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NONLINEARITY IN OSCILLATING BRIDGES

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Abstract. We first recall several historical oscillating bridges that, in some cases, led to collapses. Some of them are quite recent and show that, nowadays, oscillations in suspension bridges are not yet well understood. Next, we survey some attempts to model bridges with differential equations. Although these equations arise from quite different scientific communities, they display some common features. One of them, which we believe to be incorrect, is the acceptance of the linear Hooke law in elasticity. This law should be used only in presence of small deviations from equilibrium, a situation which does not occur in widely oscillating bridges. Then we discuss a couple of recent models whose solutions exhibit self-excited oscillations, the phenomenon visible in real bridges. This suggests a different point of view in modeling equations and gives a strong hint how to modify the existing models in order to obtain a reliable theory. The purpose of this paper is precisely to highlight the necessity of revisiting the classical models, to introduce reliable models, and to indicate the steps we believe necessary to reach this target.

1. Introduction

The story of bridges is full of many dramatic events, such as uncontrolled oscillations which, in some cases, led to collapses. To get into the problem, we invite the reader to have a look at the videos [101, 103, 104, 105]. These failures have to be attributed to the action of external forces, such as the wind or traffic loads, or to macroscopic mistakes in the projects. From a theoretical point of view, there is no satisfactory mathematical model which, up to nowadays, perfectly describes the complex behavior of bridges. And the lack of a reliable analytical model precludes precise studies both from numerical and engineering points of views.

The main purpose of the present paper is to show the necessity of revisiting the existing models since they fail to describe the behavior of actual bridges. We will explain which are the weaknesses of the so far considered equations and suggest some possible improvements according to the fundamental rules of classical mechanics. Only with some nonlinearity and with a sufficiently large number of degrees of freedom several behaviors may be modeled. We do not claim to have a perfect model, we just wish to indicate the way to reach it. Much more work is needed and we explain what we believe to be the next steps.

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We first survey and discuss some historical events, we recall what is known in elasticity theory, and we describe in full detail the existing models. With this database at hand, our purpose is to analyse the oscillating behavior of certain bridges, to determine the causes of oscillations, and to give an explanation to the possible appearance of different kinds of oscillations, such as torsional oscillations. Due to the lateral sustaining cables, suspension bridges most emphasise these oscillations which, however, also appear in other kinds of bridges: for instance, light pedestrian bridges display similar behaviors even if their mechanical description is much simpler.

According to [46], chaos is a disordered and unpredictable behavior of solutions in a dynamical system. With this characterization, there is no doubt that chaos is somehow present in the disordered and unpredictable oscillations of bridges. From [46, Section 11.7] we recall a general principle (GP) of classical mechanics:

(GP) The minimal requirements for a system of first-order equations to exhibit chaos is that they be nonlinear and have at least three variables.

This principle suggests that *any model aiming to describe oscillating bridges should be nonlinear and with enough degrees of freedom.*

Most of the mathematical models existing in literature fail to satisfy (GP) and, therefore, must be accordingly modified. We suggest possible modifications of the corresponding differential equations and we believe that, if solved, this would lead to a better understanding of the underlying phenomena and, perhaps, to several practical actions for the plans of future bridges, as well as remedial measures for existing structures. One of the scopes of this paper is to convince the reader that linear theories are not suitable for the study of bridges oscillations whereas, although they are certainly too naive, some recent nonlinear models do display self-excited oscillations as visible in bridges.

In Section 2, we collect a number of historical events and observations about bridges, both suspended and not. A complete story of bridges is far beyond the scopes of the present paper and the choice of events is mainly motivated by the phenomena that they displayed. The description of the events is accompanied by comments of engineers and of witnesses, and by possible theoretical explanations of the observed phenomena. This enables us to figure out a common behavior of oscillating bridges; in particular, a quite evident nonlinear behavior is manifested. Recent events testify that the problems of controlling and forecasting bridges oscillations is still unsolved.

In Section 3, we discuss several equations appearing in literature as models for oscillating bridges. Most of them use in some point the well-known linear Hooke law ($\mathcal{LHL}$ in the sequel) of elasticity. This is what we believe to be a major weakness, but not the only one, of all these models. This is also the opinion of McKenna [64, p.16]:

> We doubt that a bridge oscillating up and down by about 10 meters every 4 seconds obeys Hooke's law.

From [30], we recall what is known as $\mathcal{LHL}$.

> The linear Hooke law ($\mathcal{LHL}$) of elasticity, discovered by the English scientist Robert Hooke in 1660, states that for relatively small deformations of an object, the displacement or size of the deformation is directly proportional to the deforming force or load. Under

> these conditions the object returns to its original shape and size upon removal of the load ... At relatively large values of applied force, the deformation of the elastic material is often larger than expected on the basis of $\mathcal{LHL}$, even though the material remains elastic and returns to its original shape and size after removal of the force. $\mathcal{LHL}$ describes the elastic properties of materials only in the range in which the force and displacement are proportional.

Hence, by no means one should use $\mathcal{LHL}$ in presence of large deformations. In such case, the restoring elastic force f is "more than linear". Instead of having the usual form $f(s) = ks$, where s is the displacement from equilibrium and $k > 0$ depends on the elasticity of the deformed material, it has an additional superlinear term $\varphi(s)$ which becomes negligible for small displacements s. More precisely,

$$f(s) = ks + \varphi(s) \quad \text{with } \lim_{s\to 0} \frac{\varphi(s)}{s} = 0\,.$$

The superlinear term may be arbitrarily small and should be chosen in such a way to describe with more precision the elastic behavior of a material when larger displacements are involved. As we shall see, this apparently harmless and tiny nonlinear perturbation has devastative effects on the models and, moreover, it is amazingly useful to display self-excited oscillations as the ones visible in actual bridges. On the contrary, linear models prevent to view the real phenomena which occur in bridges, such as the sudden increase of the width of their oscillations and the switch to different ones.

The necessity of dealing with nonlinear models is by now quite clear also in more general elasticity problems; from the preface of the book by Ciarlet [23], let us quote

> ... it has been increasingly acknowledged that the classical linear equations of elasticity, whose mathematical theory is now firmly established, have a limited range of applicability, outside of which they should be replaced by genuine nonlinear equations that they in effect approximate.

To model bridges, the most natural way is to view the roadway as a thin narrow rectangular plate. In Section 3.1, we quote several references which show that classical linear elastic models for thin plates do not describe with a sufficient accuracy large deflections of a plate. But even linear theories present considerable difficulties and a further possibility is to view the bridge as a one dimensional beam; this model is much simpler but, of course, it prevents the appearance of possible torsional oscillations. This is the main difficulty in modeling bridges: find simple models which, however, display the same phenomenon visible in real bridges.

In Section 3.2 we survey a number of equations arising from different scientific communities. The first equations are based on engineering models and mainly focus the attention on quantitative aspects such as the exact values of the parameters involved. Some other equations are more related to physical models and aim to describe in full details all the energies involved. Finally, some of the equations are purely mathematical models aiming to reach a prototype equation and proving some qualitative behavior. All these models have to face a delicate choice: either consider uncoupled behaviors between vertical and torsional oscillations of the roadway or simplify the model by decoupling these two phenomena. In the former case, the equations have many degrees of freedom and become terribly complicated: hence, very few results can be obtained. In the latter case, the model fails to satisfy the requirements of (GP) and appears too far from the real world.

As a compromise between these two choices, in Section 4 we recall the model introduced in [43, 45] which describes vertical oscillations and torsional oscillations of the roadway within the same simplified beam equation. The solution to the equation exhibits self-excited oscillations quite similar to those observed in suspension bridges. We do not believe that the simple equation considered models the complex behavior of bridges but we do believe that it displays the same phenomena as in more complicated models closer related to bridges. In particular, finite time blow up occurs with wide oscillations. These phenomena are typical of differential equations of at least fourth order since they do not occur in lower order equations, see [43]. We also show that the same phenomenon is visible in a 2×2 system of nonlinear ODE's of second order related to a system suggested by McKenna [64].

Putting all together, in Section 5 we afford an explanation in terms of the energies involved. Starting from a survey of milestone historical sources [13, 87], we attempt a qualitative energy balance and we attribute the appearance of torsional oscillations in bridges to some "switch" of energy from vertical modes to torsional modes. This phenomenon is usually called in literature *flutter speed* and has to be attributed to Bleich [12]; in our opinion, the flutter speed should be seen as a *critical energy threshold* which, if exceeded, gives rise to uncontrolled phenomena such as the appearance of torsional oscillations. We give some hints on how to determine the critical energy threshold, depending on the eigenvalues and eigenfunctions which describe the oscillating modes of the roadway. This part is incomplete and certainly needs further work.

In bridges one should always expect vertical oscillations and, in case they become very large, also torsional oscillations; in order to display the possible transition between these two kinds of oscillations, in Section 5.5 we suggest a new model equation for suspension bridges, see (5.15). This problem does not seem to fit in any classical solving scheme, we do not even know if it is well-posed; it is well-known that equations modeling suspension bridges may be ill-posed, displaying multiple solutions [68]. However, we hope (5.15) to become the starting point for future fruitful discussions.

With all the results and observations at hand, in Section 6.1 we attempt a detailed description of what happened on November 7, 1940, the day when the Tacoma Narrows Bridge collapsed. As far as we are aware a universally accepted explanation of this collapse in not yet available. Our explanation fits with all the material developed in the present paper. This allows us to suggest a couple of precautions when planning future bridges, see Section 6.2.

We recently had the pleasure to participate to a conference on bridge maintenance, safety and management, see [51]. There were engineers from all over the world, the atmosphere was very enjoyable and the problems discussed were extremely interesting. And there was a large number of basic questions still unsolved, most of the results and projects had some percentage of incertitude. Many talks were devoted to suggest new equations to model the studied phenomena and to forecast the impact of new structural issues: even apparently simple questions are still without an answer. We believe this should be a strong motivation for mathematicians (from mathematical physics, analysis, numerics) to get more interested in bridges modeling, experiments, and performances. Throughout the paper we suggest a number of open problems which, if solved, could be a good starting point to reach a deeper understanding of oscillations in bridges.

2. What has been observed in bridges

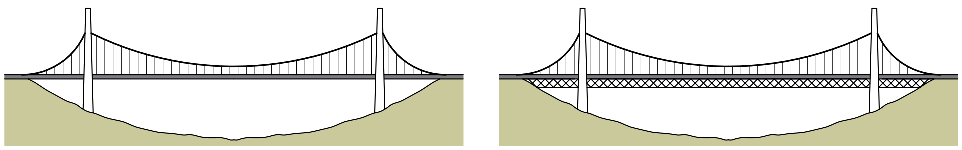

Figure 1. Suspension bridges without girder and with girder

A simplified picture of a suspension bridge can be sketched as in Figure 1 where one sees the difference between the elastic structure of a bridge without girder and the more stiff structure of a bridge with girder. Although the first design of a suspension bridge is due to the Italian engineer Verantius around 1615, see [94] and [75, p.7] or [53, p.16], the first suspension bridges were built only about two centuries later.

The Menai Straits Bridge was built in 1826 and it collapsed in 1839 due to a hurricane. In that occasion, unexpected oscillations appeared and Provis [78] provided the following description:

> ...the character of the motion of the platform was not that of a simple undulation, as had been anticipated, but the movement of the undulatory wave was oblique, both with respect to the lines of the bearers, and to the general direction of the bridge.

Also the Broughton Suspension Bridge was built in 1826. It collapsed in 1831 due to mechanical resonance induced by troops marching over the bridge in step. As a consequence of the incident, the British Army issued an order that troops should "break step" when crossing a bridge.

Figure 2. Destruction of the Brighton Chain Pier

A further event deserving to be mentioned is the collapse of the Brighton Chain Pier, built in 1823. It collapsed a first time in 1833, it was rebuilt and partially destroyed once again in 1836. Both the collapses are attributed to violent windstorms. For the second collapse a witness, William Reid, reported valuable observations and sketched a picture illustrating the destruction [79, p.99], see Figure 2 which is taken from [82]. These pictures are complemented with a report whose most interesting part says:

> For a considerable time, the undulations of all the spans seemed nearly equal ... but soon after midday the lateral oscillations of the third span increased to a degree to make it doubtful whether the work could withstand the storm; and soon afterwards the oscillating motion across the roadway, seemed to the eye to be lost in the undulating one, which in the third span was much greater than in the other three; the undulatory motion which was along the length of the road is that which is shown in the first sketch; but there was also an oscillating motion of the great chains across the work, though the one seemed to destroy the other ...

From the above accidents we learn that different kinds of oscillations may appear and some of them are considered destructive. Some decades earlier, at the end of the eighteenth century, the German physicist Ernst Chladni was touring Europe and showing, among other things, the nodal line patterns of vibrating plates, see Figure 3.

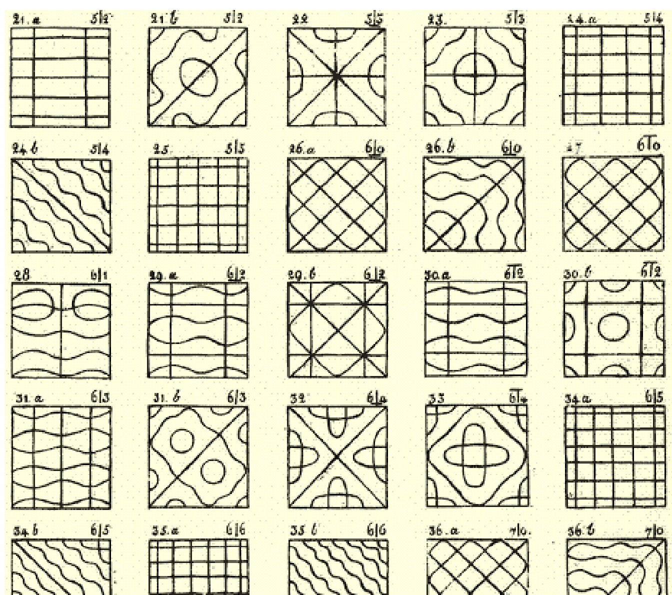

FIGURE 3. Chladni patterns in a vibrating plate

Chladni's technique, first published in [22], consisted of creating vibrations in a square-shaped metal plate whose surface was covered with light sand. The plate was bowed until it reached resonance, when the vibration caused the sand to concentrate along the nodal lines of vibrations, see [102] for the nowadays experiment. In Figure 3 we see how complicated may be the vibrations of a thin plate and hence, see Section 3.1, of a bridge. And, indeed, the above described events testify that, besides the somehow expected vertical oscillations, also different kinds of oscillations may appear. The description of different coexisting forms of oscillations is probably the most important open problem in suspension bridges.

The Tacoma Narrows Bridge collapse, occurred in 1940 just a few months after its opening, is certainly the most celebrated bridge failure both because of the impressive video [104] and because of the large number of studies that it has inspired starting from the reports [4, 13, 32, 33, 34, 87, 96]. Let us recall some observations made on the Tacoma collapse. Since we were unable to find the Federal Report [4] that we repeatedly quote below, we refer to it by trusting the valuable historical research by Scott [86] and by McKenna and coauthors, see in particular [58, 64, 65, 69]. A good starting point to describe the Tacoma collapse is... the Golden Gate Bridge, inaugurated a few years earlier, in 1937. This bridge is usually classified as "very flexible" although it is strongly stiffened by a thick girder. The bridge can swing more than an amazing 8 meters and flex about 3 meters under big loads, which explains why the bridge is classified as very flexible. The huge mass involved and these large distances from equilibrium explain why $\mathcal{LHL}$ certainly fails. Due to high winds around 120 kilometers per hour, the Golden Gate Bridge has been closed, without suffering structural damage, only three times: in 1951, 1982, and 1983. Moreover, in 1938 important vertical oscillations appeared: in [4, Appendix IX], the chief engineer of the Golden Gate Bridge writes

> ...I observed that the suspended structure of the bridge was undulating vertically in a wavelike motion of considerable amplitude ...

see also the related detailed description in [69, Section 1]. We sketch pure vertical oscillations (similar to traveling waves) in the first picture in Figure 4.

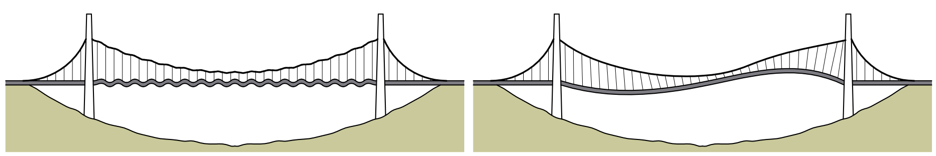

Figure 4. Vertical and torsional oscillations in bridges without girder

So, vertical oscillations show up also in apparently stiff structures. And in presence of extremely flexible structures, these oscillations can transform into the more dangerous torsional oscillations, see the second picture in Figure 4.

