not depend on n. On the other hand, several integration by parts (see (29) and (30) in the Appendix) show that

$$\int_0^\infty \eta^{n+b-1} f_n(\eta)\, d\eta = 4\,(2-b)\,(n-2+b) \int_0^\infty \eta^{n+b-3} f_{n+2}(\eta)\, d\eta \qquad \text{for all } b > 2-n\ .$$

In terms of $\mathcal{M}_b$ this reads

$$\mathcal{M}_b\,(n) = 2\,(2-b)\,(n+b-2)\,\mathcal{M}_{b-4}(n+2) \qquad \text{for all } b > 2-n$$

which, combined with (22), gives

$$\mathcal{M}_b\,(n) = (2-b)\,(n+b-2)\,(n+b-4)\,\mathcal{M}_{b-4}(n) \qquad \text{for all } b > 4-n\ . \tag{23}$$

Theorem 2.4 follows by using repeatedly (21), (22), (23), and by taking into account that the map $(b,n) \mapsto \mathcal{M}_b\,(n)$ is continuous in $\{(b,n) \in \mathbb{R}^2\ :\ n \geq 1,\ b > -n\}$.

3.5. **Proof of Theorem 2.5.** First, notice that

$$M_{0,u_0}(0) = \int_{\mathbb{R}^n} u_0\, dx$$

and

$$M'_{0,u_0}(t) = \int_{\mathbb{R}^n} v_t\, dx = -\int_{\mathbb{R}^n} \mathcal{L}\, v\, dx = 0\ .$$

This proves that

$$M_{0,u_0}(t) \equiv \int_{\mathbb{R}^n} u_0(x)\, dx\ . \tag{24}$$

Next, integrations by parts yield

$$M'_{2,u_0}(t) = \int_{\mathbb{R}^n} |x|^2\, v_t\, dx = -\int_{\mathbb{R}^n} |x|^2\, \mathcal{L}\, v\, dx = n \int_{\mathbb{R}^n} |x|^2\, v\, dx + \int_{\mathbb{R}^n} |x|^2\, (x \cdot \nabla v)\, dx$$

so that a further integration by parts gives

$$M'_{2,u_0}(t) = n \int_{\mathbb{R}^n} |x|^2\, v\, dx - (n+2) \int_{\mathbb{R}^n} |x|^2\, v\, dx = -\,2\, M_{2,u_0}(t)\ .$$

By integrating this first order linear ODE we obtain

$$M_{2,u_0}(t) = e^{-2t} \int_{\mathbb{R}^n} |x|^2\, u_0\, dx\ . \tag{25}$$

Moreover, we recall that for any smooth radially symmetric function $w = w(r)$, with $r = |x|$, we have

$$\Delta^2 w(r) = w''''(r) + \frac{2(n-1)}{r}\, w'''(r) + \frac{(n-1)(n-3)}{r^2}\, w''(r) - \frac{(n-1)(n-3)}{r^3}\, w'(r)\,.$$

Therefore, if $b \geq 4$, we get

$$\Delta^2(|x|^b) = b\,(b-2)\,(b+n-2)\,(b+n-4)\,|x|^{b-4} =: \gamma(b,n)\,|x|^{b-4}$$

so that, with an integration by parts,

$$\begin{aligned} M'_{b,u_0}(t) &= \int_{\mathbb{R}^n} |x|^b\, v_t\, dx = \int_{\mathbb{R}^n} |x|^b\, [-\Delta^2 v + \nabla\cdot(x\, v)]\, dx \\ &= -\,\gamma(b,n) \int_{\mathbb{R}^n} v\, |x|^{b-4}\, dx - b \int_{\mathbb{R}^n} |x|^b\, v\, dx\ . \end{aligned}$$

Then, by recalling the definition of M_{b,u_0} and M_{b-4,u_0}, we obtain

$$M'_{b,u_0}(t) + b\, M_{b,u_0}(t) = -\,b\,(b-2)\,(b+n-2)\,(b+n-4)\, M_{b-4,u_0}(t) \quad \text{for all } b \in [4,\infty)$$

which proves (12). The latter equation may be rewritten as

$$\frac{d}{dt}\left(e^{bt}\, M_{b,u_0}(t)\right) = -\,b\,(b-2)\,(b+n-2)\,(b+n-4)\, M_{b-4,u_0}(t)\, e^{bt} \quad \text{for all } b \in [4,\infty)\,. \tag{26}$$

We now assume that $b = 2k$ for some $k \in \mathbb{N}$ and we prove the representation formula (14). We proceed by induction on k.

