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REMARKS ON QUASILINEAR ELLIPTIC EQUATIONS AS MODELS FOR ELEMENTARY PARTICLES

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Abstract. We study a class of quasilinear elliptic equations suggested by C.H. Derrick in 1964 as models for elementary particles. For scalar fields we prove some new nonexistence results. For vector-valued fields the situation is different as shown by recent results concerning the existence of solitary waves with a topological constraint.

1. **Introduction.** For any (scalar or vector-valued) field $\psi = \psi(x,t)$, with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(|\nabla\psi|^2 - \left|\frac{\partial\psi}{\partial t}\right|^2\right) - F(\psi)$$

where ∇ denotes the gradient with respect to the space variables x and F is a smooth function. The Euler-Lagrange equation of the corresponding action functional

$$\mathcal{A} = \iint \mathcal{L}\,dxdt$$

is the semilinear wave equation

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + F'(\psi) = 0 \qquad \text{in } \mathbb{R}^{n+1}. \tag{1}$$

In a celebrated paper concerning nonlinear wave equations as models for elementary particles, C.H. Derrick [7] raised the following question: *Can (1) have stable, time-independent, localized solutions in $n = 3$ dimensions?*

The static solutions $\psi(x,t) = u(x)$ of (1) solve the semilinear elliptic equation

$$-\Delta u + F'(u) = 0 \qquad \text{in } \mathbb{R}^n. \tag{2}$$

By *localized* solution Derrick means one such that both the following integrals

$$\int_{\mathbb{R}^n} |\nabla u|^2\,dx, \qquad \int_{\mathbb{R}^n} F(u)dx$$

converge. For physical reasons we restrict our attention to the class of *solitary waves*, that is of solutions vanishing at infinity. Concerning (2), Derrick proved the following result.

Theorem 1. *Assume that $F \in C^1(\mathbb{R})$,*

$$F(s) \geq F(0) = 0 \quad \forall s \in \mathbb{R} \tag{3}$$

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and let u be a solitary wave of (2) having finite energy, namely

$$\mathcal{E} = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\, dx + \int_{\mathbb{R}^n} F(u)dx < +\infty\ ,$$

then necessarily $u \equiv 0$.

Assumption (3) implies that $u \equiv 0$ is a solution (the vacuum) and that *"the energy density (for static solutions) has the desirable feature of being everywhere positive"*. Theorem 1 is proved in [7] in the case $n = 3$ by means of a simple rescaling argument; the very same proof may be used for any $n \geq 2$. If instead of (3) we only assume $F'(0) = 0$, then any static, localized solution of (1) is unstable.

This is the starting point of Derrick's discussion: *"We are thus faced with the disconcerting fact that no equation of type (1) has any time independent solutions which could be interpreted as elementary particles."* Then he suggests *"some possible ways out of this difficulty"*. The first one consists in the transformation of the semilinear equations (1) and (2) to quasilinear equations. More precisely, he considers the Lagrangian density

$$\mathcal{L} = -\frac{1}{p}\left(|\nabla\psi|^2 - \left|\frac{\partial\psi}{\partial t}\right|^2\right)^{p/2} - F(\psi)$$

and he points out that his nonexistence proof fails for $p > n$. *"Such a Lagrangian, however, leads to a very complicated differential equation."*

In the static case, the above Lagrangian yields the quasilinear elliptic equation

$$-\Delta_p u + F'(u) = 0 \qquad \text{in } \mathbb{R}^n \tag{4}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator.

Forty years after Derrick's paper, we attempt to study the existence of solitary waves of finite energy. In this paper we consider equation (4) assuming (3) and $n, p > 1$. In Section 2 we deal with the scalar equation and we establish some nonexistence results which partially extend Theorem 1 to the quasilinear case; these results are new in the case $p > n$. In Section 3 we recall some recent existence results obtained in the vector-valued case. The solutions we consider are called topological solitary waves (for short *topological solitons*): not only they vanish at infinity but they are also characterized by a topological invariant, the charge. In some sense, these solutions give an answer to the question raised by Derrick concerning elementary particles.

