Z. angew. Math. Phys. 56 (2005) 763–782
0044-2275/05/050763-20
DOI 10.1007/s00033-005-3092-9

Zeitschrift für angewandte Mathematik und Physik ZAMP

On a long-standing conjecture by Pólya–Szegö and related topics

Graziano Crasta, Ilaria Fragalà and Filippo Gazzola

Abstract. The electrostatic capacity of a convex body is usually not simple to compute. We discuss two possible approximations of it. The first one is related to a long-standing conjecture by Pólya–Szegö. It states that, among all convex bodies, the "worst shape" for the approximation exists and is the planar disk. We prove the first part of this conjecture, and we establish some related results which give further evidence for the validity of the second part. We also suggest some complementary conjectures and open problems. The second approximation we study is based on the use of web functions.

Mathematics Subject Classification (2000). 49Q10, 31A15.

Keywords. Capacity, surface measure, shape optimization.

1. Introduction

The electrostatic capacity of a bounded domain $\Omega \subset \mathbf{R}^3$ is given by

$$\mathrm{Cap}(\Omega) = \frac{1}{4\pi} \inf \left\{ \int_{\mathbf{R}^3} |\nabla u|^2 ;\ u \in \mathcal{D}^{1,2}(\mathbf{R}^3),\ u = 1 \text{ in } \Omega \right\}, \tag{1}$$

where $\mathcal{D}^{1,2}(\mathbf{R}^3)$ is the closure of the space of smooth compactly supported functions with respect to the Dirichlet norm. Besides this variational definition, $\mathrm{Cap}(\Omega)$ may also be recovered through the asymptotic expansion

$$U_\Omega(x) = \mathrm{Cap}(\Omega)|x|^{-1} + O(|x|^{-2}) \qquad \text{for } |x| \to +\infty,$$

where U_Ω is the *equilibrium potential* of Ω, namely the unique function which solves the Euler–Lagrange equation corresponding to the infimum problem (1):

$$\Delta U_\Omega = 0 \quad \text{in } \mathbf{R}^3 \setminus \Omega, \qquad U_\Omega = 1 \quad \text{on } \partial\Omega, \qquad \lim_{|x|\to\infty} U_\Omega(x) = 0. \tag{2}$$

From a physical point of view, U_Ω represents the potential energy of the electrical field induced when the body Ω is a conductor, normalized so that the voltage difference between $\partial\Omega$ and infinity is 1. In this respect, as first rigorously defined by Kirchhoff [21] in 1869, $\mathrm{Cap}(\Omega)$ represents the total electric charge needed to induce the potential U_Ω. Standard references for potential theory are [20, 22].

The exact value of the capacity is of great interest not only in electrostatics. Indeed, capacity is linked to many physical processes and properties related to the origin of the Laplace equation in describing heat, electrical and fluid flow. For instance, Hubbard and Douglas [17] have shown that the Stokes friction of a Brownian particle is proportional to the capacity a very good approximation, and this relation becomes exact for ellipsoids.

Unfortunately, except for some special domains, the explicit solution of the boundary value problem (2) is not known, so that the capacity cannot be exactly determined. Moreover, its computation is not simple even from a numerical point of view, since one has to deal with an *exterior* problem. As suggested by Bouwkamp [3], one can try to evaluate the capacity by means of a Kelvin transformation. More precisely, one can switch the original problem (2) into a new problem in a bounded domain (the transformation of Ω by reciprocal radii), so that the capacity of Ω is just the value of the corresponding solution at the origin. This transformation allows the implementation of numerical procedures (see e.g. [4, 14, 15] and references therein), but in many cases they are not sufficiently stable. Moreover, the Kelvin transformation cannot be performed on planar domains which, as we will see, are of crucial importance for our purposes.

Therefore, as suggested in several pioneering works [9, 24, 27, 28, 29], one is led to seek approximate formulae for the capacity. In this paper, we mainly focus our attention on a captivating approximate formula, introduced almost one century ago by Aichi-Russell [1, 31], which involves only the 2-dimensional measure of the boundary of the convex body. More precisely, up to a multiplicative factor, the *Russell capacity* is simply the squareroot of the surface area of $\partial\Omega$, see (4).

As an alternative approximation, we also consider what we call the *web capacity*: roughly speaking, it is obtained by restricting the admissible class in the variational problem (1) to those functions in $\mathcal{D}^{1,2}(\mathbf{R}^3)$ which depend only on the distance from $\partial\Omega$. In this subclass, the solution can be explicitly determined, yielding an "approximate web potential", see (13). The resulting approximate formula for $\mathrm{Cap}(\Omega)$ is written in terms of the surface area of $\partial\Omega$ and of its total mean curvature, see (12).

In order to evaluate the precision of these approximations, one is led in a natural way to study shape optimization problems such as

$$\inf \mathcal{E}(\Omega), \tag{3}$$

where the cost functional $\mathcal{E}$ is the ratio between the capacity and the approximate capacity, and Ω varies in a suitable class of admissible domains (which for several reasons need to be *convex*).

The basic questions are: is the infimum in (3) strictly positive? if yes, is it attained? if yes, what is an optimal shape (which actually represents the "worst domain" for the approximation)?

In particular, we discuss a long-standing conjecture by Pólya–Szegö [29]. It says that, when considering the Russell capacity, the infimum of $\mathcal{E}$ (which is known to be strictly positive) is attained by the 2-dimensional disk. In this paper, we prove that an optimal shape does exist. In favour of its identification with a disk, we give some related theoretical and numerical results; in turn, they give rise to some further open problems which seem to be of some interest.

Concerning the web capacity, we prove that the infimum in (3) is strictly positive. Moreover, we give some numerical results: they allow to conclude that in this case the disk is not optimal, and at the same time they suggest that the infimum is attained.

This paper is organized as follows. Sections 2 and 3 contain respectively the results about Russell capacity and web capacity. All the proofs are postponed to Section 5. The numerical experiments for both kinds of approximations are collected in Section 4.

2. Russell approximation of the capacity

In this section we consider what is known nowadays as *Russell capacity* (or surface radius), namely

$$\mathrm{Cap}_{\mathcal{R}}(\Omega) = \sqrt{\frac{S(\Omega)}{4\pi}}, \tag{4}$$

where $S(\Omega)$ denotes the surface measure of $\partial\Omega$ (its 2-dimensional Hausdorff measure); we understand that if Ω is a planar set having 2-dimensional Hausdorff measure $\mathcal{H}^2(\Omega) > 0$, then $S(\Omega) = 2\mathcal{H}^2(\Omega)$. In fact, Russell himself [31] attributes this approximation to Aichi [1]. The Russell capacity is well-known to physicists, who call "shape factor" of Ω the following ratio (see e.g. [5]):

$$\mathcal{E}_{\mathcal{R}}(\Omega) = \frac{\mathrm{Cap}(\Omega)}{\mathrm{Cap}_{\mathcal{R}}(\Omega)}.$$

