Counterexamples to Symmetry for Partially Overdetermined Elliptic Problems

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Abstract

We exhibit several counterexamples showing that the famous symmetry Serrin's result for semilinear elliptic overdetermined problems may not hold for *partially* overdetermined problems, that is when both Dirichlet and Neumann boundary conditions are prescribed only *on part* of the boundary. Our counterexamples enlighten subsequent positive symmetry results obtained by the first two authors for such partially overdetermined systems and justify their assumptions as well.

1 Introduction

Let Ω be an open bounded connected subset of $\mathbb{R}^n$ with smooth enough boundary, and let Γ be a nonempty connected (relatively) open subset of $\partial\Omega$. Let also ν denote the unit outer normal to $\partial\Omega$, c be a positive constant and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. We call *overdetermined problem* any boundary value problem of the following kind:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 \quad \text{and} \quad u_\nu = -c & \text{on } \Gamma \\ u = 0 & \text{on } \partial\Omega \setminus \Gamma \,, \end{cases} \tag{1}$$

or

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 \quad \text{and} \quad u_\nu = -c & \text{on } \Gamma \\ |\nabla u| = c & \text{on } \partial\Omega \setminus \Gamma \,, \end{cases} \tag{2}$$

where u_ν denotes the normal derivative of u on $\partial\Omega$. Here and in the sequel, by a solution u to problem (1) (resp. (2)), we always mean that $u \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^1(\Omega \cup \Gamma) \cap \mathcal{C}^2(\Omega)$ (resp. $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$).

The choice of the word "overdetermined" is justified by the presence of both the Dirichlet and Neumann conditions on a same nonempty part Γ of the boundary in problems (1)-(2): this makes them in general not well-posed. Thus the existence of a solution to (1) or (2) is not always guaranteed, and, if existence happens to hold, it is actually supposed to imply some severe geometric constraint on Ω.

This kind of problem was studied by Serrin [14]. His celebrated result states that, in the case of *totally* overdetermined problems, that is when $\Gamma \equiv \partial\Omega$, then existence of a solution implies that Ω is a ball (and u is radially symmetric).

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More recently, the case of *partially* overdetermined problems, that is when $\Gamma \subsetneq \partial\Omega$, has been studied by the first two authors in [8], where they investigate the following natural question:

"If $\Gamma \subsetneq \partial\Omega$, can we still conclude that Ω is a ball whenever (1) or (2) admits a solution?"

The answer is trivially ***no*** without any extra natural geometric restriction on Ω. Assume, for instance, that Ω is an annulus, that is $\Omega = \{x \in \mathbb{R}^n; 0 < a < |x| < b\}$. Then, the solution of $-\Delta u = 1$ on Ω, with $u = 0$ on its boundary, is radially symmetric. Therefore, u_ν is equal to a constant on each piece of the boundary, but with different constants for each of them.

On the other hand, if $\partial\Omega$ is assumed to be connected, the problem becomes much more significant and delicate. In fact there are many different situations where the answer to the above question is ***yes***, so that Serrin's symmetry result continues to hold. This occurs under suitable additional assumptions, involving both regularity and geometric features, on the source term f and the overdetermined region Γ: for the detailed statements, as well as for a more extensive bibliography about overdetermined problems, we refer to [8].

The goal of this note is to show that there are nontrivial cases (meaning in particular that $\partial\Omega$ is connected) when the requirements of [8] are not satisfied and problems like (1)-(2) admit a solution in domains Ω different from a ball.

