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Existence of Solutions for Singular Critical Growth Semilinear Elliptic Equations[1]

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A semilinear elliptic problem containing both a singularity and a critical growth term is considered in a bounded domain of $\mathbb{R}^n$: existence results are obtained by variational methods. The solvability of the problem depends on the space dimension n and on the coefficient of the singularity; the results obtained describe the behavior of critical dimensions and nonresonant dimensions when the Brezis–Nirenberg problem is modified with a singular term. © 2001 Elsevier Science

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1. INTRODUCTION

In this paper we consider the semilinear elliptic problem

$$(1.1)\qquad \begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \lambda u + |u|^{2^*-2}\, u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) is an open bounded domain with smooth boundary $\partial\Omega$ and containing the origin 0, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, $0 \leqslant \mu < \bar{\mu} = (n-2)^2/4$ and $\lambda > 0$. When $\mu = 0$, (1.1) simply becomes

$$(1.2)\qquad \begin{cases} -\Delta u = \lambda u + u\, |u|^{2^*-2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

this equation has been widely studied in recent years but it still has several points of interest: a somehow surprising phenomenon is that the existence of nontrivial solutions of (1.2) depends not only on λ but on the couple

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(n, λ). In particular, a crucial role is played by the spectrum σ_μ of the operator $-\Delta-\mu/|x|^2$ with Dirichlet boundary conditions: as $\mu<\bar{\mu}$, in view of [E], σ_μ is discrete, contained in the positive semiaxis and each eigenvalue λ_k $(k\geqslant 1)$ is isolated and has finite multiplicity, the smallest eigenvalue λ_1 being simple and $\lambda_k\to+\infty$ as $k\to\infty$; moreover, all eigenfunctions (for any such μ) belong to the space $H_0^1(\Omega)$.

The starting point for (1.2) is the celebrated paper by Breziz–Nirenberg [BN] where it is shown that:

— if $n\geqslant 4$, then (1.2) admits a positive solution if and only if $\lambda\in(0, \lambda_1)$.

— if $n=3$, there exist constants $\lambda_1>\lambda^*(\Omega)\geqslant\lambda^{**}(\Omega)>0$ (presumably the same) such that (1.2) admits a positive solution if $\lambda\in(\lambda^*, \lambda_1)$ and not if $\lambda\in(0, \lambda^{**}]$.

It is well-known that if $\Omega=B$ (the unit ball) positive solutions of (1.2) are radially symmetric; in this case, when $n=3$ we have $\lambda^*=\lambda^{**}=\lambda_1/4$, see [BN]. Subsequently, Capozzi–Fortunato–Palmieri [CFP] (see also [AS, GR, Z]) considered the case $\lambda\geqslant\lambda_1$ and proved the following results:

— if $n=4$, $\lambda>0$ and $\lambda\notin\sigma_0$, then (1.2) admits a nontrivial solution.

— if $n\geqslant 5$, for all $\lambda>0$ (1.2) admits a nontrivial solution.

Therefore, the solvability of (1.2) appears to be different in the three cases $n=3$, $n=4$ and $n\geqslant 5$. These phenomena involving the space dimension also appear for more general operators as the polyharmonic operator or the p-Laplacian. In particular, a conjecture by Pucci–Serrin [PS2] states that the nonexistence result for radially symmetric solutions of (1.2) when $\Omega=B$ in dimension $n=3$ "bifurcates" for the corresponding critical growth problem relative to the operator $(-\Delta)^K$ $(K\geqslant 1)$ to the space dimensions $n=2K+1, ..., 4K-1$: Pucci–Serrin call these dimensions *critical*. This conjecture is proved in a slightly weaker form by Grunau [Gr2]. It is also known that the critical dimensions for the p-Laplacian are $n\in(p, p^2)$, see [E]. Recently, an attempt was made to explain this phenomenon by means of local summability of the fundamental solutions [J, M] and with the presence of linear remainder terms in Sobolev inequalities with optimal constants [GG1]. An even more surprising fact is that up to now it is not known if (1.2) admits nontrivial solutions when $n=4$ and $\lambda\in\sigma_0$; some partial (positive) results are found by Fortunato–Jannelli [FJ] in domains having some symmetries. It seems natural to ask whether there exists indeed a difference between the dimensions $n=4$ and $n\geqslant 5$ or if it is only a technical problem due to the particular proofs developed in [CFP, GR, Z]: in agreement with [Ga], we name $n=4$ *nonresonant dimension*. It has been found independently in [Ga, Gr1] that the nonresonant dimensions for the polyharmonic operator $(-\Delta)^K$ are $n \in [4K, (2+2\sqrt{2})\, K]$, while for the p-Laplacian they are $n \in [p^2, (p^2+p\sqrt{p^2+4})/2]$, see [AG]. By exploiting the asymptotic analysis of [ABP], Gazzola–Grunau [GG2] characterize nonresonant dimensions and define them in a more rigorous way: they also give an interpretation of the limit value $n = 2+2\sqrt{2}$.

Much less is known for equation (1.1) when $\mu > 0$; as far as we are aware, only a paper by Jannelli [J] treats this problem. Among other results he shows that:

— if $\mu \leqslant \bar{\mu} - 1$, then (1.1) admits a positive solution for all $\lambda \in (0, \lambda_1)$.

— if $\bar{\mu} - 1 < \mu < \bar{\mu}$ and $\Omega = B$, then there exists $\lambda_* \in (0, \lambda_1)$ such that (1.1) admits a positive solution if and only if $\lambda \in (\lambda_*, \lambda_1)$.

Therefore, it seems that critical situations (in the sense of [PS2]) relative to (1.1) correspond to $\bar{\mu} - 1 < \mu < \bar{\mu}$. We also mention that a different but somehow related problem is studied in [E].

In this paper we pursue further the study of (1.1); first of all, we extend Theorem 1.A in [J] to the case where λu in (1.1) is replaced by a more general subcritical perturbation $g(x, u)$. Then, in the spirit of [CFS], we study (1.1) for $\lambda \geqslant \lambda_1$ and we prove an existence result whenever λ belongs to a left neighborhood (of fixed width) of any eigenvalue λ_k ($k \geqslant 1$). Further, we improve this result in the case of the noncritical situations: more precisely, we show that if $0 \leqslant \mu \leqslant \bar{\mu} - 1$, $\lambda > 0$ and $\lambda \notin \sigma_\mu$, then (1.1) admits a nontrivial solution (note that $[0, \bar{\mu} - 1] \neq \emptyset$ if and only if $n \geqslant 4$). Finally, we deal with the nonresonant situations; in the case of (1.2), this problem is studied with three different approaches in [CFP, GR, Z]: however, all these approaches rely on boundedness of eigenfunctions of $-\Delta$. Of course, if $\mu > 0$, one does not expect eigenfunctions of $-\Delta - \mu/|x|^2$ to be bounded, and all three of the just mentioned approaches fail. We overcome this difficulty only in the particular situation where $\Omega = B$ and $\lambda = \lambda_1$ by applying the asymptotic analysis of [CM]: nevertheless, even if we do not have a more general statement, this result is sufficient to conclude that the nonresonant situations are when $(\frac{n-2}{2})^2 - (\frac{n+2}{n})^2 \leqslant \mu \leqslant (\frac{n-2}{2})^2 - 1$ (and $\mu \geqslant 0$), see the comments and figure following Theorem 4 below.

The proof of our results are obtained with critical point theory: however, standard variational arguments do not apply because of a lack of compactness, the action functional does not satisfy the Palais–Smale condition (PS condition in the sequel). In [BN] it is shown that the action functional corresponding to (1.2) satisfies the PS condition only in a suitable "compactness range": then, existence results are obtained by constructing minimax levels within this range. This is also the method which we will use here, combined with the orthogonalization technique introduced in [GR].

This paper is organized as follows. In next section we state our existence results and we comment them with the aid of a figure which shows how the critical and nonresonant behavior for $\mu = 0$ relative to (1.2) can be continued for $\mu > 0$ corresponding to (1.1). In Section 4 we describe the variational procedure used in the proof: we reduce the problem of determining nontrivial solutions of (1.1) to that of finding a PS sequence in the compactness range for the corresponding action functional. The proofs of our results are given in the subsequent sections. Finally, in Section 9 we list a number of open problems which seem interesting in view of a deeper understanding of the features of (1.1). A preliminary version of part of these results may be found in [F].

2. NOTATIONS AND EXISTENCE RESULTS

For all $\mu \in [0, \bar{\mu})$, consider the Hilbert space H_μ endowed with the scalar product

$$(u, v)_{H_\mu} = \int_\Omega \nabla u \, \nabla v \, dx - \mu \int_\Omega \frac{uv}{|x|^2} \, dx \qquad \forall u, v \in H_\mu$$

and define the constant

$$S_\mu = \inf_{u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx - \mu \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{2/2^*}};$$

S_μ is independent of $\Omega \subset \mathbb{R}^n$ in the sense that if

$$S_\mu(\Omega) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx - \mu \int_\Omega \frac{u^2}{|x|^2} \, dx}{\left(\int_\Omega |u|^{2^*} \, dx \right)^{2/2^*}}$$

then $S_\mu(\Omega) = S_\mu(\mathbb{R}^n)$; see [F]. We consider the norm obtained from the scalar product $(\cdot, \cdot)_{H_\mu}$ and we denote it by $\| \cdot \|_{H_\mu}$. This norm is equivalent to the Dirichlet norm in $H_0^1(\Omega)$ by Hardy's inequality.

We state our results concerning (1.1) in a slightly more general form; we deal with the problem

$$
(2.1)\qquad \begin{cases} -\Delta u-\mu \dfrac{u}{|x|^2}=g(x,u)+|u|^{2^*-2}\,u & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases}
$$

where $g(x,\cdot)$ has subcritical growth at infinity. More precisely, we assume that

(2.2)
$g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that
$\forall \varepsilon>0\ \exists a_\varepsilon\in L^{\frac{2n}{n+2}}$ s.t. $|g(x,s)|\leqslant a_\varepsilon(x)+\varepsilon\,|s|^{\frac{n+2}{n-2}}$ for a.e. $x\in\Omega$ and $\forall s\in\mathbb{R}$.