Notice that if $\mathcal{H}^2(\Omega) = 0$, then $\mathrm{Cap}(\Omega) = \mathrm{Cap}_{\mathcal{R}}(\Omega) = 0$, so that the quotient $\mathcal{E}_{\mathcal{R}}(\Omega)$ is well-defined if and only if $\mathcal{H}^2(\Omega) > 0$. Notice also that $\mathcal{E}_{\mathcal{R}}(\Omega)$ is invariant under dilations (rotations and translations) of Ω since both its numerator and denominator are homogeneous of degree 1. Clearly, the closer $\mathcal{E}_{\mathcal{R}}(\Omega)$ is to 1, the better the Russell capacity $\mathrm{Cap}_{\mathcal{R}}(\Omega)$ approximates the electrostatic capacity $\mathrm{Cap}(\Omega)$; for instance, if $\Omega = B_R$ (a ball of radius $R > 0$) then $\mathrm{Cap}(B_R) = \mathrm{Cap}_{\mathcal{R}}(B_R) = R$ and $\mathcal{E}_{\mathcal{R}}(B_R) = 1$. In order to evaluate the error made when approximating the capacity with the Russell capacity, we are led to focus our attention on the optimization problems

$$\inf \mathcal{E}_{\mathcal{R}}(\Omega) \qquad \text{and} \qquad \sup \mathcal{E}_{\mathcal{R}}(\Omega),$$

where Ω varies in a suitable class of domains with $\mathcal{H}^2(\Omega) > 0$. As already mentioned by Pólya–Szegö [29], if we allow *any* bounded domain in $\mathbf{R}^3$, we have $\inf \mathcal{E}_{\mathcal{R}} = 0$ and $\sup \mathcal{E}_{\mathcal{R}} = +\infty$. The first equality is obtained by noticing that a nonconvex domain contained into a fixed sphere may have an arbitrarily large surface area, whereas its capacity remains bounded from above. The second equality is obtained by considering a sequence of domains which tend to become 1-dimensional (see the behaviour of thinning prolate ellipsoids in Section 4.1). Such sequence also shows that, even if we restrict admissible domains to the class of *convex bodies*

$$\mathcal{K} := \{\Omega \subset \mathbf{R}^3;\ \Omega \text{ bounded and convex}, \mathcal{H}^2(\Omega) > 0\},$$

then $\sup_{\mathcal{K}} \mathcal{E}_{\mathcal{R}} = +\infty$. On the contrary, the infimum becomes strictly positive.

Theorem 1. (Positive lower bound for $\mathcal{E}_{\mathcal{R}}$)
We have

$$\inf_{\mathcal{K}} \mathcal{E}_{\mathcal{R}} \geq 2/\pi.$$

Proof. See [29, (4), p. 165]. □

In view of Theorem 1, it is natural to seek the exact value of $\inf_{\mathcal{K}} \mathcal{E}_{\mathcal{R}}$ and to inquire if it is attained. The comparison between $\mathrm{Cap}(\Omega)$ and $\mathrm{Cap}_{\mathcal{R}}(\Omega)$ is quite delicate because they have a similar behaviour under several aspects. For instance, they decrease under symmetrization and admit the same bounds in terms of the volume and the mean width, see [29, §1.13].

More than half a century ago, Pólya–Szegö [29, §I.1.18] made the following:

Conjecture 2. *Let D be a 2-dimensional disk. Then*

$$\mathcal{E}_{\mathcal{R}}(\Omega) \geq \inf_{\mathcal{K}} \mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}(D) = \frac{2\sqrt{2}}{\pi} \approx 0.9.$$

Moreover, $\mathcal{E}_{\mathcal{R}}(\Omega) = \inf_{\mathcal{K}} \mathcal{E}_{\mathcal{R}}$ if and only if Ω is a disk.

If this conjecture were true, then we would immediately have the new isoperimetric inequality

$$\mathrm{Cap}(\Omega) \geq \sqrt{\frac{2S(\Omega)}{\pi^3}} \qquad \forall \Omega \in \mathcal{K},$$

with equality if and only if Ω is a 2-dimensional disk.

Conjecture 2 is based on a previous conjecture by Lord Rayleigh [30, Vol. 2, p. 179]: it states that, among all conducting plates of given area, the disk has the minimum electrostatic capacity. Rayleigh conjecture was proved by Pólya–Szegö by means of Steiner symmetrization:

Theorem 3. (Optimal planar shape)
Let Ω be a planar domain (not necessarily convex). Then $\mathrm{Cap}(\Omega) \geq \mathrm{Cap}(D)$, *where D is a planar disk such that $\mathcal{H}^2(\Omega) = \mathcal{H}^2(D)$. Moreover, equality holds if and only if Ω is a disk.*

Proof. See [28, p. 14] and [29, § VII.7.3, p. 157]. □

Thanks to Theorem 3, Conjecture 2 is proved if the following two facts hold true:

(i) The functional $\Omega \mapsto \mathcal{E}_{\mathcal{R}}(\Omega)$ achieves its minimum in $\mathcal{K}$.

(ii) The minimum of $\mathcal{E}_{\mathcal{R}}$ over $\mathcal{K}$ is not attained in $\mathcal{K}_o := \{\Omega \in \mathcal{K};\ \Omega$ with nonempty interior$\}$.

One of the aims of this paper, is to prove the first statement.

Theorem 4. (Existence of a minimizer)
The infimum $\inf_{\mathcal{K}} \mathcal{E}_{\mathcal{R}}$ *is attained.*

Proof. See Section 5.1. □

In order to shed some light on statement (ii), we determine a necessary condition for minimality. We say that $\Omega \in \mathcal{K}_o$ is *stationary* for $\mathcal{E}_{\mathcal{R}}$ if:

$$\frac{d}{dt}\mathcal{E}_{\mathcal{R}}(\Omega + tD)|_{t=0} = 0 \qquad \forall D \in \mathcal{K}_o, \tag{5}$$

where $\Omega + tD$ denotes the Minkowski sum $\{x + ty : x \in \Omega, y \in D\}$. Of course, if there exists an optimal shape $\Omega \in \mathcal{K}_o$ which minimizes $\mathcal{E}_{\mathcal{R}}$ over $\mathcal{K}$, Ω must be stationary according to the above definition. To give an explicit characterization of stationary bodies in $\mathcal{K}_o$, we point out that given $\Omega \in \mathcal{K}_o$ there exist a *curvature measure* μ_Ω and a *capacitary measure* ν_Ω on S^2 which yield the integral representation formulae:

$$S(\Omega) = \frac{1}{2}\int_{S^2} V_\Omega \, d\mu_\Omega \quad \text{and} \quad \mathrm{Cap}(\Omega) = \int_{S^2} V_\Omega \, d\nu_\Omega,$$

where $V_\Omega : \mathbf{R}^3 \to \mathbf{R}^3$ is the *support function* of Ω (for the precise definitions of μ_Ω, ν_Ω and V_Ω, we refer to Section 5.2). Then we prove:

Theorem 5. (Stationarity condition in $\mathcal{K}_o$)
A domain $\Omega \in \mathcal{K}_o$ *is stationary for* $\mathcal{E}_{\mathcal{R}}$ *if and only if the following equality between measures on* S^2 *holds:*

$$\nu_\Omega = \frac{\mathrm{Cap}(\Omega)}{2S(\Omega)}\mu_\Omega. \tag{6}$$

Proof. See Section 5.2. □

It seems reasonable to expect that a stationary domain $\Omega \in \mathcal{K}_o$ must gain *a priori* some regularity from the fact that it satisfies (6). Indeed, exploiting the properties of the support of the curvature measure μ_Ω [33, §4], it looks possible to exclude the presence of "nonsmooth parts" in $\partial\Omega$ and to prove that $\partial\Omega \in \mathcal{C}^2$. Moreover, (6) enables us to show that the boundary of a smooth stationary domain $\Omega \in \mathcal{K}_o$ cannot have "flat parts". More precisely, denoting by $g^\Omega : \partial\Omega \to S^2$ the Gauss map associated with $\Omega \in \mathcal{K}_o$ (*cf.* Section 5.2), we prove

Theorem 6. (Nonexistence of stationary domains with faces)
Let $\Omega \in \mathcal{K}_o$*, with* $\partial\Omega \in \mathcal{C}^2$*. Assume that* $\partial\Omega$ *has some face, that is,* $\mathcal{H}^2\big((g^\Omega)^{-1}(\xi) \cap \partial\Omega\big) > 0$ *for some vector* $\xi \in S^2$*. Then* Ω *is not a stationary domain for* $\mathcal{E}_{\mathcal{R}}$*.*

Proof. See Section 5.2. □

Domains with faces fall outside $\mathcal{C}^2_+$, the subclass of bodies $\Omega \in \mathcal{K}_0$ whose boundary is $\mathcal{C}^2$ and *strictly convex*. Within such class, the characterization of stationary bodies becomes much simpler, hence the interest of Theorem 6. Actually, for $\Omega \in \mathcal{C}^2_+$, since the Gauss map g^Ω is a diffeomorphism, condition (6) turns into a pointwise relation on $\partial\Omega$, which incidentally entails further regularity:

Theorem 7. (Stationarity condition in $\mathcal{C}^2_+$)
A domain $\Omega \in \mathcal{C}^2_+$ is stationary for $\mathcal{E}_\mathcal{R}$ if and only if the following pointwise identity holds:

$$|\nabla U_\Omega(x)|^2 = \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} H_{\partial\Omega}(x) \qquad \forall x \in \partial\Omega, \tag{7}$$

where U_Ω is the equilibrium potential of Ω, and $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$ (namely, $H_{\partial\Omega} = (k_1 + k_2)/2$, being k_i the principal curvatures of $\partial\Omega$). In particular, a stationary domain $\Omega \in \mathcal{C}^2_+$ has a C^∞ boundary.