For $k = 0$ only statement (i) needs to be proved and this follows from the explicit (constant) form of M_{0,u_0} given in (24). For $k = 1$, only statements (i) and (ii) have to be proved and these follow by (24)-(25).

Assume now that (14) has been proved for M_{2k,u_0}, for some $k \geq 0$ with the constants a_j^k as in the statement. Then, since $2k+4 \geq 4$, we may use (26) with $b = 2k+4$ to obtain

$$\frac{d}{dt}\left(e^{(2k+4)t} M_{2k+4,u_0}(t)\right) = -\,(2k+4)\,(2k+2)\,(2k+n+2)\,(2k+n) \sum_{j=0}^{k} a_j^k\, e^{(2k+4-2j)t}\,.$$

By integrating over $[0,t]$ we get

$$\begin{aligned} &e^{(2k+4)t} M_{2k+4,u_0}(t) = \\ &M_{2k+4,u_0}(0) - (2k+4)\,(2k+2)\,(2k+n+2)\,(2k+n) \sum_{j=0}^{k} \frac{a_j^k}{2k+4-2j}\, (e^{(2k+4-2j)t} - 1)\,. \end{aligned}$$

Therefore,

$$M_{2k+4,u_0}(t) =$$
$$\left(M_{2k+4,u_0}(0) + (2k+4)\,(k+1)\,(2k+n+2)\,(2k+n)\sum_{j=0}^{k}\frac{a_j^k}{k+2-j}\right)e^{-(2k+4)t}$$
$$-\,(2k+4)\,(k+1)\,(2k+n+2)\,(2k+n)\sum_{j=0}^{k}\frac{a_j^k}{k+2-j}\,e^{-2jt}\ .$$

This establishes (14) for M_{2k+4,u_0}. The proof covers all even integers $b = 2k$.

Finally, since (11) holds, by (14) we know that

$$\lim_{t\to+\infty} M_{b,u_0}(t) = a_0^k\ .$$

Moreover, by iterated applications of (iii) for $j = 0$ we see that $a_0^k = 0$ if k is even, whereas a_0^k equals a constant times $\int_{\mathbb{R}^n} u_0\,dx$ if k is odd. One then finds that $a_0^k = \mathcal{M}_{2k}$. Since also v_∞ is a (stationary) solution to (3), the same also holds for v_∞. This proves (13).

4. **Appendix.**

4.1. **Basic properties of the biharmonic heat kernels.**

The kernel of the linear operator $u \mapsto u_\tau + \Delta^2 u$ in $\mathbb{R}_+ \times \mathbb{R}^n$ is given by

$$\begin{gathered} g(\tau,x) = \alpha_n\,\frac{f_n(\eta)}{\tau^{n/4}}\ , \qquad \eta = \frac{|x|}{\tau^{1/4}}\ , \\ f_n(\eta) = \eta^{1-n}\int_0^\infty e^{-s^4}(\eta\,s)^{n/2}\,J_{(n-2)/2}(\eta\,s)\,ds\ , \end{gathered} \tag{27}$$

where J_ν denotes the ν-th Bessel function and $\alpha_n > 0$ is a normalization constant. More precisely, if ω_n denotes the surface measure of the n-dimensional unit ball (with the convention $\omega_1 = 2$), then

$$\alpha_n^{-1} = \omega_n\int_0^\infty r^{n-1}f_n(r)\,dr = \int_{\mathbb{R}^n} f_n(|x|)\,dx\ . \tag{28}$$