Of course, the girder gives more stiffness to the bridge; this is certainly the main reason why at the Golden Gate Bridge no torsional oscillation was ever detected. The Tacoma Bridge was rebuilt in 1950 with a thick girder acting as a strong stiffening structure, see [34] for some remarks on the project, and still stands today as the westbound lanes of the present-day twin bridge complex; the eastbound lanes opened in 2007. Figure 5 - picture by Michael Goff, Oregon Department of Transportation, USA - shows the striking difference between the original Tacoma Bridge collapsed in 1940 and the twin bridges as they are today.

Let us go back to the original Tacoma Bridge: even if it was more flexible, the reason of the appearance of torsional oscillations is still unclear. Scanlan [84, p.841]

FIGURE 5. The collapsed Tacoma Bridge and the current twins Tacoma Bridges

discards the possibility of the appearance of von Kármán vortices and raises doubts on the appearance of a resonance. It is reasonable to expect resonance in presence of a single-mode solicitation, such as for the Broughton Bridge. But for the Tacoma Bridge, Lazer-McKenna [58, Section 1] raise the question

> ... the phenomenon of linear resonance is very precise. Could it really be that such precise conditions existed in the middle of the Tacoma Narrows, in an extremely powerful storm?

So, no plausible explanation is available nowadays. While describing the Tacoma collapse in a letter, Farquharson [31] wrote that

> ... a violent change in the motion was noted. This change appeared to take place without any intermediate stages and with such extreme violence ... The motion, which a moment before had involved nine or ten waves, had shifted to two.

All this happened under not extremely strong winds, about 80km/h, and under a relatively high frequency of oscillation, see [32, p.23]. See also [64, Section 2.3] for more details and for the conclusion that

> there is no consensus on what caused the sudden change to torsional motion.

Besides the lack of consensus on the causes of the switch between vertical and torsional oscillations, all the above comments highlight a strong instability of the vertical oscillations as if, after reaching some critical energy threshold, an impulse (a Dirac delta) generated a new unexpected oscillation. We refer to Section 6 for our own interpretation of this phenomenon. Roughly speaking, we believe that part of the energy responsible of vertical oscillations switches to another energy which generates torsional oscillations; the switch occurs without intermediate stages. In order to explain the "switch of oscillations" several mathematical models were suggested in literature: in next section we survey some of these models.

The Tacoma Bridge collapse is just the most celebrated and dramatic evidence of oscillating bridge but bridges oscillations are still not well understood nowadays. On May 2010, the Russian authorities closed the Volgograd Bridge to all motor traffic due to its strong vertical oscillations (traveling waves) caused by windy conditions, see [105] for the BBC report and video.

As already observed, the wind is not the only possible external source which generates bridges oscillations which also appear in pedestrian bridges where lateral swaying is the counterpart of torsional oscillation. In June 2000, the very same day when the London Millennium Bridge opened and the crowd streamed on it, the bridge started to sway from side to side, see [103]. Many pedestrians fell spontaneously into step with the vibrations, thereby amplifying them. According to Sanderson [83], the bridge wobble was due to the way people balanced themselves, rather than the timing of their steps. Therefore, the pedestrians acted as negative dampers, adding energy to the bridge's natural sway. Macdonald [61, p.1056] explains this phenomenon by writing

> ... above a certain critical number of pedestrians, this negative damping overcomes the positive structural damping, causing the onset of exponentially increasing vibrations.

Although we have some doubts about the real meaning of "exponentially increasing vibrations" we have no doubts that this description corresponds to a superlinear behavior. The Millennium Bridge was made secure by adding some lateral dampers.

Another pedestrian bridge, the Assago Bridge in Milan (310m long), had a similar problem. In February 2011, just after a concert the publics crossed the bridge and, suddenly, swaying became so violent that people could hardly stand, see [35] and [101]. Even worse was the subsequent panic effect when the crowd started running in order to escape from a possible collapse; this amplified swaying but, quite luckily, nobody was injured. In this case, the project did not take into account that a large number of people would go through the bridge just after the events; when swaying started there were about 1.200 pedestrians on the footbridge. Also this problem was solved by adding positive dampers, see [88].

It is not among the scopes of this paper to give the complete story of bridges collapses for which we refer to [13, Section 1.1], to [82, Chapter IV], to [24, 32, 50, 98], to the recent monographs [3, 53], and also to [52] for a complete database. Let us just mention that between 1818 and 1889, ten suspension bridges suffered major damages or collapsed in windstorms, see [32, Table 1, p.13]. The story of bridges, suspended and not, contains many further dramatic events, an amazing amount of bridges had troubles for different reasons such as the wind, the traffic loads, or macroscopic mistakes in the project, see e.g. [49, 76]. According to [52], around 400 recorded bridges failed for several different reasons and the ones who failed after year 2000 are more than 70. We also refer to the book by Akesson [3] for the technical analysis of these failures.

As we have seen, the reasons of failures in bridges are of different kinds. Firstly, strong and/or continued winds: these may cause wide vertical oscillations which may switch to different kinds of oscillations. Especially for suspension bridges the latter phenomenon appears quite evident, due to the many elastic components (cables, hangers, towers, etc.) which appear in it. A second cause are traffic loads, such as some precise resonance phenomenon, or some unpredictable synchronised behavior, or some unexpected huge load; these problems are quite common in many different kinds of bridges. Finally, a third cause are mistakes in the project; these are both theoretical, for instance assuming $\mathcal{LHL}$, and practical, such as wrong assumptions on the possible maximum external actions.

Trusses or dampers do not solve completely the problem and torsional oscillations may still appear but, of course, only in presence of very large energy inputs. In this respect, we quote from [32, p.13] a comment on suspension bridges strengthened by stiffening trusses:

> That significant motions have not been recorded on most of these bridges is conceivably due to the fact that they have never been subjected to optimum winds for a sufficient period of time.

So, it is expected that under prolonged winds, not necessarily hurricanes, or heavy and synchronized traffic loads, a stiffening truss may become useless. Moreover, Steinman [89] writes that

> It is more scientific to eliminate the cause than to build up the structure to resist the effect.

Therefore we can say that *instead of just solving the problem, one should understand the problem.*

And precisely in order to understand the problem, we described above some events which displayed the pure elastic behavior of bridges. These were mostly suspension bridges without girders and were free to oscillate. This is a good reason why the Tacoma collapse should be further studied for deeper knowledge: it displays the pure motion without stiffening constraints which hide the elastic features of bridges. Finite elements methods may be fruitfully used to quantify the role of trusses, see e.g. [36]. The next step is to find correct mathematical models able to reproduce these oscillations and to explain what causes them. In particular, the above events and deep studies in [19, 55] show that suspension bridges behave nonlinearly and that nonlinear models have to be considered, as suggested by (GP).

3. How to model bridges

The amazing number of failures described in the previous section shows that the existing theories and models are not adequate to describe the statics and the dynamics of oscillating bridges. In this section we survey different points of view, different models, and we underline their main weaknesses. We also suggest how to modify them in order to fulfill the requirements of (GP).

3.1. **A quick overview on elasticity: from linear to semilinear models.**

A quite natural way to describe the bridge roadway is to view it as a thin rectangular plate. This is also the opinion of Rocard [82, p.150]:

> The plate as a model is perfectly correct and corresponds mechanically to a vibrating suspension bridge.

In this case, a commonly adopted theory is the linear one by Kirchhoff-Love [54, 60], see also [42, Section 1.1.2], which we briefly recall. The bending energy of a plate involves curvatures of the surface. Let κ_1, κ_2 denote the principal curvatures of the graph of a smooth function u representing the deformation of the plate, then a simple model for the bending energy of the deformed plate Ω is

$$\mathbb{E}(u) = \int_\Omega \Big(\frac{\kappa_1^2}{2} + \frac{\kappa_2^2}{2} + \sigma\kappa_1\kappa_2\Big)\,dx_1\,dx_2 \tag{3.1}$$

where σ denotes the Poisson ratio defined by $\sigma = \lambda/(2(\lambda+\mu))$ with the so-called Lamé constants λ, μ that depend on the material. For physical reasons it holds that $\mu > 0$ and usually $\lambda \geq 0$ so that $0 \leq \sigma < 1/2$. In the linear theory of elastic plates, for small deformations u the terms in (3.1) are considered to be purely quadratic with respect to the second order derivatives of u. More precisely, for small deformations u, one has

$$(\kappa_1+\kappa_2)^2 \approx (\Delta u)^2\,, \quad \kappa_1\kappa_2 \approx \det(D^2u) = (u_{x_1x_1}u_{x_2x_2} - u_{x_1x_2}^2)\,,$$

and therefore

$$\frac{\kappa_1^2}{2}+\frac{\kappa_2^2}{2}+\sigma\kappa_1\kappa_2\approx\frac{1}{2}(\Delta u)^2+(\sigma-1)\det(D^2u).$$

Then (3.1) yields

$$\mathbb{E}(u)=\int_\Omega\Big(\frac{1}{2}(\Delta u)^2+(\sigma-1)\det(D^2u)\Big)\,dx_1\,dx_2\,. \tag{3.2}$$

Note that for $-1<\sigma<1$ the functional $\mathbb{E}$ is convex; it is also coercive in suitable Sobolev spaces such as $H^2_0(\Omega)$ or $H^2\cap H^1_0(\Omega)$. This modern variational formulation appears in [39], while a discussion for a boundary value problem for a thin elastic plate in a somehow old fashioned notation is made by Kirchhoff [54]. And precisely the choice of the boundary conditions is quite delicate since it depends on the physical model considered.

Destuynder-Salaun [28, Section I.2] describe this modeling by

> ... Kirchhoff and Love have suggested to assimilate the plate to a collection of small pieces, each one being articulated with respect to the other and having a rigid-body behavior. It looks like these articulated wooden snakes that children have as toys. Hence the transverse shear strain remains zero, while the planar deformation is due to the articulation between small blocks. But this simplified description of a plate movement can be acceptable only if the components of the stress field can be considered to be negligible.

The above comment says that $\mathcal{LHL}$ should not be adopted if the components of the stress field are not negligible. An attempt to deal with large deflections for thin plates is made by Mansfield [62, Chapters 8-9]. He first considers approximate methods, then he deals with three classes of asymptotic plate theories: membrane theory, tension field theory, inextensional theory. Roughly speaking, the three theories may be adopted according to the ratio between the thickness of the plate and the typical planar dimension: for the first two theories the ratio should be less than 10^{-3}, whereas for the third theory it should be less than 10^{-2}. Since a roadway has a length of the order of 1km, the width of the order of 10m, even for the less stringent inextensional theory the thickness of the roadway should be less than 10cm which, of course, appears unreasonable. Once more, this means that $\mathcal{LHL}$ should not be adopted in bridges. In this respect, Mansfield [62, p.183] writes

> The exact large-deflection analysis of plates generally presents considerable difficulties...

Destuynder-Salaun [28, Section I.2] also revisit an alternative model due to Naghdi [73] by using a mixed variational formulation. They refer to [71, 80, 81] for further details and modifications, and conclude by saying that none between the Kirchhoff-Love model or one of these alternative models is always better than the others. Moreover, also the definition of the transverse shear energy is not universally accepted: from [28, p.149], we quote

> ... this discussion has been at the origin of a very large number of papers from both mathematicians and engineers. But to our best knowledge, a convincing justification concerning which one of the two expressions is the more suitable for numerical purpose, has never been formulated in a convincing manner. This question is nevertheless a fundamental one ...

It is clear that a crucial role is played by the word "thin". Which width is a plate allowed to have in order to be considered thin? If we assume that the width is zero like for a sheet of paper, but a quite unrealistic assumption for bridges, a celebrated two-dimensional equation was suggested by von Kármán [97]. This equation has been widely, and satisfactorily, studied from several mathematical points of view such as existence, regularity, eigenvalue problems, semilinear versions, see e.g. [42] for a survey of results. But several doubts have been raised on their physical soundness, see the objections by Truesdell [92, pp.601-602] who concludes by writing

> These objections do not prove that anything is wrong with von Kármán strange theory. They merely suggest that it would be difficult to prove that there is anything right about it.

Classical books for elasticity theory are due to Love [60], Timoshenko [90], Ciarlet [23], Villaggio [95], see also [72, 73, 91] for the theory of plates. Let us also point out a celebrated work by Ball [7] who was the first analyst to approach the real 3D boundary value problems for nonlinear elasticity. Further nice attempts to tackle nonlinear elasticity in particular situations were done by Antman [5, 6] who, however, appears quite skeptic on the possibility to have a general theory:

> ... general three-dimensional nonlinear theories have so far proved to be mathematically intractable.

The above discussion shows that classical modeling of thin plates should be carefully revisited. This suggestion is absolutely not new. In this respect, let us quote a couple of sentences written by Gurtin [47] about nonlinear elasticity:

> Our discussion demonstrates why this theory is far more difficult than most nonlinear theories of mathematical physics. It is hoped that these notes will convince analysts that nonlinear elasticity is a fertile field in which to work.

Since the previously described Kirchhoff-Love model implicitly assumes $\mathcal{LHL}$, and since quasilinear equations appear too complicated in order to give useful information, we intend to add some nonlinearity only in the source f in order to have a semilinear equation, something which appears to be a good compromise between too poor linear models and too complicated quasilinear models. This compromise is quite common in elasticity, see e.g. [23, p.322] which describes the method of asymptotic expansions for the thickness ε of a plate as a "partial linearisation"

> ... in that a system of quasilinear partial differential equations; i.e., with nonlinearities in the higher order terms, is replaced as $\varepsilon \to 0$ by a system of semilinear partial differential equations; i.e., with nonlinearities only in the lower order terms.

In Section 5.5, we suggest a new 2D mathematical model described by a semilinear fourth order wave equation. Before doing this, in next section we survey some existing models and we suggest some possible variants based on the observations listed in Section 2.

3.2. **Equations modeling suspension bridges.** Although it is oversimplified in several respects, the celebrated report by Navier [75] has been for about one century the only mathematical treatise of suspension bridges. The second milestone contribution is certainly the monograph by Melan [70]. After the Tacoma collapse, the engineering communities felt the necessity to find accurate equations in order to attempt explanations of what had occurred. A first source is certainly the work by Smith-Vincent [87] which was written precisely *with special reference to the Tacoma Narrows Bridge*. The bridge is modeled as a one dimensional beam, say on the interval $(0, L)$, and in order to obtain an autonomous equation, Smith-Vincent consider the function $\eta = \eta(x)$ representing the amplitude of the oscillation at the point $x \in (0, L)$. By linearising they obtain a fourth order linear ODE [87, (4.2)] which can be integrated explicitly. We will not write this equation because we prefer to deal with the function $v = v(x, t)$ representing the deflection at any point $x \in (0, L)$ and at time $t > 0$; roughly speaking, $v(x, t) = \eta(x) \sin(\omega t)$ for some $\omega > 0$. In this respect, a slightly better job was done in [13] although this book was not very lucky since two of the authors (McCullogh and Bleich) passed away during its preparation. Equation [13, (2.7)] coincides with [87, (4.2)]; but [13, (2.6)] considers the deflection v and reads

$$mv_{tt} + EIv_{xxxx} - H_w v_{xx} + \frac{wh}{H_w} = 0\,;\,, \quad x \in (0, L)\,,\ t > 0\,, \tag{3.3}$$

where E and I are, respectively, the elastic modulus and the moment of inertia of the stiffening girder so that EI is the stiffness of the girder; moreover, m denotes the mass per unit length, $w = mg$ is the weight which produces a cable stress whose horizontal component is H_w, and h is the increase of H_w as a result of the additional deflection v. In particular, this means that h depends on v although [13] does not emphasize this fact and considers h as a constant.

An excellent source to derive the equation of vertical oscillations in suspension bridges is [82, Chapter IV] where all the details are well explained. The author, the French physicist Yves-André Rocard (1903-1992), also helped to develop the atomic bomb for France. Consider again that a long span bridge roadway is a beam of length $L > 0$ and that it is oscillating; let $v(x, t)$ denote the vertical component of the oscillation for $x \in (0, L)$ and $t > 0$. The equation derived in [82, p.132] reads

$$mv_{tt} + EIv_{xxxx} - \big(H_w + \gamma(v)\big)v_{xx} + \frac{w}{H_w}\gamma(v) = f(x, t)\,, \quad x \in (0, L)\,,\ t > 0, \tag{3.4}$$

where H_w, EI and m are as in (3.3), $\gamma(v)$ is the variation h of H_w supposed to vary linearly with v, and f is an external forcing term. Note that a nonlinearity appears here in the term $\gamma(v)v_{xx}$. In fact, (3.4) is closely related to an equation suggested much earlier by Melan [70, p.77] but it has not been subsequently attributed to him.