The other assumptions are imposed on the primitive $G(x,s)=\int_0^s g(x,t)\,dt$: we first assume that

$$
(2.3)\qquad G(x,s)\geqslant 0 \qquad \text{for a.e. } x\in\Omega,\quad \forall s\in\mathbb{R}.
$$

Next, assume that there exist $k\in\mathbb{N}$, $\delta>0$, $\eta\in(0,\lambda_{k+1}-\lambda_k)$ such that

$$
(2.4)\qquad G(x,s)\geqslant \tfrac{1}{2}(\lambda_k+\eta)\,s^2 \qquad \text{for a.e. } x\in\Omega,\quad \forall\,|s|\leqslant\delta;
$$

and there exist $C\geqslant 0$, $\theta\in(2,2^*)$, $\Psi\in L^{q(\theta)}(\Omega)$ and $\nu\in(\lambda_k,\lambda_{k+1})$ such that

$$
(2.5)\qquad G(x,s)\leqslant \tfrac{1}{2}\nu s^2+\Psi(x)\,|s|^\theta+C\,|s|^{2^*} \qquad \text{for a.e. } x\in\Omega,\quad \forall\, s\in\mathbb{R}
$$

with $q(\theta)=\frac{2n}{2n+(2-n)\,\theta}$. Furthermore, we assume that (η as in (2.4))

$$
(2.6)\qquad G(x,s)\geqslant \frac{1}{2}(\lambda_k+\eta)\,s^2-\frac{1}{2^*}\,|s|^{2^*} \qquad \text{for a.e. } x\in\Omega,\quad \forall s\in\mathbb{R}.
$$

If $\bar{\mu}-1<\mu<\bar{\mu}$ we also need a growth condition at infinity:

(2.7) there exists an open nonempty subset $\Omega_0\subset\Omega$ such that $0\in\Omega_0$ and

$$
\lim_{s\to+\infty}\frac{G(x,s)}{s^p}=+\infty \qquad \text{uniformly w.r.t. } x\in\Omega_0,
$$

where $p=2(n-2\sqrt{\bar{\mu}-\mu})/(n-2)$.

In the sequel, by *solution* of (2.1) we mean a function $u\in H_\mu$ satisfying

$$
\int_\Omega \nabla u\,\nabla v\,dx-\mu\int_\Omega \frac{uv}{|x|^2}\,dx=\int_\Omega g(x,u)\,v\,dx+\int_\Omega |u|^{2^*-2}\,uv\,dx \qquad \forall v\in H_\mu.
$$

Define the functional $J\colon H_\mu \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2}\,dx - \int_\Omega G(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx;$$

we have $J \in C^1(H_\mu, \mathbb{R})$ and critical points of the functional J correspond to (weak) solutions of equation (2.1). In order to avoid possible confusions, in the particular case where $g(x,s) = \lambda s$ we denote the functional with a different letter:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2}\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx.$$

With the above assumptions we first prove a result when, roughly speaking, $g(x,s)$ stays below $\lambda_1 s$ in a neighborhood of $s=0$:

THEOREM 1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $0 \in \Omega$ and let $\mu \geqslant 0$.*

For $n \geqslant 4$ and $\mu \leqslant \bar{\mu}-1$ assume (2.2)–(2.5) (*with* $k=0$, $\lambda_0 = 0$), *for $\bar{\mu}-1 < \mu < \bar{\mu}$ assume* (2.2)–(2.5) (*with* $k=0$, $\lambda_0=0$) *and* (2.7); *then equation* (2.1) *admits a positive solution.*

Similarly, if $g(x,s)$ stays above $\lambda_1 s$, we prove:

THEOREM 2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $0 \in \Omega$ and let $\mu \geqslant 0$.*

For $n \geqslant 4$ and $\mu \leqslant \bar{\mu}-1$ assume (2.2)–(2.6) (*with* $k \geqslant 1$), *for $\bar{\mu}-1 < \mu < \bar{\mu}$ assume* (2.2)–(2.7) (*with* $k \geqslant 1$); *then equation* (2.1) *admits a nontrivial solution.*

Note that for $g(x,s) = \lambda s$ the previous results yield

COROLLARY 1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $0 \in \Omega$.*

If $n \geqslant 4$ and $0 \leqslant \mu \leqslant \bar{\mu}-1$ then equation (1.1) *admits a nontrivial solution for all $\lambda > 0$ such that $\lambda \notin \sigma_\mu$.*

Theorems 1 and 2 nothing say about (1.1) in the case where $\bar{\mu}-1 < \mu < \bar{\mu}$; in the next result we establish that the solutions exist whenever λ belongs to a left neighborhood of constant width of any eigenvalue:

THEOREM 3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $0 \in \Omega$; assume that $\mu \geqslant 0$, $\bar{\mu}-1 < \mu < \bar{\mu}$ and that there exists $\lambda_k \in \sigma_\mu$ such that*

$$\lambda \in (\lambda_k - S_\mu\, |\Omega|^{-2/n}, \lambda_k);$$

then (1.1) *admits* ν_k *pairs of nontrivial solutions, where* ν_k *denotes the multiplicity of* λ_k.

These results confirm that critical situations correspond to $\bar{\mu}-1<\mu<\bar{\mu}$. Concerning nonresonant situations we have

THEOREM 4. *Let* $\Omega = B$; *if* $n \geqslant 5$ *and*

$$0 \leqslant \mu < \bar{\mu} - \left(\frac{n+2}{n}\right)^2.$$

Then, for $\lambda = \lambda_1$, *equation* (1.1) *admits a nontrivial solution* $\bar{u} \in H_\mu$ *such that*

$$I(\bar{u}) \in \left(0, \frac{S_\mu^{n/2}}{n}\right).$$

When $g(x, s) = \lambda s$, all the nontrivial solutions we find in Theorems 1–4 are at critical level in the interval $(0, S_\mu^{n/2}/n)$, even if we specified this fact only in Theorem 4: there, the precisation that the critical level is below the threshold $S_\mu^{n/2}/n$ is crucial. Indeed, it is known [C, FJ] that in domains having some symmetries (e.g. balls), nontrivial solutions of (1.2) exist for any $\lambda \in \sigma_0$ and in any dimension $n \geqslant 3$: however, these solutions are at high critical levels. Therefore, even if stated in a particular situation, Theorem 4

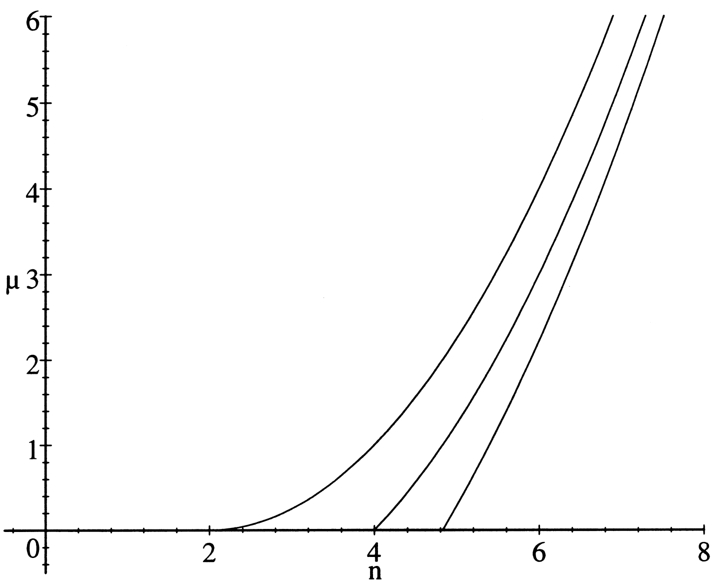

FIGURE 1

establishes that the nonresonant situation for (1.1) is whenever $\bar{\mu}-(\frac{n+2}{n})^2 \leqslant \mu \leqslant \bar{\mu}-1$. Figure 1 shows how the phenomena relative to (1.2) for $\mu=0$ propagate for all $\mu>0$.

These three curves, going from left to right, have respectively equations $\mu=(\frac{n-2}{2})^2$, $\mu=(\frac{n-2}{2})^2-1$, $\mu=(\frac{n-2}{2})^2-(\frac{n+2}{n})^2$; the intersection of these curves with the axis $\mu=0$ are $n=2$, $n=4$ and $n=2+2\sqrt{2}$. Between the first two curves we have critical behavior, between the second and the third we have nonresonant behavior: note that as $n\to\infty$ the nonresonant behavior tends to disappear.

3. SOME TECHNICAL ASYMPTOTIC ESTIMATES

Fix $k\in\mathbb{N}$ and for all $i\in\mathbb{N}$ denote by e_i an L^2 normalized eigenfunction relative to $\lambda_i\in\sigma_\mu$; let H^- denote the space spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_1, ..., \lambda_k$ and $H^+ := (H^-)^\perp$, and let $P_k\colon H_\mu\to H^-$ denote the orthogonal projection. Take always $m\in\mathbb{N}$ large enough so that $B_{1/m}\subset\Omega$ where $B_{1/m}$ denotes the ball of radius $1/m$ with center in 0; in the case of assumption (2.7) assume also that m is so large that $B_{1/m}\subset\Omega_0$. Consider the functions $\zeta_m\colon\Omega\to\mathbb{R}$ defined by

$$\zeta_m(x) := \begin{cases} 0 & \text{if} \quad x\in B_{1/m} \\ m\,|x|-1 & \text{if} \quad x\in A_m = B_{2/m}\setminus B_{1/m} \\ 1 & \text{if} \quad x\in\Omega\setminus B_{2/m}. \end{cases}$$

Then, as in [GR], define the approximating eigenfunctions $e_i^m := \zeta_m e_i$ and the space

$$H_m^- := \operatorname{span}\{e_i^m;\, i=1, ..., k\}.$$

We prove that the functions e_i^m converge to the eigenfunctions e_i and we estimate the approximation error:

