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NODAL SOLUTIONS TO CRITICAL GROWTH ELLIPTIC PROBLEMS UNDER STEKLOV BOUNDARY CONDITIONS

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Abstract. We study elliptic problems at critical growth under Steklov boundary conditions in bounded domains. For a second order problem we prove existence of nontrivial nodal solutions. These are obtained by combining a suitable linking argument with fine estimates on the concentration of Sobolev minimizers on the boundary. When the domain is the unit ball, we obtain a multiplicity result by taking advantage of the explicit form of the Steklov eigenfunctions. We also partially extend the results in the ball to the case of fourth order Steklov boundary value problems.

1. **Introduction and results.**

In a celebrated paper, Pohozaev [26] proved that the semilinear elliptic equation

$$-\Delta u = |u|^{2^*-2}u \qquad \text{in } \Omega \tag{1}$$

admits no positive solutions in a bounded smooth starshaped domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) under homogeneous Dirichlet boundary conditions. In fact, in these domains, Pohozaev's identity combined with the unique continuation property rules out also the existence of nodal solutions (see [20]) so that (1) admits only the trivial solution $u \equiv 0$. Here $2^* = \frac{2n}{n-2}$ denotes the critical exponent for the embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$. Since then, in order to obtain existence results for the *Dirichlet* problem associated to (1), many attempts were made to modify the geometry (topology) of the domain Ω or to perturb the critical nonlinearity $|u|^{2^*-2}u$ in (1). It appears an impossible task to exhaust all the related literature. In these papers, existence of nontrivial solutions to (1) was obtained.

Brezis [10, Section 6.4] suggested to study (1) under *Neumann* boundary conditions:

$$u_\nu = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where u_ν denotes the outer normal derivative of u on $\partial\Omega$. In fact, problem (1)-(2) is a particular case of the following (second order) elliptic problem with purely critical

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growth and *Steklov* boundary conditions:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega \\ u_\nu = \delta u & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Here, $\delta \in \mathbb{R}$ and (3) becomes the Neumann problem when $\delta = 0$ whereas it tends to the Dirichlet problem as $\delta \to -\infty$. We say that a function $u \in H^1(\Omega)$ is a *weak* solution of (3) if

$$\int_\Omega \nabla u \nabla v - \delta \int_{\partial\Omega} uv = \int_\Omega |u|^{2^*-2}uv \qquad \text{for all } v \in H^1(\Omega)\ .$$

It can be shown that weak solutions are in fact strong (classical) solutions, see [11]. As far as we are aware, existence results for (3) have been obtained only for $\delta \le 0$. In this respect, a crucial role is played by the maximal mean curvature of the boundary, namely

$$H_{\max} := \max_{x\in\partial\Omega} H(x), \tag{4}$$

where $H(x)$ is the mean curvature of $\partial\Omega$ at x. We collect some known results in the following statement:

Proposition 1. [1, 15, 16] *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 3$) be a smooth bounded domain.*
(i) If $\delta \in \left(\frac{2-n}{2} H_{\max}, 0\right)$, then (3) admits a positive solution.
(ii) If $\delta \ge 0$, then (3) admits no positive solutions.
(iii) If $\delta = 0$ and $n \ge 4$, then (3) admits a nontrivial nodal solution.
(iv) If $\delta = 0$, $n = 3$ and Ω is symmetric with respect to a plane, then (3) admits a nontrivial nodal solution.

One of the purposes of the present paper is to study the case where $\delta > 0$. We prove

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ ($n \ge 3$) be a smooth bounded domain and let $\{\delta_i\}_{i\ge 0}$ be the sequence of positive Steklov eigenvalues (see Proposition 4). Then problem (3) admits a pair of nontrivial nodal solutions for all $\delta > 0$ if $n \ge 4$, and for all $\delta > 0$ with $\delta \ne \delta_i$ if $n = 3$.*

We conjecture that if $n = 3$ and $\delta = \delta_i$ the existence of solutions might depend on the domain and that any possible solution (if ever) should be at high energy level.

The difference between the cases $\delta < 0$ and $\delta \ge 0$ relies on the geometric properties of the related action functional. The variational characterization of its critical points is of mountain-pass type in the first case and of linking type in the latter. And, as far as linking arguments are required, it is well-known that in order to lower the energy level of Palais-Smale sequences one needs to estimate "mixed terms" which are difficult to bound, see [14, 18]. To overcome this difficulty, in our proof we adapt ideas from [2, 3, 14, 18, 25] and combine a careful estimate of the mixed critical growth term with concentration phenomena of Sobolev minimizers on $\partial\Omega$.

When $\Omega = B$ (the unit ball), the previous results may be improved. The next (known) statement shows that the lower bound $\delta > \frac{2-n}{2} H_{\max}$ in Proposition 1 is not sharp:

Proposition 2. [15, 32] *Let $\Omega = B$ (the unit ball of $\mathbb{R}^n$, $n \ge 3$), then:*
(i) If $\delta \le 2 - n$, then (3) admits no positive radial solutions.

(ii) If $\delta \in (2-n,0)$*, then* (3) *admits a unique positive radial solution* u_δ *which is explicitly given by*

$$u_\delta(x) = \frac{[n(n-2)C_{\delta,n}]^{\frac{n-2}{4}}}{(C_{\delta,n}+|x|^2)^{\frac{n-2}{2}}},$$

where $C_{\delta,n} := \frac{2-n}{\delta} - 1$*.*

(iii) If $\delta = 0$*, then* (3) *admits infinitely many solutions.*

In the unit ball, Theorem 1.1 states (in particular) that (3) has nontrivial solutions for all $\delta \in (0,1)$. We improve this statement with a multiplicity result. For all $n \geq 3$, we put

$$h(n) := (n-2)\left[\frac{n^2}{\Gamma(n)}\right]^{2/n} \left[\frac{\Gamma(\frac{n}{2})}{2}\right]^{1+2/n} \left[\frac{(n+2)\,\Gamma(\frac{n+2}{2(n-2)})}{\sqrt{\pi}\,\Gamma(\frac{n^2}{2(n-2)})}\right]^{1-2/n}, \tag{5}$$

and we prove

Theorem 1.2. *Assume that* $\Omega = B$ *(the unit ball of* $\mathbb{R}^n$*,* $n \geq 3$*). If* $\delta \in (1-h(n),1)$*, then problem* (3) *admits at least* n *pairs of nontrivial nodal solutions.*

Figure 1 displays the plot of the function $h = h(n)$.

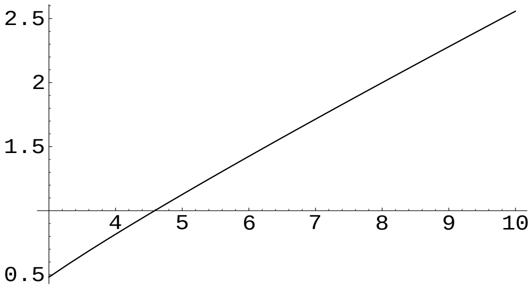

Figure 1: the map $h = h(n)$.

In particular, since $h(n) > 1$ for all $n \geq 5$, Theorem 1.2 yields nontrivial nodal solutions also for some values of $\delta < 0$. Clearly, nodal solutions cannot be radially symmetric since then they would solve the Dirichlet problem for (1) in the smaller ball defined by the nodal region containing the origin, against Pohozaev nonexistence result.

A further goal of this paper is to highlight the nonstandard variational structure of (3). The space spanned by the eigenfunctions of the linear boundary value problem does not exhaust all the functional space under consideration. Therefore, the linking argument used for its study has a more complicated behaviour. We collect the main properties concerning the linear Steklov (second and fourth order) problem in Section 2.

The last main objective of the present work is the comparison between the variational structure of (3) and that of the corresponding fourth order critical growth problem

$$\begin{cases} \Delta^2 u = |u|^{2_*-2}u & \text{in } \Omega \\ u = 0, \quad \Delta u = du_\nu & \text{on } \partial\Omega \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$ is a smooth bounded domain, $d \in \mathbb{R}$ and $2_* = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding $H^2(\Omega) \subset L^{2_*}(\Omega)$. We say that a function $u \in H^2 \cap H^1_0(\Omega)$ is a *weak* solution of (6) if

$$\int_\Omega \Delta u \Delta v - d \int_{\partial\Omega} u_\nu v_\nu = \int_\Omega |u|^{2_*-2} uv \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega)\ .$$

Also for this fourth order equation, weak solutions are in fact strong (classical) solutions, see [7, Proposition 23]. We refer to [27, 28] for a corresponding nonexistence result based on Pohozaev identity and to [6, 8] for a survey of existence results under different kinds of boundary conditions. The boundary conditions in (6) are again named after *Steklov*; they become the *Navier* boundary conditions when $d = 0$ and tend to *Dirichlet* boundary conditions as $d \to -\infty$.

Although (6) has the same variational structure as (3), it exhibits several different features. In particular, one cannot expect to go below the compactness threshold by concentrating Sobolev minimizers on the boundary since $u = 0$ on $\partial\Omega$. Therefore, the extension of Theorem 1.1 to (6) seems out of reach. We only consider the case where $\Omega = B$ so that the first two Steklov eigenvalues are $d_1 = n$ and $d_2 = n+2$, see [7] and Proposition 7 below. The eigenvalue d_1 plays the same role as the eigenvalue $\delta_0 = 0$ for (3).

When $d < n$, some results are already known. For $n \geq 5$, let

$$\sigma_n = \begin{cases} n - (n-4)(n^2-4)\frac{\Gamma(\frac{n}{2})}{2^{\frac{8}{n}+1}} \left(\frac{n\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{4}{n}} \left(\frac{\Gamma(\frac{2n}{n-4})}{\Gamma(\frac{n^2}{2(n-4)})}\right)^{1-\frac{4}{n}} & \text{if } n = 5 \text{ or } 6 \\ \frac{4(n-3)}{n-4} & \text{if } n \geq 7\ . \end{cases}$$

In particular, $\sigma_5 \approx 4.5$ and $\sigma_6 \approx 5.2$, see [4]. Concerning positive solutions, we have

Proposition 3. [8] *Assume that $\Omega = B$ (the unit ball of $\mathbb{R}^n$, $n \geq 5$).*
(i) If $d \leq 4$ or $d \geq n$, then (6) admits no positive solution.
(ii) If $d \in (\sigma_n, n)$ problem (6) admits a radial positive solution.
(iii) For every $d \in \mathbb{R}$, problem (6) admits no radial nodal solutions.

Now, for $n \geq 5$, we put

$$g(n) := \frac{n^2(n-2)}{2} \left[\frac{(n-4)(n+2)}{\Gamma(n)}\right]^{4/n} \left[\frac{\Gamma(\frac{n}{2})}{2}\right]^{1+4/n} \left[\frac{(n+4)\Gamma(\frac{2n}{n-4})\Gamma(\frac{n+4}{2(n-4)})}{\sqrt{\pi}\,\Gamma(\frac{n^2+2n}{2(n-4)})}\right]^{1-4/n} \tag{7}$$

Then, in some dimensions, we can prove existence and multiplicity results for $d \geq n$:

Theorem 1.3. *Assume that $\Omega = B$ (the unit ball of $\mathbb{R}^n$) and let $n = 5, 6, 8$. If $d \in (n+2-g(n), n+2)$, then problem (6) admits at least n pairs of nontrivial solutions.*

Figure 2 displays the plot of the function $g = g(n)$.

