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Best constants and minimizers for embeddings of second order Sobolev spaces

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Abstract

By considering the kernels of the first two traces, four different second order Sobolev spaces may be constructed. For these spaces, embeddings into Lebesgue spaces, the best embedding constant and the possible existence of minimizers are studied. The Euler equation corresponding to some of these minimization problems is a semilinear biharmonic equation with boundary conditions involving third order derivatives: it is shown that the complementing condition is satisfied.

Keywords: Sobolev embeddings; Biharmonic operator; Complementing condition

1. Introduction and main results

We are interested in best constants and existence of minimizers for embeddings of second order Hilbertian Sobolev spaces. For a smooth open domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 5$), not necessarily bounded, we consider the space

$$H^2(\Omega) = \{u \in L^2(\Omega);\ D^2u \in L^2(\Omega)\}$$

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where D^2u denotes the (generalized) Hessian matrix of u. It is well known [1] that if $\Omega \not\equiv \mathbb{R}^n$, then any function $u \in H^2(\Omega)$ admits some *traces* on the boundary $\partial\Omega$. In particular, there exist two linear continuous operators

$$\gamma_0 : H^2(\Omega) \to H^{3/2}(\partial\Omega) \quad \text{and} \quad \gamma_\nu : H^2(\Omega) \to H^{1/2}(\partial\Omega)$$

such that $\gamma_0 u = u|_{\partial\Omega}$ and $\gamma_\nu u = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$; here and in the sequel ν denotes the unit outward normal to $\partial\Omega$. The kernels of these traces give rise to proper subspaces of $H^2(\Omega)$ which we denote by

$$H_0^2(\Omega) = \ker\gamma_0 \cap \ker\gamma_\nu, \qquad H^2 \cap H_0^1(\Omega) = \ker\gamma_0, \qquad H_\nu^2(\Omega) = \ker\gamma_\nu.$$

If $\Omega = \mathbb{R}^n$ the traces are undefined and $H_0^2(\mathbb{R}^n) = H^2 \cap H_0^1(\mathbb{R}^n) = H_\nu^2(\mathbb{R}^n) = H^2(\mathbb{R}^n)$. For general domains Ω, we consider embeddings of these spaces into $L^p(\Omega)$ for $2 < p \leqslant 2^* = 2n/(n-4)$ (the critical Sobolev exponent). More precisely, we seek some properties of the best embedding constants and we investigate the existence or nonexistence of minimizers for the corresponding ratio.

Due to a lack of compactness, this problem becomes much more difficult when $p = 2^*$. In this case we only have partial answers and several problems are left open. When $\Omega = \mathbb{R}^n$, we denote by $\mathcal{D}^{2,2}$ the closure of the space of smooth compactly supported functions in $\mathbb{R}^n$ with respect to the norm $\|D^2 \cdot\|_2$; here and in the sequel, $\|\cdot\|_q$ represents the usual L^q-norm. Two integration by parts show that $\|D^2u\|_2^2 = \|\Delta u\|_2^2$ for all $u \in \mathcal{D}^{2,2}$. Then, the best constant for the embedding $\mathcal{D}^{2,2} \subset L^{2^*}(\mathbb{R}^n)$ may be characterized by

$$S = \inf\{\|D^2u\|_2^2;\ u \in \mathcal{D}^{2,2},\ \|u\|_{2^*} = 1\} = \inf\{\|\Delta u\|_2^2;\ u \in \mathcal{D}^{2,2},\ \|u\|_{2^*} = 1\}. \quad (1)$$

Up to translations and nontrivial real multiples, the infimum in (1) is achieved only by the functions

$$u_\varepsilon(x) = \frac{[(n-4)(n-2)n(n+2)]^{(n-4)/8}\varepsilon^{(n-4)/2}}{(\varepsilon^2 + |x|^2)^{(n-4)/2}}, \quad (2)$$

for any $\varepsilon > 0$, see [6, Theorem 2.1] and also [12, Theorem 4], [8, Theorem 1.3], [7, Lemma 2].