LEMMA 1. *As $m\to\infty$ we have*

$$e_i^m \to e_i \quad \textit{in } H_\mu \qquad \forall i\in\mathbb{N}.$$

Furthermore,

(i) *if $H_m^- = span\{e_i^m;\, i=1, ..., k\}$, we have*

$$\max_{\{u\in H_m^-/\|u\|_{L^2}=1\}} \|u\|_{H_\mu}^2 \leqslant \lambda_k + o(1)$$

(ii) *if* $\Omega = B$ *and* $H_m^- = span\{e_1^m\}$, *we have*

$$\max_{\{u \in H_m^- / \|u\|_{L^2}=1\}} \|u\|_{H_\mu}^2 \leqslant \lambda_1 + cm^{-2\sqrt{\bar{\mu}-\mu}}.$$

Proof. To show the convergence in H_μ, it suffices to show the convergence in H_0^1, thanks to the equivalence of the two norms. We have

$$\int_\Omega |\nabla(e_i^m - e_i)|^2 \, dx = \int_\Omega |e_i \nabla \zeta_m + (\zeta_m - 1) \nabla e_i|^2 \, dx$$

$$= \int_{A_m} |\nabla \zeta_m|^2 (e_i)^2 \, dx + 2 \int_{A_m} \nabla \zeta_m (\zeta_m - 1) \, e_i \nabla e_i \, dx$$

$$+ \int_{B_{2/m}} (\zeta_m - 1)^2 |\nabla e_i|^2 \, dx.$$

We first show that $\int |\nabla \zeta_m|^2 (e_i)^2 \to 0$; indeed using Hölder's inequality, we have:

$$\int_{A_m} |\nabla \zeta_m|^2 (e_i)^2 \, dx = m^2 \int_{A_m} (e_i)^2 \, dx < m^2 \int_{B_{2/m}} (e_i)^2 \, dx$$

$$\leqslant m^2 \left(\int_{B_{2/m}} |e_i|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \left(\int_{B_{2/m}} dx \right)^{2/n}$$

$$= m^2 \left(\int_{B_{2/m}} |e_i|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} C \left(\frac{2}{m} \right)^2$$

$$= C \left(\int_{B_{2/m}} |e_i|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \to 0$$

as $m \to \infty$, by the absolute continuity of the integral.

Similarly, $\int \nabla \zeta_m (\zeta_m - 1) \, e_i \nabla e_i \to 0$; indeed,

$$\left| \int_{A_m} \nabla \zeta_m (\zeta_m - 1) \, e_i \nabla e_i \, dx \right|$$

$$\leqslant m \left(\int_{A_m} |e_i|^{2^*} dx \right)^{1/2^*} \left(\int_{A_m} |\nabla e_i|^2 \, dx \right)^{1/2} \left(\int_{A_m} dx \right)^{1/n}$$

$$= C \left(\int_{B_{2/m}} |e_i|^{2^*} dx \right)^{1/2^*} \left(\int_{B_{2/m}} |\nabla e_i|^2 \, dx \right)^{1/2} \to 0 \quad \text{as} \quad m \to \infty.$$

Finally,

$$\int_{B_{2/m}} (\zeta_m - 1)^2 |\nabla e_i|^2 \, dx \leqslant \int_{B_{2/m}} |\nabla e_i|^2 \, dx \to 0 \qquad \text{as} \quad m \to \infty,$$

and the first part of the lemma is proved.

In order to prove (i), let

$$\widetilde{e_i^m} = \frac{e_i^m}{\|e_i^m\|_{L^2}};$$

since $e_i^m \to e_i$ in H_μ, we have $\|e_i^m\|_{L^2} \to \|e_i\|_{L^2} = 1$. With this we can prove that $\widetilde{e_i^m} \to e_i$ in H_μ: indeed,

$$\text{(3.1)} \qquad \begin{aligned} \|\widetilde{e_i^m} - e_i\|_{H_\mu} &= \left\| \frac{e_i^m}{\|e_i^m\|_{L^2}} - \frac{e_i}{\|e_i^m\|_{L^2}} + \frac{e_i}{\|e_i^m\|_{L^2}} - e_i \right\|_{H_\mu} \\ &\leqslant \frac{1}{\|e_i^m\|_{L^2}} \|e_i^m - e_i\|_{H_\mu} + \left(\frac{1}{\|e_i^m\|_{L^2}} - 1 \right) \|e_i\|_{H_\mu} \to 0 \end{aligned}$$

as $m \to \infty$. Now let $u_m \in H_m^- \cap \partial B$ (here $\partial B = \{u \in H_\mu ; \|u\|_{L^2} = 1\}$) be such that

$$\max_{H_m^- \cap \partial B} \|u\|_{H_\mu}^2 = \|u_m\|_{H_\mu}^2;$$

then, there exist $\alpha_1^m, \ldots, \alpha_k^m$ such that $u_m = \sum_{i=1}^k \alpha_i^m \widetilde{e_i^m}$ and

$$\text{(3.2)} \qquad 1 = \|u_m\|_{L^2}^2 = \sum_{i=1}^k (\alpha_i^m)^2 + 2 \sum_{1 \leqslant i < j \leqslant k} \alpha_i^m \alpha_j^m (\widetilde{e_i^m}, \widetilde{e_j^m})_{L^2}.$$

Furthermore, we have

$$\begin{aligned} &|(\widetilde{e_i^m}, \widetilde{e_j^m})_{L^2} - (e_i, e_j)_{L^2}| \\ &\quad \leqslant |(\widetilde{e_i^m}, \widetilde{e_j^m} - e_j)_{L^2}| + |(\widetilde{e_i^m} - e_i, e_j)_{L^2}| \\ &\quad \leqslant \|\widetilde{e_i^m}\|_{L^2} \|\widetilde{e_j^m} - e_j\|_{L^2} + \|\widetilde{e_i^m} - e_i\|_{L^2} \|e_j\|_{L^2} \to 0 \qquad \text{as} \quad m \to \infty \end{aligned}$$

and hence

$$(\widetilde{e_i^m}, \widetilde{e_j^m})_{L^2} \to (e_i, e_j)_{L^2} = 0,$$

which shows that $(\widetilde{e_i^m}, \widetilde{e_j^m})_{L^2} = o(1)$ as $m \to \infty$. So, by (3.2), we have

$$\text{(3.3)} \qquad 1 = \|u_m\|_{L^2}^2 = \sum_{i=1}^k (\alpha_i^m)^2 + o(1).$$

Similarly, one obtains that

$$(\widetilde{e_i^m}, \widetilde{e_j^m})_{H_\mu} \to (e_i, e_j)_{H_\mu} = 0 \quad \text{as} \quad m \to \infty. \tag{3.4}$$

Using (3.1), (3.3) and (3.4) we have:

$$\begin{aligned}
\|u_m\|_{H_\mu}^2 &= \sum_{i=1}^{k} (\alpha_i^m)^2 \, \|\widetilde{e_i^m}\|_{H_\mu}^2 + 2 \sum_{1 \leqslant i<j \leqslant k} \alpha_i^m \alpha_j^m (\widetilde{e_i^m}, \widetilde{e_j^m})_{H_\mu} \\
&= \sum_{i=1}^{k} (\alpha_i^m)^2 \, (\|e_i\|_{H_\mu}^2 + o(1)) + o(1) = \sum_{i=1}^{k} (\alpha_i^m)^2 \, \lambda_i \, \|e_i\|_{L^2}^2 + o(1) \\
&\leqslant \lambda_k \sum_{i=1}^{k} (\alpha_i^m)^2 + o(1) = \lambda_k + o(1)
\end{aligned}$$

which proves (i).

In the case (ii), since $\mu \geqslant 0$, by Theorems 2.2 and 2.7 in [AL] we know that the first eigenfunction e_1 is radially symmetric, $e_1 = e_1(r)$ $(r = |x|)$. Therefore, [CM, Lemma 3.1] tells us that we have the following asymptotic behavior:

$$e_1(r) \approx r^{1-\frac{n}{2}+\sqrt{\bar{\mu}-\mu}} \quad \text{and} \quad e_1'(r) \approx r^{-\frac{n}{2}+\sqrt{\bar{\mu}-\mu}} \quad \text{as} \quad r \to 0. \tag{3.5}$$

Thanks to these estimates it is possible to determine the rate of convergence of e_1^m as $m \to \infty$ by arguing in radial coordinates. We have

$$\begin{aligned}
\|e_1^m\|_{H_\mu}^2 - \|e_1\|_{H_\mu}^2 &= \int_{A_m} (|me_1 \, \nabla |x| + (m|x|-1) \, \nabla e_1|^2 - |\nabla e_1|^2) \, dx \\
&\quad - \int_{B_{1/m}} |\nabla e_1|^2 \, dx - \mu \int_{A_m} \frac{m^2 |x|^2 - 2m|x|}{|x|^2} \, e_1^2 \, dx \\
&\quad + \mu \int_{B_{1/m}} \frac{e_1^2}{|x|^2} \, dx \\
&\leqslant \int_{A_m} \left[\left(m^2 e_1^2 + 2m\mu \frac{e_1^2}{|x|} \right) + 2m(m|x|-1) \, e_1 \, |\nabla e_1| \right. \\
&\quad \left. + (m^2 |x|^2 - 2m|x|) \, |\nabla e_1|^2 \right] dx + \mu \int_{B_{1/m}} \frac{e_1^2}{|x|^2} \, dx \\
&\leqslant Cm^2 \int_{B_{2/m}} e_1^2 \, dx + Cm \int_{B_{2/m}} e_1 \, |\nabla e_1| \, dx + \mu \int_{B_{1/m}} \frac{e_1^2}{|x|^2} \, dx \\
&\leqslant Cm^2 \int_0^{2/m} r^{1+2\sqrt{\bar{\mu}-\mu}} \, dr + Cm \int_0^{2/m} r^{2\sqrt{\bar{\mu}-\mu}} \, dr \\
&\quad + C \int_0^{1/m} r^{-1+2\sqrt{\bar{\mu}-\mu}} \, dr \leqslant Cm^{-2\sqrt{\bar{\mu}-\mu}};
\end{aligned}$$

therefore,

$$\|e_1^m\|_{H_\mu}^2 \leqslant \|e_1\|_{H_\mu}^2 + Cm^{-2\sqrt{\bar{\mu}-\mu}} = \lambda_1 + Cm^{-2\sqrt{\bar{\mu}-\mu}}. \tag{3.6}$$