The counterexamples we construct for problems of type (1) or (2) are of different kind. Problems of type (1) are treated in Section 2 by an approach based on shape optimization and domain derivative. More precisely, we consider the problem of minimizing the Dirichlet energy of domains with prescribed volume and confined in a planar box, that is

$$|\Omega^*| = \alpha,\ \Omega^* \subset D,\ J(\Omega^*) = \min_{|\Omega|=\alpha, \Omega\subset D} J(\Omega), \tag{3}$$

where $D = (-1,1)^2$ and

$$J(\Omega) := \inf_{v \in H_0^1(\Omega)} \left\{ \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - v \right) dx \right\}. \tag{4}$$

Choosing α in a suitable range and applying the regularity results in [1, 2], we obtain that (3) admits an optimal open shape Ω^* with a nonempty smooth "free boundary" $\partial\Omega^* \cap D$. Then, writing down the optimality conditions by using shape derivatives, we are lead to a problem of type (1) on Ω^*, with $f \equiv 1$ and $\Gamma = \partial\Omega^* \cap D$.

Problems of type (2) are treated in Section 3 by a different approach, which works in any dimension $n \geq 2$. In this case, the counterexamples are derived through some explicit computations. They are based on the idea of studying the zero level surfaces of radial functions u built so as to satisfy both an elliptic equation of the type $-\Delta u = f(u)$ on the whole $\mathbb{R}^n$ and the eikonal equation $|\nabla u| = c$ on the complement of a ball. Such construction can be adapted to treat also the case of a partially overdetermined problem similar to (2), but stated on an exterior domain (see Section 3.2).

2 Counterexamples using shape optimization

In this section we use shape optimization in order to prove the following.

Theorem 2.1 *There exists an open starshaped planar domain $\Omega \subset (-1,1)^2$, different from a disk, such that, for a nonempty connected analytic subset Γ of $\partial\Omega$, the problem*

$$\begin{cases} -\Delta u &= 1 & in & \Omega \\ u &= 0 & on & \partial\Omega \\ u_\nu &= -c & on & \Gamma, \end{cases} \tag{5}$$

admits a solution.

The interest of this negative result should be considered in the light of the following extension of Serrin's result proved in [8]:

Proposition 2.2 *Let Ω be open and bounded with $\partial\Omega$ connected. Let $\Gamma \subset \partial\Omega$ nonempty and (relatively) open. Assume there exists an open set $\widetilde{\Omega}$ with a connected* analytic *boundary containing Γ. If there exists a solution u of* (1) *with f analytic, then $\Omega = \widetilde{\Omega}$, Ω is a ball, and u is radially symmetric.*

In particular, Proposition 2.2 implies that the analytic piece Γ of the boundary of Ω found in Theorem 2.1 cannot be continued into a globally analytic closed "curve" (namely the boundary of another open set $\widetilde{\Omega}$). In the counterexample provided here, $\partial\Omega$ is piecewise analytic and globally at most $\mathcal{C}^{1,\frac{1}{2}}$ as analyzed in [13].

Proof of Theorem 2.1. Let $D = (-1,1)^2$ and $\alpha \in (\pi, 4)$. We will construct Ω as an optimal set for the shape minimization problem (3).

From [4, Theorem 2.4.6] (see also [10]), we know there exists a quasi-open optimal set Ω^* which solves problem (3). In view of [2, Corollary 1.2], Ω^* is in fact an open set. It is known that, for any open bounded set Ω (and in particular for Ω^*), the functional J defined in (4) satisfies

$$J(\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla u_\Omega|^2 - u_\Omega\right) dx$$

where u_Ω denotes the unique solution of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{6}$$

Since $\alpha < 4$, Ω^* cannot be equal to D so that the free boundary $\Gamma := \partial\Omega^* \cap D$ is nonempty. Moreover, by [1, Section 5], we infer that Γ is analytic because $f \equiv 1$ is positive and analytic. On this "free boundary" Γ, using the notion of shape derivative (see for instance [10]), we classically obtain the Euler-Lagrange equation of problem (3), namely, (6) with $\Omega = \Omega^*, u = u_{\Omega^*}$ together with

$$|\nabla u_{\Omega^*}| = \Lambda > 0 \text{ on } \partial\Omega^* \cap D. \tag{7}$$

Since $f(u) = 1 > 0$, the positivity of the Lagrange multiplier Λ follows from [1, Proposition 6.1]. By elliptic regularity, we know that there exists a unique solution $u_{\Omega^*} \in \mathcal{C}^\infty(\Omega \cup \Gamma)$ to (7).