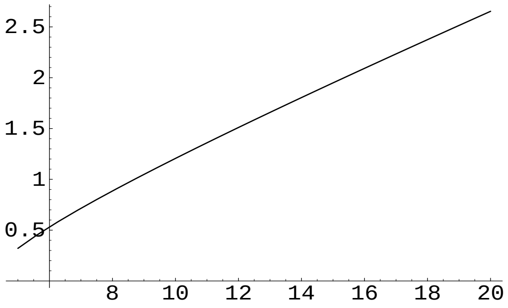

Figure 2: the map $g = g(n)$.

As we explain in Section 6, even if we do not have a complete proof, we believe that Theorem 1.3 holds for every $n \geq 5$. If this is true, since $g(n) \geq 2$ for $n \geq 16$, this means that the existence result, for n large, covers the whole range between $d_1 = n$ and $d_2 = n + 2$. Hence, in this case it is reasonable to conjecture that (6) admits solutions for any $d > \sigma_n$.

We conclude this section by pointing out that all the solutions we find (in Theorems 1.1, 1.2 and 1.3) are at low energy level, below the compactness threshold. This explains why we obtain stronger results in large space dimensions. It remains open and interesting to investigate existence results for high energy solutions and nonexistence results for low energy solutions.

2. **Some results about the eigenvalue problems.** In this section we collect some facts about the two boundary eigenvalue problems

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_\nu = \delta u & \text{on } \partial\Omega \end{cases} \tag{8}$$

and

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = \Delta u - d u_\nu = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{9}$$

Here and in the sequel, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$-norm $(1 \leq p \leq \infty)$, and we put

$$\|u\|_\partial^2 = \int_{\partial\Omega} u^2 \qquad \text{for } u \in H^1(\Omega)\ , \qquad\qquad \|u\|_{\partial_\nu}^2 = \int_{\partial\Omega} u_\nu^2 \qquad \text{for } u \in H^2 \cap H^1_0(\Omega).$$

Consider first (8); its smallest eigenvalue is $\delta_0 = 0$. This turns (8) into a Neumann problem which is solved by any constant function in Ω. Consider the space $H^1(\Omega)$ endowed with the scalar product

$$(u, v)_1 := \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv \qquad \text{for all } u, v \in H^1(\Omega) \tag{10}$$

and the induced norm

$$\|u\|^2 := \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} |u|^2 \qquad \text{for all } u \in H^1(\Omega). \tag{11}$$

We define

$$X(\Omega) := \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} u = 0 \right\} \setminus H^1_0(\Omega)$$

and

$$\delta_1 := \inf_{u \in X(\Omega)} \frac{\|\nabla u\|_2^2}{\|u\|_\partial^2}\ ,$$

so that δ_1 is the first nontrivial Steklov eigenvalue of $-\Delta$. Consider the space

$$Z_1 = \{v \in C^\infty(\overline{\Omega}) : \Delta u = 0 \text{ in } \Omega\}$$

and denote by V its completion with respect to the norm (11). Then, we have:

Proposition 4. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an open bounded domain with smooth boundary. Then:*

– Problem (8) admits infinitely many (countable) eigenvalues.

– The first eigenvalue $\delta_0 = 0$ is simple, it is associated to constant eigenfunctions and eigenfunctions of one sign necessarily correspond to δ_0.

– The set of eigenfunctions forms a complete orthonormal system in V.

– *Any eigenfunction* φ *of* (8) *corresponding to a positive eigenvalue satisfies* $\int_{\partial\Omega}\varphi = 0$.

– *The space* $H^1(\Omega)$ *endowed with* (10) *admits the following orthogonal decomposition*

$$H^1(\Omega) = V \oplus H^1_0(\Omega). \tag{12}$$

– *If* $v \in H^1(\Omega)$ *and if* $v = v_1 + v_2$ *is the corresponding orthogonal decomposition with* $v_1 \in V$ *and* $v_2 \in H^1_0(\Omega)$*, then* v_1 *and* v_2 *are weak solutions of*

$$\begin{cases} \Delta v_1 = 0 & \text{in } \Omega \\ v_1 = v & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta v_2 = \Delta v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega\ . \end{cases}$$

Proof. With the scalar product (10) we decompose the space $H^1(\Omega)$ as

$$H^1(\Omega) = H^1_0(\Omega) \oplus H^1_0(\Omega)^\perp.$$

Thus, every $v \in H^1(\Omega)$ may be written in a unique way as $v = v_1 + v_2$, where $v_2 \in H^1_0(\Omega)$ and v_1 satisfies

$$v_1 = v \quad \text{on } \partial\Omega \qquad \text{and} \qquad \int_\Omega \nabla v_1 \nabla v_0 = 0 \quad \text{for all } v_0 \in H^1_0(\Omega).$$

Hence, v_1 weakly solves the problem

$$\begin{cases} \Delta v_1 = 0 & \text{in } \Omega \\ v_1 = v & \text{on } \partial\Omega \end{cases}$$

and $v_2 = v - v_1$ weakly solves

$$\begin{cases} \Delta v_2 = \Delta v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega\ . \end{cases}$$

The kernel of the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is $H^1_0(\Omega)$ so that γ is an isomorphism between $H^1_0(\Omega)^\perp$ and $H^{1/2}(\partial\Omega)$. Therefore, the linear map

$$\begin{array}{rccc} I_1 : & H^1_0(\Omega)^\perp & \to & L^2(\partial\Omega) \\ & u & \mapsto & \gamma u \end{array}$$

is compact. Next, let $I_2 : L^2(\partial\Omega) \to (H^1_0(\Omega)^\perp)'$ be the linear continuous operator such that

$$\langle I_2 u, v\rangle = \int_{\partial\Omega} uv \quad \text{for all } u \in L^2(\partial\Omega),\ v \in H^1_0(\Omega)^\perp$$

and let $L : H^1_0(\Omega)^\perp \to (H^1_0(\Omega)^\perp)'$ be the linear continuous operator defined by:

$$\langle Lu, v\rangle = \int_\Omega \nabla u \nabla v + \int_{\partial\Omega} uv \quad \text{for all } u, v \in H^1_0(\Omega)^\perp.$$

Then, L is an isomorphism and the linear operator $K = L^{-1} I_2 I_1 : H^1_0(\Omega)^\perp \to H^1_0(\Omega)^\perp$ is a compact self-adjoint operator with strictly positive eigenvalues, $H^1_0(\Omega)^\perp$ admits an othonormal basis of eigenfunctions of K and the set of eigenvalues of K can be ordered in a strictly decreasing sequence $\{\lambda_i\}_{i\geq 1}$ which converges to zero. Thus, problem (8) admits infinitely many eigenvalues given by $\delta_i + 1 = \frac{1}{\lambda_i}$ and the eigenfunctions coincide with the eigenfunctions of K. Hence, $H^1_0(\Omega)^\perp \equiv V$.

By the divergence Theorem, we see that any solution u of (8) with $\delta > 0$ satisfies $\int_{\partial\Omega} u = 0$. To conclude the proof it remains to show that the unique eigenvalue corresponding to a positive eigenfunction is $\delta_0 = 0$. To see this, let $\delta \geq 0$ be an eigenvalue corresponding to a positive eigenfunction $\varphi > 0$ in Ω. By definition, we know that φ satisfies

$$\int_\Omega \nabla\varphi\nabla v = \delta \int_{\partial\Omega} \varphi\, v \quad \text{for all } v \in H^1(\Omega).$$

Choosing $v \equiv 1$ and recalling that $\varphi \in V$, the above identity shows that necessarily $\delta = 0$. □

For $i = 0, 1, ...$, we denote with φ_i^ℓ the eigenfunctions corresponding to δ_i, where $\ell = 1, 2, ...N_i$ and N_i is the multiplicity of δ_i. Now, by the property of the φ_i^ℓ, we have:

$$\int_\Omega \nabla\varphi_i^\ell \nabla\varphi_j^k = \delta_i \int_{\partial\Omega} \varphi_i^\ell \varphi_j^k = \delta_j \int_{\partial\Omega} \varphi_i^\ell \varphi_j^k, \quad \text{for } \ell = 1, 2, ...N_i,\ k = 1, 2, ...N_j.$$

On the other hand, by the orthogonality in the scalar product (10) we also have

$$\int_\Omega \nabla\varphi_i^\ell \nabla\varphi_j^k = -\int_{\partial\Omega} \varphi_i^\ell \varphi_j^k,$$

so that

$$\int_\Omega \nabla\varphi_i^\ell \nabla\varphi_j^k = \int_{\partial\Omega} \varphi_i^\ell \varphi_j^k = 0, \qquad \text{for all } i \neq j. \tag{13}$$

A similar argument yields

$$\int_\Omega \nabla\varphi_i^\ell \nabla\varphi_i^k = \int_{\partial\Omega} \varphi_i^\ell \varphi_i^k = 0, \qquad \text{for all } \ell \neq k. \tag{14}$$

It is readily verified that the same relations hold by replacing φ_i^ℓ with any $u_0 \in H_0^1(\Omega)$, namely

$$\int_\Omega \nabla u_0 \nabla\varphi_i^k = \int_{\partial\Omega} u_0 \varphi_i^k = 0, \qquad \text{for all } i \text{ and all } u_0 \in H_0^1(\Omega). \tag{15}$$

This means that the subspaces in the direct sum (12) are also orthogonal with respect to the inner products associated to the Dirichlet norm and to the L^2 norm on the boundary $\partial\Omega$.

When $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (8). To this end, consider the spaces of harmonic polynomials [4, Sect. 9.3-9.4]:

$$\mathcal{D}_k := \{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\ P \text{ is homogeneous polynomial of degree } k\}.$$

Also, denote by μ_k the dimension of $\mathcal{D}_k$ so that [4, p.450]

$$\mu_k = \frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}.$$

Then, from [9, p.160] we easily infer

Proposition 5. [9]

If $n \geq 2$ and $\Omega = B$, then for all $k = 0, 1, 2, ...$:
(i) the eigenvalues of (8) are $\delta_k = k$;
(ii) the multiplicity N_k of δ_k equals μ_k;
(iii) any $\varphi_k^\ell \in \mathcal{D}_k$, with $\ell = 1, 2, ..., N_k$, is an eigenfunction corresponding to δ_k.