Problem 3.1. Study oscillations and possible blow up in finite time for traveling waves to (3.4) having velocity $c > 0$, $v = v(x, t) = y(x - ct)$ for $x \in \mathbb{R}$ and $t > 0$, in the cases where $f \equiv 1$ is constant and where f depends superlinearly on v. Assuming that $\gamma(v) = \gamma v$ and putting $\tau = x - ct$ one is led to find solutions to the ODE

$$EIy''''(\tau) - \big(\gamma y(\tau) + H_w - mc^2\big)y''(\tau) + \frac{w\gamma}{H_w}y(\tau) = 1\,, \quad \tau \in \mathbb{R}\,.$$

By letting $w(\tau) = y(\tau) - \frac{H_w}{w\,\gamma}$ and normalising some constants, we arrive at

$$w''''(\tau) - \big(\alpha w(\tau) + \beta\big)\, w''(\tau) + w(\tau) = 0\,, \quad \tau \in \mathbb{R}\,, \tag{3.5}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$; we expect different behaviors depending on α and β. It would be interesting to see if local solutions to (3.5) blow up in finite time with wide oscillations. Moreover, one should also consider the more general problem

$$w''''(\tau) - \big(\alpha w(\tau) + \beta\big) w''(\tau) + f(w(\tau)) = 0\,, \quad \tau \in \mathbb{R}\,,$$

with f being superlinear, for instance $f(s) = s + \varepsilon s^3$ with $\varepsilon > 0$ small. Incidentally, we note that such f satisfies (3.13) and (4.4)-(4.5) below.

Rocard [82, pp.166-167] also studies the possibility of simultaneous excitation of different bending and torsional modes and obtains a coupled system of linear equations of the kind of (3.4). With few variants, equations (3.3) and (3.4) seem nowadays to be well-accepted among engineers, see e.g. [24, Section VII.4]; moreover, quite similar equations are derived to describe related phenomena in cable-stayed bridges [20, (1)] and in arch bridges traversed by high-speed trains [56, (14)-(15)].

Let $v(x,t)$ and $\theta(x,t)$ denote respectively the vertical and torsional components of the oscillation of the bridge, then the following system is derived in [25, (1)-(2)] for the linearised equations of the elastic combined vertical-torsional oscillation motion:

$$\begin{gathered} m v_{tt} + EI v_{xxxx} - H_w v_{xx} + \frac{w^2}{H_w^2}\frac{EA}{L}\int_0^L v(z,t)\,dz = f(x,t) \\ I_0\theta_{tt} + C_1\theta_{xxxx} - (C_2 + H_w\ell^2)\theta_{xx} + \frac{\ell^2 w^2}{H_w^2}\frac{EA}{L}\int_0^L \theta(z,t)\,dz = g(x,t) \\ x \in (0,L)\,,\ t > 0, \end{gathered} \tag{3.6}$$

where m, w, H_w are as in (3.3), EI, C_1, C_2, EA are respectively the flexural, warping, torsional, extensional stiffness of the girder, I_0 the polar moment of inertia of the girder section, 2ℓ the roadway width, $f(x,t)$ and $g(x,t)$ are the lift and the moment for unit girder length of the self-excited forces. The linearisation here consists in dropping the term $\gamma(v)v_{xx}$ but a preliminary linearisation was already present in (3.4) in the zero order term. And the nonlocal linear term $\int_0^L v$, which replaces the zero order term in (3.4), is obtained by assuming $\mathcal{LHL}$. The nonlocal term in (3.6) represents the increment of energy due to the external wind during a period of time; this will be better explained in Section 5.1.

A special mention is deserved by an important paper by Abdel-Ghaffar [1] where variational principles are used to obtain the combined equations of a suspension bridge motion in a fairly general nonlinear form. The effect of coupled vertical-torsional oscillations as well as cross-distortional of the stiffening structure is clarified by separating them into four different kinds of displacements: the vertical displacement v, the torsional angle θ, the cross section distortional angle ψ, the warping displacement u, although u can be expressed in terms of θ and ψ. These displacements are well described in Figure 6 which are taken from [1, Figure 2].

A careful analysis of the energies involved is made, reaching up to fifth derivatives in the equations, see [1, (15)]. Higher order derivatives are then neglected and the following nonlinear system of three PDE's of fourth order in the three unknown displacements v, θ, ψ is obtained, see [1, (28)-(29)-(30)]:

$$\begin{aligned} &\frac{w}{g}v_{tt} + EIv_{xxxx} - \Big(2H_w + H_1(t) + H_2(t)\Big)v_{xx} + \frac{b}{2}\Big(H_1(t) - H_2(t)\Big)(\theta_{xx} + \psi_{xx}) \\ &+ \frac{w}{2H_w}\Big(H_1(t) + H_2(t)\Big) - \frac{w_s r^2}{g}\Big(1 + \frac{EI}{2G\mu r^2}\Big)v_{xxtt} + \frac{w_s^2 r^2}{4gG\mu}v_{tttt} = 0\,, \end{aligned}$$

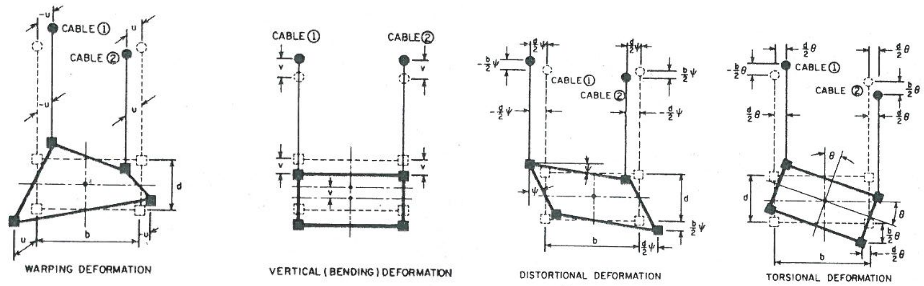

FIGURE 6. The four different kinds of displacements

$$
\begin{aligned}
& I_m\theta_{tt} + E\Gamma\theta_{xxxx} - GJ\theta_{xx} - \frac{H_w b^2}{2}(\theta_{xx}+\psi_{xx}) - \frac{\gamma\Gamma}{g}\theta_{xxtt} \\
& - \frac{b^2}{4}\Big(H_1(t)+H_2(t)\Big)(\theta_{xx}+\psi_{xx}) + \frac{b}{2}\Big(H_1(t)-H_2(t)\Big)v_{xx} \\
& - \frac{\gamma\lambda}{g}\psi_{xxtt} + \frac{b\,w}{4H_w}\Big(H_2(t)-H_1(t)\Big) + E\Lambda\,\psi_{xxxx} + \frac{w_c b^2}{4g}\,\psi_{tt} = 0\,, \\
& \frac{w_c b^2}{4g}(\psi_{tt}+\theta_{tt}) + \frac{EAb^2d^2}{4}\psi_{xxxx} - \frac{H_w b^2}{2}(\psi_{xx}+\theta_{xx}) \\
& - \frac{\gamma A b^2 d^2}{4g}\,\psi_{xxtt} - \frac{\gamma\lambda}{g}\theta_{xxtt} + E\Lambda\theta_{xxxx} - \frac{b^2}{4}\Big(H_1(t)+H_2(t)\Big)(\theta_{xx}+\psi_{xx}) \\
& + \frac{b}{2}\Big(H_1(t)-H_2(t)\Big)v_{xx} + \frac{wb}{4H_w}\Big(H_2(t)-H_1(t)\Big) = 0\,.
\end{aligned}
$$

We will not explain here what is the meaning of all the constants involved: some of the constants have a clear meaning, for the interpretation of the remaining ones, we refer to [1]. Let us just mention that H_1 and H_2 represent the vibrational horizontal components of the cable tension and depend on v, θ, ψ, and their first derivatives, see [1, (3)]. We wrote these equations in order to convince the reader that the behavior of the bridge is modeled by terribly complicated equations. After making such huge effort, Abdel-Ghaffar simplifies the problem by neglecting the cross section deformation, the shear deformation and rotatory inertia; he obtains a coupled nonlinear vertical-torsional system of two equations in the two unknowns functions v and θ. These equations are finally linearised, by neglecting H_1 and H_2 which are considered small when compared with the initial tension H_w. Then the coupling effect disappears and equations (3.6) are recovered, see [1, (34)-(35)]. What a pity, an accurate modeling ended up with a linearisation! But there was no choice, how can one imagine to get any kind of information from the above system?

After the previously described pioneering models from [13, 70, 75, 82, 87] there has not been much work among engineers about alternative differential equations; the attention has turned to improving performances through design factors, see e.g. [48], or on how to solve structural problems rather than how to understand them more deeply. In this respect, from [64, p.2] we quote a personal discussion between McKenna and a distinguished civil engineer who said

> ... having found obvious and effective physical ways of avoiding the problem, engineers will not give too much attention to the mathematical solution of this fascinating puzzle ...

Only modeling modern footbridges has attracted some interest from a theoretical point of view. As already mentioned, pedestrian bridges are extremely flexible and display elastic behaviors similar to suspension bridges, although the oscillations are of different kind. When a suspension bridge is attacked by the wind its starts oscillating, but soon afterwards the wind itself modifies its behavior according to the bridge oscillation; so, the wind amplifies the oscillations by blowing synchronously. A qualitative description of this phenomenon was already attempted by Rocard [82, p.135]:

> ... it is physically certain and confirmed by ordinary experience, although the effect is known only qualitatively, that a bridge vibrating with an appreciable amplitude completely imposes its own frequency on the vortices of its wake. It appears as if in some way the bridge itself discharges the vortices into the fluid with a constant phase relationship with its own oscillation. ...

This reminds the above described behavior of footbridges where *pedestrians fall spontaneously into step with the vibrations*: in both cases, external forces synchronise their effect and amplify the oscillations of the bridge. This is one of the reasons why self-excited oscillations appear in suspension and pedestrian bridges.

In [18] a simple 1D model was proposed in order to describe the crowd-flow phenomena occurring when pedestrians walk on a flexible footbridge. The resulting equation [18, (2)] reads

$$\big(m_s(x)+m_p(x,t)\big)u_{tt}+\delta(x)u_t+\gamma(x)u_{xxxx}=g(x,t) \tag{3.7}$$

where x is the coordinate along the beam axis, t the time, $u=u(x,t)$ the lateral displacement, $m_s(x)$ is the mass per unit length of the beam, $m_p(x,t)$ the linear mass of pedestrians, $\delta(x)$ the viscous damping coefficient, $\gamma(x)$ the stiffness per unit length, $g(x,t)$ the pedestrian lateral force per unit length. In view of the superlinear behavior for large displacements observed for the London Millennium Bridge, see Section 2, we wonder if instead of a linear model one should consider a lateral force also depending on the displacement, $g=g(x,t,u)$, being superlinear with respect to u.

Problem 3.2. Study (3.7) modified as follows

$$u_{tt}+\delta u_t+\gamma u_{xxxx}+f(u)=g(x,t)\quad (x\in\mathbb{R}\,,\ t>0)$$

where $\delta>0$, $\gamma>0$ and $f(s)=s+\varepsilon s^3$ for some $\varepsilon>0$ small. One could first consider the Cauchy problem

$$u(x,0)=u_0(x)\,,\quad u_t(x,0)=u_1(x)\quad (x\in\mathbb{R})$$

with $g\equiv 0$. Then one could seek traveling waves such as $u(x,t)=w(x-ct)$ which solve the ODE

$$\gamma w''''(\tau)+c^2w''(\tau)+\delta cw'(\tau)+f(w(\tau))=0\quad (x-ct=\tau\in\mathbb{R}).$$

Finally, one could also try to find properties of solutions in a bounded interval $x\in(0,L)$.

Scanlan-Tomko [85] introduce a model in which the torsional angle θ of the roadway section satisfies the equation

$$I[\theta''(t)+2\zeta_\theta\omega_\theta\theta'(t)+\omega_\theta^2\theta(t)]=A\theta'(t)+B\theta(t)\,, \tag{3.8}$$

where I, ζ_θ, ω_θ are, respectively, associated inertia, damping ratio, and natural frequency. The right-hand side of (3.8) represents the aerodynamic force and was postulated to depend linearly on both θ' and θ with the positive constants A and B depending on several parameters of the bridge. Since (3.8) may be seen as a two-variables first order linear system, it fails to fulfil both the requirements of (GP). Hence, (3.8) is not suitable to describe the disordered behavior of a bridge. And indeed, elementary calculus shows that if A is sufficiently large, then solutions to (3.8) are positive exponentials times trigonometric functions which do not exhibit a sudden appearance of self-excited oscillations, they merely blow up in infinite time. In order to have a more reliable description of the bridge, in Section 4 we consider the fourth order nonlinear ODE $w''''+kw''+f(w)=0$ ($k\in\mathbb{R}$). We will see that solutions to this equation blow up in finite time with self-excited oscillations appearing suddenly, without any intermediate stage.

That linearization yields wrong models is also the opinion of McKenna [64, p.4] who comments (3.8) by writing

> This is the point at which the discussion of torsional oscillation starts in the engineering literature.

He claims that the problem is in fact nonlinear and that (3.8) is obtained after an incorrect linearisation. McKenna concludes by noticing that *Even in recent engineering literature ... this same mistake is reproduced.* The mistake claimed by McKenna is that the equations are often linearized by taking $\sin\theta=\theta$ and $\cos\theta=1$ also for large amplitude torsional oscillations θ. The corresponding equation then becomes linear and the main torsional phenomenon disappears. Avoiding this rude approximation, but considering the cables and hangers as linear springs obeying $\mathcal{LHL}$, McKenna reaches an uncoupled second order system for the functions representing the vertical displacement y of the barycenter B of the cross section of the roadway and the deflection from horizontal θ, see Figure 7. Here, 2ℓ denotes the width of the roadway whereas C_1 and C_2 denote the two lateral hangers which have opposite extension behaviors.

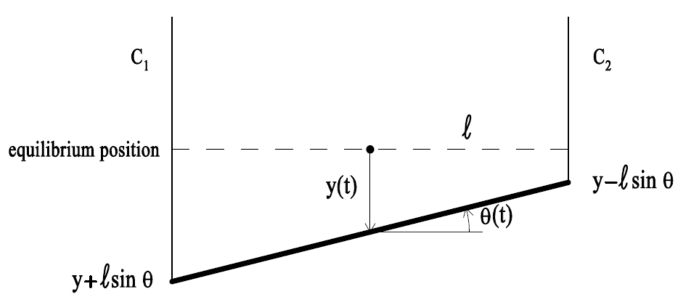

FIGURE 7. Vertical and torsional displacements of the cross section of the roadway

McKenna-Tuama [67] suggest a slightly different model. They write:

> ... there should be some torsional forcing. Otherwise, there would be no input of energy to overcome the natural damping of the system ... we expect the bridge to behave like a stiff spring, with a restoring force that becomes somewhat superlinear.

McKenna-Tuama end up with the following coupled second order system

$$\begin{aligned} \frac{m\ell^2}{3}\theta'' &= \ell\cos\theta\Big(f(y-\ell\sin\theta)-f(y+\ell\sin\theta)\Big)\,,\\ my'' &= -\Big(f(y-\ell\sin\theta)+f(y+\ell\sin\theta)\Big)\,, \end{aligned} \tag{3.9}$$

see again Figure 7. The delicate point is the choice of the superlinearity f which [67] take first as $f(s)=(s+1)^+-1$ and then as $f(s)=e^s-1$ in order to maintain the asymptotically linear behavior as $s\to 0$. Using (3.9), [64, 67] were able to numerically replicate the phenomenon observed at the Tacoma Bridge, namely the sudden transition from vertical oscillations to torsional oscillations. They found that if the vertical motion was sufficiently large to induce brief slackening of the hangers, then numerical results highlighted a rapid transition to a torsional motion. By commenting the results in [64, 67], McKenna-Moore [66, p.460] write that

> ... the range of parameters over which the transition from vertical to torsional motion was observed was physically unreasonable ... the restoring force due to the cables was oversimplified ... it was necessary to impose small torsional forcing.