We now prove the geometric properties of solutions of (3). First, since $\alpha > \pi$, Ω^* is not a disk. Second, we show that Ω^* is starshaped, or at least that it may be replaced by an optimal starshaped set. To this end, we introduce $\widetilde{\Omega} := S_Y S_X(\Omega^*)$, where S_X and S_Y denote the Steiner symmetrization about the axes OX and OY respectively, see e.g. [10], [12]. Because of the symmetry of the square D with respect to these axes, we have $\widetilde{\Omega} \subset D$. Moreover, $|\widetilde{\Omega}| = |\Omega^*| = \alpha$ and, by well-known properties of Steiner symmetrization, $J(\widetilde{\Omega}) \le J(\Omega^*)$. Therefore, $\widetilde{\Omega}$ is also a solution of the shape optimization problem (3) so that, as for any optimal set, $\widetilde{\Gamma} = \partial\widetilde{\Omega} \cap D$ is smooth and $u_{\widetilde{\Omega}}$ satisfies (5). To verify that it is starshaped, we may denote

$$\forall x \in [-1,1],\ \ A(x) := \{y \in [-1,1];\ (x,y) \in S_X(\Omega^*)\}.$$

As a consequence of the definition of the Steiner symmetrization, we have $[0 \le x \le \hat{x}] \Rightarrow [A(\hat{x}) \subset A(x)]$. We may also write

$$S_Y S_X(\Omega^*) = \left\{(x,y); |y| \le \frac{1}{2} meas A(x)\right\}.$$

Since $x \in [0,1] \to meas A(x)$ is nonincreasing, we have

$$\left[|y| \le \frac{1}{2} meas A(x), \lambda \in (0,1)\right] \ \Rightarrow\ \left[|\lambda y| \le \frac{1}{2} meas A(x) \le \frac{1}{2} meas A(\lambda x)\right].$$

This proves that $\widetilde{\Omega}$ is starshaped.

Therefore, $\Omega = \widetilde{\Omega}, \Gamma = \partial\Omega \cap D, u = u_{\widetilde{\Omega}}, c = \Lambda$ satisfy the statement of Theorem 2.1. □

We conclude this section by mentioning some possible extensions of Theorem 2.1.

Remark 2.3 The construction done in the proof of Theorem 2.1 is valid in any dimension and one finds as well an optimal *open* set $\Omega^* \subset (-1,1)^n$ (see [1] for a proof), which is different from a ball if $\alpha > |B_{\mathbb{R}^n}(0,1)|$. But, the full regularity of the boundary is not proved -and probably does not hold- in any dimension. According to some recent papers ([5, 7, 15, 16]), it is very likely that full regularity of the boundary may be extended to dimensions greater than 2 (up to 6? but not more?).

However, as proved in [1], the *reduced boundary* of this Ω^* is an analytic hypersurface and this regular part of the boundary is of positive $(n-1)$-Hausdorff measure if $\alpha < 2^n$, whereas Ω^* is not a ball if $\alpha > \omega_n$ (the measure of the unit ball). Therefore, this also provides a (generalized) counterexample in any dimension by choosing Γ to be this reduced boundary.

Remark 2.4 In view of [3] (see also [9, Section 3.4]), it is possible to extend the statement of Theorem 2.1 to the case when J is replaced by the shape functional $\Omega \to \lambda_1(\Omega)$, the first eigenvalue of the Laplace operator on Ω with homogeneous Dirichlet boundary conditions. This provides one more example of an optimal domain Ω^* where u_{Ω^*}, the first normalized eigenfunction, solves (1) with $f(u) = \lambda u$ (here, $\lambda = \lambda_1(\Omega^*)$). The proof is similar and we do not reproduce it here. It is possible that one could go further and extend the same construction to more general sources $f(u)$, for instance of power-type such as $f(u) = u^p$.