We now turn to the fourth order problem (9). Consider the space $H^2 \cap H_0^1(\Omega)$ endowed with the scalar product

$$(u, v)_2 := \int_\Omega \Delta u \Delta v \qquad \text{for all } u, v \in H^2 \cap H_0^1(\Omega) \tag{16}$$

and the induced norm

$$|||u|||^2 := \int_\Omega |\Delta u|^2 \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega). \tag{17}$$

Let $\mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$. The smallest (positive) eigenvalue d_1 of (9) is characterized variationally as

$$d_1 \; := \; \inf_{u \in \mathcal{H}(\Omega)} \frac{\|\Delta u\|_2^2}{\|u\|^2_{\partial_\nu}}.$$

Hence, d_1 is the largest constant satisfying

$$\|\Delta u\|_2^2 \geq d_1 \|u\|^2_{\partial_\nu} \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega)$$

and $d_1^{-1/2}$ is the norm of the compact linear operator $H^2 \cap H^1_0(\Omega) \to L^2(\partial\Omega)$, $u \mapsto u_\nu$.

Consider the space

$$Z_2 = \left\{ v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0, \; u = 0 \text{ on } \partial\Omega \right\}$$

and denote by W its completion with respect to the norm (17). Then, we have

Proposition 6. [17]

Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with smooth boundary. Then:

– Problem (9) admits infinitely many (countable) eigenvalues.

– The first eigenvalue d_1 is simple and eigenfunctions of one sign necessarily correspond to d_1.

– The set of eigenfunctions forms a complete orthonormal system in W.

– The space $H^2 \cap H^1_0(\Omega)$ endowed with (16) admits the following orthogonal decomposition

$$H^2 \cap H^1_0(\Omega) = W \oplus H^2_0(\Omega).$$

– If $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the corresponding orthogonal decomposition with $v_1 \in W$ and $v_2 \in H^2_0(\Omega)$, then v_1 and v_2 are weak solutions of

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \\ (v_2)_\nu = 0 & \text{on } \partial\Omega \; . \end{cases}$$

Again, when $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (9):

Proposition 7. [17]

If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, \ldots$:
(i) the eigenvalues of (9) are $d_k = n + 2(k-1)$;
(ii) the multiplicity N_k of d_k equals μ_{k-1};
(iii) for all $\psi^\ell_k \in \mathcal{D}_{k-1}$, with $\ell = 1, 2, \ldots, N_k$, the function $\phi^\ell_k(x) := (1 - |x|^2)\psi^\ell_k(x)$ is an eigenfunction corresponding to d_k.

Let us mention that the fourth order Steklov eigenvalue problem (9) was first studied in the two dimensional case [21, 24] where only partial results about the first eigenvalue were obtained.

3. The Palais-Smale condition.

Let

$$S_2 = \min_{u\in\mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} = \pi n(n-2)\left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2/n}, \tag{18}$$

where for the last equality we refer to [31]. In order to obtain some compactness for the second order problem (3), a crucial role is played by an inequality due to Li-Zhu [22]: there exists $M = M(\Omega) > 0$ such that

$$\frac{S_2}{2^{2/n}}\|u\|_{2^*}^2 \le \|\nabla u\|_2^2 + M\|u\|_\partial^2 \qquad \text{for all } u\in H^1(\Omega). \tag{19}$$

Consider the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\delta}{2}\int_{\partial\Omega} u^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} \tag{20}$$

whose critical points are weak solutions of (3). We prove

Lemma 3.1. *The functional I satisfies the Palais-Smale condition at levels $c \in (-\infty, \frac{S_2^{n/2}}{2n})$, that is, if $\{u_m\}_{m\ge 0}\subset H^1(\Omega)$ is such that*

$$I(u_m)\to c < \frac{S_2^{n/2}}{2n}, \quad dI(u_m)\to 0 \quad \textit{in} \quad (H^1(\Omega))', \tag{21}$$

then there exists $u\in H^1(\Omega)$ such that $u_m\to u$ in $H^1(\Omega)$, up to a subsequence.

Proof. To deduce that $\{u_m\}_{m\ge 0}$ is bounded in $H^1(\Omega)$ we follow [29, Theorem 4.12]. Let $\{\delta_j\}_{j\ge 0}$ be the set of Steklov eigenvalues of $-\Delta$ and denote with M_j the eigenspace associated to δ_j. If $\delta = \delta_k$, for some $k\ge 0$, we define:

$$H_+ := \overline{\bigoplus_{j\ge k+1} M_j}\bigoplus H_0^1(\Omega), \quad H_0 := M_k \quad \text{and} \quad H_- := \bigoplus_{j\le k-1} M_j$$

and, in view of Proposition 4, we have

$$H^1(\Omega) = H_+\oplus H_0\oplus H_-.$$

Thus we may decompose $u_m = u_m^+ + u_m^0 + u_m^-$, where $u_m^+\in H_+$, $u_m^0\in H_0$ and $u_m^-\in H_-$. If $\delta_k<\delta<\delta_{k+1}$, for $k\ge 0$, we just have the two spaces H_+ and H_- but the decomposition works similarly. By (21) and arguing as in [29], one can prove that each of the components of u_m, and in turn u_m, is bounded in $H^1(\Omega)$. By this we conclude that (up to a subsequence) there exists $u\in H^1(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_m\to u \quad \text{a.e. in } \Omega. \tag{22}$$

Hence, by compactness of the map $H^1(\Omega)\to L^2(\partial\Omega)$ defined by $u\mapsto u|_{\partial\Omega}$, we have:

$$u_m\,|_{\partial\Omega}\to u\,|_{\partial\Omega} \quad \text{in} \quad L^2(\partial\Omega). \tag{23}$$

We apply (19) to the function u_m-u and, in view of (23), we get

$$\frac{S_2}{2^{2/n}}\|u_m-u\|_{2^*}^2 \le \|\nabla(u_m-u)\|_2^2 + o(1). \tag{24}$$

On the other hand, by the Brezis-Lieb Lemma [12], we know that

$$\|u_m\|_{2^*}^{2^*} - \|u\|_{2^*}^{2^*} = \|u_m-u\|_{2^*}^{2^*} + o(1). \tag{25}$$

Exploiting (21), (22), (23) and (25) we have

$$\begin{aligned}
o(1) &= \langle dI(u_m), u_m - u\rangle \\
&= \int_\Omega |\nabla u_m|^2 - \int_\Omega \nabla u_m \cdot \nabla u - \delta \int_{\partial\Omega} u_m(u_m - u) - \int_\Omega |u_m|^{2^*-2} u_m (u_m - u) \\
&= \int_\Omega (|\nabla u_m|^2 - 2\nabla u_m \cdot \nabla u + |\nabla u|^2) - \int_\Omega |u_m|^{2^*} + \int_\Omega |u|^{2^*} + o(1) \\
&= \int_\Omega |\nabla (u_m - u)|^2 - \int_\Omega |u_m - u|^{2^*} + o(1),
\end{aligned}$$

so that

$$\|\nabla(u_m - u)\|_2^2 = \|u_m - u\|_{2^*}^{2^*} + o(1). \tag{26}$$

By (21) we also get that

$$o(1) = \langle dI(u_m), u_m\rangle = \|\nabla u_m\|_2^2 - \delta \|u_m\|_\partial^2 - \|u_m\|_{2^*}^{2^*},$$

that is,

$$\|u_m\|_{2^*}^{2^*} = \|\nabla u_m\|_2^2 - \delta \|u_m\|_\partial^2 + o(1). \tag{27}$$

Inserting (27) into (21) we obtain

$$\frac{1}{n}\|\nabla u_m\|_2^2 - \frac{\delta}{n}\|u_m\|_\partial^2 = c + o(1)$$

and therefore

$$\|\nabla u\|_2^2 - \delta \|u\|_\partial^2 + \|\nabla(u_m - u)\|_2^2 = nc + o(1). \tag{28}$$

On the other hand, exploiting the convergence $\langle dI(u_m), v\rangle \to \langle dI(u), v\rangle$ for any fixed $v \in H^1(\Omega)$, we deduce that u solves (3) (that is, $dI(u) = 0$) so that

$$\|\nabla u\|_2^2 - \delta \|u\|_\partial^2 = \|u\|_{2^*}^{2^*} \geq 0.$$

The last inequality combined with (28) gives

$$\|\nabla(u_m - u)\|_2^2 \leq nc + o(1) < \frac{S_2^{n/2}}{2} + o(1). \tag{29}$$

Furthermore (24) and (26) give

$$\|\nabla(u_m - u)\|_2^{2-\frac{4}{n}} \left(\frac{S_2}{2^{2/n}} - \|\nabla(u_m - u)\|_2^{\frac{4}{n}} \right) \leq o(1).$$

This, combined with (29), shows that $\|\nabla(u_m - u)\|_2 = o(1)$. And this, together with (23), proves that $u_m \to u$ in $H^1(\Omega)$. □

We now turn to the fourth order problem. Let

$$S_4 = \min_{u \in \mathcal{D}^{2,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2} = \pi^2 (n+2)n(n-2)(n-4) \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{4/n}, \tag{30}$$

(see again [31]) and consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{d}{2}\int_{\partial\Omega} u_\nu^2 - \frac{1}{2_*}\int_\Omega |u|^{2_*} \tag{31}$$

whose critical points are weak solutions of (6). We have

Lemma 3.2. *The functional J satisfies the Palais-Smale condition at levels $c \in (-\infty, \frac{2S_4^{n/4}}{n})$, that is, if $\{u_m\}_{m\geq 0} \subset H^2 \cap H_0^1(\Omega)$ is such that*

$$J(u_m) \to c < \frac{2}{n} S_4^{n/4}, \quad dJ(u_m) \to 0 \quad \text{in} \quad (H^2 \cap H_0^1(\Omega))', \tag{32}$$

then there exists $u \in H^2 \cap H_0^1(\Omega)$ such that $u_m \to u$ in $H^2 \cap H_0^1(\Omega)$, up to a subsequence.

Proof. The first step consists in showing that $\{u_m\}_{m\geq 0}$ is bounded in $H^2 \cap H_0^1(\Omega)$. As in Lemma 3.1, this follows by arguing as in Theorem 4.12 in [29], suitably adapted to this case. For the rest of the proof one can follow the same lines as the proof of Lemma 3.1 except that, now, one has to exploit the compactness of the linear map $H^2 \cap H_0^1(\Omega) \ni u \mapsto u_\nu|_{\partial\Omega} \in L^2(\partial\Omega)$ and the inequality (19) must be replaced by the Sobolev inequality: $S_4\|u\|_{2_*}^2 \leq \|\Delta u\|_2^2$, for all $u \in H^2 \cap H_0^1(\Omega)$. ☐

4. **Proof of Theorem 1.1.** We prove Theorem 1.1 by showing that there exists a critical level for the functional (20) below the compactness threshold found in Lemma 3.1. In order to do this, we need some asymptotic estimates and a suitable linking geometry.