In fact, McKenna-Tuama [67] numerically show that a purely vertical forcing in (3.9) may create a torsional response. Therefore, (3.9) seems to be the first model able to reproduce the behavior of the Tacoma Bridge but, perhaps, it may be improved. First, one could allow the nonlinearity to appear before the possible slackening of the hangers. Second, the restoring force and the parameters involved should be chosen carefully.

Problem 3.3. Try a doubly superlinear term f in (3.9). For instance, take $f(s)=s+\varepsilon s^3$ with $\varepsilon>0$ small, so that (3.9) becomes

$$\begin{gathered} \frac{m\ell^2}{3}\theta''+2\ell^2\cos\theta\sin\theta\Big(1+3\varepsilon y^2+\varepsilon\ell^2\sin^2\theta\Big)=0\\ my''+2\Big(1+3\varepsilon\ell^2\sin^2\theta\Big)y+2\varepsilon y^3=0\,. \end{gathered} \tag{3.10}$$

It appears challenging to determine some features of the solution (y,θ) to (3.10) and also to perform numerical experiments to see what kind of oscillations are displayed by the solutions.

System (3.9) is a 2×2 system which should be considered as a nonlinear fourth order model; therefore, it fulfills the necessary conditions of the general principle (GP). Another fourth order differential equation was suggested in [57, 68, 69] as a one-dimensional model for a suspension bridge, namely a beam of length L suspended by hangers. When the hangers are stretched there is a restoring force which is proportional to the amount of stretching, according to $\mathcal{LHL}$. But when the beam moves in the opposite direction, there is no restoring force exerted on it. Under suitable boundary conditions, if $u(x,t)$ denotes the vertical displacement of the beam in the downward direction at position x and time t, the following nonlinear beam equation is derived

$$u_{tt} + u_{xxxx} + \gamma u^+ = W(x,t)\,, \quad x \in (0,L)\,,\ t > 0\,, \tag{3.11}$$

where $u^+ = \max\{u, 0\}$, γu^+ represents the force due to the cables and hangers which are considered as a linear spring with a one-sided restoring force, and W represents the forcing term acting on the bridge, including its own weight per unit length, the wind, the traffic loads, or other external sources. After some normalisation, by seeking traveling waves $u(x,t) = 1 + w(x - ct)$ to (3.11) and putting $k = c^2 > 0$, McKenna-Walter [69] reach the ODE

$$w''''(\tau) + kw''(\tau) + f(w(\tau)) = 0 \quad (x - ct = \tau \in \mathbb{R}) \tag{3.12}$$

where $k \in (0, 2)$ and $f(s) = (s + 1)^+ - 1$. Subsequently, in order to maintain the same behavior but with a smooth nonlinearity, Chen-McKenna [21] suggest to consider (3.12) with $f(s) = e^s - 1$. For later discussion, we notice that both these nonlinearities satisfy

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\,, \quad f(s)\,s > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}. \tag{3.13}$$

Hence, when $W \equiv 0$, (3.11) is just a special case of the more general semilinear fourth order wave equation

$$u_{tt} + u_{xxxx} + f(u) = 0\,, \quad x \in (0,L)\,,\ t > 0\,, \tag{3.14}$$

where the natural assumptions on f are (3.13) plus further conditions, according to the model considered. Traveling waves to (3.14) solve (3.12) with $k = c^2$ being the squared velocity of the wave. Recently, for $f(s) = (s+1)^+ - 1$ and its variants, Benci-Fortunato [8] proved the existence of special solutions to (3.12) deduced by solitons of the beam equation (3.14).

Problem 3.4. It could be interesting to insert into the wave-type equation (3.14) the term corresponding to the beam elongation; that is,

$$\int_0^L \Big(\sqrt{1 + u_x(x,t)^2} - 1\Big)\, dx.$$

This would lead to a quasilinear equation such as

$$u_{tt} + u_{xxxx} - \Big(\frac{u_x}{\sqrt{1 + u_x^2}}\Big)_x + f(u) = 0$$

with f satisfying (3.13). What can be said about this equation? Does it admit oscillating solutions in a suitable sense? One should first consider the case of an unbounded beam ($x \in \mathbb{R}$) and then the case of a bounded beam ($x \in (0,L)$) complemented with some boundary conditions.

Motivated by the fact that it appears unnatural to ignore the motion of the main sustaining cable, a slightly more sophisticated and complicated string-beam model was suggested by Lazer-McKenna [58, Section 3.4]. They treat the cable as a vibrating string, coupled with the vibrating beam of the roadway by piecewise linear springs that have a given spring constant k if expanded, but no restoring force if compressed. The sustaining cable is subject to some forcing term such as the wind or the motions in the towers. This leads to the system

$$\begin{aligned} &v_{tt} - c_1 v_{xx} + \delta_1 v_t - k_1(u - v)^+ = f(x,t) \quad x \in (0,L)\,,\ t > 0\,,\\ &u_{tt} + c_2 u_{xxxx} + \delta_2 u_t + k_2(u - v)^+ = W_0 \quad x \in (0,L)\,,\ t > 0\,, \end{aligned}$$

where v is the displacement from equilibrium of the cable and u is the displacement of the beam, both measured in the downwards direction. The constants c_1 and c_2 represent the relative strengths of the cables and roadway respectively, whereas k_1 and k_2 are the spring constants and satisfy $k_2 \ll k_1$. The two damping terms can possibly be set to 0, while f and W_0 are the forcing terms. We also refer to [2] for a study of the same problem in a rigorous functional analytic setting.

In a series of recent papers, Bochicchio-Giorgi-Vuk [14, 15, 16, 17], generalized the above model by taking into account the midplane stretching of the roadway due to its elongation. They consider a beam identified with the interval $(0, L)$ and they end up with the nonlinear system

$$\begin{gathered} v_{tt} - v_{xx} + bv_t - F(u-v, u_t - v_t) = f \quad x \in (0,L)\,,\ t > 0\,, \\ u_{tt} + u_{xxxx} + au_t + \Big(p - M(\|u_x\|^2_{L^2(0,1)})\Big)u_{xx} + F(u-v, u_t - v_t) = g \\ x \in (0,L)\,,\ t > 0\,, \end{gathered}$$

where u is the downward deflection of the beam, v is the vertical displacement of the sustaining cable, f and g are external forcing, a and b are damping constants, p is the axial force acting at one end of the beam which is negative when the beam is stretched and is positive when the beam is compressed; $F(u-v, u_t-v_t)$ represents the nonlinear response of the hangers connecting the beam with the cable and the term $M(\|u_x\|^2_{L^2(0,1)})$ takes into account the geometric nonlinearity of the beam due to its stretching.

Classical linear models viewing a suspension bridge as a beam connected to a sustaining cable go back to Biot-von Kármán [11]. They are still used nowadays by engineers for first qualitative information on the plans, see [99]. The above nonlinear problems are by far more precise. However, if one wishes to view torsional oscillations, the bridge cannot be considered as a one dimensional beam. In this respect, Rocard [82, p.148] states

> Conventional suspension bridges are fundamentally unstable in the wind because the coupling effect introduced between bending and torsion by the aerodynamic forces of the lift.

Hence, if some model wishes to display any possible instability of bridges, it should necessarily take into account more degrees of freedom of the roadway. To be exhaustive one should consider vertical oscillations y of the roadway, its torsional angle θ, and coupling with the two sustaining cables u and v. This model was suggested by Matas-Očenášek [63] who consider the hangers as linear springs and obtain a system of four equations; three of them are second order wave-type equations, the last one is again a fourth order equation such as

$$my_{tt} + k\,y_{xxxx} + \delta y_t + E_1(y - u - \ell\sin\theta) + E_2(y - v + \ell\sin\theta) = W(x) + f(x,t)\,;$$

we refer to [29, (SB_4)] for an interpretation of the parameters involved.

In our opinion, any model which describes the bridge as a one dimensional beam is too simplistic, unless the model takes somehow into account the possible appearance of a torsional motion. In [43] it was suggested to maintain the one dimensional model provided one also allows displacements below the equilibrium position and these displacements replace the deflection from horizontal of the roadway of the bridge; in other words,

$$\text{the unknown function } w \text{ represents the upwards vertical displacement when } w > 0 \text{ and the deflection from horizontal, computed in a suitable unity measure, when } w < 0. \tag{3.15}$$

In this setting, instead of (3.11) one should consider the more general semilinear fourth order wave equation (3.14) with f satisfying (3.13) plus further conditions which make $f(s)$ superlinear and unbounded when both $s \to \pm\infty$; hence, $\mathcal{LHL}$ is dropped by allowing f to be as close as one may wish to a linear function but eventually superlinear for large displacements. The superlinearity assumption is justified both by the observations in Section 2 and by the fact that more the position of the bridge is far from the horizontal equilibrium position, more the action of the wind becomes relevant because the wind hits transversally the roadway of the bridge. If ever the bridge would reach the limit vertical position, in case the roadway is torsionally rotated of a right angle, the wind would hit it orthogonally, that is, with full power.

In this section we listed a number of attempts to model bridges mechanics by means of differential equations. The sources for this list are very heterogeneous. However, except for some possible small damping term, none of them contains odd derivatives. Moreover, none of them is acknowledged by the scientific community to perfectly describe the complex behavior of bridges. Some of them fail to satisfy the requirements of (GP) and, in our opinion, must be accordingly modified. Some others seem to better describe the oscillating behavior of bridges but still need some improvements.

4. Blow up oscillating solutions to some fourth order differential equations

If the trivial solution to some dynamical system is unstable one may hope to magnify self-excitement phenomena through finite time blow up. In this section we survey and discuss several results about solutions to (3.12) which blow up in finite time. Let us rewrite the equation with a different time variable, namely

$$w''''(t) + kw''(t) + f(w(t)) = 0 \quad (t \in \mathbb{R})\,. \tag{4.1}$$

We first recall the following results proved in [9].

Theorem 4.1. *Let $k \in \mathbb{R}$ and assume that f satisfies* (3.13).

(i) *If a local solution w to* (4.1) *blows up at some finite $R \in \mathbb{R}$, then*

$$\liminf_{t\to R} w(t) = -\infty \quad \text{and} \quad \limsup_{t\to R} w(t) = +\infty\,. \tag{4.2}$$

(ii) *If f also satisfies*

$$\limsup_{s\to+\infty} \frac{f(s)}{s} < +\infty \quad \text{or} \quad \limsup_{s\to-\infty} \frac{f(s)}{s} < +\infty, \tag{4.3}$$

then any local solution to (4.1) *exists for all $t \in \mathbb{R}$.*

If both the conditions in (4.3) are satisfied then global existence follows from classical theory of ODE's; but (4.3) merely requires that f is "one-sided at most linear" so that statement (ii) is far from being trivial and, as shown in [43], it does not hold for equations of order at most 3. On the other hand, Theorem 4.1 (i) states that, under the sole assumption (3.13), the only way that finite time blow up can occur is with "wide and thinning oscillations" of the solution w; again, in [43] it was shown that this kind of blow up is a phenomenon typical of at least fourth order problems such as (4.1) since it does not occur in related lower order equations. Note that assumption (4.3) includes, in particular, the cases where f is either concave or convex.

Theorem 4.1 does not guarantee that the blow up described by (4.2) indeed occurs. For this reason, we assume further that

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}), \quad f'(s) \geq 0 \quad \forall s \in \mathbb{R}, \quad \liminf_{s \to \pm\infty} |f''(s)| > 0 \tag{4.4}$$

and the growth conditions: There exist $p > q \geq 1$, $\alpha \geq 0$, $0 < \rho \leq \beta$ such that

$$\rho |s|^{p+1} \leq f(s)s \leq \alpha |s|^{q+1} + \beta |s|^{p+1} \quad \forall s \in \mathbb{R}. \tag{4.5}$$

Notice that (4.4)-(4.5) strengthen (3.13). In [45] the following sufficient conditions for the finite time blow up of local solutions to (4.1) has been proved.

Theorem 4.2. *Let $k \leq 0$, $p > q \geq 1$, $\alpha \geq 0$, and assume that f satisfies (4.4) and (4.5). Assume that $w = w(t)$ is a local solution to (4.1) in a neighborhood of $t = 0$ which satisfies*

$$w'(0)w''(0) - w(0)w'''(0) - kw(0)w'(0) > 0. \tag{4.6}$$

Then, w blows up in finite time for $t > 0$; that is, there exists $R \in (0, +\infty)$ such that (4.2) holds.

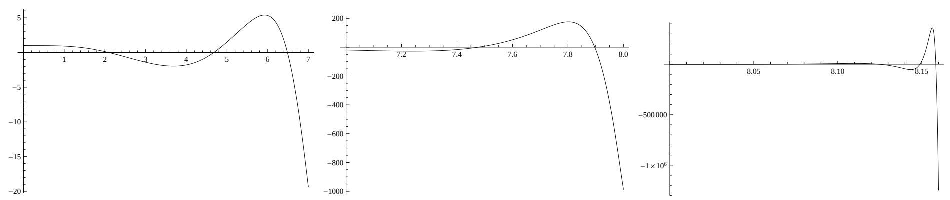

FIGURE 8. Solution to (4.1) for $[w(0), w'(0), w''(0), w'''(0)] = [1, 0, 0, 0]$, $k = 3$, $f(s) = s + s^3$. The three intervals are $t \in [0, 7]$, $t \in [7, 8]$, $t \in [8, 8.16]$

Unfortunately, the oscillations displayed by the solutions to (4.1) cannot be prevented since they arise suddenly after a long time of apparent calm. In Figure 8, we display the plot of a solution to (4.1). It can be observed that the solution has oscillations with increasing amplitude and rapidly decreasing "nonlinear frequency"; numerically, the blow up seems to occur at $t = 8.164$. Even more impressive appears the plot in Figure 9.

Here the solution has "almost regular" oscillations between -1 and $+1$ for $t \in [0, 80]$. Then the amplitude of oscillations nearly doubles in the interval $[80, 93]$ and, suddenly, it violently amplifies after $t = 96.5$ until the blow up which seems to occur only slightly later at $t = 96.59$. We also refer to [43, 44, 45] for further plots.

We also refer to [43, 45] for numerical results and plots of solutions to (4.1) with nonlinearities $f = f(s)$ having different growths as $s \to \pm\infty$. In such case, the solution still blows up according to (4.2) but, although its "limsup" and "liminf" are respectively $+\infty$ and $-\infty$, the divergence occurs at different rates.

Traveling waves to (3.14) which propagate at some velocity $c > 0$, depending on the elasticity of the material of the beam, solve (4.1) with $k = c^2 > 0$. Further

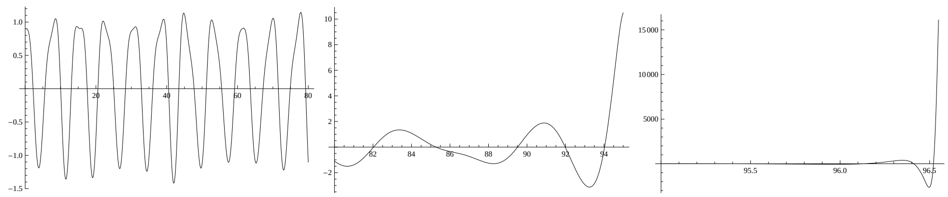

FIGURE 9. Solution to (4.1) for $[w(0), w'(0), w''(0), w'''(0)] = [0.9, 0, 0, 0]$, $k = 3.6$, $f(s) = s + s^3$. The three intervals are $t \in [0, 80]$, $t \in [80, 95]$, $t \in [95, 96.55]$

numerical results obtained in [43, 45] suggest that a statement similar to Theorem 4.2 also holds for $k > 0$ and, as expected, that the blow up time R is decreasing with respect to the initial height $w(0)$ and increasing with respect to k. Since $k = c^2$ and c represents the velocity of the traveling wave, this means that the time of blow up is an increasing function of k. In turn, since the velocity of the traveling wave depends on the elasticity of the material used to construct the bridge (larger c means less elastic), this tells us that more the bridge is stiff more it will survive to exterior forces such as the wind and/or traffic loads.

Problem 4.3. Prove Theorem 4.2 when $k > 0$. This would allow to show that traveling waves to (3.14) blow up in finite time. Numerical results in [43, 45] suggest that a result similar to Theorem 4.2 also holds for $k > 0$.

Problem 4.4. Prove that the blow up time of solutions to (4.1) depends increasingly with respect to $k \in \mathbb{R}$. The interest of an analytical proof of this fact relies on the important role played by k within the model.

Problem 4.5. The blow up time R of solutions to (4.1) may be related to the expectation of life of the oscillating bridge. Provide an estimate of R in terms of f and of the initial data.