Proof. See Section 5.2. □

Using (7), we immediately infer:

Corollary 8. (Stationarity of balls)
Balls satisfy the stationarity condition (7).

Proof. See Section 5.2. □

In view of the numerical results of Section 4, we may exclude the (even local) minimality of balls, see Figure 1. However, we think that balls might have the following property, whose validity would prove Conjecture 2:

Conjecture 9. *Balls are the only stationary bodies for $\mathcal{E}_\mathcal{R}$ in $\mathcal{K}_o$.*

Thanks to the above considerations on the regularity of stationary domains, it is also of some interest to consider the following weakened version of Conjecture 9.

Conjecture 10. *Let $\Omega \in \mathcal{C}^2_+$ and suppose that for some $R > 0$ there exists a solution to the overdetermined problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbf{R}^3 \setminus \Omega \\ u = 1 & \text{on } \partial\Omega \\ |\nabla u(x)|^2 = \dfrac{H_{\partial\Omega}(x)}{R} & \text{on } \partial\Omega \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{8}$$

Then Ω is a ball of radius R.

System (8) is not covered by the literature about overdetermined problems on exterior domains, see [10, 13, 16, 26]. Let us stress that, for any $\Omega \in \mathcal{C}^2_+$, there exists necessarily some point $x_0 \in \partial\Omega$ where the equilibrium potential U_Ω satisfies

$$|\nabla U_\Omega(x_0)|^2 = \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} H_{\partial\Omega}(x_0) \tag{9}$$

(see the end of Section 5.2 for a proof). Conjecture 10 states that (9) holds at *every* point $x_0 \in \partial\Omega$ only if Ω is a ball of radius $R = S(\Omega)/4\pi\mathrm{Cap}(\Omega)$, and proving this assertion seems to be a challenging problem. Actually, powerful tools such as the moving planes by Serrin [34], the P-function by Payne–Philippin [25], rearrangement and comparison arguments adapted from Talenti [36], the concavity property of the maps $t \mapsto \sqrt{S(\Omega_t)}$ [6, Lemma 4.2] and $t \mapsto \mathrm{Cap}(\Omega_t)$ [2] do not work, at least applied in a standard way.

3. An alternative approximation: the web capacity

In this section we consider a different approximation for the capacity, the *web capacity*, namely

$$\mathrm{Cap}_{\mathcal{W}}(\Omega) := \frac{1}{4\pi} \inf \left\{ \int_{\mathbf{R}^3} |\nabla u|^2;\ u \in \mathcal{W},\ u = 1 \text{ in } \Omega \right\} \tag{10}$$

where $\mathcal{W} = \mathcal{W}(\Omega) \subset \mathcal{D}^{1,2}(\mathbf{R}^3)$ is the subspace of web functions, that is, functions which on $\mathbf{R}^3 \setminus \Omega$ have the same level lines as the distance function $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$ from the boundary of Ω. In a previous paper [7] we used web functions in order to estimate the torsional rigidity of planar bounded convex domains; in such case, *inner* parallel sets are necessary, while for the exterior problem considered here, *outer* parallel sets are in order. More precisely, for $t > 0$, we consider the parallel body $\Omega_t = \Omega + B_t$, where B_t is the ball of radius t centered at the origin. If we set $S_t := S(\Omega_t)$ then, under the crucial assumption $\Omega \in \mathcal{K}$, Steiner formula reads (see for example [32, §4.2])

$$S_t = S + 2Mt + 4\pi t^2, \qquad t \geq 0, \tag{11}$$

where M is the total mean curvature of $\partial\Omega$. For Ω open and smooth, we have $M = \int_{\partial\Omega} H_{\partial\Omega}(x)\,dx$; for arbitrary convex domains M coincides with $2\pi B$, where B is the mean width of Ω (see [32, (5.3.12)]). Alternatively, one may compute M by finding first S_t and then exploiting (11) (for instance, in the case of polyhedra). We also recall that, by the quadratic Minkowskian inequalities (see e.g. [32, p. 322]) we have $M^2 \geq 4\pi S$, with equality only if Ω is a ball. We now establish that the web capacity may be expressed only in terms of M and S.

Theorem 11. (Explicit form of the web capacity and potential)
If B_R denotes a ball of radius $R > 0$ then $\mathrm{Cap}_{\mathcal{W}}(B_R) = R$, while for every $\Omega \in \mathcal{K}$

different from a ball there holds

$$\mathrm{Cap}_{\mathcal{W}}(\Omega) = \frac{1}{4\pi} \frac{\sqrt{M^2 - 4\pi S}}{\mathrm{Arcosh}\frac{M}{2\sqrt{\pi S}}}. \tag{12}$$

Moreover, the unique minimizer ψ *of* (10) *(i.e. the "web conductor potential") is given by*

$$\psi(x) = 1 - \left(\int_0^\infty \frac{dt}{S_t} \right)^{-1} \cdot \int_0^{d_\Omega(x)} \frac{dt}{S_t}. \tag{13}$$

Proof. See Section 5.3. □

Formula (12) is due to Szegö [35], see also [29, §3.1]. We prove it in a different and more constructive way, which also allows us to obtain the explicit expression (13) of the web minimizer. This fact is a strong argument in favour of the web capacity with respect to the Russell one, which yields no approximate conductor potential. In order to estimate how fine the approximation is, it would be interesting to evaluate the difference $U_\Omega - \psi$ in a suitable norm (*e.g.*, in L^∞ or $\mathcal{D}^{1,2}$). However, also for the web capacity, we may consider the quotient functional

$$\mathcal{E}_{\mathcal{W}}(\Omega) = \frac{\mathrm{Cap}(\Omega)}{\mathrm{Cap}_{\mathcal{W}}(\Omega)}.$$

By Theorem 11 we know that $\mathrm{Cap}_{\mathcal{W}}(\Omega) > 0$ whenever $\Omega \in \mathcal{K}$ and therefore the ratio $\mathcal{E}_{\mathcal{W}}(\Omega)$ is well-defined for all such Ω. Moreover, since $\mathcal{W} \subset \mathcal{D}^{1,2}(\mathbf{R}^3)$, the value of $\mathcal{E}_{\mathcal{W}}(\Omega)$ falls into the interval $(0, 1]$. So, when compared to the Russell capacity, the web capacity has the advantage of being always larger than the true capacity. On the other hand, by using the inequality $x < \cosh(\sqrt{x^2 - 1})$ on $(1, +\infty)$, one can easily check that

$$\mathrm{Cap}_{\mathcal{W}}(\Omega) \geq \mathrm{Cap}_{\mathcal{R}}(\Omega) \qquad \forall\, \Omega \in \mathcal{K},$$

so that the Russell capacity yields a better approximation of the capacity whenever $\mathcal{E}_{\mathcal{R}}(\Omega) < 1$. By analogy to Conjecture 2, the following questions naturally arise.

Problem 12. Is $\inf_{\mathcal{K}} \mathcal{E}_{\mathcal{W}}$ attained? And, if affirmative, which is the optimal shape?

We show that the first question above is meaningful thanks to the following result.