4.1. **Some asymptotic estimates.** In this section, we prove some asymptotic estimates of the norms of the Sobolev minimizers which concentrate on $\partial\Omega$. We take into account the effect of the curvature of the boundary $\partial\Omega$, following an idea from [1]. Since Ω is smooth and bounded, there exists $\overline{x} \in \partial\Omega$ such that in a neighborhood of $\overline{x}$, Ω lies on one side of the tangent hyperplane at $\overline{x}$ and the mean curvature with respect to the unit outward normal at $\overline{x}$ is positive. Furthermore, there exists a ball of radius $R_0 > 0$ such that $\Omega \subset B_{R_0}$. With a change of coordinates, we may assume that $\overline{x} = 0$ (the origin), that the tangent hyperplane coincides with $x_n = 0$ and that Ω lies in $\mathbb{R}_+^n = \{x = (x', x_n);\, x_n > 0\}$. More precisely, there exists $R > 0$ and a smooth function $\rho : \omega \to \mathbb{R}_+$ (where $\omega = \{x' \in \mathbb{R}^{n-1};\, |x'| < R\}$) such that

$$(x', x_n) \in \Omega \cap B_R \Leftrightarrow x_n > \rho(x')\ , \qquad (x', x_n) \in \partial\Omega \cap B_R \Leftrightarrow x_n = \rho(x').$$

Furthermore, since the curvature is positive at 0, there exist λ_i $(i = 1, ..., n-1)$ such that

$$\Lambda := \sum_{i=1}^{n-1} \lambda_i > 0 \qquad \text{and} \qquad \rho(x') = \sum_{i=1}^{n-1} \lambda_i x_i^2 + O(|x'|^3) \quad \text{as } x' \to 0. \tag{33}$$

Let $\Sigma := \{x \in B_R;\, 0 < x_n < \rho(x')\}$. Finally we set

$$U_\epsilon(x) := \frac{\epsilon^{\frac{n-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{n-2}{2}}} \tag{34}$$

and

$$K' := \int_{\mathbb{R}^n} |\nabla U_\epsilon(x)|^2\, dx, \quad K'' := \int_{\mathbb{R}^n} |U_\epsilon(x)|^{2^*}\, dx.$$

Recall that $S_2 = K'/(K'')^{2/2^*}$ (see (18)). Now let $\omega_n := |\partial B| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, we prove that, as $\varepsilon \to 0$, the following asymptotic estimates hold:

$$\int_\Omega |\nabla U_\epsilon(x)|^2\, dx = \frac{K'}{2} - \Lambda \frac{\omega_{n-1}(n-2)^2}{2(n-1)} \begin{cases} \epsilon\,|\log\epsilon| + o(\epsilon\,|\log\epsilon|) & \text{if} n = 3, \\ \frac{\Gamma((n+3)/2)\Gamma((n-3)/2)}{\Gamma(n)}\,\epsilon + o(\epsilon) & \text{if} n \geq 4, \end{cases} \tag{35}$$

$$\int_\Omega |\nabla U_\epsilon(x)|\,dx = O(\epsilon^{\frac{n-2}{2}}) \qquad \text{for any } n \geq 3, \tag{36}$$

$$\int_\Omega |U_\epsilon(x)|^{2^*}\,dx = \frac{K''}{2} - \Lambda\,\frac{\omega_{n-1}}{2(n-1)} \begin{cases} O(\epsilon) & \text{if} \quad n=3, \\ \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n)}\,\epsilon + o(\epsilon) & \text{if} \quad n \geq 4, \end{cases} \tag{37}$$

$$\int_{\partial\Omega} U_\epsilon(x)\,d\sigma = O(\epsilon^{\frac{n-2}{2}}) \qquad \text{for any } n \geq 3, \tag{38}$$

$$\int_{\partial\Omega} |U_\epsilon(x)|^2\,d\sigma = b(n) \begin{cases} \epsilon\,|\log\epsilon| + o(\epsilon\,|\log\epsilon|) & \text{if} \quad n=3, \\ \epsilon + o(\epsilon) & \text{if} \quad n \geq 4, \end{cases} \tag{39}$$

where $b(n)$ is defined as in [1, (3.9)]:

$$b(n) := \begin{cases} \omega_2/2 & \text{if} \quad n=3, \\ \omega_{n-1}\displaystyle\int_0^{+\infty} \frac{r^{n-2}}{(1+r^2)^{n-2}}\,dr & \text{if} \quad n \geq 4 \end{cases}$$

Proof of (35). A direct computation shows that

$$\begin{aligned} \int_\Omega |\nabla U_\epsilon(x)|^2\,dx &= \frac{1}{2}\int_{B_R} |\nabla U_\epsilon(x)|^2\,dx - \int_\Sigma |\nabla U_\epsilon(x)|^2\,dx + \int_{\Omega\setminus B_R} |\nabla U_\epsilon(x)|^2\,dx \\ &= \frac{K'}{2} - O(\epsilon^{n-2}) - \int_\Sigma |\nabla U_\epsilon(x)|^2\,dx + \int_{\Omega\setminus B_R} |\nabla U_\epsilon(x)|^2\,dx. \end{aligned}$$

Furthermore, since $\nabla U_\epsilon(x) = -\frac{(n-2)\,x\,\epsilon^{\frac{n-2}{2}}}{(\epsilon^2+|x|^2)^{\frac{n}{2}}}$,

$$\int_{\Omega\setminus B_R} |\nabla U_\epsilon(x)|^2\,dx \leq \int_{B_{R_0}\setminus B_R} |\nabla U_\epsilon(x)|^2\,dx = O(\epsilon^{n-2}),$$

while we may also exploit [1, (2.17)] (with minor changes) to deduce

$$\int_\Sigma |\nabla U_\epsilon(x)|^2\,dx = \Lambda\,\frac{\omega_{n-1}(n-2)^2}{2(n-1)} \begin{cases} \epsilon\,|\log\epsilon| + o(\epsilon|\log\epsilon|) & \text{if} \quad n=3, \\ \frac{\Gamma((n+3)/2)\Gamma((n-3)/2)}{\Gamma(n)}\,\epsilon + o(\epsilon) & \text{if} \quad n \geq 4, \end{cases}$$

and (35) follows.

Proof of (36). In view of the explicit form of ∇U_ϵ (see above), we have

$$\int_\Omega |\nabla U_\epsilon(x)|\,dx = c\epsilon^{\frac{n-2}{2}} \int_\Omega \frac{|x|}{(\epsilon^2+|x|^2)^{\frac{n}{2}}}\,dx \leq c\epsilon^{\frac{n-2}{2}} \int_\Omega \frac{dx}{|x|^{n-1}} = c\epsilon^{\frac{n-2}{2}}$$

and (36) is proved.

Proof of (37). We have

$$\begin{aligned} \int_\Omega |U_\epsilon(x)|^{2^*}\,dx =& \frac{1}{2}\int_{B_R} |U_\epsilon(x)|^{2^*}\,dx - \int_\Sigma |U_\epsilon(x)|^{2^*}\,dx + \int_{\Omega\setminus B_R} |U_\epsilon(x)|^{2^*}\,dx \\ =& \frac{K''}{2} - O(\epsilon^n) - \int_\Sigma |U_\epsilon(x)|^{2^*}\,dx + \int_{\Omega\setminus B_R} |U_\epsilon(x)|^{2^*}\,dx. \end{aligned}$$

Furthermore

$$\int_{\Omega\setminus B_R} |U_\epsilon(x)|^{2^*}\,dx \leq \int_{B_{R_0}\setminus B_R} |U_\epsilon(x)|^{2^*}\,dx = O(\epsilon^n)$$

and the integral over Σ can be estimated as in [1, (2.18)]:

$$\int_{\Sigma} |U_\epsilon(x)|^{2^*}\, dx = \Lambda \frac{\omega_{n-1}}{2(n-1)} \begin{cases} O(\epsilon) & \text{if} \quad n=3, \\ \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n)}\,\epsilon + o(\epsilon) & \text{if} \quad n \geq 4. \end{cases}$$

Proof of (38). This follows as for (36), namely

$$\int_{\partial\Omega} U_\epsilon(x)\, d\sigma \leq c\epsilon^{\frac{n-2}{2}} \int_{\partial\Omega} \frac{dx}{|x|^{n-2}} = c\epsilon^{\frac{n-2}{2}}.$$

Proof of (39). We have

$$\begin{aligned} \int_{\partial\Omega} |U_\epsilon(x)|^2\, d\sigma &= \int_{\partial\Omega\cap B_R} |U_\epsilon(x)|^2\, d\sigma + \int_{\partial\Omega\setminus B_R} |U_\epsilon(x)|^2\, d\sigma \\ &= \int_{\partial\Omega\cap B_R} |U_\epsilon(x)|^2\, d\sigma + O(\epsilon^{n-2}). \end{aligned}$$

The first term of the above sum can be estimated as in [1, (3.10)]:

$$\int_{\partial\Omega\cap B_R} |U_\epsilon(x)|^2\, d\sigma = b(n) \begin{cases} \epsilon\,|\log\epsilon| + o(\epsilon|\log\epsilon|) & \text{if} \quad n=3, \\ \epsilon + o(\epsilon) & \text{if} \quad n \geq 4. \end{cases}$$

We conclude with the following estimates:

$$\begin{aligned} I_\alpha \equiv \int_\Omega |U_\epsilon(x)|^\alpha dx &= \int_{\Omega\cap B_R} |U_\epsilon(y)|^\alpha dy + \int_{\Omega\setminus B_R} |U_\epsilon(y)|^\alpha dy \\ &\leq C\epsilon^{\alpha\frac{n-2}{2}} \int_{B_R} \frac{dy}{(\epsilon^2+|y|^2)^{\alpha\frac{n-2}{2}}} + O(\epsilon^{\alpha\frac{n-2}{2}}) = \qquad (y=\epsilon z,\ |z|=\rho) \\ &= C\epsilon^{n-\alpha\frac{n-2}{2}} \int_0^{R/\epsilon} \frac{\rho^{n-1}}{(1+\rho^2)^{\alpha\frac{n-2}{2}}} d\rho + O(\epsilon^{\alpha\frac{n-2}{2}}) \\ &\leq C\epsilon^{n-\alpha\frac{n-2}{2}} \Big(C_0 + \int_1^{R/\epsilon} \rho^{n-1-\alpha(n-2)} d\rho \Big) + O(\epsilon^{\alpha\frac{n-2}{2}}) \\ &\leq \begin{cases} C_1\epsilon^{n-\alpha\frac{n-2}{2}} + C_2\epsilon^{\alpha\frac{n-2}{2}} & \text{for} \quad \alpha \neq \frac{n}{n-2} \\ \epsilon^{n/2}(C_1 + C_2|\ln\epsilon|) & \text{for} \quad \alpha = \frac{n}{n-2}. \end{cases} \end{aligned}$$

In particular, we get

$$I_{(2^*-1)} = I_{\frac{n+2}{n-2}} = O(\epsilon^{(n-2)/2}), \qquad I_1 = O(\epsilon^{(n-2)/2}). \tag{40}$$

$$I_{(2^*-2)} = \begin{cases} I_4 = O(\epsilon) & \text{if} \quad n=3 \\ I_2 = O(\epsilon^2 \ln\epsilon) & \text{if} \quad n=4 \\ I_{\frac{4}{n-2}} = O(\epsilon^2) & \text{if} \quad n \geq 5. \end{cases} \tag{41}$$