Next, using again (3.5), we estimate

$$\begin{aligned}\|e_1^m\|_{L^2}^2 &= \int_\Omega e_1^2\,dx - \int_\Omega (1-\zeta_m^2)\,e_1^2\,dx \geqslant 1 - \int_{B_{2/m}} e_1^2\,dx \\ &\geqslant 1 - C\int_0^{2/m} r^{1+2\sqrt{\bar{\mu}-\mu}}\,dr \geqslant 1 - Cm^{-2-2\sqrt{\bar{\mu}-\mu}},\end{aligned}$$

which, inserted into (3.6), gives

$$\max_{u\in H_m^-\cap\partial B} \|u\|_{H_\mu}^2 = \frac{\|e_1^m\|_{H_\mu}^2}{\|e_1^m\|_{L^2}^2} \leqslant \frac{\lambda_1 + Cm^{-2\sqrt{\bar{\mu}-\mu}}}{1-Cm^{-2-2\sqrt{\bar{\mu}-\mu}}} \leqslant \lambda_1 + Cm^{-2\sqrt{\bar{\mu}-\mu}},$$

that is, the result. ▮

As in [J] we consider the family of functions

$$u_\varepsilon^*(x) := \frac{C_\varepsilon}{[\varepsilon^2\,|x|^{\gamma'/\sqrt{\bar{\mu}}} + |x|^{\gamma/\sqrt{\bar{\mu}}}]^{\sqrt{\bar{\mu}}}} \qquad (\varepsilon > 0), \tag{3.7}$$

where $C_\varepsilon = (4\varepsilon^2 n(\bar{\mu}-\mu)/n-2)^{(n-2)/4}$, $\gamma = \sqrt{\bar{\mu}} + \sqrt{\bar{\mu}-\mu}$ and $\gamma' = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu}$; for all $\varepsilon > 0$ the function u_ε^* solves the equation

$$-\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*-2}\,u \qquad \text{in} \quad \mathbb{R}^n \setminus \{0\}$$

and satisfies $\|u_\varepsilon^*\|_{H_\mu}^2 = \|u_\varepsilon^*\|_{L^{2^*}}^{2^*} = S_\mu^{n/2}$; see [F] for the details. Since u_ε^* is a radial function we can view it also as a function defined on $\mathbb{R}^+$; when no confusion arises we denote $u_\varepsilon^*(|x|) = u_\varepsilon^*(x)$.

For all $m \in \mathbb{N}$ and $\varepsilon > 0$ consider also the shifted functions

$$u_\varepsilon^m(x) = \begin{cases} u_\varepsilon^*(x) - \dfrac{C_\varepsilon}{\left[\varepsilon^2\left(\dfrac{1}{m}\right)^{\gamma'/\sqrt{\bar{\mu}}} + \left(\dfrac{1}{m}\right)^{\gamma/\sqrt{\bar{\mu}}}\right]^{\sqrt{\bar{\mu}}}} & \text{if} \quad x \in B_{1/m}\setminus\{0\} \\ 0 & \text{if} \quad x \in \Omega \setminus B_{1/m}. \end{cases}$$

We have the following estimates, in the spirit of Lemma 1.1 in [BN]:

There exist $C_1, C_2, K > 0$ such that if $\varepsilon^{n-2} m^{2\sqrt{\bar{\mu}-\mu}} < K$ then

$$\|u_\varepsilon^m\|_{H_\mu}^2 \leqslant S_\mu^{n/2} + C_1\,\varepsilon^{n-2} m^{2\sqrt{\bar{\mu}-\mu}} \tag{3.8}$$

$$\|u_\varepsilon^m\|_{L^{2^*}}^{2^*} \geqslant S_\mu^{n/2} - C_2\,\varepsilon^n m^{\frac{2n}{n-2}\sqrt{\bar{\mu}-\mu}}. \tag{3.9}$$

Proof. In this proof we denote all positive constants by C.
First note that

$$\int_{\Omega} |\nabla u_\varepsilon^m|^2\,dx = \int_{\mathbb{R}^n} |\nabla u_\varepsilon^*|^2\,dx - \int_{\mathbb{R}^n\setminus B_{1/m}} |\nabla u_\varepsilon^*|^2\,dx \leqslant \int_{\mathbb{R}^n} |\nabla u_\varepsilon^*|^2\,dx. \tag{3.10}$$

Next, consider the singular part:

$$\begin{aligned}
&\int_\Omega \frac{(u_\varepsilon^m)^2}{|x|^2}\,dx \\
&\quad = \int_{B_{1/m}} \frac{(u_\varepsilon^*)^2}{|x|^2}\,dx + \int_{B_{1/m}} \frac{C_\varepsilon^2}{\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{2\sqrt{\bar\mu}}} \frac{1}{|x|^2}\,dx \\
&\qquad -2\int_{B_{1/m}} \frac{C_\varepsilon u_\varepsilon^*}{\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}} \frac{1}{|x|^2}\,dx \\
&\quad \geqslant \int_{\mathbb{R}^n} \frac{(u_\varepsilon^*)^2}{|x|^2}\,dx - C\int_{1/m}^\infty \frac{\varepsilon^{2\sqrt{\bar\mu}}}{[\varepsilon^2 r^{\gamma'/\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}}]^{2\sqrt{\bar\mu}}} \frac{1}{r^2} r^{n-1}\,dr \\
&\qquad -C\int_0^{1/m} \frac{\varepsilon^{2\sqrt{\bar\mu}}}{[\varepsilon^2 r^{\gamma'/\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}}]^{\sqrt{\bar\mu}}\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}} r^{n-3}\,dr.
\end{aligned}$$

Since we have

$$C\int_{1/m}^\infty \frac{\varepsilon^{2\sqrt{\bar\mu}}}{[\varepsilon^2 r^{\gamma'/\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}}]^{2\sqrt{\bar\mu}}} r^{n-3}\,dr \leqslant C\varepsilon^{2\sqrt{\bar\mu}} m^{2\sqrt{\bar\mu-\mu}}$$

and

$$\int_0^{1/m} \frac{\varepsilon^{2\sqrt{\bar\mu}}}{[\varepsilon^2 r^{\gamma'/\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}}]^{\sqrt{\bar\mu}}\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}} r^{n-3}\,dr \leqslant C\varepsilon^{2\sqrt{\bar\mu}} m^{2\sqrt{\bar\mu-\mu}},$$

we obtain

$$\int_\Omega \frac{(u_\varepsilon^m)^2}{|x|^2}\,dx \geqslant \int_{\mathbb{R}^n} \frac{(u_\varepsilon^*)^2}{|x|^2}\,dx - C\varepsilon^{2\sqrt{\bar\mu}} m^{2\sqrt{\bar\mu-\mu}}$$

which, together with (3.10), shows that

$$\|u_\varepsilon^m\|_{H_\mu}^2 = \int_\Omega |\nabla u_\varepsilon^m|^2\,dx - \mu\int_\Omega \frac{(u_\varepsilon^m)^2}{|x|^2}\,dx \leqslant \|u_\varepsilon^*\|_{H_\mu}^2 + C\varepsilon^{2\sqrt{\bar\mu}} m^{2\sqrt{\bar\mu-\mu}}$$

and (3.8) follows.

In order to prove (3.9) note that

$$\|u_\varepsilon^m\|_{L^{2^*}}^{2^*} = \int_{B_{1/m}} |u_\varepsilon^m|^{2^*}\,dx$$

$$\geqslant \int_{B_{1/m}} |u_\varepsilon^*|^{2^*}\,dx - 2^* \int_{B_{1/m}} |u_\varepsilon^*|^{2^*-1} \frac{C_\varepsilon}{\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}}\,dx$$

$$= \int_{\mathbb{R}^n} |u_\varepsilon^*|^{2^*}\,dx - \int_{\mathbb{R}^n\setminus B_{1/m}} |u_\varepsilon^*|^{2^*}\,dx$$

$$- \int_{B_{1/m}} \frac{2^*\,|u_\varepsilon^*|^{2^*-1}\,C_\varepsilon}{\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}}\,dx.$$

We estimate the second integral by

$$\int_{\mathbb{R}^n\setminus B_{1/m}} |u_\varepsilon^*|^{2^*}\,dx = C \int_{1/m}^{\infty} \frac{C_\varepsilon^{2^*}}{[\varepsilon^2 r^{\gamma'/\sqrt{\bar\mu}} + r^{\gamma/\sqrt{\bar\mu}}]^n}\, r^{n-1}\,dr \leqslant C\varepsilon^n m^{\frac{2n}{n-2}\sqrt{\bar\mu-\mu}}$$

and the third integral by

$$\int_{B_{1/m}} \frac{|u_\varepsilon^*|^{2^*-1}\,C_\varepsilon}{\left[\varepsilon^2\left(\frac{1}{m}\right)^{\gamma'/\sqrt{\bar\mu}} + \left(\frac{1}{m}\right)^{\gamma/\sqrt{\bar\mu}}\right]^{\sqrt{\bar\mu}}}\,dx \leqslant C\varepsilon^n m^{\frac{2n}{n-2}\sqrt{\bar\mu-\mu}}.$$

Hence, $\|u_\varepsilon^m\|_{L^{2^*}}^{2^*} \geqslant \|u_\varepsilon^*\|_{L^{2^*}}^{2^*} - C\varepsilon^n m^{(2n/(n-2))\sqrt{\bar\mu-\mu}}$ and (3.9) follows. ▮

4. THE VARIATIONAL CHARACTERIZATION

The variational characterization is based either on a mountain-pass [AR] or on a linking [R] argument.