Theorem 13. (Positive lower bound for $\mathcal{E}_{\mathcal{W}}$)
We have

$$\inf_{\mathcal{K}} \mathcal{E}_{\mathcal{W}} > 0.$$

Proof. See Section 5.4. □

Remarks. **(i)** The plots represented in Figures 2 and 4 of Section 4 seem to suggest that an optimal shape for the quotient $\mathcal{E}_{\mathcal{W}}$ does exist; indeed, sequences of ellipsoids or ellipses which tend to degenerate into a segment turn out to be maximizing for $\mathcal{E}_{\mathcal{W}}$.

(ii) Concerning the second question in Problem 12, in Section 4 we show that $\mathcal{E}_{\mathcal{W}}$ *does not* attain its minimum on the unit disk: we find a planar ellipse which gives a lower value for $\mathcal{E}_{\mathcal{W}}$.

(iii) It is worth noticing that $\mathcal{E}_{\mathcal{W}}(\Omega) = 1$ if and only if Ω is a ball. Indeed, if Ω is a ball, then the conductor potential is radially symmetric, so that the capacity coincides with the web capacity (notice that, when Ω is a ball, both its numerator and denominator in (12) vanish). For the converse implication, we refer to [10].

4. Exact value of $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{W}}$ for some simple sets

In this section, we give the value of the errors $\mathcal{E}_{\mathcal{R}}(\Omega)$ and $\mathcal{E}_{\mathcal{W}}(\Omega)$ for some particular convex bodies Ω. As we will see, in many significant cases they are close to one, showing that they yield good approximations for the capacity.

4.1. Ellipsoids

Denote by

$$E_{a,b,c} = \left\{(x,y,z) \in \mathbf{R}^3;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\right\} \qquad a \geq b \geq c > 0.$$

As extremal cases we find planar ellipses

$$E_{a,b,0} = \left\{(x,y,0) \in \mathbf{R}^3;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\right\} \qquad a \geq b > 0.$$

For ellipsoids with three different semi-axes $a > b > c > 0$ the explicit value of the capacity is known [22, (4,15) p. 38] and is given by

$$\mathrm{Cap}(E_{a,b,c}) = 2\left(\int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}\right)^{-1}, \tag{14}$$

a formula which involves elliptic integrals. In particular, for planar ellipses with semi-axes $a > b > 0$ we have

$$\mathrm{Cap}(E_{a,b,0}) = 2\left(\int_0^\infty \frac{ds}{\sqrt{s(a^2+s)(b^2+s)}}\right)^{-1}.$$

In their full generality, also the Russell capacity and the web capacity of ellipsoids may be expressed only by means of elliptic integrals; therefore, we focus our attention on the case where two of the three axes are equal.

Prolate ellipsoids. Thanks to rescaling we may restrict to ellipsoids $E_{a,1,1}$ with $a > 1$. Let us recall that [22, (4,18) p. 39]:

$$\mathrm{Cap}(E_{a,1,1}) = \frac{\sqrt{a^2-1}}{\mathrm{Arcosh} a} \qquad a > 1.$$

Moreover,

$$S(E_{a,1,1}) = 2\pi\left(1 + \frac{a^2}{\sqrt{a^2-1}}\mathrm{Arcsin}\frac{\sqrt{a^2-1}}{a}\right)$$

and

$$M(E_{a,1,1}) = 2\pi\left(a + \frac{\mathrm{Arctanh}\frac{\sqrt{a^2-1}}{a}}{\sqrt{a^2-1}}\right).$$

Therefore, by (4),

$$\mathrm{Cap}_{\mathcal{R}}(E_{a,1,1}) = \sqrt{\frac{1}{2}\left(1 + \frac{a^2}{\sqrt{a^2-1}}\mathrm{Arcsin}\frac{\sqrt{a^2-1}}{a}\right)}.$$

Consider now the web capacity. By Theorem 11 we infer

$$\mathrm{Cap}_{\mathcal{W}}(E_{a,1,1}) = \frac{\sqrt{\left(a + \frac{\mathrm{Arctanh}\frac{\sqrt{a^2-1}}{a}}{\sqrt{a^2-1}}\right)^2 - 2\left(1 + \frac{a^2}{\sqrt{a^2-1}}\mathrm{Arcsin}\frac{\sqrt{a^2-1}}{a}\right)}}{2\mathrm{Arcosh}\,\frac{a + \frac{\mathrm{Arctanh}\frac{\sqrt{a^2-1}}{a}}{\sqrt{a^2-1}}}{\sqrt{2\left(1 + \frac{a^2}{\sqrt{a^2-1}}\mathrm{Arcsin}\frac{\sqrt{a^2-1}}{a}\right)}}}.$$

Oblate ellipsoids. Again, we may restrict to ellipsoids $E_{1,1,a}$ with $0 < a < 1$. In view of [22, (4,19) p. 39], we have

$$\mathrm{Cap}(E_{1,1,a}) = \frac{\sqrt{1-a^2}}{\mathrm{Arcos} a} \qquad 0 < a < 1.$$

Moreover,

$$S(E_{1,1,a}) = 2\pi\left(1 + \frac{a^2}{\sqrt{1-a^2}}\mathrm{Arcosh}\frac{1}{a}\right)$$

and

$$M(E_{1,1,a}) = 2\pi\left(a + \frac{\mathrm{Arctan}\frac{\sqrt{1-a^2}}{a}}{\sqrt{1-a^2}}\right).$$

Therefore,

$$\mathrm{Cap}_{\mathcal{R}}(E_{1,1,a}) = \sqrt{\frac{1}{2}\left(1 + \frac{a^2}{\sqrt{1-a^2}}\mathrm{Arcosh}\frac{1}{a}\right)}.$$

In particular, for the unit 2-dimensional disk $D = E_{1,1,0}$ we find

$$\mathrm{Cap}_{\mathcal{R}}(D) = \frac{1}{\sqrt{2}}, \qquad \mathrm{Cap}(D) = \frac{2}{\pi}, \qquad \mathcal{E}_{\mathcal{R}}(D) = \frac{2\sqrt{2}}{\pi} \approx 0.9.$$

Concerning the web capacity, Theorem 11 yields

$$\mathrm{Cap}_{\mathcal{W}}(E_{1,1,a}) = \frac{\sqrt{\left(a + \frac{\mathrm{Arctan}\frac{\sqrt{1-a^2}}{a}}{\sqrt{1-a^2}}\right)^2 - 2\left(1 + \frac{a^2}{\sqrt{1-a^2}}\mathrm{Arcosh}\frac{1}{a}\right)}}{2\mathrm{Arcosh}\,\frac{a + \frac{\mathrm{Arctan}\frac{\sqrt{1-a^2}}{a}}{\sqrt{1-a^2}}}{\sqrt{2\left(1 + \frac{a^2}{\sqrt{1-a^2}}\mathrm{Arcosh}\frac{1}{a}\right)}}}.$$

In Figure 1 we represent the map $f : (0, \infty) \mapsto \mathbf{R}_+$ defined by

$$f(a) = \begin{cases} \mathcal{E}_{\mathcal{R}}(E_{1,1,a}) & \text{if } 0 < a \leq 1 \\ \mathcal{E}_{\mathcal{R}}(E_{a,1,1}) & \text{if } a \geq 1. \end{cases}$$

Notice that $f(a) \to +\infty$ as $a \to +\infty$, and that the unit ball corresponds to $a = 1$, which is a flex point for f, see also Corollary 8.