4.2. **Linking argument.** Assume first that $\delta_k < \delta < \delta_{k+1}$, for some $k \geq 0$, and consider the orthogonal decomposition of $H^1(\Omega)$ relative to the scalar product (10):

$$H^1(\Omega) = H_{k_-} \oplus H_{k_+}, \tag{42}$$

where H_{k_-} is the subspace spanned by an orthonormal (with respect to (10)-(11)) set of eigenfunctions φ_i^ℓ, $\ell = 1, 2, ..., N_i$, $i = 0, 1, ..., k$, with eigenvalues $0 = \delta_0 < \delta_1 < ... < \delta_k$, see Proposition 4. Let U_ϵ be as in (34), and define

$$\bar{U}_\epsilon = U_\epsilon - z_\epsilon$$

where $z_\epsilon = P_{H_{k\_}} U_\epsilon$ is the projection of the function U_ϵ on the subspace $H_{k\_}$. Then,

$$z_\epsilon = \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (\varphi_i^\ell, U_\epsilon)_1 \ \varphi_i^\ell,$$

and, by the property of the eigenfunctions φ_i^ℓ, we get

$$\begin{aligned} |(\varphi_i^\ell, U_\epsilon)_1| &= \Big| \int_\Omega \nabla\varphi_i^\ell \nabla U_\epsilon \, dx + \int_{\partial\Omega} \varphi_i^\ell U_\epsilon \, d\sigma \Big| \le (\delta_i + 1) \int_{\partial\Omega} |\varphi_i^\ell U_\epsilon| \, d\sigma \\ &\le C_i \int_{\partial\Omega} U_\epsilon \, d\sigma = O(\epsilon^{\frac{n-2}{2}}), \end{aligned} \tag{43}$$

where the last equality follows by (38). Therefore, we also have

$$\|z_\epsilon\|_\infty = O(\epsilon^{\frac{n-2}{2}}). \tag{44}$$

In turn, by (40) and (44) we obtain

$$\begin{aligned} \int_\Omega |\bar{U}_\epsilon|^{2^*} dx &= \int_\Omega |U_\epsilon|^{2^*} dx - 2^* \int_0^1 dt \int_\Omega |U_\epsilon - t z_\epsilon|^{2^*-2} (U_\epsilon - t z_\epsilon) z_\epsilon \, dx \\ &= \int_\Omega |U_\epsilon|^{2^*} dx + O(\epsilon^{n-2}), \end{aligned}$$

which, together with (37), gives

$$\int_\Omega |\bar{U}_\epsilon(x)|^{2^*} \, dx = \frac{K''}{2} - \Lambda \, \frac{\omega_{n-1}}{2(n-1)} \begin{cases} O(\epsilon) & \text{if} \quad n = 3, \\ \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n)} \, \epsilon + o(\epsilon) & \text{if} \quad n \ge 4. \end{cases} \tag{45}$$

Moreover, by (13) we have

$$\begin{aligned} \int_\Omega |\nabla \bar{U}_\epsilon|^2 \, dx = \int_\Omega |\nabla U_\epsilon|^2 \, dx &- 2 \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (\varphi_i^\ell, U_\epsilon)_1 \int_\Omega \nabla U_\epsilon \nabla \varphi_i^\ell \, dx \\ &+ \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (\varphi_i^\ell, U_\epsilon)_1^2 \int_\Omega |\nabla \varphi_i^\ell|^2 \, dx \ . \end{aligned}$$

Further, by Hölder inequality and (36) we get

$$\Big| \int_\Omega \nabla U_\epsilon \nabla \varphi_i^\ell \, dx \Big| \le \|\nabla U_\epsilon\|_1 \cdot \|\nabla \varphi_i^\ell\|_\infty = O(\epsilon^{\frac{n-2}{2}}) \ .$$

By combining this with (35) and (43), we infer

$$\int_\Omega |\nabla \bar{U}_\epsilon|^2 \, dx = \frac{K'}{2} - \Lambda \, \frac{\omega_{n-1}(n-2)^2}{2(n-1)} \begin{cases} \epsilon \, |\log \epsilon| + o(\epsilon \, |\log \epsilon|) & \text{if} \quad n = 3, \\ \frac{\Gamma((n+3)/2)\Gamma((n-3)/2)}{\Gamma(n)} \, \epsilon + o(\epsilon) & \text{if} \quad n \ge 4. \end{cases} \tag{46}$$

Finally, by using again (13) we obtain

$$\begin{aligned} \int_{\partial\Omega} |\bar{U}_\epsilon|^2 \, d\sigma = \int_{\partial\Omega} |U_\epsilon|^2 \, d\sigma &- 2 \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (\varphi_i^\ell, U_\epsilon)_1 \int_{\partial\Omega} U_\epsilon \varphi_i^\ell \, d\sigma \\ &+ \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (\varphi_i^\ell, U_\epsilon)_1^2 \int_{\partial\Omega} (\varphi_i^\ell)^2 \, d\sigma \ . \end{aligned}$$

Arguing as above and using (38), (39) and (43), we infer

$$\int_{\partial\Omega} |\bar{U}_\epsilon|^2 \, d\sigma = b(n) \begin{cases} \epsilon \, |\log \epsilon| + o(\epsilon \, |\log \epsilon|) & \text{if} \quad n = 3, \\ \epsilon + o(\epsilon) & \text{if} \quad n \geq 4. \end{cases} \tag{47}$$

Let I be the functional defined in (20) and let

$$\Sigma_k = \Big\{ u \in H_{k_+} \; : \; \|u\| = \rho \Big\},$$

where $\rho > 0$ is chosen small so that one has $\inf_{v \in \Sigma_k} I(v) = \alpha_k > 0$. Now let

$$Q_k = \Big\{ s\bar{U}_\epsilon + u_-, \quad 0 \leq s \leq R_1, \quad u_- \in H_{k_-} \quad \|u_-\| \leq R_2 \Big\},$$

where $R_1 > \rho$ and $R_2 > 0$ are independent from ϵ. More precisely, R_1 is chosen sufficiently large to satisfy $I(R_1 \bar{U}_\epsilon) < 0$ and so that Σ_k and ∂Q_k link, see [30, Example 8.3]. The choice of R_2 is explained below.

By writing $u_- = \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} c_i^\ell \varphi_i^\ell$ and using (13), we have

$$\begin{aligned} I(s\bar{U}_\epsilon + u_-) = & \frac{s^2}{2} \Big[\int_\Omega |\nabla \bar{U}_\epsilon|^2 dx - \delta \int_{\partial\Omega} |\bar{U}_\epsilon|^2 d\sigma \Big] + \frac{1}{2} \int_\Omega |\nabla u_-|^2 \, dx \\ & - \frac{\delta}{2} \int_{\partial\Omega} |u_-|^2 \, d\sigma - \frac{1}{2^*} \int_\Omega |s\bar{U}_\epsilon + u_-|^{2^*} dx = \frac{s^2}{2} \Big[\int_\Omega |\nabla \bar{U}_\epsilon|^2 dx - \delta \int_{\partial\Omega} |\bar{U}_\epsilon|^2 d\sigma \Big] \\ & - \sum_{i=0}^{k} \frac{\delta - \delta_i}{2} \sum_{\ell=1}^{N_i} (c_i^\ell)^2 \int_{\partial\Omega} |\varphi_i^\ell|^2 \, d\sigma - \frac{1}{2^*} \int_\Omega |s\bar{U}_\epsilon + u_-|^{2^*} dx \\ \leq & \frac{s^2}{2} \Big[\int_\Omega |\nabla \bar{U}_\epsilon|^2 dx - \delta \int_{\partial\Omega} |\bar{U}_\epsilon|^2 d\sigma \Big] - \frac{s^{2^*}}{2^*} \int_\Omega |\bar{U}_\epsilon|^{2^*} dx - D(\delta - \delta_k) \, c^2 \\ & - \int_0^1 dt \int_\Omega u_- |s\bar{U}_\epsilon + tu_-|^{2^*-2} (s\bar{U}_\epsilon + tu_-) dx, \end{aligned} \tag{48}$$

where

$$D = \frac{1}{2} \min_{i,\,\ell} \|\varphi_i^\ell\|_\partial^2 \quad \text{and} \quad c^2 := \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} (c_i^\ell)^2, \quad \text{with} \quad c > 0.$$

By using the inequality $(\alpha+\beta+\gamma)^{2^*-2} \leq K(\alpha^{2^*-2} + \beta^{2^*-2} + \gamma^{2^*-2})$, for $\alpha, \beta, \gamma \geq 0$, we estimate :

$$\begin{aligned} & \Big| \int_\Omega u_- |s\bar{U}_\epsilon + tu_-|^{2^*-2} \bar{U}_\epsilon dx \Big| \\ \leq & \int_\Omega |u_-| |sU_\epsilon - sz_\epsilon + tu_-|^{2^*-2} U_\epsilon dx + \int_\Omega |u_-| |sU_\epsilon - sz_\epsilon + tu_-|^{2^*-2} |z_\epsilon| dx \\ \leq & K \Big\{ s^{2^*-2} \Big[\int_\Omega |u_-| |U_\epsilon|^{2^*-1} dx + \int_\Omega |u_-| |z_\epsilon|^{2^*-2} U_\epsilon \, dx + \int_\Omega |u_-| |z_\epsilon| |U_\epsilon|^{2^*-2} dx \\ & + \int_\Omega |z_\epsilon|^{2^*-1} |u_-| \, dx \Big] + t^{2^*-2} \Big[\int_\Omega |u_-|^{2^*-1} U_\epsilon \, dx + \int_\Omega |u_-|^{2^*-1} |z_\epsilon| \, dx \Big] \Big\}. \end{aligned}$$

Then, by (40), (41), (44), by the bound $\|u_-\|_\infty \leq c \sum_{i=0}^{k} \sum_{\ell=1}^{N_i} \|\varphi_i^\ell\|_\infty$ and recalling that $s \leq R_1$, we can estimate the last term in (48):

$$\Big| \int_0^1 dt \int_\Omega u_- |s\bar{U}_\epsilon + tu_-|^{2^*-2} s\bar{U}_\epsilon \, dx \Big| \leq \Psi(\epsilon) \big(c + c^{2^*-1} \big),$$

where $\Psi(\epsilon) = O(\epsilon^{(n-2)/2})$. Thus we can write:

$$I(s\bar{U}_\epsilon+u_-) \le I(s\bar{U}_\epsilon)-\int_0^1 t\,dt\int_\Omega u_-^2|s\bar{U}_\epsilon+tu_-|^{2^*-2}dx+\Psi(\epsilon)(c+c^{2^*-1})-D(\delta-\delta_k)\,c^2\,, \tag{49}$$

with $\max_{c\ge0}[\Psi(\epsilon)(c+c^{2^*-1})-D(\delta-\delta_k)\,c^2] = O(\epsilon^{n-2})$. Taking into account that

$$\max_{t\ge0}\big(at-bt^{\frac{n}{n-2}}\big)=\Big(\frac{n-2}{n}\Big)^{\frac{n-2}{2}}\frac{2}{n}\frac{a^{n/2}}{b^{(n-2)/2}},\qquad\text{for all}\quad a,b>0, \tag{50}$$

we finally obtain from (49)

$$I(s\bar{U}_\epsilon+u_-) \le \frac{1}{n}\left(\frac{\|\nabla\bar{U}_\epsilon\|_2^2-\delta\|\bar{U}_\epsilon\|_\partial^2}{\|\bar{U}_\epsilon\|_{2^*}^2}\right)^{\frac{n}{2}}-\int_0^1 t\,dt\int_\Omega u_-^2|s\bar{U}_\epsilon+tu_-|^{2^*-2}dx+O(\epsilon^{n-2}) \tag{51}$$