Remark 2.5 The minimal shape Ω^* for the second Dirichlet eigenvalue $\lambda_2(\Omega)$ of the Laplace operator, among all planar *convex* domains of given area, is also a natural candidate for another nice counterexample. It is expected that Ω^* looks like a "stadium" (the convex envelope of two identical tangent balls), without nevertheless being exactly a stadium, see [11]. If so, and as explained in [8], the strictly convex part of $\partial\Omega^*$ would provide the expected analytic part Γ. But it is not obvious that Γ be nonempty. The exact regularity of Ω^* is still to be completely understood: see [11, Theorems 4,6,8] and [13].

Remark 2.6 In the proof of Theorem 2.1, we started with some optimal shape Ω^* and adapted it so that it satisfies the required conditions. We may wonder whether *all optimal shapes* have the same symmetry properties. This question is related to the nontrivial question of equality case in the Steiner symmetrization, namely: is it true that $J(\Omega) = J(S_X(\Omega))$ implies that $\Omega = S_X(\Omega)$ up to a translation? We refer to [6] for this question.

3 Counterexamples via explicit construction

In this section we provide an explicit example of a problem of type (2) which admits a solution on a domain different from a ball. We also exhibit a similar example for an analogous exterior problem.

3.1 A counterexample in an interior domain

We denote by B the unit ball in $\mathbb{R}^n$ and we prove

Theorem 3.1 *There exists an open bounded simply connected domain $\Omega \subset \mathbb{R}^n$ different from a ball, with $\partial\Omega$ globally $\mathcal{C}^\infty$ and containing a connected portion Γ of ∂B, and there exists a Lipschitz continuous and strictly increasing function $f : \mathbb{R} \to (0, +\infty)$ such that the problem*

$$\begin{cases} -\Delta u = f(u) & in \quad \Omega \\ |\nabla u| = 8 & on \quad \partial\Omega \\ u = 0 & on \quad \Gamma, \end{cases} \tag{8}$$

admits a solution $u \in \mathcal{C}^2(\overline{\Omega})$.

Proof. Fix an integer $n \geq 2$ and consider the function $f : \mathbb{R} \to (0, +\infty)$ defined by

$$f(s) = \begin{cases} \dfrac{64(n-1)}{8-s} & \text{if } s \leq 0 \\ \\ 4\Big[(n+2)\sqrt{s+4} - 6\Big] & \text{if } s \geq 0 \ . \end{cases}$$

Then, f is globally Lipschitz continuous and strictly increasing over $\mathbb{R}$.

Consider also the (radial) function u defined on $\mathbb{R}^n$ by

$$u(x) = \begin{cases} (3 - |x|^2)^2 - 4 & \text{if } |x| \leq 1 \\ 8(1 - |x|) & \text{if } |x| \geq 1 \ . \end{cases}$$

Then, $u \in \mathcal{C}^2(\mathbb{R}^n)$; to see this, it suffices to write $u = u(r)$ as a function of the real variable $r = |x|$ and to note that

$$u'(r) = \begin{cases} -4r(3 - r^2) & \text{if } r \leq 1 \\ -8 & \text{if } r \geq 1 \ , \end{cases} \qquad u''(r) = \begin{cases} -12 + 12r^2 & \text{if } r \leq 1 \\ 0 & \text{if } r \geq 1 \ , \end{cases}$$

are continuous functions in $[0, \infty)$. Moreover, some computations show that u satisfies

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n \ , \qquad u = 0 \quad \text{on } \partial B \ , \qquad |\nabla u| = 8 \quad \text{in } \mathbb{R}^n \setminus B \ ,$$

where B denotes the unit ball.