Problem 4.6. Condition (4.5) is a superlinearity assumption which requires that f is bounded both from above and below by the same power $p > 1$. Prove Theorem 4.2 for more general kinds of superlinear functions f.

Problem 4.7. Can assumption (4.6) be relaxed? Of course, it cannot be completely removed since the trivial solution $w(t) \equiv 0$ is globally defined, that is, $R = +\infty$. Numerical experiments in [43, 45] could not detect any nontrivial global solution to (4.1). If we put an equality in (4.6) we obtain a 3D manifold in the phase space $\mathbb{R}^4$ but since the stable manifold of $\{0\}$ is a 2D manifold, see [9, Proposition 20], one has probability 0 to find a global solution even in this case.

Problem 4.8. Study (4.1) with a damping term: $w''''(t) + kw''(t) + \delta w'(t) + f(w(t)) = 0$ for some $\delta > 0$. Study the competition between the damping term $\delta w'$ and the nonlinear self-exciting term $f(w)$.

Note that Theorems 4.1 and 4.2 ensure that there exists an increasing sequence $\{z_j\}_{j \in \mathbb{N}}$ such that:

(i) $z_j \nearrow R$ as $j \to \infty$;
(ii) $w(z_j) = 0$ and w has constant sign in (z_j, z_{j+1}) for all $j \in \mathbb{N}$.

It is also interesting to compare the rate of blow up of the displacement and of the acceleration on these intervals. By slightly modifying the proof of [45, Theorem 3] one can obtain the following result which holds for any $k \in \mathbb{R}$.

Theorem 4.9. *Let $k \in \mathbb{R}$, $p > q \geq 1$, $\alpha \geq 0$, and assume that f satisfies* (4.4) *and* (4.5). *Assume that $w = w(t)$ is a local solution to*

$$w''''(t) + kw''(t) + f(w(t)) = 0 \quad (t \in \mathbb{R})$$

which blows up in finite time as $t \nearrow R < +\infty$. Denote by $\{z_j\}$ the increasing sequence of zeros of w such that $z_j \nearrow R$ as $j \to +\infty$. Then

$$\int_{z_j}^{z_{j+1}} w(t)^2 \, dt \ll \int_{z_j}^{z_{j+1}} w''(t)^2 \, dt \,, \quad \int_{z_j}^{z_{j+1}} w'(t)^2 \, dt \ll \int_{z_j}^{z_{j+1}} w''(t)^2 \, dt \tag{4.7}$$

as $j \to \infty$. Here, $g(j) \ll \psi(j)$ means that $g(j)/\psi(j) \to 0$ as $j \to \infty$.

The estimate (4.7), clearly due to the superlinear term, has a simple interpretation in terms of comparison between blowing up energies, see Section 5.1.

Remark 4.10. Equation (4.1) also arises in several different contexts, see the book by Peletier-Troy [77] where one can find some other physical models, a survey of existing results, and further references. Moreover, besides (3.14), (4.1) may also be fruitfully used to study some other partial differential equations. For instance, one can consider nonlinear elliptic equations such as

$$\begin{gathered} \Delta^2 u + e^u = \frac{1}{|x|^4} \quad \text{in } \mathbb{R}^4 \setminus \{0\} \,, \\ \Delta^2 u + |u|^{8/(n-4)} u = 0 \quad \text{in } \mathbb{R}^n \ (n \geq 5), \\ \Delta\big(|x|^2 \Delta u\big) + |x|^2 |u|^{8/(n-2)} u = 0 \quad \text{in } \mathbb{R}^n \ (n \geq 3); \end{gathered} \tag{4.8}$$

it is known (see, e.g. [42]) that the Green function for some fourth order elliptic problems displays oscillations, differently from second order problems. Furthermore, one can also consider the semilinear parabolic equation

$$u_t + \Delta^2 u = |u|^{p-1} u \text{ in } \mathbb{R}^{n+1}_+ \,, \quad u(x,0) = u_0(x) \text{ in } \mathbb{R}^n$$

where $p > 1 + 4/n$ and u_0 satisfies suitable assumptions. It is shown in [38, 41] that the linear biharmonic heat operator has an “eventual local positivity” property: for positive initial data u_0 the solution to the linear problem with no source is eventually positive on compact subsets of $\mathbb{R}^n$ but negativity can appear at any time far away from the origin. This phenomenon is due to the sign changing properties, with infinite oscillations, of the biharmonic heat kernels. We also refer to [9, 45] for some results about the above equations and for the explanation of how they can be reduced to (4.1) and, hence, how they display self-excited oscillations.

Problem 4.11. For any $q > 0$ and parameters $a, b, k \in \mathbb{R}$, $c \geq 0$, study the equation

$$w''''(t) + aw'''(t) + kw''(t) + bw'(t) + cw(t) + |w(t)|^q w(t) = 0 \quad (t \in \mathbb{R}) \,. \tag{4.9}$$

Any reader who is familiar with the second order Sobolev space H^2 recognises the critical exponent in the first equation in (4.8). In view of Liouville-type results in [27] when $q \leq 8/(n-4)$, it would be interesting to study the equation $\Delta^2 u + |u|^q u = 0$ with the same technique. The radial form of this equation may be written as (4.1) only when $q = 8/(n-4)$ since for other values of q the transformation in [40] gives rise to the appearance of first and third order derivatives as in (4.9): this motivates

(4.9). The values of the parameters corresponding to the equation $\Delta^2 u + |u|^q u = 0$ can be found in [40].

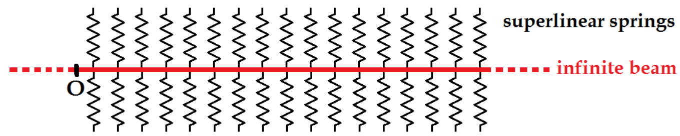

FIGURE 10. Beam subject to two-sided restoring springs

We now describe an ideal model obeying to (4.1). Consider an infinite beam fixed at some point O and subject to the restoring forces of a large number of nonlinear two-sided springs as in Figure 10. If the beam had finite length, it could model the roadway of a suspension bridge and the springs would model the hangers. Let $u = u(x)$ denote the vertical displacement of the beam. Assume that, besides the nonlinear restoring force $g = g(u)$ due to the springs, there is a uniform downwards load $p(x) \equiv p > 0$ acting on the beam, for instance, its weight per unit length. Then, the same arguments which lead to (3.5) yield the semilinear equation

$$EIu''''(x) - Tu''(x) = p - g(u(x)) \quad (x \in \mathbb{R})\,.$$

This is the general equation of an infinite beam having flexural rigidity EI, constant tension $T \geq 0$, and subject to both a downwards load p (its weight) and to the restoring action $g = g(u)$ due to some elastic springs. Take, for instance, $g(u) = u + u^3$ and let $u_p > 0$ be the unique solution of $g(u_p) = p$. Put $f(s) := g(s+u_p) - p$ so that f satisfies (4.4)-(4.5). Put $w(x) = u(x) - u_p$; then w solves the equation

$$EIw''''(x) - Tw''(x) + f(w(x)) = 0 \quad (x \in \mathbb{R})$$

and Theorem 4.2 applies.

Our next target is to reproduce the self-excited oscillations found in Theorem 4.2 in the second order system

$$x'' - f(y-x) + \beta(y+x) = 0\,, \quad y'' - f(y-x) + \delta(y+x) = 0\,, \tag{4.10}$$

where $\beta, \delta \in \mathbb{R}$ and f is a superlinear function. This will facilitate a precise study of the behavior of the solutions when the nonlinear part of f tends to vanish. To (4.10) we associate the initial value problem

$$x(0) = x_0\,, \quad x'(0) = x_1\,, \quad y(0) = y_0\,, \quad y'(0) = y_1\,. \tag{4.11}$$

The following statement holds.

Theorem 4.12. *Assume that $\beta < \delta \leq -\beta$ (so that $\beta < 0$). Assume also that $f(s) = \sigma s + cs^2 + ds^3$ with $d > 0$ and $c^2 \leq 2d\sigma$. Let $(x_0, y_0, x_1, y_1) \in \mathbb{R}^4$ satisfy*

$$(3\beta - \delta)x_0 y_1 + (3\delta - \beta)x_1 y_0 > (\beta + \delta)(x_0 x_1 + y_0 y_1)\,. \tag{4.12}$$

If (x, y) is a local solution to (4.10)-(4.11) in a neighborhood of $t = 0$, then (x, y) blows up in finite time for $t > 0$ with self-excited oscillations; that is, there exists $R \in (0, +\infty)$ such that

$$\liminf_{t\to R} x(t) = \liminf_{t\to R} y(t) = -\infty \quad \text{and} \quad \limsup_{t\to R} x(t) = \limsup_{t\to R} y(t) = +\infty\,.$$

Proof. After performing the change of variables

$$w := y - x\,, \quad z := y + x\,, \tag{4.13}$$

system (4.10) becomes

$$w'' + (\delta - \beta)z = 0\,, \quad z'' - 2f(w) + (\beta + \delta)z = 0\,,$$

which may be rewritten as a single fourth order equation

$$w''''(t) + (\beta + \delta)w''(t) + 2(\delta - \beta)f(w(t)) = 0\,. \tag{4.14}$$

Assumption (4.12) reads

$$w'(0)w''(0) - w(0)w'''(0) - (\beta + \delta)w(0)w'(0) > 0\,.$$

Furthermore, in view of the above assumptions, f satisfies (4.4)-(4.5) with $\rho = d/2$, $p = 3$, $\alpha = 2\sigma$, $q = 1$, $\beta = 3d$. Whence, Theorem 4.2 states that w blows up in finite time for $t > 0$ and that there exists $R \in (0, +\infty)$ such that

$$\liminf_{t\to R} w(t) = -\infty \quad \text{and} \quad \limsup_{t\to R} w(t) = +\infty\,. \tag{4.15}$$

Next, we remark that (4.14) admits a first integral, namely

$$\begin{aligned} E(t) &:= \frac{\beta+\delta}{2}\, w'(t)^2 + w'(t)w'''(t) + 2(\delta - \beta)F(w(t)) - \frac{1}{2}\, w''(t)^2 \\ &= \frac{\beta+\delta}{2}\, w'(t)^2 + (\beta - \delta)w'(t)z'(t) + 2(\delta - \beta)F(w(t)) - \frac{(\beta-\delta)^2}{2}\, z(t)^2 \equiv \overline{E}\,, \end{aligned} \tag{4.16}$$

for some constant $\overline{E}$. By (4.15) there exists an increasing sequence $m_j \to R$ of local maxima of w such that

$$z(m_j) = \frac{w''(m_j)}{\beta - \delta} \geq 0\,, \quad w'(m_j) = 0\,, \quad w(m_j) \to +\infty \text{ as } j \to \infty\,.$$

By plugging m_j into the first integral (4.16) we obtain

$$\overline{E} = E(m_j) = 2(\delta - \beta)F(w(m_j)) - \frac{(\beta-\delta)^2}{2}\, z(m_j)^2$$

which proves that $z(m_j) \to +\infty$ as $j \to +\infty$. We may proceed similarly in order to show that $z(\mu_j) \to -\infty$ on a sequence $\{\mu_j\}$ of local minima of w. Therefore, we have

$$\liminf_{t\to R} z(t) = -\infty \quad \text{and} \quad \limsup_{t\to R} z(t) = +\infty\,.$$

Assume for contradiction that there exists $K \in \mathbb{R}$ such that $x(t) \leq K$ for all $t < R$. Then, recalling (4.13), on the above sequence $\{m_j\}$ of local maxima for w, we would have $y(m_j) - K \geq y(m_j) - x(m_j) = w(m_j) \to +\infty$ which is incompatible with (4.16) since

$$2(\delta - \beta)F(y(m_j) - x(m_j)) - \frac{(\beta-\delta)^2}{2}(y(m_j) + x(m_j))^2 \equiv \overline{E}$$

and F has growth of order 4 with respect to its divergent argument. Similarly, by arguing on the sequence $\{\mu_j\}$, we rule out the possibility that there exists $K \in \mathbb{R}$ such that $x(t) \geq K$ for ll $t < R$. Finally, by changing the role of x and y we find that also $y(t)$ is unbounded both from above and below as $t \to R$. This completes the proof. □

Remark 4.13. Numerical results in [45] suggest that the assumption $\delta \le -\beta$ is not necessary to obtain (4.15). So, most probably, Theorem 4.12 and the results of this section hold true also without this assumption. Assumption (4.12) is related to (4.6) and is a kind of initial energy condition; it cannot be completely removed, as explained in Problem 4.7.

A special case of function f satisfying the assumptions of Theorem 4.12 is $f_\varepsilon(s) = s + \varepsilon s^3$ for any $\varepsilon > 0$. We wish to study the situation when the problem tends to become linear; that is, when $\varepsilon \to 0$. Plugging such f_ε into (4.10) gives the system

$$\begin{gathered} x'' + (\beta+1)x + (\beta-1)y + \varepsilon(x-y)^3 = 0 \\ y'' + (\delta+1)x + (\delta-1)y + \varepsilon(x-y)^3 = 0 \end{gathered} \tag{4.17}$$

so that the limit linear problem obtained for $\varepsilon = 0$ reads

$$\begin{gathered} x'' + (\beta+1)x + (\beta-1)y = 0 \\ y'' + (\delta+1)x + (\delta-1)y = 0\,. \end{gathered} \tag{4.18}$$

The theory of linear systems tells us that the shape of the solutions to (4.18) depends on the signs of the parameters

$$A = \beta + \delta\,, \quad B = 2(\delta - \beta)\,, \quad \Delta = (\beta+\delta)^2 + 8(\beta - \delta)\,.$$

Under the same assumptions of Theorem 4.12, for (4.18) we have $A \le 0$ and $B > 0$ but the sign of Δ is not known a priori and three different cases may occur.

- If $\Delta < 0$ (a case including also $A = 0$), then we have exponentials times trigonometric functions so either we have self-excited oscillations which increase amplitude as $t \to \infty$ or we have damped oscillations which tend to vanish as $t \to \infty$. Consider the case $\delta = -\beta = 1$ and $(x_0, y_0, x_1, y_1) = (1, 0, 1, -1)$, then (4.12) is fulfilled and Theorem 4.12 yields

Corollary 4.14. *For any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that the solution $(x^\varepsilon, y^\varepsilon)$ to the Cauchy problem*

$$\begin{gathered} x'' - 2y + \varepsilon(x-y)^3 = 0 \\ y'' + 2x + \varepsilon(x-y)^3 = 0 \\ x(0) = 1, \quad y(0) = 0, \quad x'(0) = 1, \quad y'(0) = -1 \end{gathered} \tag{4.19}$$

blows up as $t \to R_\varepsilon$ and satisfies

$$\liminf_{t \to R_\varepsilon} x^\varepsilon(t) = \liminf_{t \to R_\varepsilon} y^\varepsilon(t) = -\infty, \quad \limsup_{t \to R_\varepsilon} x^\varepsilon(t) = \limsup_{t \to R_\varepsilon} y^\varepsilon(t) = +\infty\,.$$

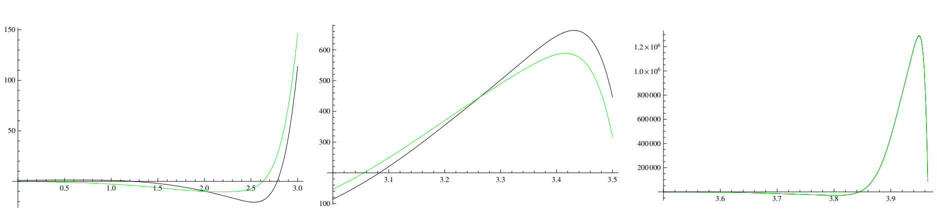

FIGURE 11. The solution x^ε (black) and y^ε (green) to (4.19) for $\varepsilon = 0.1$

A natural conjecture, supported by numerical experiments, is that $R_\varepsilon \to \infty$ as $\varepsilon \to 0$. For several $\varepsilon > 0$, we plotted the solution to (4.19) and the pictures all looked like Figure 11. When $\varepsilon = 0.1$ the blow up seems to occur at $R_\varepsilon = 4.041$. Notice that x^ε and y^ε "tend to become the same", in the third picture they are indistinguishable. After some time, when wide oscillations amplifies, x^ε and y^ε move almost synchronously. When $\varepsilon = 0$, the solution to (4.19) is explicitly given by $x^0(t) = e^t\cos(t)$ and $y^0(t) = -e^t\sin(t)$, thereby displaying oscillations blowing up in infinite time similar to those visible for solutions to (3.8).