Now we can fix $R_2 \gg R_1$ such that if $\|u_-\| = R_2$ then

$$\frac{1}{n}\left(\frac{\|\nabla\bar{U}_\epsilon\|_2^2-\delta\|\bar{U}_\epsilon\|_\partial^2}{\|\bar{U}_\epsilon\|_{2^*}^2}\right)^{n/2}-\int_0^1 t\,dt\int_\Omega u_-^2|s\bar{U}_\epsilon+tu_-|^{2^*-2}dx<0\quad\text{for all } s\in[0,R_1]$$

uniformly with respect to ϵ. Subsequently, we take ϵ sufficiently small (say $\epsilon < \overline{\epsilon}$) so that

$$I(s\bar{U}_\epsilon \pm u_-) \le 0 \qquad \text{for } \|u_-\| = R_2 \quad \text{and for all } s\in[0,R_1].$$

Moreover, by the definition of I and H_{k_-} we have $I(u_-) \le 0$ for every $u_- \in H_{k_-}$ whereas by definition of R_1 we have that $I(R_1\bar{U}_\epsilon) < 0$; this, combined with (49), allows to conclude that $I(R_1\bar{U}_\epsilon + u_-) \le 0$ for every $\|u_-\| \le R_2$, provided ϵ is sufficiently small. We have so proved that

$$\alpha_k = \inf_{v\in\Sigma_k} I(v) > \sup_{v\in\partial Q_k} I(v) = 0.$$

Now, by defining

$$\Gamma_k = \{h\in C^0(H^1,H^1)\,;\, h|_{\partial Q_k} = I\},$$

it follows, from [30, Theorem 8.4], that the number

$$\beta_k = \inf_{h\in\Gamma_k}\sup_{v\in Q_k} I(h(v))$$

is a critical value of I, whenever $\beta_k < S_2^{n/2}/2n$. Since $\beta_k \le \sup_{v\in Q_k} I(v) \equiv \overline{\beta}_k$, it is sufficient to prove that $\overline{\beta}_k < S_2^{n/2}/2n$. To this end, we remark that the estimates (45)-(46)-(47) and (51) yield

$$\begin{aligned}I(s\bar{U}_\epsilon+u_-) &\le \frac{1}{n}\left(\frac{\|\nabla\bar{U}_\epsilon\|_2^2-\delta\|\bar{U}_\epsilon\|_\partial^2}{\|\bar{U}_\epsilon\|_{2^*}^2}\right)^{n/2}+O(\epsilon^{n-2})\\ &\le \frac{1}{n}\left[\frac{S_2}{2^{2/n}}-\left\{\begin{array}{lll}\epsilon|\log\epsilon|K'+o(\epsilon|\log\epsilon|) & \text{if} & n=3\\ \epsilon k\big(\delta+\gamma\frac{n-2}{2}\big)+o(\epsilon) & \text{if} & n\ge4\end{array}\right.\right]^{n/2}+O(\epsilon^{n-2}),\end{aligned} \tag{52}$$

where $k > 0$. We find that indeed $\overline{\beta}_k < S_2^{n/2}/2n$ provided ϵ is small enough. This completes the proof of Theorem 1.1 when $\delta_k < \delta < \delta_{k+1}$.

Assume now that $n \ge 4$ and $\delta = \delta_k$ for some $k \ge 1$. We consider first the case $n \ge 5$. In the estimate (49) the term $-D(\delta-\delta_k)c^2$ is no longer there so that (52) becomes

$$I(s\bar{U}_\epsilon+u_-) \le \frac{1}{n}\Big[\frac{S_2}{2^{2/n}}-\epsilon k\big(\delta+\gamma\frac{n-2}{2}\big)+o(\epsilon)\Big]^{n/2}+O(\epsilon^{\frac{n-2}{2}}),$$

proving again that $\overline{\beta}_k < S_2^{n/2}/2n$ for ϵ sufficiently small.

Let now $n = 4$. Here $2^* = 4$ and by arguing as in [14, Lemma 2.2] we deduce that

$$\left| \|u_- + s\bar{U}_\epsilon\|_4^4 - \|s\bar{U}_\epsilon\|_4^4 - \|u_-\|_4^4 \right| \leq c' \left[\|s\bar{U}_\epsilon\|_3^3 \|u_-\|_2 + \|s\bar{U}_\epsilon\|_1 \|u_-\|_4^3 \right]$$

which, together with (40) and (44), yields

$$\|u_- + s\bar{U}_\epsilon\|_4^4 \geq \|s\bar{U}_\epsilon\|_4^4 + \frac{1}{2}\|u_-\|_4^4 - c\, s^4 \epsilon^{\frac{4}{3}}, \qquad \text{for every } s > 0.$$

Recalling that $s \leq R_1$ and inserting this into (48), we conclude that

$$I(s\bar{U}_\epsilon + u_-) \leq \frac{s^2}{2}\left[\|\nabla \bar{U}_\epsilon\|_2^2 - \delta \|\bar{U}_\epsilon\|_\partial^2\right] - \frac{s^4}{4}\|\bar{U}_\epsilon\|_4^4 - \frac{1}{8}\|u_-\|_4^4 + c'\, \epsilon^{\frac{4}{3}}$$

$$\leq \frac{1}{4}\left(\frac{\|\nabla \bar{U}_\epsilon\|_2^2 - \delta \|\bar{U}_\epsilon\|_\partial^2}{\|\bar{U}_\epsilon\|_4^2}\right)^2 + c'\, \epsilon^{\frac{4}{3}} \leq \frac{1}{4}\left[\frac{S_2}{2^{1/2}} - \epsilon k\big(\delta + \gamma\big) + o(\epsilon)\right]^2 + c'\, \epsilon^{\frac{4}{3}}.$$

Hence, if ϵ is sufficiently small, we obtain $\overline{\beta}_k < S_2^2/8$. The proof of Theorem 1.1 is so complete also in the resonance case $\delta = \delta_k$, provided $n \geq 4$.

5. **Proof of Theorem 1.2.**

In this section and in the next one, an important role is played by the explicit value of the measure of ∂B, namely

$$\omega_n := |\partial B| = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{53}$$

For $j \geq 0$, we denote by M_j the eigenspace associated to δ_j (the Steklov eigenvalues of $-\Delta$ in B) and we define

$$M_+ := \overline{\bigoplus_{j \geq 1} M_j} \quad \text{and} \quad M_- := M_0 \bigoplus M_1.$$

By Proposition 5 we have

$$M_0 = span\{\varphi_0\} \quad \text{and} \quad M_1 = span\{\varphi_1^i\}_{1 \leq i \leq n},$$

where $\varphi_0(x) = 1$ and $\varphi_1^i(x) = x_i$ for $i = 1, ..., n$ (notice that $N_0 = 1$ and $N_1 = n$). We set

$$Q(u) := \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}\ , \qquad K_2 := \sup_{M_-} Q(u)\ , \tag{54}$$

and we prove

Lemma 5.1. *For any $n \geq 3$, $K_2 = Q(\varphi_1^1) = \frac{\omega_n}{n}\left[\frac{\omega_{n-1}}{n-1}\beta(\frac{3n-2}{2(n-2)}, \frac{n+1}{2})\right]^{(2/n)-1}$.*

Proof. First we note that

$$\|\nabla \varphi_1^i\|_2^2 = |B| = \frac{\omega_n}{n} \quad \text{for all} \quad i = 1, ..., n, \quad \|\nabla \varphi_0\|_2^2 = 0. \tag{55}$$

Next, take $u \in M_1$ so that $u(x) = \sum_1^n \alpha_i \varphi_1^i(x)$, where the α_i are the components of a real vector $\alpha \in \mathbb{R}^n$. We denote by $\{y_i\}_{1 \leq i \leq n}$ a complete orthonormal system of coordinates in $\mathbb{R}^n$, obtained as image of $\{x_i\}_{1 \leq i \leq n}$ through a rotation R such that $R(\frac{\alpha}{|\alpha|}) = (1, 0, ..., 0)$. Then, in view of (13), we get

$$Q(u) = \frac{\sum_1^n \alpha_i^2 \|\nabla \varphi_1^i\|_2^2}{\left(\int_B |\sum_1^n \alpha_i x_i|^{2^*}\, dx\right)^{2/2^*}} = \frac{\omega_n |\alpha|^2}{n\left(\int_B |\alpha|^{2^*} |y_1|^{2^*}\, dy\right)^{2/2^*}} = Q(\varphi_1^1),$$

for all $u \in M_1$. Similarly, one can prove that $\|u+t\varphi_0\|_{2^*}^{2^*} = \|\varphi_1^1+t\varphi_0\|_{2^*}^{2^*}$, for all $t \geq 0$ and all $u \in M_1$ such that $|\alpha| = 1$. This, combined with (55), shows that it suffices to study the real function

$$t \mapsto Q(\varphi_1^1+t\varphi_0) = \frac{1}{\|\varphi_1^1+t\varphi_0\|_{2^*}^2}, \quad t \geq 0$$

and prove that it attains its maximum at $t = 0$. In turn, we may consider the function

$$g(t) := \|\varphi_1^1+t\varphi_0\|_{2^*}^{2^*}, \quad t \geq 0$$

and show that

$$\min_{t\geq 0} g(t) = g(0). \tag{56}$$

Writing $x = (x_1, x')$, where $x' \in \mathbb{R}^{n-1}$, and denoting with B_r the ball in $\mathbb{R}^{n-1}$ of radius r and center 0, we deduce:

$$\begin{aligned} g(t) &= \int_B |x_1+t|^{2^*}\,dx = \int_{-1}^1 \int_{B_{(1-x_1^2)^{1/2}}} |x_1+t|^{2^*}\,dx'\,dx_1 \\ &= \frac{\omega_{n-1}}{n-1}\int_{-1}^1 |x_1+t|^{2^*}(1-x_1^2)^{\frac{n-1}{2}}\,dx_1\ . \end{aligned}$$

Therefore,

$$g'(t) = \frac{2^*\omega_{n-1}}{n-1}\int_{-1}^1 |x_1+t|^{2^*-2}(x_1+t)(1-x_1^2)^{\frac{n-1}{2}}\,dx_1\ ,$$

$$g''(t) = \frac{2^*(2^*-1)\omega_{n-1}}{n-1}\int_{-1}^1 |x_1+t|^{2^*-2}(1-x_1^2)^{\frac{n-1}{2}}\,dx_1\ .$$

This readily shows that $g'(0) = 0$ and $g''(t) > 0$ for all $t \geq 0$; and this proves that $g'(t) > 0$ for all $t > 0$ so that (56) follows. □

Lemma 5.2. *Let K_2 be as in (54). If*

$$\delta > 1 - \frac{S_2}{2^{2/n}K_2}\ , \tag{57}$$

then

$$\mu := \sup_{u\in M_-} I(u) < \frac{S_2^{n/2}}{2n}.$$

Moreover, there exist $\rho, \eta > 0$ such that

$$I(u) \geq \eta, \quad \text{for all} \quad u \in M_+ \oplus H_0^1(B): \quad \|u\| = \rho.$$

Proof. Let $u \in M_-$ and let K_2 be as in (54). Since $\delta_1 = 1$ (see Proposition 5), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\|\nabla u\|_2^2 - \delta\|u\|_\partial^2\right) - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \leq \frac{1-\delta}{2}\|\nabla u\|_2^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \\ &\leq \frac{1-\delta}{2}K_2\,\|u\|_{2^*}^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \leq \frac{(1-\delta)^{n/2}\,K_2^{n/2}}{n}, \end{aligned}$$

where the last inequality follows from (50). Therefore,

$$\mu \leq \frac{(1-\delta)^{n/2}\,K_2^{n/2}}{n} < \frac{S_2^{n/2}}{2n}\ ,$$

where the second inequality is ensured by (57).