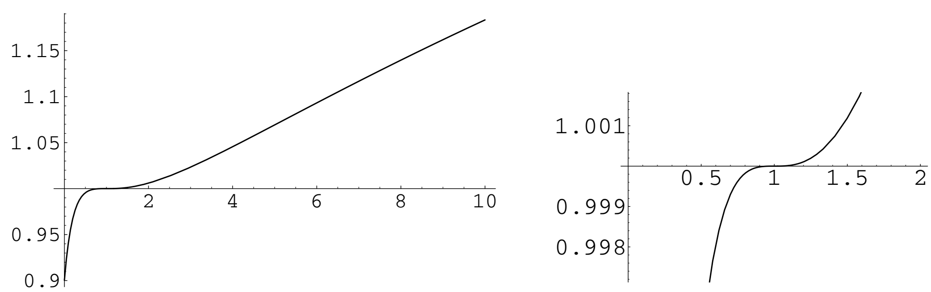

Figure 1. the plot of $a \mapsto f(a)$ on $(0, +\infty)$ and near $a = 1$

In Figure 2 we represent the map $g : (0, \infty) \mapsto \mathbf{R}_+$ defined by

$$g(a) = \begin{cases} \mathcal{E}_{\mathcal{W}}(E_{1,1,a}) & \text{if } 0 < a \leq 1 \\ \mathcal{E}_{\mathcal{W}}(E_{a,1,1}) & \text{if } a \geq 1. \end{cases}$$

We have an absolute maximum for $a = 1$, which corresponds to the unit ball. Quite surprisingly, we also find a relative minimum for $a \approx 39.457$: we have no explanation of this fact. Notice also that $g(a) \to 1$ as $a \to +\infty$.

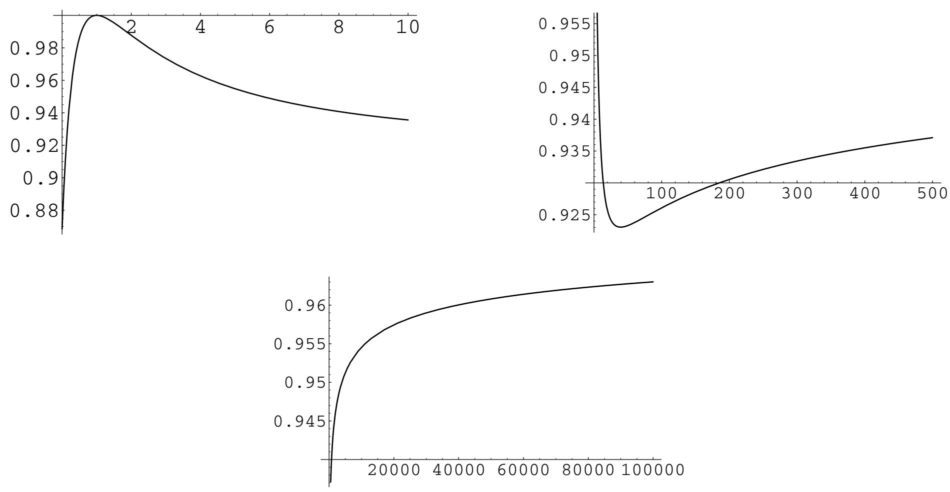

Figure 2. the plot of $a \mapsto g(a)$ on $(0, 10)$, on $(1, 500)$, and on $(500, 10^5)$

Planar ellipses. We now represent the errors $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{W}}$ for planar ellipses.

Consider first the map $a \mapsto \mathcal{E}_{\mathcal{R}}(E_{1,a,0})$ for $a \in (0, 1]$. As may also be seen in the proof of Theorem 4 (see Section 5.1), the error tends to infinity as the convex body tends to a 1-dimensional object ($a \to 0$), see Figure 3. Moreover, for $a = 1$ we find the 2-dimensional disk D which by Theorem 3 is the absolute minimum for $\mathcal{E}_{\mathcal{R}}$ over $\mathcal{K} \setminus \mathcal{K}_o$. Finally, notice that $\mathcal{E}_{\mathcal{R}}$ is monotonic and $\mathcal{E}_{\mathcal{R}} = 1$ for $a \approx 0.266$.

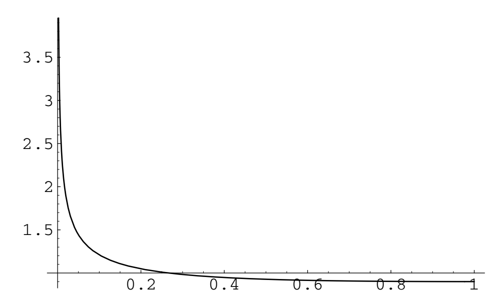

Figure 3. the plot of $a \mapsto \mathcal{E}_{\mathcal{R}}(E_{1,a,0})$ for $a \in (0, 1]$

Consider now the map $a \mapsto \mathcal{E}_{\mathcal{W}}(E_{1,a,0})$ for $a \in (0,1]$. Its plot in Figure 4 shows that $\mathcal{E}_{\mathcal{W}}(E_{1,a,0}) \to 1$ as $a \to 0$, and that the unit disk *is not* the absolute minimum for $\mathcal{E}_{\mathcal{W}}$ (not even among convex planar domains).

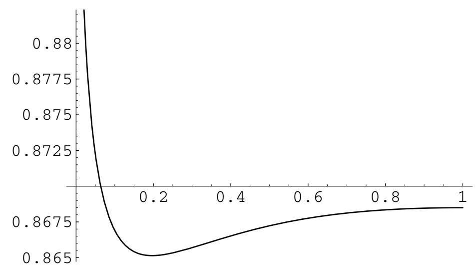

Figure 4. the plot of $a \mapsto \mathcal{E}_{\mathcal{W}}(E_{1,a,0})$ for $a \in (0,1)$

4.2. Half ball

Denote by $\Sigma = \{(x,y,z) \in \mathbf{R}^3;\ x^2+y^2+z^2<1,\ z>0\}$. By [23, §II.3.14] we know that

$$\mathrm{Cap}(\Sigma) = 2\left(1-\frac{1}{\sqrt{3}}\right) \approx 0.845.$$

Moreover, $S(\Sigma) = 3\pi$ and $M(\Sigma) = \dfrac{\pi}{2}(\pi+4)$. Hence,

$$\mathrm{Cap}_{\mathcal{R}}(\Sigma) = \frac{\sqrt{3}}{2} \approx 0.866 \qquad \text{and} \qquad \mathcal{E}_{\mathcal{R}}(\Sigma) = 4\left(\frac{1}{\sqrt{3}}-\frac{1}{3}\right) \approx 0.976.$$

On the other hand, by Theorem 11 we deduce

$$\mathrm{Cap}_{\mathcal{W}}(\Sigma) = \frac{\sqrt{\pi^2+8\pi-32}}{8\mathrm{Arcosh}\,\frac{\pi+4}{4\sqrt{3}}} \qquad \text{and} \qquad \mathcal{E}_{\mathcal{W}}(\Sigma) \approx 0.966.$$

4.3. Regular polyhedra

We summarize the results in Table 1. All polyhedra Ω have inradius $R_\Omega = 1$. The approximate value of the capacity is taken from [4, Table 2], the value of the Russell capacity follows from well-known formulae for the surface measure of polyhedra, the web capacity is taken from [24, Table 4].

Ω	$\mathrm{Cap}(\Omega)$	$\mathrm{Cap}_{\mathcal{R}}(\Omega)$	$\mathrm{Cap}_{\mathcal{W}}(\Omega)$	$\mathcal{E}_{\mathcal{R}}(\Omega)$	$\mathcal{E}_{\mathcal{W}}(\Omega)$
Tetrahedron	1.745	$\sqrt{\frac{6\sqrt{3}}{\pi}}$	1.956	0.959	0.892
Cube	1.322	$\sqrt{\frac{6}{\pi}}$	1.421	0.957	0.93
Octahedron	1.249	$\sqrt{\frac{3\sqrt{3}}{\pi}}$	1.337	0.971	0.934
Dodecahedron	1.112	$\sqrt{\frac{30\sqrt{25+10\sqrt{5}}}{\pi(25+11\sqrt{5})}}$	1.163	0.966	0.956
Icosahedron	1.079	$\sqrt{\frac{30\sqrt{3}}{\pi(7+3\sqrt{5})}}$	1.117	0.982	0.966

Table 1.

5. Proofs

5.1. Proof of Theorem 4

Let us first recall the following comparison result involving ellipsoids. We will combine it with the monotonicity of capacity with respect to inclusion, and with the explicit formulae of Section 4 for the capacity of ellipsoids.