2. **Scalar case: nonexistence results.**

In this section we prove nonexistence results for the scalar equation (4). Under suitable assumptions on F, it is known [8, 15] that bounded weak solutions of (4) belong to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. We first extend Theorem 1 to the quasilinear equation (4) under an additional monotonicity assumption:

Theorem 2. *Assume that $F \in C^1(\mathbb{R})$, that F satisfies (3) and the further condition*

$$sF'(s) \geq 0 \qquad \forall s \in \mathbb{R}. \tag{5}$$

If u is a distributional solitary wave of (4) such that

$$\mathcal{E} = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p\, dx + \int_{\mathbb{R}^n} F(u)dx < +\infty \tag{6}$$

then necessarily $u \equiv 0$.

Proof. By (3) and (6) we deduce that $u \in D^{1,p}(\mathbb{R}^n)$. Consider a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^n)$ such that $u_k(x)u(x) \geq 0$ for a.e. $x \in \mathbb{R}^n$ and such that $u_k \to u$ in $D^{1,p}(\mathbb{R}^n)$. Then, from (4) we get

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla u_k dx + \int_{\mathbb{R}^n} F'(u)u_k dx = 0.$$

Hence, letting $k \to +\infty$, by Fatou's Lemma (recall (5) and $u_k u \geq 0$) we infer

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx + \int_{\mathbb{R}^n} F'(u)u dx \leq 0.$$

By (5), this shows at once that $u \equiv 0$. □

If we remove assumption (5) and the finite energy condition (6), we can prove nonexistence of nontrivial solutions of (4) in the restricted class of *radial* solitary waves. Strictly related to previous results in [9], we have

Theorem 3. *Assume that $F \in C^1(\mathbb{R})$ and that F satisfies (3). If $u \in C^1(\mathbb{R}^n)$ is a distributional radial solitary wave of (4), then necessarily $u \equiv 0$.*

Proof. We perform the proof by combining some tools introduced in [9]. Assume that (4) admits a C^1 distributional radial solitary wave $u = u(r)$ where $r = |x|$. Then, it is known [9] that $|u'|^{p-2}u' \in C^1(\mathbb{R})$ and that u solves the problem

$$\begin{cases} \Big(|u'(r)|^{p-2}u'(r)\Big)' + \dfrac{n-1}{r}|u'(r)|^{p-2}u'(r) = F'(u(r)) \qquad (r > 0) \\ u'(0) = 0 \ , \quad \lim\limits_{r\to\infty} u(r) = 0 \ , \end{cases} \tag{7}$$

where primes denote differentiation with respect to r. In this proof, all the limits are intended as $r \to \infty$. Consider the energy function

$$E(r) = \frac{p-1}{p}|u'(r)|^p - F(u(r)) \ , \qquad (r \geq 0). \tag{8}$$

By differentiating with respect to r and using equation (7) we obtain

$$E'(r) = \left[\Big(|u'(r)|^{p-2}u'(r)\Big)' - F'(u(r))\right] u'(r) = -\frac{n-1}{r}|u'(r)|^p \leq 0. \tag{9}$$

This proves that $\lim E(r)$ exists. Since $\lim F(u(r)) = 0$, (8) shows that also $\lim u'(r)$ exists; but this limit is necessarily 0 because $\lim u(r) = 0$. Hence, using again (8), we infer that

$$\lim_{r\to\infty} E(r) = 0. \tag{10}$$

Moreover, by (3) we have $E(0) = -F(u(0)) \leq 0$. This, combined with (9) and (10) proves that $F(u(0)) = 0$ and $E(r) \equiv 0$. In turn, this yields $E'(r) \equiv 0$ which, in view of (9), shows that $u'(r) \equiv 0$. Since u vanishes at infinity, we finally get $u \equiv 0$. □

Several remarks about Theorem 3 are in order. First of all, note that as soon as we drop the sign assumption (3) on F, for (4) existence results for nonnegative radial solitary waves may be obtained, see [10].