Next, notice that for all $u \in M_+ \oplus H_0^1(B)$ we have

$$I(u) = \frac{1-\delta}{4}\|u\|^2 + \frac{1+\delta}{4}(\|\nabla u\|_2^2 - \|u\|_\partial^2) - \frac{1}{2^*}\|u\|_{2^*}^{2^*} \geq \frac{1-\delta}{4}\|u\|^2 - C\|u\|^{2^*} \ ,$$

for some $C > 0$, according to (19). Therefore, the existence of $\rho, \eta > 0$ as in the statement follows. □

Let K_2 be as in (54). By Lemma 5.1, we have

$$K_2 = Q(\varphi_1^1) = \frac{\omega_n}{n\left(\int_B |y_1|^{2^*}\, dy\right)^{2/2^*}} \ .$$

Notice that

$$\int_B |y_1|^{2^*}\, dy = \int_{-1}^1 \int_{B_{(1-y_1^2)^{1/2}}} |y_1|^{2^*}\, dy'\, dy_1 = \frac{\omega_{n-1}}{n-1}\,\beta\left(\frac{3n-2}{2(n-2)}, \frac{n+1}{2}\right)$$

so, by using (53) and exploiting the properties of the beta functions, we deduce that

$$K_2 = \frac{2\,\pi}{n\,\Gamma\left(\frac{n}{2}\right)}\left[\frac{n^2\sqrt{\pi}\,\Gamma(\frac{n^2}{2(n-2)})}{(n+2)\,\Gamma(\frac{n+2}{2(n-2)})}\right]^{1-2/n} .$$

Lemma 5.2 allows us to apply a result by Bartolo-Benci-Fortunato [5, Theorem 2.4] from which we deduce that, if $1-\delta < S_2/(2^{2/n}K_2)$, then I admits at least n (the multiplicity of δ_1) pairs of critical points at levels below $S_2^{n/2}/2n$. Set $h(n) := S_2/(2^{2/n}K_2)$ and compute, using (18), to obtain (5).

6. **Proof of Theorem 1.3.**

For $j \geq 1$, we denote by M_j the eigenspace associated to d_j, where the d_j's are the positive Steklov eigenvalues of Δ^2 in the ball and we define

$$M_+ := \overline{\bigoplus_{j\geq 2} M_j} \quad \text{and} \quad M_- := M_1 \bigoplus M_2.$$

By Proposition 7 we have

$$M_1 = span\{\phi_1\} \quad \text{and} \quad M_2 = span\{\phi_2^i\}_{1\leq i\leq n},$$

where $\phi_1(x) = (1-|x|^2)$ and $\phi_2^i(x) = x_i(1-|x|^2)$ for $i = 1, ..., n$. We set

$$Q(u) := \frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2} \ , \quad K_4 := \sup_{M_-} Q(u) \tag{58}$$

and we prove

Lemma 6.1. *If $n = 5, 6, 8$, then $K_4 = Q(\phi_2^1)$ and*

$$K_4 = \frac{4(n+2)\omega_n}{n}\left[\frac{\omega_{n-2}}{2}\beta(\frac{n-1}{2}, \frac{3n-4}{n-4})\beta(\frac{3n-4}{2(n-4)}, \frac{n^2+n-4}{2(n-4)})\right]^{(4/n)-1} .$$

Proof. First we note that

$$\|\Delta\phi_2^i\|_2^2 = 4\frac{n+2}{n}\omega_n \quad \text{for all} \quad i = 1, ..., n, \quad \|\Delta\phi_1\|_2^2 = 4n\omega_n. \tag{59}$$

Next, let $u \in M_2$ so that $u(x) = \sum_1^n \alpha_i\phi_2^i(x)$, where the α_i are the components of a real vector $\alpha \in \mathbb{R}^n$. We denote by $\{y_i\}_{1\leq i\leq n}$ a complete orthonormal system of coordinates in $\mathbb{R}^n$, obtained as image of $\{x_i\}_{1\le i\le n}$ through a rotation R such that $R(\frac{\alpha}{|\alpha|}) = (1,0,...,0)$. Then, we get

$$Q(u) = \frac{\sum_1^n \alpha_i^2 \|\Delta\phi_2^i\|_2^2}{\left(\int_B |\sum_1^n \alpha_i x_i|^{2_*}(1-|x|^2)^{2_*}\right)^{2/2_*}} = \frac{4\frac{n+2}{n}\omega_n|\alpha|^2}{\left(\int_B |\alpha y_1|^{2_*}(1-|y|^2)^{2_*}\right)^{2/2_*}} = Q(\phi_2^1),$$

for all $u \in M_2$. Similarly, one can prove that $\|u+t\phi_1\|_{2_*}^{2_*} = \|\phi_2^1+t\phi_1\|_{2_*}^{2_*}$, for all $t \ge 0$ and all $u \in M_2$ such that $|\alpha| = 1$. This, combined with (59), shows that it suffices to study the real function

$$f(t) = Q(\phi_2^1 + t\phi_1) = \frac{\|\Delta\phi_2^1\|_2^2 + t^2\|\Delta\phi_1\|_2^2}{\|\phi_2^1+t\phi_1\|_{2_*}^2}, \quad t \ge 0$$

and prove that

$$\max_{t\ge 0} f(t) = f(0). \tag{60}$$

Let us simplify (60). Writing $x = (x_1, x')$, where $x' \in \mathbb{R}^{n-1}$, and denoting with B_r the ball in $\mathbb{R}^{n-1}$ of radius r and center 0, we deduce:

$$\begin{aligned}
\|\phi_2^1+t\phi_1\|_{2_*}^{2_*} &= \int_B (1-|x|^2)^{2_*}|x_1+t|^{2_*}\,dx \\
&= \int_{-1}^1 \int_{B_{(1-x_1^2)^{1/2}}} (1-x_1^2-|x'|^2)^{2_*}|x_1+t|^{2_*}\,dx'\,dx_1 \\
&= \omega_{n-1}\left(\int_{-1}^1 |x_1+t|^{2_*}\int_0^{(1-x_1^2)^{1/2}} (1-x_1^2-\rho^2)^{2_*}\rho^{n-2}\,d\rho\,dx_1\right)[\rho=(1-x_1^2)^{1/2}r] \\
&= \omega_{n-1}\left(\int_{-1}^1 |x_1+t|^{2_*}(1-x_1^2)^{2_*+(n-1)/2}\,dx_1\right)\left(\int_0^1 (1-r^2)^{2_*}r^{n-2}\,dr\right) \\
&= \frac{\omega_{n-1}}{2}\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\left(\int_{-1}^1 |s+t|^{2_*}(1-s^2)^{\frac{n^2-n+4}{2(n-4)}}\,ds\right) \\
&=: \frac{\omega_{n-1}}{2}\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\varphi(t).
\end{aligned}$$

We have so found that $f(t) = C_nF(t)$, where $C_n = \frac{8\omega_n}{n2^{4/n}\left(\omega_{n-1}\beta\left(\frac{n-1}{2},\frac{3n-4}{n-4}\right)\right)^{2/2_*}}$ and

$$F(t) = \frac{n+2+n^2t^2}{(\varphi(t))^{2/2_*}}.$$

The claim 60 becomes

$$\max_{t\ge 0} F(t) = F(0). \tag{61}$$

When $n = 5, 6, 8$, the number 2_* is an even integer so that we may expand the term $|s+t|^{2_*}$ and write φ as a polynomial.

Case $n = 5$. Here, $2_* = 10$ and

$$\begin{aligned}
\varphi(t) &= \int_{-1}^1 (s+t)^{10}(1-s^2)^{12}\,ds = \sum_{k=0}^{10}\binom{10}{k}t^k\int_{-1}^1 s^{10-k}(1-s^2)^{12}\,ds \\
&= \frac{\beta(\frac{1}{2},13)}{29667}\left(1+175t^2+3850t^4+23870t^6+49445t^8+29667t^{10}\right)
\end{aligned}$$

so that

$$F(t)=C_5\frac{7+25t^2}{(1+175t^2+3850t^4+23870t^6+49445t^8+29667t^{10})^{\frac{1}{5}}},$$

where $C_5:=\left(\frac{29667}{\beta(\frac{1}{2},13)}\right)^{\frac{1}{5}}$. Let now

$$\widetilde{F}(t):=\frac{F(\sqrt{t})}{C_5}=\frac{7+25t}{(1+175t+3850t^2+23870t^3+49445t^4+29667t^5)^{\frac{1}{5}}},$$

so that by direct computations we get

$$\widetilde{F}'(t)=4\frac{9889t^4-9548t^3-10626t^2-1820t-55}{(1+175t+3850t^2+23870t^3+49445t^4+29667t^5)^{\frac{6}{5}}}.$$

Consider the function

$$g(t):=9889t^4-9548t^3-10626t^2-1820t-55,\quad t\geq 0,$$

we have $g'(t)=4\left(9889t^3-7161t^2-5313t-455\right)$ and $g''(t)=132(161t^2-434t-899)$. Therefore there exists a unique $\overline{t}>0$ such that

$$g''(t)<0\quad\text{if}\quad t<\overline{t},\quad g''(\overline{t})=0,\quad g''(t)>0\quad\text{if}\quad t>\overline{t}.$$

This, together with $g'(0)<0$ and $\lim_{t\to+\infty}g'(t)=+\infty$, shows that g' has a global minimum at $\overline{t}$ and $g'(\overline{t})<0$. Hence, there exists a unique $\sigma>\overline{t}$ such that

$$g'(t)<0\quad\text{if}\quad t<\sigma,\quad g'(\sigma)=0,\quad g'(t)>0\quad\text{if}\quad t>\sigma.$$

Similarly, since $g(0)<0$ and $\lim_{t\to+\infty}g(t)=+\infty$, we know that g has a global minimum at σ and $g(\sigma)<0$. This proves that there exists a unique $\tau>\sigma$ such that

$$g(t)<0\quad\text{if}\quad t<\tau,\quad g(\tau)=0,\quad g(t)>0\quad\text{if}\quad t>\tau.$$

Finally, this shows that $\widetilde{F}$ has a global minimum at τ, whereas F has a global minimum at $\sqrt{\tau}$. Since $F(0)=7C_5>\lim_{t\to+\infty}F(t)=25C_5(29667)^{-1/5}$, this proves that (61) holds when $n=5$.