We recall that a sequence $\{u_m\} \subset H_\mu$ is called a PS sequence for J at level c if $J(u_m) \to c$ and $J'(u_m) \to 0$ in $(H_\mu)'$, the dual space of H_μ; we have

Lemma 2. *Assume* (2.2) *and let* $\{u_m\} \subset H_\mu$ *be a PS sequence for J; then there exists* $u \in H_\mu$ *such that* $u_m \rightharpoonup u$, *up to a subsequence, and* $J'(u) = 0$. *Moreover, if* $J(u_m) \to c$ *with* $c \in (0, S_\mu^{n/2}/n)$ *then* $u \not\equiv 0$ *and hence u is a non-trivial solution of* (2.1).

Proof. The proof is standard, see [BN]: we briefly sketch it. Let $f(x, s) = g(x, s) + |s|^{2^*-2} s$ and $F(x, s) = \int_0^s f(x, t)\, dt$; since (2.2) holds, we have

$$\exists \vartheta \in (0, \tfrac{1}{2}) \quad \exists \bar{s} > 0 \qquad \text{such that}$$
$$F(x, s) \leqslant \vartheta f(x, s)\, s \quad \text{for a.e. } x \in \Omega \quad \forall\, |s| \geqslant \bar{s}:$$

therefore $\{u_m\}$ is bounded and there exists u such that $u_m \rightharpoonup u$, up to a subsequence. Furthermore, $J'(u) = 0$ by weak continuity of J'.

Assume $c \in (0, S_\mu^{n/2}/n)$ and, by contradiction, $u \equiv 0$; as the term $g(x, u_m)\, u_m$ is subcritical, we infer from $J'(u_m)[u_m] = o(1)$ that

$$\|u_m\|_{H_\mu}^2 - \|u_m\|_{L^{2^*}}^{2^*} = o(1). \tag{4.1}$$

By the definition of S_μ we have $\|u\|_{H_\mu}^2 \geqslant S_\mu\, \|u\|_{L^{2^*}}^2$ for all $u \in H_\mu$; then we obtain

$$o(1) \geqslant \|u_m\|_{H_\mu}^2 (1 - S_\mu^{-2^*/2}\, \|u_m\|_{H_\mu}^{2^*-2}).$$

If $\|u_m\|_{H_\mu} \to 0$ we contradict $c > 0$; therefore, $\|u_m\|_{H_\mu}^2 \geqslant S_\mu^{n/2} + o(1)$ and by (4.1) we get

$$J(u_m) = \frac{1}{n}\, \|u_m\|_{H_\mu}^2 + \frac{n-2}{2n}\, (\|u_m\|_{H_\mu}^2 - \|u_m\|_{L^{2^*}}^{2^*}) + o(1) \geqslant \frac{1}{n}\, S_\mu^{n/2} + o(1)$$

which contradicts $c < \frac{1}{n} S_\mu^{n/2}$. ∎

By Lemma 2, in order to prove Theorems 1–4 it suffices to build a PS sequence for J at a level strictly between 0 and $S_\mu^{n/2}/n$. We first deal with the case where the functional J has a mountain-pass geometry: since we are looking for positive solutions we set $g(x, s) = 0$ for all $s \leqslant 0$ and we obtain

Lemma 3. *Assume* (2.3), (2.5) *then the functional J admits a PS sequence in the cone of positive functions at level*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], H_\mu);\ \gamma(0) = 0,\ J(\gamma(1)) < 0\}$.

Proof. We want to prove that the functional J satisfies all the hypotheses of the mountain pass theorem except for the PS condition. Obviously $J(0) = 0$ and there exist $\alpha, \rho > 0$ such that

$$J(v) \geqslant \alpha \qquad \forall v \in \partial B_\rho \cap H_\mu.$$

Indeed by (2.5) and Hölder's inequality and using $\|v\|_{H_\mu}^2 \geqslant \lambda_1 \|v\|_{L^2}^2$ for any $v \in H_\mu$, we have

$$J(v) = \frac{1}{2}\|v\|_{H_\mu}^2 - \int_\Omega G(x, v)\,dx - \frac{1}{2^*}\|v\|_{L^{2^*}}^{2^*}$$
$$\geqslant \frac{1}{2}\|v\|_{H_\mu}^2 - \frac{1}{2}\nu\|v\|_{L^2}^2 - \int_\Omega |\Psi(x)|\,|v|^\theta\,dx - C\|v\|_{L^{2^*}}^{2^*} - \frac{1}{2^*}\|v\|_{L^{2^*}}^{2^*}$$
$$\geqslant \frac{1}{2}\|v\|_{H_\mu}^2 - \frac{\nu}{2}\frac{\|v\|_{H_\mu}^2}{\lambda_1} - \left(\int_\Omega |\Psi(x)|^{q(\theta)}\,dx\right)^{\frac{1}{q(\theta)}}\|v\|_{L^{2^*}}^\theta - \left(C + \frac{1}{2^*}\right)\|v\|_{L^{2^*}}^{2^*}$$
$$\geqslant C_1\|v\|_{H_\mu}^2 - C_2\|v\|_{H_\mu}^\theta - C_3\|v\|_{H_\mu}^{2^*} \qquad \text{with} \quad C_1, C_2, C_3 > 0.$$

Furthermore, for any $v \in H_\mu$ there exists $t > 0$ such that $J(tv) < 0$; indeed by (2.3) we have

$$J(tv) \leqslant \frac{t^2}{2}\|v\|_{H_\mu}^2 - \frac{t^{2^*}}{2^*}\|v\|_{L^{2^*}}^{2^*}.$$

Therefore, by Theorem 2.2 in [BN] we infer that J admits a PS sequence at level c; such sequence may be chosen in the cone of positive functions because $J(|u|) \leqslant J(u)$ for all $u \in H_\mu$. ▮

Next we deal with the case where the functional J has a linking geometry:

LEMMA 4. *Assume* (2.3), (2.5), (2.6); *let* $Q_m^\varepsilon := [(\overline{B_R} \cap H_m^-) \oplus [0, R]\{u_\varepsilon\}]$ *and let* $\Gamma := \{h \in C(Q_m^\varepsilon, H_\mu) : h(v) = v, \ \forall v \in \partial Q_m^\varepsilon\}$; *then* J *admits a PS sequence at level*

$$c = \inf_{h \in \Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)).$$

Proof. For $v \in H_m^- \oplus \mathbb{R}^+\{u_\varepsilon^m\}$ we may write $v = w + \alpha u_\varepsilon^m$, where by definition

$$|\text{supp}(u_\varepsilon^m) \cap \text{supp}(w)| = 0. \tag{4.2}$$

Claim 1. If (2.5) holds then there exist $\alpha, \rho > 0$ such that

$$J(v) \geqslant \alpha \qquad \forall v \in \partial B_\rho \cap H^+.$$

This follows as in the proof of Lemma 3 and by using $\|v\|_{H_\mu}^2 \geqslant \lambda_{k+1}\|v\|_{L^2}^2$ for any $v \in H^+$.

Claim 2. By definition of Q_m^ε, there exists $R > \rho$ such that $\max_{v \in \partial Q_m^\varepsilon} J(v) \leqslant \omega_m$ with $\omega_m \to 0$ as $m \to \infty$.

Indeed, by (2.6) we clearly have

$$\lim_{m \to \infty} \max_{v \in H_m^-} J(v) = 0$$

and by (2.3) $J(ru_\varepsilon^m) \leqslant \frac{1}{2} r^2 \|u_\varepsilon^m\|_{H_\mu}^2 - \frac{1}{2^*} r^{2^*} \|u_\varepsilon^m\|_{L^{2^*}}^{2^*}$ which, by (3.8)–(3.9) becomes negative if $r = R$ and R is large enough: therefore, $J(v) \leqslant \omega_m$ for all $v \in (H_m^-) \cup (H_m^- \oplus R\{u_\varepsilon^m\})$; finally, since $\max_{0 \leqslant r \leqslant R} J(ru_\varepsilon^m) < +\infty$, if $v \in [(\partial B_R \cap H_m^-) \oplus [0, R]\{u_\varepsilon^m\}]$, by (4.2) we obtain $J(v) \leqslant 0$ for large enough R.

By claims 1 and 2, the functional J satisfies all the assumptions of the linking theorem [R] except for the PS condition. Indeed, in view of Lemma 1, if m is large enough, then

$$P_k H_m^- = H^- \qquad \text{and} \qquad H_m^- \oplus H^+ = H,$$

where $P_k : H \to H^-$ is the projection introduced above; therefore, $\partial B_\rho \cap H^+$ and ∂Q_m^ε link (cf. [R]). Then by standard methods we obtain a PS sequence for J at level

$$c = \inf_{h \in \Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)). \qquad \blacksquare$$

5. PROOF OF THEOREM

By Lemmas 2 and 3 the proof of Theorem 1 follows if we show that there exists ε small enough such that

$$\max_{t \geqslant 0} J(tu_\varepsilon^m) < \frac{1}{n} S_\mu^{n/2}. \tag{5.1}$$

By contradiction, assume that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$J(t_\varepsilon u_\varepsilon^m) \geqslant \frac{1}{n} S_\mu^{n/2}. \tag{5.2}$$

We first show that t_ε is bounded as $\varepsilon \to 0$:

LEMMA 5. *If* (5.2) *holds, then* $t_\varepsilon \to t_0 > 0$, *up to a subsequence.*

Proof. By contradiction assume that $t_\varepsilon \to +\infty$ up to subsequences, then by (2.3) we have

$$J(t_\varepsilon u_\varepsilon^m) \leqslant \frac{1}{2} t_\varepsilon^2 \|u_\varepsilon^m\|_{H_\mu}^2 - \frac{1}{2^*} t_\varepsilon^{2^*} \|u_\varepsilon^m\|_{L^{2^*}}^{2^*} \to -\infty \qquad \text{as} \quad \varepsilon \to 0$$

in contradiction with (5.2). As G has subcritical growth at infinity and as $\{t_\varepsilon\}$ is bounded, we get

$$\lim_{\varepsilon \to 0} \int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx = 0. \tag{5.3}$$