Let $\Omega_1 = \{x \in B; x_1 < \frac{1}{2}\}$ and $D = \{x \in B; x_1 = \frac{1}{2}\}$. Consider a bounded domain $\Omega_2 \subset \{x \in \mathbb{R}^n; x_1 > \frac{1}{2}\}$ such that $D \subset \partial\Omega_2$ and $(\partial\Omega_2 \setminus \overline{D}) \subset (\mathbb{R}^n \setminus \overline{B})$. Let $\Omega = \Omega_1 \cup D \cup \Omega_2$; for a suitable choice of Ω_2 one has $\partial\Omega \in \mathcal{C}^\infty$. Let $\Gamma = \partial\Omega_1 \cap \partial\Omega$, then u satisfies (8) but Ω is not a ball. $\square$

Theorem 3.1 should be compared with the statements (b) in Theorems 1-3-7 in [8]. Note in particular that:

- the overdetermined part Γ is analytically continuable according to the definition in [8, Section 3.1];

- the boundary $\partial\Omega$ is smooth and Γ has large maximal mean curvature according to the definition in [8, Section 3.2];

- the overdetermined part Γ contains a hat according to the definition in [8, Section 3.3].

Therefore, Theorem 3.1 shows that if f is not analytic and/or f is increasing, then Theorems 1-3-7 (b) in [8] do not hold.

3.2 A counterexample in an exterior domain

Theorem 3.2 *There exists an open bounded simply connected domain $\Omega \subset \mathbb{R}^n$ different from a ball, with $\partial\Omega$ globally $\mathcal{C}^\infty$ and containing a connected portion Γ of ∂B, and there exists a Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ |\nabla u| = \frac{1}{2} & \text{on } \partial\Omega \\ u = 1 & \text{on } \Gamma \\ u \to 0, \ |\nabla u| \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{9}$$

admits a solution $u \in \mathcal{C}^2(\mathbb{R}^n \setminus \Omega)$.

Proof. Fix an integer $n \geq 2$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(s) = \begin{cases} \dfrac{n-1}{2(3-2s)} & \text{if } 1 \leq s < \dfrac{3}{2} \\[2ex] \dfrac{3(n-3)}{16}(3-\sqrt{9-8s})^3 - \dfrac{n-4}{16}(3-\sqrt{9-8s})^4 & \text{if } 0 < s \leq 1 \ . \end{cases}$$

Then, f is globally Lipschitz continuous over $(0, \frac{3}{2})$; moreover, if $n \geq 4$ then f is positive and strictly increasing.

Consider also the (radial) function u defined on $\mathbb{R}^n \setminus \{0\}$ by

$$u(x) = \begin{cases} \dfrac{3-|x|}{2} & \text{if } |x| \leq 1 \\[2ex] \dfrac{3}{2|x|} - \dfrac{1}{2|x|^2} & \text{if } |x| \geq 1 \ . \end{cases}$$

Then, $u \in \mathcal{C}^2(\mathbb{R}^n \setminus \{0\})$; to see this, it suffices to write $u = u(r)$ as a function of the real variable $r = |x|$ and to note that

$$u'(r) = \begin{cases} -\frac{1}{2} & \text{if } 0 < r \leq 1 \\ -\frac{3}{2r^2} + \frac{1}{r^3} & \text{if } r \geq 1 \ , \end{cases} \qquad u''(r) = \begin{cases} 0 & \text{if } 0 < r \leq 1 \\ \frac{3}{r^3} - \frac{3}{r^4} & \text{if } r \geq 1 \ , \end{cases}$$

are continuous functions in $(0, \infty)$. Moreover, some computations show that u satisfies

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n \setminus \{0\} \ , \qquad u = 1 \quad \text{on } \partial B \ , \qquad |\nabla u| = \frac{1}{2} \quad \text{in } \overline{B} \setminus \{0\} \ ,$$

where B denotes the unit ball. Take any smooth domain $\Omega \subsetneq B$ such that $0 \in \Omega$ and $\{x \in \partial B; \, x_1 < \frac{1}{2}\} \subset \partial\Omega$. Let $\Gamma = \partial\Omega \cap \partial B$, then u satisfies (9) but Ω is not a ball. □

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