Case $n=6$. Here $2_*=6$,

$$\varphi(t)=\int_{-1}^{1}(s+t)^6(1-s^2)^{\frac{17}{2}}\,ds=\frac{\beta(\frac{1}{2},\frac{19}{2})}{704}\left(1+72t^2+528t^4+704t^6\right)$$

and

$$F(t)=C_6\frac{8+36t^2}{(1+72t^2+528t^4+704t^6)^{\frac{1}{3}}},$$

where $C_6:=\left(\frac{704}{\beta(\frac{1}{2},\frac{19}{2})}\right)^{\frac{1}{3}}$. To simplify further, we set

$$\widetilde{F}(t):=\frac{F(\sqrt{t}/2)}{C_6}=\frac{8+9t}{(1+18t+33t^2+11t^3)^{\frac{1}{3}}}$$

and we compute

$$\widetilde{F}'(t)=\frac{11t^2-68t-39}{(1+18t+33t^2+11t^3)^{\frac{4}{3}}}.$$

This shows that F has a global minimum for $t=\overline{t}>0$ and no local maximum for $t>0$. Hence, since $F(0)=8C_6>\lim_{t\to+\infty}F(t)=36C_6(704)^{-1/3}$, we conclude that (61) holds when $n=6$.

Case $n = 8$. Here $2_* = 4$,

$$\varphi(t) = \int_{-1}^{1} (s+t)^4 (1-s^2)^{\frac{15}{2}}\, ds = \frac{\beta(\frac12, \frac{17}{2})}{120} \left(1 + 40t^2 + 120t^4\right)$$

and

$$F(t) = C_8 \frac{10 + 64t^2}{(1 + 40t^2 + 120t^4)^{\frac12}},$$

where $C_8 := \left(\frac{120}{\beta(\frac12, \frac{17}{2})}\right)^{\frac12}$. Consider

$$\widetilde{F}(t) =: \frac{F(\sqrt{t/2})}{2C_8} = \frac{5 + 16t}{(1 + 20t + 30t^2)^{\frac12}},$$

we have

$$\widetilde{F}'(t) = 2\frac{5t - 17}{(1 + 20t + 30t^2)^{\frac32}}.$$

Coming back to the function F, this means that F has a global minimum for $t = \overline{t} > 0$ and no local maximum for $t > 0$. Thus, since $F(0) = 10C_8 > \lim_{t\to+\infty} F(t) = 64C_8(120)^{-1/2}$, we conclude that (61) holds also when $n = 8$. ☐

Lemma 6.2. *Let K_4 be as in (58). If*

$$d > n + 2 - \frac{n+2}{K_4} S_4,$$

then

$$\mu := \sup_{u \in M_-} J(u) < \frac{2}{n} S_4^{n/4}.$$

Moreover, there exist $\rho, \eta > 0$ such that

$$J(u) \geq \eta, \quad \textit{for all} \quad u \in M_+ \oplus H_0^2(B) : \quad \|\Delta u\|_2 = \rho.$$

Proof. Let $u \in M_-$. Since $d_2 = n + 2$ (see Proposition 7), we have

$$J(u) = \frac12 \left(\|\Delta u\|_2^2 - d\|u\|_{\partial_\nu}^2\right) - \frac{1}{2_*}\|u\|_{2_*}^{2_*} \leq \frac12 \left(\frac{n+2-d}{n+2}\right) \|\Delta u\|_2^2 - \frac{1}{2_*}\|u\|_{2_*}^{2_*}$$

$$\leq \frac12 \left(\frac{n+2-d}{n+2}\right) K_4 \, \|u\|_{2_*}^2 - \frac{1}{2_*}\|u\|_{2_*}^{2_*} \leq \frac{2}{n}\left(\frac{n+2-d}{n+2} K_4\right)^{\frac{n}{4}},$$

where the last inequality follows from

$$\max_{s\geq 0} \left(as - bs^{\frac{n}{n-4}}\right) = \left(\frac{n-4}{n}\right)^{\frac{n-4}{4}} \frac{4}{n} \frac{a^{n/4}}{b^{(n-4)/4}}, \quad \text{for all} \quad a, b > 0.$$

Therefore,

$$\mu \leq \frac{2}{n}\left(\frac{n+2-d}{n+2} K_4\right)^{\frac{n}{4}}.$$

To conclude we observe that $\mu < \frac{2}{n} S_4^{\frac{n}{4}}$ for $n + 2 - d < \frac{S_4(n+2)}{K_4}$. Let now $u \in M_+ \oplus H_0^2(B)$ and $\rho = S_4^{\frac{n}{8}} \left(\frac{n+2-d}{n+2}\right)^{\frac{n-4}{8}}$, for $\|\Delta u\|_2 = \rho$ we have

$$J(u) \geq \frac12 \left(\frac{n+2-d}{n+2}\right) \|\Delta u\|_2^2 - \frac{1}{2_* S_4^{n/(n-4)}} \|\Delta u\|_2^{2_*} = \frac{2}{n}\left(\frac{n+2-d}{n+2} S_4\right)^{\frac{n}{4}} =: \eta.$$

The proof is now complete. ☐

Lemma 6.2 allows us to apply [5, Theorem 2.4] from which we deduce that, if $n+2-d < S_4(n+2)/K_4$, then J admits at least n (the multiplicity of d_2) pairs of critical points at levels below $(2/n)\, S_4^{n/4}$. Set $g(n) := \frac{S_4(n+2)}{K_4}$ and, using (30) and (53), compute to obtain (7).

7. Remarks on Theorem 1.3 in general dimensions.

As already mentioned, we do not have a proof of Theorem 1.3 in general dimensions $n \geq 5$. However, we make the following

Conjecture 1. *Assume that* $\Omega = B$, *the unit ball of* $\mathbb{R}^n$ *with* $n \geq 5$. *If* $d \in (n+2-g(n), n+2)$, *problem* (6) *admits at least* n *pairs of nontrivial solutions.*

Let us explain the three main reasons why we believe this conjecture to be true. First, we notice that what is missing for the proof of this conjecture is Lemma 6.1. In turn, this reduces to show that $F(0) \geq F(t)$, for every $t \geq 0$, or that $G(t) \geq 0$, where

$$G(t) := (n+2)^{\frac{n}{n-4}}\varphi(t) - \varphi(0)(n+2+n^2t^2)^{\frac{n}{n-4}} = (n+2)^{\frac{n}{n-4}}\varphi(t) - b(n+2+n^2t^2)^{\frac{n}{n-4}} \tag{62}$$

and $b := \beta\left(\frac{3n-4}{2(n-4)}, \frac{n^2+n-4}{2(n-4)}\right)$.

We can prove this property only locally:

Lemma 7.1. *For any* $n \geq 5$, *we have* $G(0) = G'(0) = 0$ *and* $G''(0) > 0$.

Proof. Consider first the function φ. We have

$$\varphi'(t) = 2_* \int_{-1}^{1} |s+t|^{2_*-2}(s+t)(1-s^2)^a\, ds > 0 \quad \text{for } t > 0 \quad \text{and } \varphi'(0) = 0,$$

$$\varphi''(t) = 2_*(2_*-1) \int_{-1}^{1} |s+t|^{2_*-2}(1-s^2)^a\, ds > 0 \quad \text{for } t \geq 0,$$

where $a := \frac{n^2-n+4}{2(n-4)}$. Thus φ is an increasing and convex function. Since

$$G'(t) = (n+2)^{\frac{n}{n-4}}\varphi'(t) - b\, 2_* n^2 t (n+2+n^2t^2)^{\frac{4}{n-4}},$$

we have $G(0) = G'(0) = 0$. On the other hand,

$$G''(t) = (n+2)^{\frac{n}{n-4}}\varphi''(t) - b\, 2_* n^2 (n+2+n^2t^2)^{\frac{8-n}{n-4}}(n+2+n^2t^2+4n2_*t^2),$$

so that

$$G''(0) = (n+2)^{\frac{n}{n-4}}\varphi''(0) - b\, 2_* n^2 (n+2)^{\frac{4}{n-4}} = \frac{8n^2(n+2)^{\frac{4}{n-4}}(2n+1)}{(n-4)^2}\, b > 0,$$

where in the last step we exploited the property $\beta(p+1,q) = \frac{p}{p+q}\beta(p,q)$ to deduce that

$$\varphi''(0) = 2_*(2_*-1)\, \beta\left(\frac{n+4}{2(n-4)}, \frac{n^2+n-4}{2(n-4)}\right) = 2_*(2_*-1)\frac{n(n+2)}{n+4}\, b.$$

□

The second argument which brings some evidence in favor of Conjecture 1 is that, although we cannot prove (61), we have

Lemma 7.2. *There exists* $n_0 \in \mathbb{N}$ *such that* $F(0) > \lim_{t\to+\infty} F(t)$, *for all* $n \geq n_0$.

Proof. As $n \to +\infty$ we have

$$F(0) = \frac{n+2}{\left[2\int_0^1 s^{2_*}(1-s^2)^{\frac{n^2-n+4}{2(n-4)}}\,ds\right]^{2/2_*}} \sim \frac{n}{2\int_0^1 s^2(1-s^2)^{n/2}\,ds}\,,$$

$$\lim_{t\to+\infty} F(t) = \frac{n^2}{\left[2\int_0^1 (1-s^2)^{\frac{n^2-n+4}{2(n-4)}}\,ds\right]^{2/2_*}} \sim \frac{n^2}{2\int_0^1 (1-s^2)^{n/2}\,ds}\,.$$

An integration by parts shows that

$$\int_0^1 s^2(1-s^2)^{n/2}\,ds = \frac{1}{n+2}\int_0^1 (1-s^2)^{n/2+1}\,ds < \frac{1}{n}\int_0^1 (1-s^2)^{n/2}\,ds$$

and the statement follows. □

The last argument which brings some evidence to Conjecture 1 are the numerical plots (obtained with Mathematica) of the functions G defined in (62) when $n = 7, 9, 10, ..., 20$. Not only it seems that $G(t) \geq 0$ for all $t \geq 0$ but also that G is increasing and convex.

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