Next, a natural question that we leave as an open problem is if in Theorem 3 one can drop the radial symmetry assumption on the solution. To this end, we recall that radial symmetry of bounded *positive* solutions of (4) has been widely studied. First, it has been proved in [11] in the semilinear case $p = 2$ under an additional assumption on the decay at infinity of the solution; subsequently, this assumption was removed in [13]. The very same statement was extended to the singular case

$1 < p < 2$ in [5]. In the degenerate case $p > 2$, the situation is more delicate and radial symmetry of positive solutions of (4) is known only under the additional assumption that the solution admits a unique critical point, see [13]. On the other hand, in view of [9], if $F'(s) > 0$ in a (right) neighborhood of $s = 0$, then positive radial solutions to (4) may exist only whenever $\int_0 |F(s)|^{-1/p} ds = +\infty$. Otherwise, *nonnegative* radial solutions of (4) have compact support (a ball). In such case, radially symmetric solutions about two different centers and with disjoint supports may be "sticked" together in order to obtain multibump solutions, see again [13] and references therein. Hence, it is readily seen that radial symmetry of nonnegative solutions may fail. Moreover, as far as we are aware, nothing seems to be known about the symmetry of sign changing solutions.

Finally, note that Theorem 3 does not exclude the existence of radial solutions which do not vanish at infinity. On the other hand, if we replace (3) with the stronger assumption

$$F(s) > F(0) = 0 \text{ for all } s \neq 0 \text{ and } \liminf_{|s|\to\infty} F(s) > 0 \tag{11}$$

then any finite energy solution of (4) vanishes at infinity, provided that

$$p > n, \tag{12}$$

which is precisely the situation where Theorem 3 gives new informations. To see this, note that in such case any solution u of (4) satisfying (6) is Hölder continuous; therefore, if there exist $c > 0$ and a sequence $\{x_k\} \subset \mathbb{R}^n$ such that $|x_k| \to \infty$ and $|u(x_k)| \geq c$, then there exists $\rho, \delta > 0$ such that $F(u(x)) \geq \delta$ for all x such that $|x - x_k| \leq \rho$ for some k. This would violate (6). Taking into account this fact, Theorem 3 has the following

Corollary 1. *Assume (12), that $F \in C^1(\mathbb{R})$ and that F satisfies (11). If $u \in C^1(\mathbb{R}^n)$ is a distributional radial solution of (4) satisfying (6), then necessarily $u \equiv 0$.*

3. **Vector-valued case: topological solitons.** In this section we deal with the vector-valued equation (4) and we recall some results concerning topological solitary waves; this notion, in connection with quasilinear elliptic equations, has been introduced in [4]. Contrary to the scalar case, here the existence of solutions is guaranteed by a topological constraint. Formula (4) represents now a system of quasilinear equations where

- F is a real smooth function defined in some $\Omega \subset \mathbb{R}^{n+1}$;
- F' denotes the gradient of F;
- the solutions u are vector-valued fields $u = (u^1, \ldots, u^{n+1}) : \mathbb{R}^n \to \Omega$;
- $\Delta_p u$ denotes the vector whose j-th component is $\operatorname{div}(|\nabla u|^{p-2}\nabla u^j)$.

As suggested by Derrick, throughout this section we require that $p > n$.

We also assume that

(F1): $F \in C^1(\Omega)$ where $\Omega = \mathbb{R}^{n+1}\backslash\{\eta\}$ and $\eta \in \mathbb{R}^{n+1} \setminus \{0\}$;

(F2): $F(\xi) \geq F(0) = 0$ for every $\xi \in \Omega$; F is twice differentiable at $\xi = 0$ and its Hessian matrix is non degenerate;

(F3): there exist $c, \rho > 0$ such that, if $|\xi| < \rho$, then $F(\xi + \eta) \geq c|\xi|^{np/(n-p)}$;

(F4): $\liminf_{|\xi|\to+\infty} F(\xi) > 0$.

Consider the function space

$$H = L^2(\mathbb{R}^n) \cap D^{1,p}(\mathbb{R}^n);$$

by (12), the functions in H are bounded, Hölder continuous and vanish at infinity (see [2]). By (F1) and (F2) the real-valued energy functional

$$\mathcal{E}(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx + \int_{\mathbb{R}^n} F(u) dx$$

is well-defined on the open set

$$\Lambda = \{u \in H; \ u(x) \neq \eta \ \forall x \in \mathbb{R}^n\}.$$

Moreover, the energy functional $\mathcal{E}$ is coercive and $\mathcal{E}(u) \to +\infty$ as $u \to \partial\Lambda$, see [2]; note that the proof of coercivity can be improved taking into account the proof in [12].