Moreover, there exists $t_0 \geqslant 0$ such that $t_\varepsilon \to t_0$, up to a subsequence; if $t_\varepsilon \to 0$ then by (3.8), (3.9) and (5.3) we have

$$J(t_\varepsilon u_\varepsilon^m) = \frac{1}{2} t_\varepsilon^2 \|u_\varepsilon^m\|_{H_\mu}^2 - \int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx - \frac{1}{2^*} t_\varepsilon^{2^*} \|u_\varepsilon^m\|_{L^{2^*}}^{2^*} = o(1)$$

which contradicts (5.2) ∎

By arguing as in Lemma 4 in [GR] we obtain

$$\frac{1}{2} \|t_\varepsilon u_\varepsilon^m\|_{H_\mu}^2 - \frac{1}{2^*} \|t_\varepsilon u_\varepsilon^m\|_{L^{2^*}}^{2^*} \leqslant \frac{1}{n} S_\mu^{n/2} + c\varepsilon^{n-2} \qquad \text{as} \quad \varepsilon \to 0. \tag{5.4}$$

We now estimate the lower order term $\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx$:

LEMMA 6. *There exists a function $\tau = \tau(\varepsilon)$ such that $\lim_{\varepsilon \to 0} \tau(\varepsilon) = +\infty$ and such that for ε small enough we have*

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx \geqslant \tau(\varepsilon) \cdot \varepsilon^{n-2}.$$

Proof. In this proof, all positive constants will be denoted by C. Consider first the case $\bar{\mu} - 1 < \mu < \bar{\mu}$ and let

$$\beta = \frac{\sqrt{\bar{\mu}}}{\sqrt{\bar{\mu} - \mu}}, \qquad p = \frac{2(n - 2\sqrt{\bar{\mu} - \mu})}{n-2}.$$

If ε is small enough then $B_{\varepsilon^\beta} \subset B_{1/m} \subset \Omega_0$; by (2.7) we know that there exists a continuous function $\varphi = \varphi(s)$ with $\lim_{s \to +\infty} \varphi(s) = +\infty$ and there exists $\bar{s} \geqslant 0$ such that

$$\text{if } s \geqslant \bar{s}, \text{ then } G(x, s) \geqslant \varphi(s)\, s^p \qquad \text{for a.e.} \quad x \in \Omega. \tag{5.5}$$

For any $x \in B_{\varepsilon^\beta}$ we have (γ and γ' as in (3.7))

$$\varepsilon^2 |x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}} \leqslant 2\varepsilon^{\gamma/\sqrt{\bar\mu-\mu}}; \tag{5.6}$$

then by Lemma 5, and for ε small enough we have $t_\varepsilon u_\varepsilon^m(x) > \bar s$ for any $x \in B_{\varepsilon^\beta}$: hence, we can use (5.5) which, combined with (2.3), gives

$$\begin{aligned}\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx &\geqslant \int_{B_{\varepsilon^\beta}} G(x, t_\varepsilon u_\varepsilon^m)\, dx \\ &\geqslant C\varphi(C\varepsilon^{-\bar\mu/\sqrt{\bar\mu-\mu}}) \int_{B_{\varepsilon^\beta}} (u_\varepsilon^*(x) - u_\varepsilon^*(1/m))^p\, dx.\end{aligned} \tag{5.7}$$

Let $q = p^{1/\gamma'} > 1$; then for small enough ε we have $B_{\varepsilon^\beta/q} \subset B_{1/(qm)}$ and

$$u_\varepsilon^*(x) \geqslant u_\varepsilon^*\left(\frac{1}{qm}\right) > p u_\varepsilon^*(1/m) \qquad \forall x \in B_{\varepsilon^\beta/q};$$

hence by (5.6) and (5.7)

$$\begin{aligned}\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx &\geqslant C\varphi(C\varepsilon^{-\bar\mu/\sqrt{\bar\mu-\mu}}) \int_0^{\varepsilon^\beta/q} (u_\varepsilon^*(r))^p\, r^{n-1}\, dr \\ &\geqslant C\varepsilon^{-(\bar\mu/\sqrt{\bar\mu-\mu})\,p}\varphi(C\varepsilon^{-\bar\mu\sqrt{\bar\mu-\mu}}) \int_0^{\varepsilon^\beta/q} r^{n-1}\, dr \\ &= C\varphi(C\varepsilon^{-\bar\mu/\sqrt{\bar\mu-\mu}})\, \varepsilon^{n-2}.\end{aligned}$$

This ends the proof in the case $\bar\mu - 1 < \mu < \bar\mu$ by setting

$$\tau(\varepsilon) = \varphi(C\varepsilon^{-\bar\mu/\sqrt{\bar\mu-\mu}}).$$

Consider now the case $0 \leqslant \mu \leqslant \bar\mu - 1$: we want to apply (2.4) and we require that

$$\begin{aligned}t_\varepsilon u_\varepsilon^m(x) &= t_\varepsilon(u_\varepsilon^*(x) - u_\varepsilon^*(1/m)) \\ &\leqslant t_\varepsilon u_\varepsilon^*(x) = \frac{t_\varepsilon C_\varepsilon}{[\varepsilon^2 |x|^{\gamma'/\sqrt{\bar\mu}} + |x|^{\gamma/\sqrt{\bar\mu}}]^{\sqrt{\bar\mu}}} \leqslant \frac{t_\varepsilon C_\varepsilon}{|x|^\gamma} \leqslant \delta \qquad \forall x \in B_{1/m}.\end{aligned}$$

The last inequality holds if and only if

$$|x| \geqslant \left(\frac{t_\varepsilon}{\delta}\right)^{1/\gamma} \left(\frac{4\varepsilon^2 n(\bar\mu-\mu)}{n-2}\right)^{\sqrt{\bar\mu}/2\gamma};$$

by Lemma 5 there exists $C_1 > 0$ such that, for ε small enough

$$\left(\frac{t_\varepsilon}{\delta}\right)^{1/\gamma}\left(\frac{4\varepsilon^2 n(\bar{\mu}-\mu)}{n-2}\right)^{\sqrt{\bar{\mu}}/2\gamma} < C_1 \varepsilon^{\sqrt{\bar{\mu}}/\gamma} < \frac{1}{m},$$

so that if $|x| \geqslant C_1 \varepsilon^{\sqrt{\bar{\mu}}/\gamma}$ then $t_\varepsilon C_\varepsilon / |x|^\gamma \leqslant \delta$. Note that there exists $C_2 > 0$ such that

$$\varepsilon^2 |x|^{\gamma'/\sqrt{\bar{\mu}}} + |x|^{\gamma/\sqrt{\bar{\mu}}} \leqslant C_2 |x|^{\gamma/\sqrt{\bar{\mu}}} \qquad \forall\, |x| \geqslant C_1 \varepsilon^{\sqrt{\bar{\mu}}/\gamma}. \tag{5.8}$$

Let $q = 2^{1/\gamma'}$; we argue as in the previous case: by (2.3), (2.4), (5.8) and Lemma 5 we have

$$\begin{aligned}\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx &\geqslant C \int_{C_1\varepsilon^{\sqrt{\bar{\mu}}/\gamma}}^{1/qm} (u_\varepsilon^*(r) - u_\varepsilon^*(1/m))^2\, r^{n-1}\, dr \\ &\geqslant C \int_{C_1\varepsilon^{\sqrt{\bar{\mu}}/\gamma}}^{1/qm} (u_\varepsilon^*(r))^2\, r^{n-1}\, dr \geqslant C C_\varepsilon^2 \int_{C_1\varepsilon^{\sqrt{\bar{\mu}}/\gamma}}^{1/qm} r^{1-2\sqrt{\bar{\mu}-\mu}}\, dr.\end{aligned}$$

To continue we distinguish two cases.

Case 1. $\mu < \bar{\mu} - 1$.

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx \geqslant C\varepsilon^{2\sqrt{\bar{\mu}}}\varepsilon^{2(\sqrt{\bar{\mu}}/\gamma - \sqrt{\bar{\mu}}\sqrt{\bar{\mu}-\mu}/\gamma)} \geqslant \tau(\varepsilon)\, \varepsilon^{n-2}$$

with

$$\tau(\varepsilon) = C\varepsilon^{2(\sqrt{\bar{\mu}}/\gamma - \sqrt{\bar{\mu}}\sqrt{\bar{\mu}-\mu}/\gamma)}$$

and $\lim_{\varepsilon\to 0} \tau(\varepsilon) = +\infty$ because $\sqrt{\bar{\mu}}/\gamma - \sqrt{\bar{\mu}}\sqrt{\bar{\mu}-\mu}/\gamma < 0$.

Case 2. $\mu = \bar{\mu} - 1$.

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx \geqslant C \int_{C_1\varepsilon^{\sqrt{\bar{\mu}}/\gamma}}^{1/qm} C_\varepsilon^2 r^{-1}\, dr = C\varepsilon^{2\sqrt{\bar{\mu}}}\, |\ln(C\varepsilon^{\sqrt{\bar{\mu}}/\gamma})| = \tau(\varepsilon)\, \varepsilon^{n-2}$$

with $\tau(\varepsilon) = C\, |\ln(C\varepsilon^{\sqrt{\bar{\mu}}/\gamma})|$.