If we replace the Cauchy problem in (4.19) with

$$x(0) = 1, \quad y(0) = 0, \quad x'(0) = -1, \quad y'(0) = 1$$

then (4.12) is not fulfilled. However, for any $\varepsilon > 0$ that we tested, the corresponding numerical solutions looked like in Figure 11. In this case, the limit problem with $\varepsilon = 0$ admits as solutions $x^0(t) = e^{-t}\cos(t)$ and $y^0(t) = e^{-t}\sin(t)$ which do exhibit oscillations but, now, strongly damped.

Let us also consider the two remaining limit systems which, however, do not display oscillations.

- If $\Delta = 0$, since $A \leq 0$, there are no trigonometric functions in the limit case (4.18).
- If $\Delta > 0$, then necessarily $A < 0$ since $B > 0$, and hence only exponential functions are involved: the solution to (4.18) may blow up in infinite time or vanish at infinity.

These remarks enable us to conclude that *the linear case cannot be seen as a limit situation of the nonlinear case* since the behavior of the solution to (4.18) depends on β, δ, and on the initial conditions, while nothing can be deduced from the sequence of solutions $(x^\varepsilon, y^\varepsilon)$ to problem (4.17) as $\varepsilon \to 0$ because these solutions all behave similarly independently of β and δ. Furthermore, the solutions to the limit problem (4.18) may or may not exhibit oscillations and if they do, these oscillations may be both of increasing amplitude or of vanishing amplitude as $t \to +\infty$. All this shows that linearisation may give misleading and unpredictable answers.

The above results also explain why we believe that (3.8) is not suitable to display self-excited oscillations as the ones which appeared for the TNB. Since it has only two degrees of freedom, it fails to consider both vertical and torsional oscillations which, on the contrary, are visible in the McKenna-type system (4.10). We have seen in Theorem 4.12 that destructive self-excited oscillations may blow up in finite time, something very similar to what may be observed in [104]. Hence, (4.10) shows more realistic self-excited oscillations than (3.8). Although the blow up occurs at $t = 4.04$, the solution plotted in Figure 11 is relatively small until $t = 3.98$. This, together with the behavior displayed in Figures 8 and 9, allows us to conclude that *in nonlinear systems, self-excited oscillations appear suddenly, without any intermediate stage.*

5. Affording an explanation in terms of energies

5.1. **Energies involved.** A precise description of all the energies involved in a structure would lead to perfect models and would give all the information for correct plans. Unfortunately, bridges, as well as many other structures, do not allow simple characterizations of all the energies present in the structure and, maybe, not all possible existing energies have ever been detected. But let us make an attempt to classify the different kinds of energy.

Let Ω be either a segment (beam model) or a thin rectangle (plate model). If $v(x,t)$ denotes the vertical displacement at $x \in \Omega$ and at $t > 0$, then the total kinetic energy at time t is given by

$$\frac{m}{2}\int_\Omega v_t(x,t)^2\,dx$$

where m is the mass. This energy gives rise to the term mv_{tt} in the corresponding Euler-Lagrange equation, see Section 3.2.

Then one should consider potential energy, which is more complicated. In order to avoid confusion, in the sequel we call potential energy only the energy due to gravity which, in the case of a bridge, is computed in terms of the vertical displacement v. From [13, pp.75-76], we quote

> The potential energy is stored partly in the stiffening frame in the form of elastic energy due to bending and partly in the cable in the form of elastic stress-strain energy and in the form of an increased gravity potential.

Hence, an important role is played by stored energy which is somehow hidden. Part of the stored energy is potential energy but the largest part of the stored energy in a bridge is its elastic energy.

The distinction between elastic and potential stored energies, which in our opinion appears essential, is not highlighted with enough care in [13] nor in any subsequent treatise of suspension bridges. A further criticism about [13] is that it often makes use of $\mathcal{LHL}$, see [13, p.214]. Apart these two weak points, [13] makes a quite careful quantitative analysis of the energies involved. In particular, concerning the elastic energy, the contribution of each component of the bridge is taken into account in [13]: the chords (p.145), the diagonals (p.146), the cables (p.147), the towers (pp.164-168), as well as quantitative design factors (pp.98-103).

A detailed energy method is also introduced at p.74, as a practical tool to determine the modes of vibrations and natural frequencies of suspension bridges: the energies considered are expressed in terms of the amplitude of the oscillation $\eta = \eta(x)$ and therefore, they do not depend on time. As already mentioned, the nonlocal term in (3.6) represents the increment of energy due to the external wind during a period of time. Recalling that $v(x,t) = \eta(x)\sin(\omega t)$, [13, p.28] represents the net energy input per cycle by

$$A := \frac{w^2}{H_w^2}\,\frac{EA}{L}\int_0^L \eta(z)\,dz - C\int_0^L \eta(z)^2\,dz \tag{5.1}$$

where L is the length of the beam and $C > 0$ is a constant depending on the frequency of oscillation and on the damping coefficient, so that the second term is a quantum of energy being dissipated as heat: mechanical hysteresis, solid friction damping, aerodynamic damping, etc. It is explained in Figure 13 in [13, p.33] that

> the kinetic energy will continue to build up and therefore the amplitude will continue to increase until $A = 0$.

Hence, the larger is the input of energy $\int_0^L \eta$ due to the wind, the larger needs to be the displacement v before the kinetic energy will stop to build up. This is related to [13, pp.241-242], where an attempt is made

> to approach by rational analysis the problem proper of self-excitation of vibrations in truss-stiffened suspension bridges. ... The theory discloses the peculiar mechanism of catastrophic self-excitation in such bridges.

The word "self-excitation" suggests behaviors similar to (4.2). As shown in [45], the oscillating blow up of solutions described by (4.2) occurs in many fourth order differential equations, including PDE's, see also Remark 4.10, whereas it does not occur in lower order equations as expected from (GP). But these oscillations, and the energy generating them, are somehow hidden also in fourth order equations; let us explain qualitatively what we mean by this. Engineers usually say that *the wind feeds into the structure an increment of energy* (see [13, p.28]) and that *the bridge eats energy* but we think it is more appropriate to say that *the bridge ruminates energy.* That is, first the bridge stores the energy due to prolonged external sources. Part of this stored energy is indeed dissipated (eaten) by the structural damping of the bridge. From [13, p.211], we quote

> Damping is dissipation of energy imparted to a vibrating structure by an exciting force, whereby a portion of the external energy is transformed into molecular energy.

Every bridge has its own damping capacity defined as the ratio between the energy dissipated in one cycle of oscillation and the maximum energy of that cycle. The damping capacity of a bridge depends on several components such as elastic hysteresis of the structural material and friction between different components of the structure, see [13, p.212]. A second part of the stored energy becomes potential energy if the bridge is above its equilibrium position. The remaining part of the stored energy, namely the part exceeding the damping capacity plus the potential energy, is stored into inner elastic energy; only when this stored elastic energy reaches a critical threshold (saturation), the bridge starts "ruminating" energy and gives rise to torsional or more complicated oscillations.

When (4.2) occurs, the estimate (4.7) shows that $|w''(t)|$ blows up at a higher rate when compared to $|w(t)|$ and $|w'(t)|$. Although any student is able to see if a function or its first derivative are large just by looking at the graph, most people are unable to see if its second derivative is large. Roughly speaking, the term $\int w''(t)^2$ measures the elastic energy, the term $\int w'(t)^2$ measures the kinetic energy, whereas $\int w(t)^2$ is a measure of the potential energy due to gravity. Hence, (4.7) states that the elastic energy has a higher rate of blow up when compared to the kinetic and potential energies. But since large $|w''(t)|$ cannot be easily detected, the bridge may have large elastic energy, and hence large total energy, without revealing it; so, there seems to be some "hidden" elastic energy. This interpretation well agrees with the numerical results described in Section 4 which show that blow up in finite time for (4.1) occurs after a long waiting time of apparent calm and sudden wide oscillations.

A flavor of what we call hidden energy was already present in [13] where the energy storage capacity of a bridge is often discussed, see (p.34, p.104, p.160, p.164) for the storage capacity of the different vibrating components of the bridge. Moreover, the displayed comment just before (3.9) shows that McKenna-Tuama [67] also had the feeling that some energy could be hidden.

5.2. **Energy balance.** As far as we are aware, the first attempt for a precise quantitative energy balance in a beam representing a suspension bridge was made in [87, Chapter VII]. Although all the computations are performed with precise values of the constants, in our opinion the analysis there is not complete since it does not distinguish between different kinds of potential energies; what is called potential energy is just the energy stored in bending a differential length of the beam.

A better attempt is made in [13, p.107] where the plot displays the behavior of the stored energies: the potential energy due to gravity and the elastic energies of the cables and of the stiffening frame. Moreover, the important notion of flutter speed is first used therein. Rocard [82, p.185] attributes to Bleich [12]

> to have pointed out the connection with the flutter speed of aircraft wings ... He distinguishes clearly between flutter and the effect of the staggered vortices and expresses the opinion that two degrees of freedom (bending and torsion) at least are necessary for oscillations of this kind.

A further comment on [12] is given at [86, p.80]:

> Bleich's work ... ultimately opened up a whole new field of study. Wind tunnel tests on thin plates suggested that higher wind velocities increased the frequency of vertical oscillation while decreasing that of torsional oscillation.

The conclusion is that if the two frequencies correspond, a flutter critical velocity is reached, as manifested in a potentially catastrophic coupled oscillation. In order to define the flutter speed, [13, pp.246-247] assumes that the bridge is subject to a natural steady state oscillating motion; the flutter speed is then defined by:

> With increasing wind speed the external force necessary to maintain the motion at first increases and then decreases until a point is reached where the air forces alone sustain a constant amplitude of the oscillation. The corresponding velocity is called the critical velocity or flutter speed.

The importance of the flutter speed is then described by

> Below the critical velocity V_c an exciting force is necessary to maintain a steady-state motion; above the critical velocity the direction of the force must be reversed (damping force) to maintain the steady-state motion. In absence of such a damping force the slightest increase of the velocity above V_c causes augmentation of the amplitude.

This means that self-excited oscillations appear as soon as the flutter speed is exceeded. Also Rocard devotes a large part of [82, Chapter VI] to

> ... predict and delimit the range of wind speeds that inevitably produce and maintain vibrations of restricted amplitudes.

This task is reached by a careful study of the natural frequencies of the structure. Moreover, Rocard aims to

> ... calculate the really critical speed of wind beyond which oscillatory instability is bound to arise and will always cause fracture.

The flutter speed V_c for a bridge without damping is computed on [82, p.163] and reads

$$V_c^2 = \frac{2r^2\ell^2}{2r^2+\ell^2}\,\frac{\omega_T^2-\omega_B^2}{\alpha} \tag{5.2}$$

where 2ℓ denotes the width of the roadway, see Figure 7, r is the radius of gyration, ω_B and ω_T are the lowest modes circular frequencies of the bridge in bending and torsion respectively, α is the mass of air in a unit cube divided by the mass of steel and concrete assembled within the same unit length of the bridge; usually, $r \approx \ell/\sqrt{2}$ and $\alpha \approx 0.02$. More complicated formulas for the flutter speed are obtained in presence of damping factors. Moreover, Rocard [82, p.158] shows that, for the original Tacoma Bridge, (5.2) yields $V_c = 47$mph while the bridge collapsed under the action of a wind whose speed was $V = 42$mph; he concludes that his computations are quite reliable.

In pedestrian bridges, the counterpart of the flutter speed is the *critical number of pedestrians*, see the quoted sentence by Macdonald [61] in Section 2. For this reason, we find it more suitable to deal with energies rather than with velocities: the flutter speed V_c corresponds to a *critical energy threshold* $\overline{E}$ above which the bridge displays self-excited oscillations. We believe that

> the wind in suspension bridges and pedestrians on footbridges insert elastic energy in the structure; if the wind reaches the flutter speed or the pedestrians reach the critical number, the corresponding amount of energy is the critical energy threshold and is the threshold where the nonlinear behavior of the bridge manifests, due to sufficiently large displacements of the roadway from equilibrium.

We believe that when the total energy within the structure reaches the threshold $\overline{E}$ an impulse transfers part of the energy of the vertical oscillations to the energy of a torsional oscillation. The threshold $\overline{E}$ is a structural parameter depending only on the elasticity of the bridge, namely on the materials used for its construction. We refer to Section 5.4 for an attempt to characterize $\overline{E}$ and a possible way to determine it.

Remark 5.1. With some numerical results at hand, Lazer-McKenna [58, p.565] attempt to explain the Tacoma collapse with the following comment:

> An impact, due to either an unusual strong gust of wind, or to a minor structural failure, provided sufficient energy to send the bridge from one-dimensional to torsional orbits.

We believe that what they call *an unusual impact* is, in fact, an impulse for the transition from vertical to torsional modes.

5.3. **Oscillating modes in suspension bridges: seeking the correct boundary conditions.** Smith-Vincent [87, Section I.2] analyse the different forms of motion of a suspension bridge and write

> The natural modes of vibration of a suspension bridge can be classified as vertical and torsional. In pure vertical modes all points on a given cross section move vertically the same amount and in phase... The amount of this vertical motion varies along the longitudinal axis of the bridge as a modified sine curve.

Then, concerning torsional motions, they write

> In pure torsional modes each cross section rotates about an axis which is parallel to the longitudinal axis of the bridge and is in the same vertical plane as the centerline of the roadway. Corresponding points on opposite sides of the centerline of the roadway move equal distances but in opposite directions.

Moreover, Smith-Vincent also analyse small oscillations:

> For small torsional amplitudes the movement of any point is essentially vertical, and the wave form or variation of amplitude along a line parallel to the longitudinal centerline of the bridge ... is the same as for a corresponding pure vertical mode.

With these remarks at hand, in this section we try to set up a reliable eigenvalue problem. We consider the roadway bridge as a long narrow rectangular thin plate, simply supported on its short sides. So, let $\Omega = (0, L) \times (-\ell, \ell) \subset \mathbb{R}^2$ where L is the length of the bridge and 2ℓ is its width; a realistic assumption is that $2\ell \ll L$.

As already mentioned in Section 3.1, the choice of the boundary conditions is delicate since it depends on the physical model considered. We first recall that the boundary conditions $u = \Delta u = 0$ are the so-called Navier boundary conditions. On flat parts of the boundary where no curvature is present, they describe simply supported plates, see e.g. [42]. When x_1 is fixed, either $x_1 = 0$ or $x_1 = L$, these conditions reduce to $u = u_{x_1x_1} = 0$. And precisely on these two sides, the roadway Ω is assumed to be simply supported; this is assumed in any of the models we met. The delicate point is the determination of the boundary conditions on the other sides.

To get into the problem, we start by dealing with the linear Kirchhoff-Love theory described in Section 3.1. In view of (3.2), the elastic energy of the vertical deformation u of the rectangular plate Ω subject to a load $f = f(x_1, x_2)$ is given by

$$\mathbb{E}(u) = \int_\Omega \Big(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2u) - f\,u\Big)\,dx_1\,dx_2 \tag{5.3}$$

and yields the Euler-Lagrange equation $\Delta^2 u = f$ in Ω. For a fully simply supported plate, that is $u = u_{x_1x_1} = 0$ on the vertical sides and $u = u_{x_2x_2} = 0$ on the horizontal sides, this problem has been solved by Navier [74] in 1823, see also [62, Section 2.1]. Since the bridge is not a fully simply supported plate, different boundary conditions should be considered on the horizontal sides. The load problem on the rectangle Ω with only the vertical sides being simply supported was considered by Lévy [59], Zanaboni [100], and Nadai [72], see also [62, Section 2.2] for the analysis of different kinds of boundary conditions on the remaining two sides $x_2 = \pm\ell$. Let us also mention the more recent monograph [93, Chapter 3] for a clear description of bending of rectangular plates.

It appears natural to consider the horizontal sides to be free. If no physical constraint is present on the horizontal sides, then the boundary conditions there become (see e.g. [93, (2.40)])

$$\begin{gathered} u_{x_2x_2}(x_1, \pm\ell) + \sigma u_{x_1x_1}(x_1, \pm\ell) = 0\,, \\ u_{x_2x_2x_2}(x_1, \pm\ell) + (2 - \sigma)u_{x_1x_1x_2}(x_1, \pm\ell) = 0\,, \quad x_1 \in (0, L)\,. \end{gathered}$$

We are so led to consider the eigenvalue problem

$$
\begin{gathered}
\Delta^2 u=\lambda u \qquad x=(x_1,x_2)\in\Omega\,,\\
u(x_1,x_2)=u_{x_1x_1}(x_1,x_2)=0 \quad x\in\{0,L\}\times(-\ell,\ell)\,,\\
u_{x_2x_2}(x_1,x_2)+\sigma u_{x_1x_1}(x_1,x_2)=0 \quad x\in(0,L)\times\{-\ell,\ell\}\,,\\
u_{x_2x_2x_2}(x_1,x_2)+(2-\sigma)u_{x_1x_1x_2}(x_1,x_2)=0 \quad x\in(0,L)\times\{-\ell,\ell\}\,.
\end{gathered}
\tag{5.4}
$$

The oscillating modes of Ω are the eigenfunctions to (5.4). By separating variables, one sees that the eigenfunctions have the form

$$
\psi_m(x_2)\,\sin\Big(\frac{m\pi}{L}x_1\Big) \quad (m\in\mathbb{N}\setminus\{0\}) \tag{5.5}
$$

for some $\psi_m\in C^4(-\ell,\ell)$ satisfying a suitable linear fourth order ODE, see [37].