Lemma 14. (John Lemma)
Let $\Omega \in \mathcal{K}_o$. Then there exists an ellipsoid $E_{a,b,c}$ such that, up to a translation and rotation of Ω, we have

$$E_{a,b,c} \subset \Omega \subset 3E_{a,b,c},$$

where $3E(a,b,c) := \{3x \ : \ x \in E_{a,b,c}\}$.

Proof. See [19, p. 202]. □

We now prove Theorem 4. By density of $\mathcal{K}_o$ in $\mathcal{K}$ in the Hausdorff distance, we may take a minimizing sequence $\{\Omega_n\} \subset \mathcal{K}_o$ for $\mathcal{E}_{\mathcal{R}}$. By Lemma 14 and thanks to rescaling, there exists a sequence of ellipsoids $\{E_{1,b_2^n,b_3^n}\}$ $(1 \geq b_2^n \geq b_3^n > 0)$ such that

$$E_{1,b_2^n,b_3^n} \subset \Omega_n \subset 3E_{1,b_2^n,b_3^n}. \tag{15}$$

By Blaschke selection Theorem (see *e.g.* [32, Theorem 1.8.6]), up to a subsequence, $\{\Omega_n\}$ tends in the Hausdorff distance to a convex body Ω_∞, and both $\{b_2^n\}$ and $\{b_3^n\}$ converge. Since $\mathrm{Cap}(\Omega)$ and $\mathrm{Cap}_{\mathcal{R}}(\Omega)$ are continuous with respect to such convergence, we are done if we prove that $\Omega_\infty \in \mathcal{K}$. This is true if $b_3^n \not\to 0$ (in this case $\Omega_\infty \in \mathcal{K}_o$), or if $b_3^n \to 0$ but $b_2^n \not\to 0$ (in this case $\Omega_\infty \in \mathcal{K} \setminus \mathcal{K}_o$). Thus we have to exclude that $\lim_n b_3^n = \lim_n b_2^n = 0$. To this aim we observe that, by (15) and by monotonicity of Cap and $\mathrm{Cap}_{\mathcal{R}}$ with respect to inclusions of convex bodies, there holds

$$\mathcal{E}_{\mathcal{R}}(\Omega_n) = \frac{\mathrm{Cap}(\Omega_n)}{\mathrm{Cap}_{\mathcal{R}}(\Omega_n)} \geq \frac{1}{3}\,\frac{\mathrm{Cap}(E_{1,b_2^n,b_3^n})}{\mathrm{Cap}_{\mathcal{R}}(E_{1,b_2^n,b_3^n})} \geq \frac{1}{3}\,\frac{\mathrm{Cap}(E_{1,b_2^n,b_3^n})}{\mathrm{Cap}_{\mathcal{R}}(P_{1,b_2^n,b_3^n})} \tag{16}$$

where P_{1,b_2^n,b_3^n} is the parallelepiped having semi-axes of length 1, b_2^n, and b_3^n. Notice that (for some $c > 0$)

$$\mathrm{Cap}_{\mathcal{R}}(P_{1,b_2^n,b_3^n}) \leq c\sqrt{b_2^n}. \tag{17}$$

Moreover, by (14),

$$\begin{aligned}
\frac{1}{\mathrm{Cap}(E_{1,b_2^n,b_3^n})} &= \frac{1}{2}\int_0^\infty \frac{ds}{\sqrt{(s+1)(s+(b_2^n)^2)(s+(b_3^n)^2)}} \\
&\leq \frac{1}{2}\int_0^1 \frac{ds}{\sqrt{s(s+(b_2^n)^2)}} + \frac{1}{2}\int_1^\infty \frac{ds}{s^{3/2}} \\
&= \log\left(\frac{1+\sqrt{1+(b_2^n)^2}}{b_2^n}\right) + 1 = O(|\log b_2^n|) \qquad \text{as } b_2^n \to 0.
\end{aligned} \tag{18}$$

This, inserted with (17) into (16), yields

$$\mathcal{E}_{\mathcal{R}}(\Omega_n) \geq \frac{C}{\sqrt{b_2^n}\,|\log b_2^n|} \to +\infty \qquad \text{as } b_2^n \to 0,$$

against the assumption that $\{\Omega_n\}$ is a minimizing sequence for $\mathcal{E}_{\mathcal{R}}$.

5.2. Proofs of Theorem 5, Theorem 6, Theorem 7, and Corollary 8

First of all, we introduce some useful tools.

For any $\Omega \in \mathcal{K}$, the *support function* $V_\Omega : \mathbf{R}^3 \to \mathbf{R}$ is defined by

$$V_\Omega(\xi) := \sup\{x \cdot \xi \ : \ x \in \Omega\}.$$

Geometrically, for any $\xi \in S^2$, $V_\Omega(\xi)$ represents the distance from $P(\xi)$ to the origin, being $P(\xi)$ a support plane of Ω with unit normal ξ pointing away from Ω into $\mathbf{R}^3 \setminus \Omega$.

For a given domain $\Omega \in \mathcal{K}_o$, we denote by $g^\Omega : \partial\Omega \to S^2$ the *Gauss map*, which associates to a point $x \in \partial\Omega$ the unit outer normal to $\partial\Omega$ at x. Then g^Ω is well-defined for $\mathcal{H}^2$-a.e. $x \in \partial\Omega$. We also recall that, for any measure σ on $\partial\Omega$, its push-forward $g_*^\Omega(\sigma)$ through the mapping g^Ω defines a measure on S^2, given by

$$\int_{S^2} \varphi(x)\, dg_*^\Omega(\sigma) := \int_{\partial\Omega} \varphi(g^\Omega(x))d\sigma(x) \qquad \forall \varphi \in C^0(S^2).$$

Assume that $\Omega \in \mathcal{K}_o$ is stationary for $\mathcal{E}_{\mathcal{R}}$, let $D \in \mathcal{K}_o$ and put $\Omega_t := \Omega + tD$. Differentiating $\mathcal{E}_{\mathcal{R}}$ as a quotient, (5) becomes

$$\frac{d}{dt}\,\mathrm{Cap}(\Omega_t)|_{t=0} = \frac{\mathrm{Cap}(\Omega)}{2S(\Omega)} \cdot \frac{d}{dt}\,S(\Omega_t)|_{t=0} \qquad \forall D \in \mathcal{K}_o. \tag{19}$$

The derivatives appearing in (19) can be made explicit by means of suitable representation formulae for the first variation of the surface area and of the capacity. Indeed, there exists a *curvature measure* μ_Ω defined on S^2 such that $S(\Omega)$ and $\frac{d}{dt}S(\Omega_t)|_{t=0}$ admit the following integral representations:

$$S(\Omega) = \frac{1}{2}\int_{S^2} V_\Omega \, d\mu_\Omega, \qquad \frac{d}{dt}S(\Omega_t)|_{t=0} = \int_{S^2} V_D \, d\mu_\Omega. \tag{20}$$

For the general definition and properties of μ_Ω, we refer *e.g.* to [33] or [32, Ch. IV]. In a similar way behaves the electrostatic capacity. Precisely, there exists a *capacitary measure* ν_Ω, defined on S^2, such that

$$\mathrm{Cap}(\Omega) = \int_{S^2} V_\Omega \, d\nu_\Omega, \qquad \frac{d}{dt}\mathrm{Cap}(\Omega_t)|_{t=0} = \int_{S^2} V_D \, d\nu_\Omega. \tag{21}$$