The biharmonic heat kernels are defined in (27) by means of Bessel functions. We refer to [1] for the definition and main properties of the Bessel functions and to [13] for a power series representation of f_n defined in (27). Here we just point out that $(f_n)_{n\ge1}$ obeys the following recurrence formula:

$$f_n'(\eta) = -\,\eta\,f_{n+2}(\eta) \qquad \text{for all } n \ge 1\ . \tag{29}$$

This follows by direct computation:

$$\begin{aligned} \frac{d}{d\eta}\,f_n(\eta) &= \frac{d}{d\eta}\left[\int_0^\infty e^{-s^4}\,s^{n-1}\,(\eta\,s)^{(2-n)/2}J_{(n-2)/2}(\eta\,s)\,ds\right] \\ \{\text{by [1, (4.6.2)]}\} &= -\int_0^\infty e^{-s^4}\,s^n\,(\eta\,s)^{(2-n)/2}J_{n/2}(\eta\,s)\,ds = -\,\eta\,f_{n+2}(\eta)\ . \end{aligned}$$

From Proposition 1, we know that f_n has an exponential decay at infinity. In [13] it was also proved that the functions f_n satisfy the following ODE:

$$f_n'''(\eta) + \frac{n-1}{\eta}\,f_n''(\eta) - \frac{n-1}{\eta^2}\,f_n'(\eta) - \frac{\eta}{4}\,f_n(\eta) = 0. \tag{30}$$

In particular, (30) enables one to show that as $\eta \to \infty$, the function $\eta \mapsto f_n(\eta)$ changes sign infinitely many times. We refer again to [13] for the details. Finally, we also refer to [21] for possible extensions to higher order polyharmonic heat equations.

4.2. **The spectrum of the biharmonic Fokker-Planck operator.** We recall here some basic properties of the eigenfunctions of the operator defined by (4). Let $\mu > 0$ be the number in (6) and, for any $a \in [0, 2\mu)$, consider the function

$$\rho_a(x) = e^{a\,|x|^{4/3}} , \qquad x \in \mathbb{R}^n \tag{31}$$

so that, in particular, $\rho_a \equiv 1$ if $a = 0$. For any such function ρ_a consider the space $L^2_a(\mathbb{R}^n)$, the weighted space endowed with the scalar product and norm

$$(u, v)_{L^2_a(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho_a(x)\, u(x)\, v(x)\, dx \ , \qquad \|u\|^2_{L^2_a(\mathbb{R}^n)} = (u, u)_{L^2_a(\mathbb{R}^n)} \ . \tag{32}$$

Clearly, if $a = 0$ we have $L^2_a(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Together with the space $L^2_a(\mathbb{R}^n)$, we consider the weighted Sobolev space $H^4_a(\mathbb{R}^n)$ endowed with the scalar product

$$\langle u, v\rangle_{H^4_a(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho_a(x) \sum_{|\alpha|\le 4} D^\alpha u(x)\, D^\alpha v(x)\, dx \ .$$

In view of [11, Proposition 2.1], we know that $\mathcal{L}$ is a bounded linear operator from $H^4_a(\mathbb{R}^n)$ onto $L^2_a(\mathbb{R}^n)$. Stationary solutions v to (3) satisfy $\mathcal{L}\, v = 0$ and belong to the kernel of $\mathcal{L}$. By Proposition 1, we infer that the kernel of $\mathcal{L}$ is a one dimensional space spanned by v_∞. For a complete proof of this fact we refer to [11]. Here, we just give a heuristic justification of this statement. Stationary solutions v to (3) satisfy

$$\nabla \cdot (\nabla \Delta v - x\, v) = 0 \, , \qquad x \in \mathbb{R}^n \, .$$

By integrating this equation over the ball B_r (centered at the origin and of radius $r > 0$) and by applying the divergence theorem we obtain

$$\int_{\partial B_r} \left(\frac{\partial \Delta v}{\partial \nu} - v\,(x \cdot \nu) \right) d\sigma = 0 \qquad \text{for all } r > 0 \ .$$

If we assume that v is radially symmetric and consider it as a function of $r = |x|$, the latter reads

$$(\Delta v)' - r\, v = 0 \tag{33}$$

and therefore

$$v''' + \frac{n-1}{r}\, v'' - \frac{n-1}{r^2}\, v' - r\, v = 0 \qquad \text{for all } r > 0 \ . \tag{34}$$