In conclusion, also if $0 \leqslant \mu \leqslant \bar{\mu} - 1$ there exists $\tau = \tau(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \tau(\varepsilon) = +\infty$$

and such that for ε small enough we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\, dx \geqslant \tau(\varepsilon)\, \varepsilon^{n-2}. \qquad \blacksquare$$

The proof of Theorem 1 is now obtained using (5.4), Lemmas 5 and 6; indeed if ε is small enough we have

$$J(t_\varepsilon u_\varepsilon^m) \leqslant \frac{1}{n} S_\mu^{n/2} + (c - \tau(\varepsilon)) \cdot \varepsilon^{n-2} < \frac{1}{n} S_\mu^{n/2}$$

which contradicts (5.2). ∎

6. PROOF OF THEOREM 2

Since the identity $Id \in \Gamma$, we have

$$\inf_{h \in \Gamma} \max_{v \in Q_m^\varepsilon} J(h(v) \leqslant \max_{v \in Q_m^\varepsilon} J(v);$$

by Lemmas 2 and 4, Theorem 2 follows if we can prove that for some $\varepsilon > 0$ and $m \in \mathbb{N}$ we have

$$\sup_{v \in Q_m^\varepsilon} J(v) < \frac{1}{n} S_\mu^{n/2}. \tag{6.1}$$

By contradiction assume that

$$\forall m \in \mathbb{N}, \quad \forall \varepsilon > 0 \qquad \sup_{v \in Q_m^\varepsilon} J(v) \geqslant \frac{1}{n} S_\mu^{n/2}. \tag{6.2}$$

As the set $\{v \in Q_m^\varepsilon; J(v) \geqslant 0\}$ is compact, the supremum in (6.2) is attained. Therefore, for all $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geqslant 0$ such that, for $v_\varepsilon := w_\varepsilon + t_\varepsilon u_\varepsilon^m$, we have

$$J(v_\varepsilon) = \max_{v \in Q_m^\varepsilon} J(v) \geqslant \frac{1}{n} S_\mu^{n/2},$$

that is

$$\frac{1}{2} \|v_\varepsilon\|_{H_\mu}^2 - \int_\Omega G(x, v_\varepsilon)\, dx - \frac{1}{2^*} \|v_\varepsilon\|_{L^{2^*}}^{2^*} \geqslant \frac{1}{n} S_\mu^{n/2}, \qquad \forall \varepsilon > 0. \tag{6.3}$$

By claim 2 in the proof of Lemma 4 we immediately obtain that the sequences $\{t_\varepsilon\} \subset \mathbb{R}^+$ and $\{w_\varepsilon\} \subset H_m^-$ are bounded. Hence, up to subsequences we may assume that

$$t_\varepsilon \to t_0 \geqslant 0 \qquad w_\varepsilon \to w_0 \in H_m^-,$$

where the convergence of $\{w_\varepsilon\}$ can be viewed in any norm topology since the space H_m^- is finite dimensional. As $w_\varepsilon \in H_m^-$, by using Lemma 1(i) and (2.6) we have

(6.4)
$$\begin{aligned} J(w_\varepsilon) &= \frac{1}{2}\|w_\varepsilon\|_{H_\mu}^2 - \int_\Omega G(x, t_\varepsilon w_\varepsilon)\,dx - \frac{1}{2^*}\|w_\varepsilon\|_{L^{2^*}}^{2^*} \\ &\leqslant \frac{\lambda_k + o(1)}{2}\|w_\varepsilon\|_{L^2}^2 - \frac{\lambda_k}{2}\|w_\varepsilon\|_{L^2}^2 - \frac{\eta}{2}\|w_\varepsilon\|_{L^2}^2 + \frac{1}{2^*}\|w_\varepsilon\|_{L^{2^*}}^{2^*} - \frac{1}{2^*}\|w_\varepsilon\|_{L^{2^*}}^{2^*} \\ &= \frac{o(1)-\eta}{2}\|w_\varepsilon\|_{L^2}^2 \leqslant 0 \end{aligned}$$

for m large enough (from now on we maintain m fixed). By using (6.3) and by arguing as for Lemma 5, we have $t_\varepsilon \to t_0 > 0$, up to a subsequence. Moreover, by arguing as in Lemma 6 we have

$$\int_\Omega G(x, t_\varepsilon u_\varepsilon^m)\,dx \geqslant \tau(\varepsilon)\cdot\varepsilon^{n-2} \tag{6.5}$$

for ε small enough.

The proof of Theorem 2 is now easily completed: by (4.2), (5.4), (6.4) and (6.5) (which all hold because we assumed (6.3)) we have

$$J(v_\varepsilon) = J(w_\varepsilon) + J(t_\varepsilon u_\varepsilon^m) \leqslant \frac{1}{n} S_\mu^{n/2} + (c - \tau(\varepsilon))\cdot\varepsilon^{n-2},$$

which contradicts (6.3) for ε small enough; thus (6.1) holds. ∎

7. PROOF OF THEOREM 3

Let $\lambda_+ = \min\{\lambda_j \in \sigma_\mu; \lambda < \lambda_j\}$ and assume that $\lambda_+ - \lambda < S_\mu\,|\Omega|^{-2/n}$; for any $j \in \mathbb{N}$ let $M(\lambda_j)$ be the eigenspace corresponding to λ_j, let $M^+ = \overline{\bigoplus_{\lambda_j \geqslant \lambda_+} M(\lambda_j)}$ (closure in H_μ) and let $M^- = \bigoplus_{\lambda_j \leqslant \lambda^+} M(\lambda_j)$; then the following result holds:

LEMMA 7. *We have*

$$\beta_\lambda = \sup_{u \in M^-} I(u) \leqslant (\lambda_+ - \lambda)^{n/2}\frac{|\Omega|}{n} < \frac{1}{n} S_\mu^{n/2};$$

furthermore, there exist $\rho_\lambda > 0$ and $\delta_\lambda \in (0, \beta_\lambda)$ such that $I(u) \geqslant \delta_\lambda$ for any $u \in M^+$ with $\|u\|_{H_\mu} = \rho_\lambda$.

Proof. For any $u \in M^-$ we have $\|u\|_{H_\mu}^2 \leqslant \lambda_+ \|u\|_{L^2}^2$ and by Hölder's inequality we get

$$I(u) = \frac{1}{2}\|u\|_{H_\mu}^2 - \frac{\lambda}{2}\|u\|_{L^2}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*} \leqslant \frac{1}{2}(\lambda_+ - \lambda)\|u\|_{L^2}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*}$$

$$\leqslant \frac{1}{2}(\lambda_+ - \lambda)|\Omega|^{2/n}\|u\|_{L^{2^*}}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*};$$

since

$$\max_{\rho \geqslant 0}\left[\frac{1}{2}(\lambda_+ - \lambda)|\Omega|^{2/n}\rho^2 - \frac{1}{2^*}\rho^{2^*}\right] = \frac{1}{n}(\lambda_+ - \lambda)^{n/2}|\Omega| < \frac{1}{n}S_\mu^{n/2},$$

we have

$$\beta_\lambda \leqslant \frac{1}{n}(\lambda_+ - \lambda)^{n/2}|\Omega| < \frac{1}{n}S_\mu^{n/2}.$$

Let $u \in M^+$, by the inequalities $\lambda_+ \|u\|_{L^2}^2 \leqslant \|u\|_{H_\mu}^2$ and $S_\mu \|u\|_{L^{2^*}}^2 \leqslant \|u\|_{H_\mu}^2$ we have

$$I(u) = \frac{1}{2}\|u\|_{H_\mu}^2 - \frac{\lambda}{2}\|u\|_{L^2}^2 - \frac{1}{2^*}\|u\|_{L^{2^*}}^{2^*} \geqslant \frac{1}{2}\frac{\lambda_+ - \lambda}{\lambda_+}\|u\|_{H_\mu}^2 - \frac{1}{2^* S_\mu^{2^*/2}}\|u\|_{H_\mu}^{2^*};$$

since

$$\max_{\rho \geqslant 0}\left[\frac{1}{2}\frac{\lambda_+ - \lambda}{\lambda_+}\rho^2 - \frac{1}{2^* S_\mu^{2^*/2}}\rho^{2^*}\right] = \frac{1}{n}\left(\frac{\lambda_+ - \lambda}{\lambda_+}\right)^{n/2} S_\mu^{n/2},$$

if we take $\rho_\lambda = (((\lambda_+ - \lambda)/\lambda_+)\, S_\mu^{2^*/2})^{(n-2)/4}$ and $\delta_\lambda < \frac{1}{n}((\lambda_+ - \lambda)/\lambda_+)^{n/2} S_\mu^{n/2}$ then we have $I(u) \geqslant \delta_\lambda$ for all $u \in M^+ \cap \partial B_{\rho_\lambda}$.

It remains to prove that $\delta_\lambda < \beta_\lambda$: since $M^+ \cap M^- = M(\lambda_+)$ we have $M^+ \cap M^- \cap B_{\rho_\lambda} \neq \varnothing$ and any $u \in M^+ \cap M^- \cap B_{\rho_\lambda}$ satisfies $\delta_\lambda < J(u) \leqslant \sup_{u \in M^-} J(u) = \beta_\lambda$. ▮

Thanks to this Lemma, to complete the proof of Theorem 3 it suffices to apply Theorem 2.5 in [CFS] (which is a restatement of Theorem 2.4 in [BBF]) with $H = H_\mu$, $V = M^+$, $W = M^-$, $\beta = S_\mu^{n/2}/n$, $\beta' = \beta_\lambda$, $\delta = \delta_\lambda$, $\rho = \rho_\lambda$ and using that $\dim V - \operatorname{codim} W = \nu_k + 1$. ▮

8. PROOF OF THEOREM 4

The proof of Theorem 4 follows the same lines as that of Theorem 2; however, some refinements of the estimates are required. In order to emphasize the dependence on m we denote v_ε^m, u_ε^m, w_ε^m instead of u_ε, w_ε, v_ε.

We again want to show (6.1) and we argue by contradiction assuming that (6.2) holds: for all m large enough and all $\varepsilon > 0$ there exist $v_\varepsilon^m \in Q_m^\varepsilon$ and $t_\varepsilon \geqslant 0$ such that

$$\frac{1}{2}\|v_\varepsilon^m\|_{H_\mu}^2 - \frac{\lambda_1}{2}\|v_\varepsilon^m\|_{L^2}^2 - \frac{1}{2^*}\|v_\varepsilon^m\|_{L^{2^*}}^{2^*} \geqslant \frac{1}{n} S_\mu^{n/2}. \tag{8.1}$$

If (8.1) holds, then the sequences $\{t_\varepsilon\}$ and $\{w_\varepsilon^m\}$ satisfy again

$$t_\varepsilon \geqslant c > 0 \qquad \text{and} \qquad \|w_\varepsilon^m\|_{H_\mu} \leqslant c. \tag{8.2}$$

In order to deal only with one parameter, we set $\varepsilon = m^{-((n+2)/(n-2))\sqrt{\bar{\mu}-\mu}}$. Then, as $m \to \infty$, (3.8) and (3.9) become

$$\|u_\varepsilon^m\|_{H_\mu}^2 \leqslant S_\mu^{n/2} + C_1 m^{-n\sqrt{\bar{\mu}-\mu}} \tag{8.3}$$

$$\|u_\varepsilon^m\|_{L^{2^*}}^{2^*} \geqslant S_\mu^{n/2} - C_2 m^{-\frac{n^2}{n-2}\sqrt{\bar{\mu}-\mu}}; \tag{8.4}$$

note that $m^{-(n^2/(n-2))\sqrt{\bar{\mu}-\mu}} = o(m^{-n\sqrt{\bar{\mu}-\mu}})$. Furthermore, as $m \to \infty$, we also have

$$\|u_\varepsilon^m\|_{L^2}^2 \geqslant C_3 m^{-(n+2)}: \tag{8.5}$$

this follows by arguing as in the proof of Lemma 6, see [F].