The measure ν_Ω is characterized by

$$\nu_\Omega := \frac{1}{4\pi} g_*^\Omega(|\nabla U_\Omega|^2 \mathcal{H}^2 \llcorner \partial\Omega), \tag{22}$$

where U_Ω denotes the equilibrium potential of Ω and $\mathcal{H}^2 \llcorner \partial\Omega$ is the 2-dimensional Hausdorff measure over $\partial\Omega$. Remark that, by a theorem of Dahlberg [8], the function $|\nabla U_\Omega|^2$ is defined $\mathcal{H}^2$-a.e. on $\partial\Omega$, and it is in L^1 with respect to $\mathcal{H}^2 \llcorner \partial\Omega$. Therefore, the measure $|\nabla U_\Omega|^2 \mathcal{H}^2 \llcorner \partial\Omega$ is well-defined, and one may also consider its push-forward according to the Gauss map as in the r.h.s. of (22). Taking (22) into account, the identities in (21) may be rewritten as

$$\begin{aligned} \mathrm{Cap}(\Omega) &= \frac{1}{4\pi}\int_{\partial\Omega} V_\Omega(g^\Omega(x))|\nabla U_\Omega|^2 \, d\mathcal{H}^2, \\ \frac{d}{dt}\mathrm{Cap}(\Omega_t)|_{t=0} &= \frac{1}{4\pi}\int_{\partial\Omega} V_D(g^\Omega(x))|\nabla U_\Omega|^2 \, d\mathcal{H}^2. \end{aligned} \tag{23}$$

For smooth domains, these formulae date back to Hadamard. Recently, they have been extended to arbitrary convex bodies by Jerison (see respectively Proposition 1.5 and Corollary 3.16 in [18]).

We are now in a position to give the

Proof of Theorem 5. By (19), (20), and (21), $\Omega \in \mathcal{K}_o$ is stationary for $\mathcal{E}_\mathcal{R}$ if and only if

$$\int_{S^2} V_D \, d\nu_\Omega = \frac{\mathrm{Cap}(\Omega)}{2S(\Omega)} \int_{S^2} V_D \, d\mu_\Omega \qquad \forall D \in \mathcal{K}_o. \tag{24}$$

Using the density result of Lemma 1.7.9 in [32], the density of $\mathcal{K}_o$ in $\mathcal{K}$, and the continuity result of Lemma 1.8.10 in [32], (24) is equivalent to

$$\int_{S^2} \varphi \, d\nu_\Omega = \frac{\mathrm{Cap}(\Omega)}{2S(\Omega)} \int_{S^2} \varphi \, d\mu_\Omega \qquad \forall \varphi \in \mathcal{C}^0(S^2). \qquad \square$$

Proof of Theorem 6. Assume for contradiction that Ω is stationary for $\mathcal{E}_\mathcal{R}$. For every $\varepsilon > 0$, let $\varphi_\varepsilon \in \mathcal{C}^0(S^2;[0,1])$, with $\varphi_\varepsilon(\xi) = 1$ and support contained into a ball of radius ε centered at ξ. Testing (6) with φ_ε, and recalling (22) and (25), we obtain

$$\int_{\partial\Omega} \varphi_\varepsilon(g^\Omega(x))|\nabla U_\Omega|^2\, d\mathcal{H}^2 = \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} \int_{\partial\Omega} \varphi_\varepsilon(g^\Omega(x)) H_{\partial\Omega}\, d\mathcal{H}^2.$$

Passing to the limit as $\varepsilon \to 0$ yields

$$\int_{(g^\Omega)^{-1}(\xi)\cap\partial\Omega} |\nabla U_\Omega|^2\, d\mathcal{H}^2 = \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} \int_{(g^\Omega)^{-1}(\xi)\cap\partial\Omega} H_{\partial\Omega}\, d\mathcal{H}^2.$$

The above equation, together with the assumption $\mathcal{H}^2\big((g^\Omega)^{-1}(\xi)\cap\partial\Omega\big) > 0$, gives immediately a contradiction. Indeed, $H_{\partial\Omega}$ is identically zero on $(g^\Omega)^{-1}(\xi)\cap\partial\Omega$, whereas $|\nabla U_\Omega|$ is strictly positive on $\partial\Omega$ by Hopf boundary lemma. □

Proof of Theorem 7. Since $\partial\Omega \in \mathcal{C}^2$, the curvature measure μ_Ω is given by (see [12, §1.10])

$$\mu_\Omega = 2 g^\Omega_*(H_{\partial\Omega}\mathcal{H}^2 \llcorner \partial\Omega), \tag{25}$$

where $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega$. Hence, equalities in (20) turn into

$$S(\Omega) = \int_{\partial\Omega} V_\Omega(g^\Omega(x)) H_{\partial\Omega}(x)\, d\mathcal{H}^2, \;\; \frac{d}{dt}S(\Omega_t)|_{t=0} = 2\int_{\partial\Omega} V_D(g^\Omega(x)) H_{\partial\Omega}(x)\, d\mathcal{H}^2. \tag{26}$$

Then, using (23) and (26), and taking into account that g^Ω is a diffeomorphism, (6) is equivalent to (7).

Finally, the C^∞ regularity of $\partial\Omega$ is straightforward from (7) by using a bootstrap argument. Since $\partial\Omega \in \mathcal{C}^2$, by standard elliptic regularity [11, Theorem 9.19] $U_\Omega \in \mathcal{C}^{1,\theta}$ up to the boundary. Then by (7) $H_{\partial\Omega} \in \mathcal{C}^{0,\theta}$, hence $\partial\Omega \in \mathcal{C}^{2,\theta}$. By iteration, more regularity is gained at each step, and the result follows. □

Proof of Corollary 8. It is enough to check that, if $\Omega = B_R$, then (7) is satisfied. In fact, we have:

$$|\nabla U_\Omega(x)|^2 \equiv \frac{1}{R^2} \quad \text{and} \quad \frac{4\pi \mathrm{Cap}(\Omega)}{S(\Omega)} = H_{\partial\Omega}(x) \equiv \frac{1}{R} \qquad \forall\, x \in \partial\Omega. \qquad \square$$

Proof of (9). Multiply (7) by the nonnegative function $V_\Omega(x)$, and integrate with respect to $\mathcal{H}^2$ on $\partial\Omega$. This gives an identity *whatever* $\Omega \in \mathcal{K}_o$, by using the representation formulae for $S(\Omega)$ and $\mathrm{Cap}(\Omega)$ in (23) and (26). □

5.3. Proof of Theorem 11

Recalling definition (10) and using the coarea formula, we see that

$$\begin{aligned}\mathrm{Cap}_{\mathcal{W}}(\Omega) &= \frac{1}{4\pi}\inf\left\{\int_{\mathbf{R}^3\setminus\Omega}|\phi'(d_\Omega(x))|^2\,dx;\ \phi(0)=1,\ \|\phi'\circ d_\Omega\|_{L^2(\mathbf{R}^3\setminus\Omega)}<\infty\right\}\\ &= \frac{1}{4\pi}\inf\left\{\int_0^\infty S_t|\phi'(t)|^2\,dt;\ \phi(0)=1,\ \|S_t\phi'(t)^2\|_{L^1(0,\infty)}<\infty\right\}. \quad (27)\end{aligned}$$

The unique minimizer ψ of this variational problem solves the Euler–Lagrange equation $(S_t\psi'(t))'=0$, that is, $\psi'(t)=\frac{k}{S_t}$ for some $k\in\mathbf{R}$. Since $\psi(0)=1$, by integrating over $(0,+\infty)$ we obtain

$$k=-\frac{1}{\int_0^\infty S_\tau^{-1}d\tau}$$

and therefore

$$\psi'(t)=-\frac{1}{S_t\int_0^\infty S_\tau^{-1}d\tau},$$

which gives (13) after integration. Moreover, recalling (27), we get at once

$$\mathrm{Cap}_{\mathcal{W}}(\Omega)=\frac{1}{4\pi}\int_0^\infty S_t|\psi'(t)|^2\,dt=\frac{1}{4\pi}\,\frac{1}{\int_0^\infty S_t^{-1}dt}. \quad (28)$$

Taking into account that

$$S_t=\frac{1}{4\pi}\left(4\pi t+M+\sqrt{M^2-4\pi S}\right)\left(4\pi t+M-\sqrt{M^2-4\pi S}\right)\ ,$$

(12) follows after a simple computation.