Hence, any radially symmetric stationary solution to (3) satisfies (34) and the initial conditions

$$v(0) = \alpha \ , \qquad v'(0) = 0 \ , \qquad v''(0) = \beta$$

for some values of the parameters α and β; notice that, due to radial symmetry, $v'(0) = 0$. Therefore, the space of all radially symmetric stationary solutions to (3) is a 2-dimensional vector space identified to $\mathbb{R}^2$ through the couple of its initial values (α, β). In order to determine all solutions to (34) we need to find two linearly independent (*i.e.* non proportional) solutions.

In view of (30), one of them is $v_\infty(r) = 2^{n/2}\, \alpha_n\, f_n(\sqrt{2}\, r)$ which may be rewritten as in (5). Consider the solution $\overline{v}$ to (34) corresponding to $(\alpha, \beta) = (1, 1)$ so that

$\overline{v}$ is initially positive and increasing. If we integrate (33) and we use the initial condition $\Delta\overline{v}(0) = n > 0$, we obtain

$$\Delta\overline{v}(r) > \int_0^r t\,\overline{v}(t)\,dt > 0$$

which shows that $\overline{v}$ is subharmonic and that it cannot have a maximum point. Hence, $r \mapsto \overline{v}(r)$ is increasing on $\mathbb{R}_+$ and it does not vanish at infinity. Therefore, the 2-dimensional space of even solutions to (34) is spanned by v_∞ and $\overline{v}$ but the only solution vanishing at infinity is v_∞ (and its multiples).

Concerning nontrivial eigenvalues of the operator $\mathcal{L}$ defined in (4), we recall [11, Theorem 2.1].

Proposition 3. *The spectrum of $\mathcal{L}$ coincides with the set of nonnegative integers, $\sigma(\mathcal{L}) = \mathbb{N}$. Each eigenvalue $\lambda \in \sigma(\mathcal{L})$ has finite multiplicity and the corresponding eigenfunctions are given by*

$$D^\alpha v_\infty \qquad \text{for } |\alpha| = \lambda \in \mathbb{N}\ .$$

The set of eigenfunctions is complete in $L^2_a(\mathbb{R}^n)$ for any $a \in [0, \mu)$.

4.3. **An open problem.** Let $L^2_a(\mathbb{R}^n)$ be as in (32) and consider the (normalized) projection operator P_a by

$$P_a\,w := \left(\int_{\mathbb{R}^n} \rho_a\,w\,v_\infty\,dx\right)\frac{v_\infty}{\|v_\infty\|^2_{L^2_a(\mathbb{R}^n)}} \qquad \text{for all } w \in L^2_a(\mathbb{R}^n)\ . \tag{35}$$

Contrary to the second order heat equation, the operator $\mathcal{L}$ is not self-adjoint: we refer to [11, Section 3] for some properties of the adjoint operator $\mathcal{L}^*$. Therefore, although from Proposition 3 we know that the least nontrivial eigenvalue of $\mathcal{L}$ is 1, we *cannot obtain* the standard Poincaré-type inequality

$$\|u - P_a u\|^2_{L^2_a(\mathbb{R}^n)} \le (u, \mathcal{L}\,u)_{L^2_a(\mathbb{R}^n)} \qquad \text{for all } u \in H^4_a(\mathbb{R}^n)\ .$$

In turn, we do not know whether the following estimate holds:

$$(u, \mathcal{L}\,u)_{L^2_a(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho_a(x)\,u(x)\,\mathcal{L}\,u(x)\,dx \ge \|u\|^2_{L^2_a(\mathbb{R}^n)} \qquad \text{for all } u \in [\ker\mathcal{L}\,]^\perp\ .$$

These estimates, which have their own interest, would allow to study the convergence rates (in the weighted L^p-norm, $1 \le p < \infty$) of the solution to (3) towards its projection $P_a u$ onto the kernel, that is, onto the space spanned by v_∞.

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