In fact, for any domain $\Omega \subseteq \mathbb{R}^n$, the spaces $H_0^2(\Omega)$ and $H^2 \cap H_0^1(\Omega)$ are Banach spaces (Hilbert spaces) when endowed with the norm

$$u \mapsto \|\Delta u\|_2. \quad (3)$$

It is shown in [13] that for any smooth domain $\Omega \subset \mathbb{R}^n$ we have

$$\begin{aligned}&\inf\{\|\Delta u\|_2^2;\ u \in H_0^2(\Omega),\ \|u\|_{2^*} = 1\}\\ &\quad = \inf\{\|\Delta u\|_2^2;\ u \in H^2 \cap H_0^1(\Omega),\ \|u\|_{2^*} = 1\} = S\end{aligned}$$

although the infimum is not achieved if $\Omega \neq \mathbb{R}^n$. Hence, not only the best embedding constant is independent of the domain Ω, but it is also independent of $\ker\gamma_\nu$.

On the other hand, due to the invariance up to the addition of constants (respectively affine functions), if we consider the space $H_\nu^2(\Omega)$ (respectively the whole space $H^2(\Omega)$) then (3) is no longer a norm. In these cases, one has also to take into account the $L^2(\Omega)$-norm of the function and of *all* its second order derivatives, while first order derivatives can still be neglected thanks to interpolation theory, see, e.g., [1, Theorem 4.14]. In other words, for any $a > 0$ both $H^2(\Omega)$ and $H^2_\nu(\Omega)$ become Banach spaces when endowed with the norm

$$u \mapsto \left(\|D^2u\|_2^2 + a\|u\|_2^2\right)^{1/2}. \tag{4}$$

We are so led to introduce the embedding constants

$$\Sigma(\Omega, a) = \inf\left\{\|D^2u\|_2^2 + a\|u\|_2^2;\ u \in H^2(\Omega),\ \|u\|_{2^*} = 1\right\},$$
$$\Sigma^\nu(\Omega, a) = \inf\left\{\|D^2u\|_2^2 + a\|u\|_2^2;\ u \in H^2_\nu(\Omega),\ \|u\|_{2^*} = 1\right\} \quad (a > 0). \tag{5}$$

In Section 2 we prove

Theorem 1. *Assume that $n \geqslant 5$ and let S be as in* (1). *Then, for any $a > 0$ we have*:

(i) *$\Sigma(\mathbb{R}^n, a) = S$ and it is not achieved*;
(ii) *if $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, then $\Sigma(\Omega, a) < S/2^{4/n}$.*

In order to prove Theorem 1(ii), we strongly use the smoothness of $\partial\Omega$ and the positivity of the mean curvature in some boundary point $\bar{x}$. For this reason, and because a similar result holds in the first order case, it is reasonable to conjecture that for the half space $\mathbb{R}^n_+$ we have

$$\Sigma\left(\mathbb{R}^n_+, a\right) = \frac{S}{2^{4/n}} \quad \text{and it is not achieved.} \tag{6}$$

However, a proof of (6) seems rather difficult. This difficulty was already emphasized by van der Vorst [13, Remark, p. 267] where the simpler case of the space $H^2 \cap H^1_0$ (instead of H^2) is considered. The impossibility of using a reflection argument already present in the $H^2 \cap H^1_0$ setting, is here further complicated because we cannot make use of the maximum and the comparison principles.

We can prove a much weaker version of (6); we state this result for "flat" and "non-smooth" domains. More precisely, we consider the domains

$$\mathbb{R}^n_{k+} := (0, \infty)^k \times \mathbb{R}^{n-k}, \quad k = 1, \ldots, n,$$

and we prove

Theorem 2. *Assume that $n \geqslant 5$. For any $a > 0$ and any $k = 1, \ldots, n$ we have*

$$\Sigma^\nu\left(\mathbb{R}^n_{k+}, a\right) = \frac{S}{2^{4k/n}}$$

and the infimum is not achieved. In particular, $\Sigma^\nu(\mathbb{R}^n_+, a) = S/2^{4/n}$.