Problem 5.2. Determine all the eigenvalues and eigenfunctions to (5.4). Which subspace of $H^2(\Omega)$ is spanned by these eigenfunctions?

Then one should investigate the equilibrium of the corresponding loaded plate.

Problem 5.3. For any $f\in L^2(\Omega)$ study existence and uniqueness of a function $u\in H^4(\Omega)$ satisfying $\Delta^2u=f$ in Ω and $(5.4)_2$-$(5.4)_3$-$(5.4)_4$. Try first some particular forms of f as in [62, Sections 2.2, 2.2.2] and then general $f=f(x_1,x_2)$.

5.4. **Seeking the critical energy threshold.** The title of this section should not deceive. We will not give a precise method how to determine the energy threshold which gives rise to torsional oscillations in a plate. We do have an idea how to proceed but several steps are necessary before reaching the final goal.

Consider the plate $\Omega=(0,L)\times(-\ell,\ell)$ and the incomplete eigenvalue problem with missing conditions on the sides $x_2=\pm\ell$:

$$
\begin{gathered}
\Delta^2 u=\lambda u \quad x\in\Omega\,,\\
u(x_1,x_2)=u_{x_1x_1}(x_1,x_2)=0 \quad (x_1,x_2)\in\{0,L\}\times(-\ell,\ell)\,.
\end{gathered}
\tag{5.6}
$$

We consider first the simple case where the plate is square, $\Omega=(0,\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})$, and we complete (5.6) with the "simplest" boundary conditions, namely the Navier boundary conditions which represent a fully simply supported plate. As already mentioned the square is not a reasonable shape and these are certainly not the correct boundary conditions for a bridge, but they are quite helpful to describe the method we are going to suggest. So, consider the problem

$$
\begin{gathered}
\Delta^2 u=\lambda u \quad x\in\Omega\,,\\
u(x_1,x_2)=u_{x_1x_1}(x_1,x_2)=0 \quad (x_1,x_2)\in\{0,\pi\}\times(-\frac{\pi}{2},\frac{\pi}{2})\,,\\
u(x_1,x_2)=u_{x_2x_2}(x_1,x_2)=0 \quad (x_1,x_2)\in(0,\pi)\times\{-\frac{\pi}{2},\frac{\pi}{2}\}\,.
\end{gathered}
\tag{5.7}
$$

It is readily seen that, for instance, $\lambda=625$ is an eigenvalue for (5.7) and that there are 4 linearly independent corresponding eigenfunctions

$$
\{\sin(24x_1)\cos(7x_2),\sin(20x_1)\cos(15x_2),\sin(15x_1)\cos(20x_2),\sin(7x_1)\cos(24x_2)\}\,. \tag{5.8}
$$

It is well-known that similar facts hold for the second order eigenvalue problem $-\Delta u=\lambda u$ in the square, so what we are discussing is not surprising. What we want to emphasise here is that, associated to the same eigenvalue $\lambda=625$, we have

4 different kinds of vibrations in the x_1-direction and each one of these vibrations has its own counterpart in the x_2-direction corresponding to torsional oscillations.

Consider now a general plate $\Omega = (0, L) \times (-\ell, \ell)$ and let $f \in L^2(\Omega)$; in view of [62, Section 2.2], we expect the solution to the problem

$$\begin{gathered}\Delta^2 u = f \quad x \in \Omega\,, \\ u(x_1, x_2) = u_{x_1x_1}(x_1, x_2) = 0 \quad (x_1, x_2) \in \{0, \pi\} \times (-\frac{\pi}{2}, \frac{\pi}{2})\,, \\ \text{other boundary conditions} \quad (x_1, x_2) \in (0, \pi) \times \{-\frac{\pi}{2}, \frac{\pi}{2}\}\,,\end{gathered} \tag{5.9}$$

to be of the kind

$$u(x_1, x_2) = \sum_{m=1}^{\infty} \psi_m(x_2) \sin\big(\frac{m\pi}{L} x_1\big) \quad (x_1, x_2) \in \Omega$$

for some functions ψ_m depending on the Fourier coefficients of f. Since we have in mind small ℓ, we can formally expand ψ_m in Taylor polynomials and obtain

$$\psi_m(x_2) = \psi_m(0) + \psi_m'(0) x_2 + o(x_2) \quad \text{as } x_2 \to 0\,.$$

Hence, u may approximately be written as

$$u(x_1, x_2) \approx \sum_{m=1}^{\infty} [a_m + b_m x_2] \sin\big(\frac{m\pi}{L} x_1\big) \quad (x_1, x_2) \in \Omega$$

where $a_m = \psi_m(0)$ and $b_m = \psi_m'(0)$. If instead of a stationary problem such as (5.9), $u = u(x_1, x_2, t)$ satisfies an evolution problem with the same boundary conditions, then also its coefficients depend on time:

$$u(x_1, x_2, t) \approx \sum_{m=1}^{\infty} \Big(a_m(t) + b_m(t) x_2\Big) \sin\big(\frac{m\pi}{L} x_1\big) \quad (x_1, x_2) \in \Omega\,,\ t > 0\,. \tag{5.10}$$

We now attempt a qualitative description of what we believe to happen in the combination of vertical and torsional oscillations. We call "small" any quantity which is less than unity and "almost zero" (in symbols $\cong 0$) any quantity which has a smaller order of magnitude when compared with small quantities. Moreover, in order to avoid delicate sign arguments, we will often refer to a_m^2 and b_m^2 instead of a_m and b_m. Different situations occur according to the instantaneous total energy $\mathcal{E} = \mathcal{E}(t)$ of the bridge.

• **Small energy.** As long as $\mathcal{E}(t)$ is small one may not even see the oscillations, but if somebody stands on the bridge he might be able to feel oscillations. For instance, standing on the sidewalk of a bridge, one can feel the oscillations created by a car going through the roadway but the oscillations will not be visible to somebody watching the roadway from a point outside the bridge. For small energies $\mathcal{E}(t)$ only small oscillations appear and the corresponding solution (5.10) has small coefficients $a_m(t)$ while $b_m(t) \cong 0$. More precisely,

$$\begin{gathered}\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \mathcal{E}(t) < \delta \Longrightarrow a_m(t)^2 < \varepsilon\ \forall m\,, \\ \exists \gamma > 0 \text{ such that } \mathcal{E}(t) < \gamma \Longrightarrow b_m(t) \cong 0\ \forall m\,.\end{gathered} \tag{5.11}$$

The reason of the second of (5.11) is that even small variations of the b_m's correspond to a large variation of the total energy $\mathcal{E}$ because the huge masses of the cross sections would rotate along the large length L of the roadway. On the other hand, the first of (5.11) may be strengthened by assuming that also some of the a_m's are almost zero for small $\mathcal{E}$; in particular, we expect that this happens for large m since these coefficients correspond to higher eigenvalues: for all $\overline{m} \in \mathbb{N} \setminus \{0\}$, there exists $\mathcal{E}_{\overline{m}} > 0$ such that

$$\mathcal{E}(t) < \mathcal{E}_{\overline{m}} \implies a_m(t) \cong 0 \quad \forall m > \overline{m}\,. \tag{5.12}$$

To better understand this point, let us compute the elongation Γ_m due to the m-th mode:

$$\Gamma_m(t) := \int_0^L \left(\sqrt{1 + \frac{m^2\pi^2}{L^2}\, a_m(t)^2 \cos^2\Big(\frac{m\pi}{L} x_1\Big)} - 1 \right) dx_1\,; \tag{5.13}$$

this is part of the stretching energy, that is, the difference between the length of the roadway deformed by one single mode and the length of the roadway at rest. Due to the coefficient $\frac{m^2\pi^2}{L^2}$, it is clear that if $a_m^2 \equiv a_{m+1}^2$ then $\Gamma_m(t) < \Gamma_{m+1}(t)$. This is the reason why (5.12) holds.

• **Increasing energy.** According to (5.12), as long as $\mathcal{E}(t) < \mathcal{E}_{\overline{m}}$ one has $a_m(t) \cong 0$ for all $m > \overline{m}$. If $\mathcal{E}(t)$ increases but remains smaller than $\mathcal{E}_{\overline{m}}$, then the coefficients $a_m(t)^2$ for $m = 1, \dots, \overline{m}$ also increase. But they do not increase to infinity, when the total energy $\mathcal{E}(t)$ reaches the threshold $\mathcal{E}_{\overline{m}}$ the superlinear elastic response of the bridge forces the solution (5.10) to add one mode, so that $a_{\overline{m}+1}(t) \ncong 0$. Hence, the number of modes $\ncong 0$ is a nondecreasing function of $\mathcal{E}$.

• **Critical energy threshold.** What is described above is purely theoretical, but the bridge has several physical constraints. Of course, it cannot be stretched to infinity, it will break down earlier. In particular, the number of active modes cannot increase to infinity. The elastic properties of the bridge determine a critical (maximal) number of possible active modes, say μ. If the energy is distributed on the μ coefficients $a_1,\dots,a_\mu$, and if it increases up to $\mathcal{E}_\mu$, once again the superlinear elastic response of the bridge forces the solution (5.10) to change mode, but this time from the a_m to the b_m; due to (5.13), further stretching of the roadway would require more energy than switching oscillations on torsional modes. And which torsional modes will be activated depends on which coupled modes have the same eigenvalue; as an example, consider (5.8) which, roughly speaking, says that the motion may change from 24 to 7 oscillations in the x_1-direction with a consequent change of oscillation also in the x_2-direction.

• **Summarizing.** Let u in (5.10) describe the vertical displacement of the roadway. The bridge has several characteristic values which depend on its elastic structure.

- An integer number $\mu \in \mathbb{N}$ such that $a_m(t) \cong 0$ and $b_m(t) \cong 0$ for all $m > \mu$, independently of the value of $\mathcal{E}(t)$.
- μ different "intermediate energy thresholds" $E_1, \dots, E_\mu$, one for each vertical mode.
- The critical energy threshold $\overline{E} = E_\mu$.

Assume that $\mathcal{E}(0) = 0$, in which case $u(x_1, x_2, 0) = 0$, and that $t \mapsto \mathcal{E}(t)$ is increasing. As long as $\mathcal{E}(t) \le E_1$ we have $a_m \cong 0$ for all $m \ge 2$ and $b_m \cong 0$ for all $m \ge 1$; moreover, $t \mapsto a_1(t)^2$ is increasing. When $\mathcal{E}(t)$ reaches and exceeds E_1 there is a first switch: the function a_2^2 starts being positive while, as long as $\mathcal{E}(t) \le E_2$, we still have $a_m \cong 0$ for all $m \ge 3$ and $b_m \cong 0$ for all $m \ge 1$. And so on, until $\mathcal{E}(t) = E_\mu = \overline{E}$. At this point, the energy forces the solution to have a nonzero coefficient b_1 rather than a nonzero coefficient $a_{\mu+1}$. The impulse forces u to lower the number of modes for which $a_m \ncong 0$. For instance, the observation by

Farquharson (*The motion, which a moment before had involved nine or ten waves, had shifted to two*) quoted in Section 2 shows that, for the Tacoma Bridge, there was a change such as

$$\Big(a_m \cong 0\,\forall m \geq 11,\ b_m \cong 0\,\forall m \geq 1\Big) \longrightarrow \Big(a_m \cong 0\,\forall m \geq 3,\ b_m \cong 0\,\forall m \geq 2\Big)\,. \tag{5.14}$$

To complete the material in this section, two major problems are still to be solved.

- Find the correct boundary conditions on $x_2 = \pm\ell$, are these as in (5.4)?
- Find at which energy levels the "transfer of energy between modes" occurs.

5.5. **A new mathematical model for suspension bridges.** We suggest here a new initial-boundary value problem to model oscillations in suspension bridges; it involves dynamic nonlocal boundary conditions and, although it does not enter in any classical scheme, we feel that it might be a good starting point for propaedeutical discussions to deeper studies. We expect its solution (if any) to display both self-excited oscillations and instantaneous switch between vertical and torsional oscillations. The critical energy threshold (related to the flutter speed) appears explicitly in the model. Dynamic boundary conditions appear necessary due to the fact that the impact with the wind often occurs at the free edges of the rectangle modeling the roadway; moreover, in oscillating pedestrian bridges these free boundaries have shown to be quite flexible and appear to be the good place where to put dampers, as at the London Millennium Bridge. Let us underline that, at present time, the model suggested in this section should just be seen as a tentative idea in order to change the point of view on suspension bridges. We expect to receive criticisms and suggestions on how to improve it.

Let $\Omega = (0, L) \times (-\ell, \ell) \subset \mathbb{R}^2$, where L is the length of the bridge and 2ℓ is the width of the roadway, and consider the initial-boundary value problem

$$\begin{gathered}
u_{tt} + \Delta^2 u + \delta u_t + f(u) = \varphi(x,t) \quad x = (x_1, x_2) \in \Omega,\ t > 0,\\
u(x_1, x_2, t) = u_{x_1x_1}(x_1, x_2, t) = 0 \quad x \in \{0, L\} \times (-\ell, \ell),\ t > 0,\\
u_{x_2x_2}(x_1, x_2, t) = 0 \quad x \in (0, L) \times \{-\ell, \ell\},\ t > 0,\\
u_{x_2}(x_1, -\ell, t) = u_{x_2}(x_1, \ell, t) \quad x_1 \in (0, L),\ t > 0,\\
u_t(x_1, -\ell, t) + u(x_1, -\ell, t) = E(t)\,[u_t(x_1, \ell, t) + u(x_1, \ell, t)] \quad x_1 \in (0, L),\ t > 0,\\
u(x, 0) = u_0(x) \quad x \in \Omega,\\
u_t(x, 0) = u_1(x) \quad x \in \Omega.
\end{gathered} \tag{5.15}$$

Here, $u = u(x,t)$ represents the vertical displacement of the plate, $u_0(x)$ is its initial position while $u_1(x)$ is its initial vertical velocity. Before discussing the other terms and conditions, let us remark that smooth solutions may exist only if the following compatibility conditions for initial data are satisfied:

$$\begin{gathered}
u_0(x) = (u_0)_{x_1x_1}(x) = 0 \quad x \in \{0, L\} \times (-\ell, \ell),\\
(u_0)_{x_2x_2}(x) = 0 \quad x \in (0, L) \times \{-\ell, \ell\},\\
(u_0)_{x_2}(x_1, -\ell) = (u_0)_{x_2}(x_1, \ell) \quad x_1 \in (0, L),\\
u_1(x_1, -\ell) + u_0(x_1, -\ell) = E(0)\,[u_1(x_1, \ell) + u_0(x_1, \ell)] \quad x_1 \in (0, L)\,.
\end{gathered} \tag{5.16}$$

The function φ represents an external source, such as the wind, which inserts an energy $\mathcal{E}(t)$ into the structure: we define

$$\mathcal{E}(t) = \int_\Omega \varphi(x,t)^2\, dx\,. \tag{5.17}$$

The function E in $(5.15)_5$ is then defined by

$$E(t) = \begin{cases} 1 & \text{if } \mathcal{E}(t) \le \overline{E}_\delta \\ -1 & \text{if } \mathcal{E}(t) > \overline{E}_\delta \end{cases} \tag{5.18}$$

where $\overline{E}_\delta > 0$ is given and represents the critical energy threshold described in Section 5.2. It depends increasingly on the damping parameter δ: for instance, $\overline{E}_\delta = \overline{E}_0 + c\delta$ for some $c > 0$ and $\overline{E}_0 > 0$ being the threshold of the undamped problem. When $\mathcal{E}(t) \le \overline{E}_\delta$ the motion tends to become of pure vertical-type, that is, with $u_{x_2} \cong 0$: to see this, note that in this case $(5.15)_5$ may be written as

$$\frac{d}{dt}\big\{[u(x_1,\ell,t) - u(x_1,-\ell,t)]e^t\big\} = 0 \quad \forall x_1 \in (0,L)\,.$$

Whence, as long as $\mathcal{E}(t) \le \overline{E}_\delta$, the map $t \mapsto |u(x_1,\ell,t) - u(x_1,-\ell,t)|$ decreases so that the opposite endpoints of any cross section tend to have the same vertical displacement and to move synchronously as in a pure vertical oscillation. When $\mathcal{E}(t) > \overline{E}_\delta$ condition $(5.15)_5$ may be written as

$$\frac{d}{dt}\Big\{[u(x_1,\ell,t) + u(x_1,-\ell,t)]e^t\Big\} = 0 \quad \forall x_1 \in (0,L)\,.$$

Whence, as long as $\mathcal{E}(t) > \overline{E}_\delta$, the map $t \mapsto |u(x_1,\ell,t) + u(x_1,-\ell,t)|$ decreases so that the opposite endpoints of any cross section tend to have zero average and to move asynchronously as in a pure torsional oscillation; that is, $u(x_1,0,t) \cong \frac{1}{2}[u(x_1,\ell,t) + u(x_1,-\ell,t)] \cong 0$. In (5.15) the jump of $E(t)$ from/to ± 1 occurs simultaneously and instantaneously along all the points located on the free sides of the plate, in agreement with what has been observed at the Tacoma Bridge, see [4] and also [86, pp.50-51].