From now on, we denote by v^m, u^m, w^m the functions v_ε^m, u_ε^m, w_ε^m with the above choice of ε and with t_m the corresponding t_ε.

We first estimate $I(t_m u^m)$; here, the assumption $\mu < \bar{\mu} - (\frac{n+2}{n})^2$ is crucial:

Lemma 8. *If m is large enough we have*

$$I(t_m u^m) \leqslant \frac{1}{n} S_\mu^{n/2} - C m^{-(n+2)}.$$

Proof. By (8.3)–(8.5) we have

$$\begin{aligned}
I(t_m u^m) &= \frac{1}{2}\|t_m u^m\|_{H_\mu}^2 - \frac{\lambda_1}{2}\|t_m u^m\|_{L^2}^2 - \frac{1}{2^*}\|t_m u^m\|_{L^{2^*}}^{2^*} \\
&\leqslant \frac{1}{2} t_m^2 (S_\mu^{n/2} + C m^{-n\sqrt{\bar{\mu}-\mu}}) - C m^{-(n+2)} - \frac{1}{2^*} t_m^{2^*} (S_\mu^{n/2} - C m^{-\frac{n^2}{n-2}\sqrt{\bar{\mu}-\mu}}) \\
&= S_\mu^{n/2}\left(\frac{t_m^2}{2} - \frac{t_m^{2^*}}{2^*}\right) + C m^{-n\sqrt{\bar{\mu}-\mu}} - C m^{-(n+2)} + C m^{-\frac{n^2}{n-2}\sqrt{\bar{\mu}-\mu}} \\
&\leqslant \frac{S_\mu^{n/2}}{n} - C m^{-(n+2)},
\end{aligned}$$

where we used the facts that

$$\max_{s\geqslant 0}\left(\frac{s^2}{2}-\frac{s^{2^*}}{2^*}\right)=\frac{1}{n}$$

and

$$n+2<n\sqrt{\bar{\mu}-\mu}<\frac{n^2}{n-2}\sqrt{\bar{\mu}-\mu}$$

in which, the first inequality is a consequence of the assumption

$$0\leqslant\mu<\bar{\mu}-\left(\frac{n+2}{n}\right)^2. \qquad \blacksquare$$

Next we estimate the part of the functional relative to w^m:

LEMMA 9. *If m is large enough we have*

$$I(w^m)\leqslant cm^{-n\sqrt{\bar{\mu}-\mu}}.$$

Proof. By Lemma 1(ii) and Hölder's inequality we have

$$I(w^m)=\frac{1}{2}\|w^m\|^2_{H_\mu}-\frac{\lambda_1}{2}\|w^m\|^2_{L^2}-\frac{1}{2^*}\|w^m\|^{2^*}_{L^{2^*}}\leqslant C_1m^{-2\sqrt{\bar{\mu}-\mu}}\|w^m\|^2_{L^2}-C_2\|w^m\|^{2^*}_{L^2}.$$

By elementary calculus, we know that

$$\max_{s\geqslant 0}\,[C_1m^{-2\sqrt{\bar{\mu}-\mu}}s^2-C_2s^{2^*}]=Cm^{-n\sqrt{\bar{\mu}-\mu}}$$

and the result follows. $\blacksquare$

The proof of Theorem 4 is now obtained by (4.2) and Lemmas 8 and 9:

$$I(v^m)=I(t_mu^m)+I(w^m)\leqslant\frac{1}{n}S_\mu^{n/2}+c_1m^{-n\sqrt{\bar{\mu}-\mu}}-c_2m^{-(n+2)}<\frac{S_\mu^{n/2}}{n} \tag{8.6}$$

for m sufficiently large; indeed from the assumption

$$0\leqslant\mu<\bar{\mu}-\left(\frac{n+2}{n}\right)^2,$$

we deduce again $n+2<n\sqrt{\bar{\mu}-\mu}$.

The inequality (8.6) contradicts (8.1), and the proof of Theorem 4 is complete.

9. OPEN PROBLEMS

9.1. *Pohožaev nonexistence result.* A formal application of Pohožaev type identities [P, PS1] shows that if Ω is star-shaped and $\lambda \leqslant 0$ then (1.1) has only the trivial solution $u \equiv 0$. Indeed, assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solves (1.1) and let

$$F(x, u, \nabla u) = \frac{1}{2} |\nabla u|^2 - \frac{\mu}{2} \frac{u^2}{|x|^2} - \frac{\lambda}{2} u^2 - \frac{1}{2^*} |u|^{2^*};$$

then, by taking $a = \frac{n}{2} - 1$ in (5) of [PS1] we infer that

$$\lambda \int_\Omega u^2 = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \, (x \cdot \nu),$$

where $\nu = \nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x. Therefore, if Ω is star-shaped with respect to the origin and $\lambda < 0$, then (1.1) admits no nontrivial solutions $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. When $\lambda = 0$, Hopf's boundary point Lemma shows that (1.1) admits no *positive* solutions with such regularity. Since solutions of (1.1) are not expected to be smooth, one should wonder if it is still possible to apply this identity without smoothness assumptions on the solutions. Let us mention that a similar problem arises for the p-Laplacian operator for which, in general, one does not have more than $C^{1,\alpha}$-regularity: in this case, the problem has been solved in [GV, Theorem 1.1].

9.2. *What happens if $\mu < 0$?* Throughout this paper we have assumed that $\mu \geqslant 0$; this is used at two distinct points. First of all, we recall that symmetrization leaves the L^2-norm of functions unchanged, increases the L^2-norm with the singular weight $|x|^{-2}$ (see e.g. [AL, Theorem 2.2]) and decreases the L^2-norm of the gradient (see [AL, Theorem 2.7]): therefore, when $\mu \geqslant 0$ the constant S_μ is attained by some entire radially symmetric function. Of course, this allows us to reduce the corresponding Euler equation to an ODE and to determine the minimizer explicitly: it is precisely the function u_ε^* introduced in (3.7), see [F] for the details. Is this still true when $\mu < 0$?

Similarly, when $\Omega = B$, the variational characterization of the first eigenvalue λ_1 and the same arguments as above allow us to conclude that the first eigenfunction is radially symmetric; then, we apply the asymptotic estimates (3.5) (which are known to hold only for radial eigenfunctions) in order to prove Theorem 4: is the first eigenfunction radially symmetric also when $\mu < 0$?

9.3. *Asymptotic behavior of eigenfunctions.* As already mentioned, the very particular situation considered in Theorem 4 is due to the fact that the asymptotic behavior (as $|x| \to 0$) of eigenfunctions of the operator $-\Delta - \mu/|x|^2$ is known ony for radial eigenfunctions whenever $\Omega = B$. If a similar behavior also holds for every eigenfunction in any bounded domain $\Omega \ni 0$, then we would immediately have the following extension of Theorem 4:

THEOREM 4′. *Let $\Omega \ni 0$ be an open bounded domain, $\Omega \subset \mathbb{R}^n$ $(n \geqslant 5)$ and assume that $0 \leqslant \mu < (\frac{n-2}{2})^2 - (\frac{n+2}{n})^2$; then, for all $\lambda > 0$ problem* (1.1) *admits a nontrivial solution with critical level in the range* $(0, S_\mu^{n/2}/n)$.

9.4. *Nonresonant situations.* Assume that $n > 2 + 2\sqrt{2}$ so that $\frac{n-2}{2} > \frac{n+2}{n}$; according to the definition given in [GG2], in order to verify that the nonresonant situation is precisely when $(\frac{n-2}{2})^2 - (\frac{n+2}{n})^2 \leqslant \mu \leqslant (\frac{n-2}{2})^2 - 1$, one should perform an asymptotic analysis as in [ABP]. More precisely, consider the ODE problem $(0 < r < 1)$

$$u'' + \frac{n-1}{r} u' + \frac{\mu}{r^2} u + \lambda u + |u|^{2^*-2} u = 0 \qquad (9.1)$$
$$u'(0) = u(1) = 0,$$

for which one is interested in solutions u_λ having exactly one zero in the interval $[0, 1)$. One should prove that:

(i) there exists $\delta > 0$ such that if

$$\left(\frac{n-2}{2}\right)^2 - \left(\frac{n+2}{n}\right)^2 - \delta < \mu < \left(\frac{n-2}{2}\right)^2 - \left(\frac{n+2}{n}\right)^2$$

then $\lambda \to \lambda_1^-$ as $u_\lambda(0) \to \infty$.

(ii) if

$$\left(\frac{n-2}{2}\right)^2 - \left(\frac{n+2}{n}\right)^2 \leqslant \mu \leqslant \left(\frac{n-2}{2}\right)^2 - 1$$

then $\lambda \to \lambda_1^+$ as $u_\lambda(0) \to \infty$.

Perhaps, one could set $v(r) = r^\alpha u(r)$ with

$$\alpha = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \mu}$$

so that the equation in (9.1) becomes

$$v'' + \frac{\nu - 1}{r} v' + \lambda v + \frac{|v|^{2^* - 2} v}{r^{\beta}} = 0,$$

where

$$\nu = 2 + \sqrt{(n-2)^2 - 4\mu} \qquad \text{and} \qquad \beta = -2 + \frac{2}{n-2} \sqrt{(n-2)^2 - 4\mu}:$$

with this change of variables, we eliminated one term and the singular term is also the only nonlinear term.

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