5.4. Proof of Theorem 13

We argue as in the proof of Theorem 4. Consider a minimizing sequence $\{\Omega_n\}\subset\mathcal{K}_o$ for $\mathcal{E}_{\mathcal{W}}$. By Lemma 14 and rescaling we find again (15). By (28) we see that $\mathrm{Cap}_{\mathcal{W}}$ is increasing with respect to inclusions; therefore, by (15) we have

$$\mathcal{E}_{\mathcal{W}}(\Omega_n)\geq\frac{\mathcal{E}_{\mathcal{W}}(E_{1,b_2^n,b_3^n})}{3}\qquad(1\geq b_2^n\geq b_3^n>0). \quad (29)$$

By Blaschke selection Theorem, up to a subsequence, $\{\Omega_n\}$ converges in the Hausdorff distance to a convex body Ω_∞. Since both $\Omega\mapsto\mathrm{Cap}(\Omega)$ and $\Omega\mapsto\mathrm{Cap}_{\mathcal{W}}(\Omega)$ are continuous with respect to such convergence, we are done if $\Omega_\infty\in\mathcal{K}$. Thus, it remains to study the behaviour of $\mathcal{E}_{\mathcal{W}}(E_{1,b_2^n,b_3^n})$ when $\lim_n b_3^n=\lim_n b_2^n=0$. In this case, by (29) and by monotonicity of Cap and $\mathrm{Cap}_{\mathcal{W}}$ with respect to inclusions of convex bodies, we have

$$\mathcal{E}_{\mathcal{W}}(\Omega_n)\geq\frac{1}{6}\,\frac{\mathrm{Cap}(E_{1,b_2^n,b_3^n})}{\mathrm{Cap}_{\mathcal{W}}(P_{1,b_2^n,b_3^n})}, \quad (30)$$

where P_{1,b_2^n,b_3^n} is now the parallelepiped having axes of length 1, b_2^n, and b_3^n. By Theorem 11 we have

$$\mathrm{Cap}_{\mathcal{W}}(P_{1,b_2^n,b_3^n}) = \frac{1}{2\pi}\,\frac{\sqrt{\pi^2(1+b_2^n+b_3^n)^2-8\pi(b_2^n+b_3^n+b_2^nb_3^n)}}{\mathrm{Arcosh}\frac{\sqrt{\pi}(1+b_2^n+b_3^n)}{2\sqrt{2(b_2^n+b_3^n+b_2^nb_3^n)}}} \le \frac{C}{|\log b_2^n|},$$

which, together with (18) and (30), proves Theorem 13.

Acknowledgements

We are grateful to N. Garofalo for his kind interest and some discussions about Conjecture 10. We also thank S. Campi, A. Colesanti, P. Gronchi, and P. Salani for stimulating discussions during an informal meeting.

References

[1] K. Aichi, Note on the capacity of a nearly spherical conductor and especially of an ellipsoidal conductor, *Proceedings of the Physico-Mathematical Society of Tokyo* **4** (1908), 243–246.
[2] C. Borell, Capacitary inequalities of the Brunn-Minkowski type, *Math. Ann.* **263** (1983), 179–184.
[3] C. J. Bouwkamp, A simple method of calculating the electrostatic capacity, *Physica* **24** (1958), 538–542.
[4] C. S. Brown, Capacity of the regular polyhedra, *Computers Math. Appl.* **20** (1990), 43–56.
[5] Y. L. Chow, M. M. Yovanovich, The shape factor of the capacitance of a conductor, *J. Appl. Phys.* **53** (1982), 8470–8475.
[6] G. Crasta, Variational problems for a class of functionals on convex domains, *J. Differential Equations* **178** (2002), 608–629.
[7] G. Crasta, I. Fragalà, F. Gazzola, A sharp upper bound for the torsional rigidity of rods by means of web functions, *Arch. Rat. Mech. Anal.* **164** (2002), 189–211.
[8] B. E. J. Dahlberg, Estimates for harmonic measure, *Arch. Rat. Mech. Anal.* **65** (1977), 275–283.
[9] P. R. Garabedian, M. Schiffer, On estimation of electrostatic capacity, *Proc. Amer. Math. Soc.* **5** (1954), 206–211.
[10] N. Garofalo, E. Sartori, Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates, *Adv. Diff. Eq.* **4** (1999), 137–161.
[11] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin 2001.
[12] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Boston 1984.
[13] A. Greco, Radial symmetry and uniqueness for an overdetermined problem, *Math. Methods Appl. Sci.* **24** (2001), 103–115.
[14] D. Greenspan, *Lectures on the Numerical Solution of Linear, Singular, and Nonlinear Differential Equations*, Prentice-Hall Inc., Englewood Cliffs, N.J. 1968.
[15] D. Greenspan, *Discrete Numerical Methods in Physics and Engineering*, Academic Press, New York 1974.
[16] A. Henrot, G. A. Philippin, H. Prébet, Overdetermined problems on ring-shaped domains, *Adv. Math. Sci. Appl.* **9** (1999), 737–747.
[17] J. B. Hubbard, J. F. Douglas, Hydrodynamic friction of arbitrarily shaped Brownian particles, *Phys. Rev. E* **47** (1993), 2983–2986.

[18] D. Jerison, A Minkowski problem for electrostatic capacity, *Acta Math.* **176** (1996), 1–47.
[19] F. John, *Extremum Problems with Inequalities as Subsidiary Conditions*, Studies and essays presented to R. Courant in his 60th birthday, Interscience Publisher, New York 1948, 187–204.
[20] O.D. Kellogg, *Foundations of Potential Theory*, Dover, New York 1929.
[21] G. Kirchhoff, Über die Bewegung eines Rotationskörpers in einer Flüssingkeit, *J. F. Math.* (Crelle) **71** (1869), 237.
[22] L. Landau, E. Lifchitz, *Électrodynamique des Milieux Continus*, Eds. Mir, Moscou 1969.
[23] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin 1972.
[24] W. E. Parr, Upper and lower bounds for the capacitance of the regular solids, *J. Soc. Indust. Appl. Math.* **9** (1961), 334–386.
[25] L.E. Payne, G.A. Philippin, One some maximum principles involving harmonic functions and their derivatives, *Siam J. Math. Anal.* **10** 1979, 96–104.
[26] L. E. Payne, G. A. Philippin, Some overdetermined boundary value problems for harmonic functions, *Zeit. Angew. Math. Phys.* **42** (1991), 864–873.
[27] L. E. Payne, H. F. Weinberger, New bounds in harmonic and biharmonic problems, *J. Math. Phys.* **33** (1955), 291–307.
[28] G. Pólya, G. Szegö, Inequalities for the capacity of a condenser, *American J. Math.* **67** (1945), 1–32.
[29] G. Pólya, G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, 1951.
[30] Lord Rayleigh, *The Theory of Sound*, 2nd edition, London 1894/96.
[31] A. Russell, The eighth Kelvin lecture, *J. Institution Electrical Engineers* **55** (1916), 1–17.
[32] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge Univ. Press, 1993.
[33] R. Schneider, *Convex Surfaces, Curvature and Surface Area Measures*, Handbook of convex geometry, Vol. A, B, 273–299, North-Holland, Amsterdam 1993.
[34] J. Serrin, A symmetry problem in potential theory, *Arch. Rat. Mech. Anal.* **43** (1971), 304–318.
[35] G. Szegö, Über einige neue Extremaleigenschaften der Kugel, *Math. Z.* **33** (1931), 419–425.
[36] G. Talenti, Elliptic equations and rearrangements, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3** (1976), 697–718.

Graziano Crasta
Dipartimento di Matematica
Università La Sapienza
Piazzale Aldo Moro 2
00185 Roma
Italy

Ilaria Fragalà
Dipartimento di Matematica del Politecnico
piazza Leonardo da Vinci 32
20133 Milano
Italy

Filippo Gazzola
Dipartimento di Matematica del Politecnico
piazza Leonardo da Vinci 32
20133 Milano
Italy

(Received: September 29, 2003)