For every $u \in \Lambda$ we define the topological charge.

Definition 1. For every $u \in \Lambda$, let

$$S(u) = \{x \in \mathbb{R}^n; \ |u(x)| > |\eta|\}.$$

Then, if $S(u) \neq \emptyset$, we define the **topological charge** of u as the integer number

$$\mathrm{ch}(u) = \deg(P \circ u, S(u), N), \tag{13}$$

where P is the projection of $\mathbb{R}^n \setminus \{\eta\}$ on the sphere $\Sigma = \{\xi \in \mathbb{R}^{n+1}; \ |\xi - \eta| = |\eta|\}$ and $N = 2\eta$. If $S(u) = \emptyset$, we set $\mathrm{ch}(u) = 0$.

Remark 1. In [4] and [2] two different definitions of topological charge have been given. In the Appendix of [6] it has been shown that they are equivalent.

Although the functional $\mathcal{E}$ is weakly lower semicontinuous in Λ, we cannot minimize it over the connected components

$$\Lambda_k = \{u \in \Lambda; \ \mathrm{ch}(u) = k\} \qquad (k \in \mathbb{Z})$$

since the domain $\mathbb{R}^n$ is not compact and the Λ_k are not weakly closed.

The first existence result concerning (4) was proved in [2]; it is based on a Splitting Lemma (for energy and charge), in the spirit of the Concentration-Compactness Lemma in unbounded domains. The result is as follows:

Theorem 4. *Assume (12) and that F satisfies (F1)-(F2)-(F3)-(F4). Then there exists a weak solitary wave of (4) obtained as minimum of $\mathcal{E}$ in*

$$\Lambda^* = \{u \in \Lambda; \ \mathrm{ch}(u) \neq 0\}.$$

Remark 2. Let u^* denote the minimizer of the energy $\mathcal{E}$ in Λ^*. It is still an open problem to evaluate $\mathrm{ch}(u^*)$. By [4], we know that either $\mathrm{ch}(u^*) = 1$, or there are at least two nontrivial local minimizers of the energy.

If we introduce a suitable invariance property for F, we also have the existence of a solution of (4) for every fixed value of the charge. In the target space $\mathbb{R}^{n+1}$ we choose a reference frame so that

$$\eta = (1, 0, \dots, 0).$$

In this reference frame, for every $\xi \in \mathbb{R}^{n+1}$ we write

$$\xi = \left(\xi_0, \tilde{\xi}\right)$$

with $\xi_0 \in \mathbb{R}$ and $\tilde{\xi} \in \mathbb{R}^n$. Using this notation we require

(F5): for every $\xi = \left(\xi_0, \tilde{\xi}\right) \in \mathbb{R}^{n+1}$ and for every $g \in O(n)$ (orthogonal group)

$$F(\xi_0, g\tilde{\xi}) = F(\xi_0, \tilde{\xi}).$$

Then, the following multiplicity result holds, see [12].

Theorem 5. *Assume (12) and that F satisfies (F1)-(F2)-(F3)-(F5). Then, for every $k \in \mathbb{Z}$ there exists $u_k \in \Lambda_k$, weak solitary wave of (4).*

We point out that these solutions are not true minima of the energy $\mathcal{E}$; they are constrained minima on $\Lambda_k \cap Fix$ where

$$Fix = \{u = (u_0, \tilde{u}) \in H;\ u_0(gx) = u_0(x), \tilde{u}(gx) = g\tilde{u}(x)\ \forall g \in O(n), \forall x \in \mathbb{R}^n\}.$$

The subspace Fix, which is the set of fixed points for a suitable $O(n)$-action introduced by Skyrme (see [14]), is a natural constraint for finding critical points of $\mathcal{E}$. On the other hand, in Fix there is no loss of compactness because we have a uniform decay to zero at infinity of bounded sequences (Radial Lemma).

Remark 3. If we interpret these solitary waves as elementary particles and the topological charge as electric charge, it is natural to study the interaction between solitons of this kind and their own electromagnetic field; this has been done in [3] and [6]. The same techniques introduced in [4] have been used for a class of nonlinear perturbations of Schroedinger equation; for these equations, time-depending solutions have been studied, deriving also the equations of the Bohmian version of Quantum Mechanics (see [1] and references therein).

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