The differential operator in (5.15) is derived according to the linear Kirchhoff-Love model for a thin plate and, as already mentioned in Section 3.1, we follow here a compromise and consider a semilinear problem. The function f should vanish at 0 and should be superlinear. For instance, $f(u) = u + \varepsilon u^3$ with $\varepsilon > 0$ small could be a possible choice; alternatively, one could take $f(u) = a(e^{bu} - 1)$ as in [67] for some $a, b > 0$. In the first case the hangers are sought as ideal springs and gravity is somehow neglected, in the second case more relevance is given to gravity and to the possibility of slackening hangers.

Conditions $(5.15)_3$ and $(5.15)_4$ describe the tendency of a cross section of the bridge to remain straight: since $\ell \ll L$, this appears a quite natural assumption. Finally, δu_t is a damping term representing the positive structural damping of the structure, including internal frictions; its role is to higher the threshold $\overline{E}_\delta$ and to weaken the effect of the nonlinear term $f(u)$, see Problem 5.6.

As far as we are aware, there is no standard theory for problems like (5.15). It is a nonlocal problem since it links behaviors on different parts of the boundary and involves the function $E(t)$ in (5.18). It also has dynamic boundary conditions which are usually delicate to handle. We hope it to be the starting point for future fruitful discussions.

Problem 5.4. Prove that if $\varphi(x,t) \equiv 0$ then (5.15) only admits the trivial solution $u \equiv 0$. The usual trick of multiplying the equation in (5.15) by u or u_t and integrating over Ω does not allow to get rid of all the boundary terms. Note that, in this case, $E(t) \equiv 1$. As a first attempt, one could try the linear case $f(u) = u$.

Problem 5.5. Study existence and uniqueness results for (5.15); prove continuous dependence results with respect to the data φ, u_0, u_1. Of course, the delicate conditions to deal with are $(5.15)_4$ and $(5.15)_5$. If problem (5.15) were ill-posed, how can we modify these conditions in order to have a well-posed problem?

In this respect, let us emphasize that the equation modeling suspension bridges suggested in [68] is ill-posed, displaying multiple solutions both of small amplitude (close to equilibrium) and large. Uniqueness was subsequently obtained by adding suitable damping terms, see [10].

Problem 5.6. Study (5.15) with no damping, that is, $\delta = 0$: does the solution u display oscillations such as (4.2) when t tends to some finite blow up instant? Then study the competition between the damping term δu_t and the self-exciting term $f(u)$: for a given f is it true that if δ is sufficiently large then the solution u is global in time? We believe that the answer is negative and that the only effect of the damping term is to delay the blow up time.

Problem 5.7. Determine the regularity of the solutions u to (5.15) and study both the roles of the discontinuity of $E(t)$ and of the compatibility conditions (5.16).

Problem 5.8. Consider the case "with memory" where (5.17) is replaced by

$$\mathcal{E}(t) = \int_{t-\sigma}^{t} \int_{\Omega} \varphi(x,\tau)^2 \, dx \, d\tau \quad \text{for some } \sigma > 0 \,,$$

which would model a situation where if the wind blows for too long then at some critical time, instantaneously, a torsional motion appears. Note that the problem with (5.17) is much simpler because it is local in time.

In (5.15) we have neglected the stretching elastic energy which is small in plates where parts of the boundary are not fixed. If one wishes to make some corrections, one should add a further nonlinear term $g(\nabla u, D^2 u)$ and the equation would become quasilinear.

Problem 5.9. Insert into the equation (5.15) the stretching energy, something like

$$g(\nabla u, D^2 u) = -\Big(\frac{u_{x_1}}{\sqrt{1+u_{x_1}^2}}\Big)_{x_1} - \gamma\Big(\frac{u_{x_1}}{\sqrt{1+u_{x_1}^2}}\Big)_{x_2}$$

with $\gamma > 0$ small. Then prove existence, uniqueness and continuous dependence results.

An important tool to study (5.15) would be the eigenvalues and eigenfunctions of the corresponding stationary problem. In view of the dynamic boundary conditions $(5.15)_5$, a slightly simpler model could be considered, see (5.4) and subsequent discussion in Section 5.3.

6. Conclusions and future perspectives

In this article we observed phenomena displayed by real structures, we discussed models, and we recalled some theoretical results. We have emphasized the necessity of models fulfilling the requirements of (GP) since, otherwise, the solution will not display the phenomena visible in actual bridges. We also suggested to analyze oscillations in suspension bridges in terms of the energies involved; in particular, we suggested that torsional oscillations may appear whenever the internal energy becomes larger that a certain critical threshold. In this section we take advantage from all this work and we reach several conclusions.

6.1. **A possible explanation of the Tacoma collapse.** Here, we put together all our beliefs and we afford an explanation of the Tacoma collapse.

On November 7, 1940, for some time before 10:00 AM, the bridge was oscillating as it did many other times before. The wind was apparently more violent than usual and, moreover, it continued for a long time. The oscillations of the bridge were similar to those displayed in the interval $(0, 80)$ of the plot in Figure 9. Since the energy involved was quite large, also the vertical oscillations were quite large. The roadway was far from its equilibrium position and, consequently, the restoring force due to the sustaining cables and to the hangers did not obey $\mathcal{LHL}$. The oscillations were governed by a fairly complicated differential equation of the kind of (5.15). As the wind was keeping on blowing stronger than the internal damping effect, the total energy $\mathcal{E}$ in the bridge was increasing; after some time, it became larger than the critical energy threshold $\overline{E}$ of the bridge. The Tacoma Bridge had 11 different intermediate energy thresholds and therefore $\overline{E} = E_{11}$. As soon as $\mathcal{E} > \overline{E}$, the function $E(t)$ in (5.18) switched from $+1$ to -1, a torsional elastic energy appeared and gave rise, almost instantaneously, to a torsional motion. As described in (5.14), the energy switched to the first torsional mode b_1 rather than to a further vertical mode a_{11}; so, an impulse forced u to lower the number of modes for which $a_m \not\cong 0$ and the motion, which a moment before had involved nine or ten waves, shifted to two. At that moment, the graph of the function $w = w(t)$, describing the bridge according to (3.15), reached time $t = 95$ in the plot in Figure 9. Oscillations immediately went out of control and after some more oscillations the bridge collapsed.

This explanation is consistent with all the material developed in the present paper. One should compare this description with the original one from the Report [4], see also [32, pp.26-29] and [86, Chapter 4].

6.2. **What about future bridges?** Equation (4.1) is a simple prototype equation for the description of self-excited oscillations. None of the previously existing mathematical models ever displayed this phenomenon which is also visible in oscillating bridges. The reason is not that the behavior of the bridge is too complicated to be described by a differential equation but mainly because they fail to satisfy (GP). In order to avoid bad surprises as in the past, many projects nowadays include stiffening trusses or strong dampers. This has the great advantage to maintain the bridge much closer to its equilibrium position and to justify $\mathcal{LHL}$. But this also has disadvantages, see [53]. First of all, they create an artificial stiffness which can give rise to the appearances of cracks in the more elastic structure of the bridge. Second, damping effects and stiffening trusses significantly increase the weight and the cost of the whole structure. Moreover, in extreme conditions, they may become useless: under huge external solicitations the bridge would again be too far from its equilibrium position and would again violate $\mathcal{LHL}$. So, hopefully, one should find alternative solutions, see again [53].

One can act both on the structure and on the model. In order to increase the flutter speed, some suggestions on how to modify the design were made by Rocard [82, pp.169-173]: he suggests how to modify the components of the bridge in order to raise the right hand side of (5.2). More recently, some attempts to improve bridges performances can be found in [48] where, in particular, a careful analysis of the role played by the hangers is made. But much work has still to be done; from [48, p.1624], we quote

> Research on the robustness of suspension bridges is at the very beginning.

From a theoretical point of view, one should first determine a suitable equation satisfying (GP). Our own suggestion is to consider (5.15) or some variants of it, where one should choose a reliable nonlinearity f and add coefficients to the other terms, according to the expected structural features of the bridge: its length, its width, its weight, the materials used for the construction, the expected external solicitations, the structural damping ... Since we believe that the solution to this kind of equation may display blow up, which means a strong instability, a crucial role is played by all the constants which appear in the equation: a careful measurement of these parameters is necessary. Moreover, a sensitivity analysis for the continuous dependence of the solution on the parameters should be performed. One should try to estimate, at least numerically, the critical energy threshold and the possible blow up time.

In a “perfect model” for a suspension bridge, one should also take into account the role of the sustaining cables and of the towers. Each cable links all the hangers on the same side of the bridge, its shape is a catenary of given length and the hangers cannot all elongate at the same time. The towers link the two cables and, in turn, all the hangers on both sides of the roadway. In this paper we have not introduced these components but, in the next future, we will do so.

An analysis of the more flexible parts of the roadway should also be performed; basically, this consists in measuring the “instantaneous local energy” defined by

$$\mathbf{E}(u(t),\omega)=\int_\omega\Big[\Big(\frac{|\Delta u(t)|^2}{2}+(\sigma-1)\det(D^2u(t))\Big)+\frac{u_t(t)^2}{2}+F(u(t))\Big]\,dx_1\,dx_2 \quad (6.1)$$

for solutions $u=u(t)$ to (5.15), for any $t\in(0,T)$, and for any subregion $\omega\subset\Omega$ of given area. In (6.1) we recognize the first term to be as in the Kirchhoff-Love model, see (3.2); moreover, $F(s)=\int_0^s f(\sigma)d\sigma$.

Problem 6.1. Let $\Omega=(0,L)\times(-\ell,\ell)$, consider problem (5.15) and let $u=u(t)$ denote its solution provided it exists and is unique, see Problem 5.5. For given lengths $a<L$ and $b<2\ell$ consider the set $\Re$ of rectangles entirely contained in Ω and whose sides have lengths a (horizontal) and b (vertical). Let $\mathbf{E}$ be as in (6.1) and consider the maximisation problem

$$\max_{\omega\in\Re}\ \mathbf{E}(u(t),\omega)\,.$$

Using standard tools from calculus of variations, prove that there exists an optimal rectangle and study its dependence on $t\in(0,T)$; the natural conjecture is that, at least as $t\to T$, it is the “middle rectangle” $(\frac{L-a}{2},\frac{L+a}{2})\times(-\frac{b}{2},\frac{b}{2})$. Then one should find out if there exists some optimal ratio a/b. Finally, it would be extremely useful to find the dependence of the energy $\mathbf{E}(u(t),\omega)$ on the measure ab of the rectangle ω; this would allow to minimize costs for reinforcing the plate. We do not expect analytical tools to be able to locate optimal rectangles nor to give exact answers to the above problems so that a good numerical procedure could be of great help.

In [87, Chapter IV] an attempt to estimate the impact of stiffening trusses is made, although only one kind of truss design is considered. In order to determine the best way to display the truss, one should solve the following simplified problems from calculus of variations. A first step is to consider the linear model.

Problem 6.2. Assume that the rectangular plate $\Omega = (0, L) \times (-\ell, \ell)$ is simply supported on all its sides and that it is submitted to a constant load $f \equiv 1$. In the linear theory by Kirchhoff-Love model, see (5.3), its elastic energy is given by

$$E_0(\Omega) = -\min_{u \in H^2 \cap H^1_0(\Omega)} \int_\Omega \Big(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2 u) - u\Big)\, dx_1\, dx_2\,.$$

Here, $H^2 \cap H^1_0(\Omega)$ denotes the usual Hilbertian Sobolev space which, since we are in the plane, is embedded into $C^{0,\alpha}(\overline{\Omega})$. The unique minimiser u solves the corresponding Euler-Lagrange equation which reads

$$\Delta^2 u = 1 \text{ in } \Omega\,, \quad u = \Delta u = 0 \text{ on } \partial\Omega$$

and which may be reduced to a system involving the torsional rigidity of Ω:

$$-\Delta u = v\,,\ -\Delta v = 1 \text{ in } \Omega\,, \quad u = v = 0 \text{ on } \partial\Omega\,.$$

Let $\lambda > 0$ and denote by Γ_λ the set of connected curves γ contained in $\overline{\Omega}$, such that $\gamma \cap \partial\Omega \neq \emptyset$, and whose length is λ: the curves γ represent the stiffening truss to be put below the roadway. For any $\gamma \in \Gamma_\lambda$ the elastic energy of the reinforced plate $\Omega \setminus \gamma$ is given by

$$E_\gamma(\Omega) = -\min_{u \in H^2 \cap H^1_0(\Omega \setminus \gamma)} \int_\Omega \Big(\frac{1}{2}\,(\Delta u)^2 + (\sigma - 1)\det(D^2 u) - u\Big)\, dx_1\, dx_2\,,$$

and this energy should be minimized among all possible $\gamma \in \Gamma_\lambda$:

$$\min_{\gamma \in \Gamma_\lambda}\ E_\gamma(\Omega)\,.$$

Is there an optimal γ_λ for any $\lambda > 0$? In fact, for a realistic model, one should further require that all the endpoints of γ_λ lie on the boundary $\partial\Omega$. Finally, since the stiffening truss has a cost $C > 0$ per unit length, one should also solve the minimisation problem

$$\min_{\lambda \geq 0} \big\{C\lambda + \min_{\gamma \in \Gamma_\lambda} E_\gamma(\Omega)\big\}\,;$$

if C is sufficiently small, we believe that the minimum exists.

Problem 6.2 is just a simplified version of the "true problem" which ... should be completed!

Problem 6.3. Let $\Omega = (0, L) \times (-\ell, \ell)$ and fix some $\lambda > 0$. Denote by Γ_λ the set of connected curves contained in $\overline{\Omega}$ whose length is λ and whose endpoints belong to $\partial\Omega$. Study the minimization problem

$$\min_{\gamma \in \Gamma_\lambda} \Big| \min_{u \in \mathcal{H}(\Omega \setminus \gamma)} \int_\Omega \Big(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)\det(D^2 u) - u\Big)\, dx_1\, dx_2 \Big|\,,$$

where

$$\mathcal{H}(\Omega \setminus \gamma) = \{u \in H^2(\Omega \setminus \gamma);\ (5.4)_2 \text{ holds, + something on } x_2 = \pm\ell \text{ and on } \gamma\}\,.$$

First of all, instead of "something", one should find the correct conditions on $x_2 = \pm\ell$ and on γ. This could also suggest to modify the energy function to be minimized with an additional boundary integral. The questions are similar. Is there an optimal γ_λ for any $\lambda > 0$? Is there an optimal $\lambda > 0$ if one also takes into account the cost?

Then one should try to solve the same problems with variable loads.

Problem 6.4. Solve Problems 6.2 and 6.3 with nonconstant loads $f \in L^2(\Omega)$, so that the minimum problem becomes

$$\min_{\gamma\in\Gamma_\lambda}\Big|\min_{u\in\mathcal{H}(\Omega\setminus\gamma)}\int_\Omega\Big(\frac{1}{2}(\Delta u)^2+(\sigma-1)\det(D^2u)-fu\Big)\,dx_1\,dx_2\Big|.$$

What happens if $f \notin L^2(\Omega)$? For instance, if f is a delta Dirac mass concentrated at some point $x_0 \in \Omega$.

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