Next, for any bounded domain Ω and any $2 < p \leqslant 2^*$, we consider the following numbers:

$$\Sigma_p(\Omega,a)=\inf_{u\in H^2(\Omega)\setminus\{0\}}\frac{\|D^2u\|_2^2+a\|u\|_2^2}{\|u\|_p^2},$$
$$\Sigma_p^\nu(\Omega,a)=\inf_{u\in H^2_\nu(\Omega)\setminus\{0\}}\frac{\|D^2u\|_2^2+a\|u\|_2^2}{\|u\|_p^2}. \tag{7}$$

Then, we prove

Theorem 3. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n\geqslant 5)$, that $a>0$ and that $2<p<2^*$. Then the infima in* (7) *are achieved. Moreover, there exists $\bar a(p)\geqslant 0$ such that if $a\geqslant \bar a(p)$, then the minimizers of* (7) *are nonconstant.*

In Section 5 we show that, up to a Lagrange multiplier, minimizers u of $\Sigma_p(\Omega,a)$ (respectively $\Sigma_p^\nu(\Omega,a)$) as defined in (7), satisfy

$$\int_\Omega \big(D^2uD^2v+auv-|u|^{p-2}uv\big)=0 \quad \text{for all } v\in H^2(\Omega) \tag{8}$$

(respectively for all $v\in H^2_\nu(\Omega)$), where D^2uD^2v denotes the "scalar product" between Hessian matrices, namely

$$D^2uD^2v=\sum_{i,j=1}^n\frac{\partial^2u}{\partial x_i\partial x_j}\frac{\partial^2v}{\partial x_i\partial x_j}\quad \text{for all } u,v\in H^2(\Omega).$$

For all j, let $\nu_j=\frac{\partial\nu}{\partial x_j}=\cos(\nu,x_j)$ denote the jth component of ν, the normal vector to the boundary $\partial\Omega$. Let $\{t_k=t_k(x);\ k=1,\dots,n-1;\ x\in\partial\Omega\}$ denote a system of local tangential coordinates to $\partial\Omega$ so that $\{t_1,\dots,t_{n-1},\nu\}$ is a complete orthonormal system diffeomorphic to $\{x_1,\dots,x_n\}$. As pointed out by P.L. Lions [10, p. 76], the boundary conditions associated to (8) do not depend on the choice of the system $\{t_1,\dots,t_{n-1}\}$. Then, we have

Theorem 4. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n\geqslant 5)$, that $a>0$ and that $2<p\leqslant 2^*$. Assume that $u\in H^2(\Omega)$ (or $u\in H^2_\nu(\Omega)$) satisfies* (8). *Then, $u\in C^{4,\alpha}(\overline{\Omega})$ and u is a classical solution of*

$$\Delta^2u+au=|u|^{p-2}u \quad \textit{in } \Omega \tag{9}$$

satisfying the boundary conditions (if $u\in H^2(\Omega)$)

$$\frac{\partial^2u}{\partial\nu^2}=2\frac{\partial\Delta u}{\partial\nu}-\frac{\partial^3u}{\partial\nu^3}+\sum_{i,j=1}^n\sum_{k=1}^{n-1}\frac{\partial^2u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)=0\quad \textit{on } \partial\Omega,^1 \tag{10}$$

or the boundary conditions (if $u\in H^2_\nu(\Omega)$)

$$\frac{\partial u}{\partial\nu}=2\frac{\partial\Delta u}{\partial\nu}-\frac{\partial^3u}{\partial\nu^3}+\sum_{i,j=1}^n\sum_{k=1}^{n-1}\frac{\partial^2u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)=0\quad \textit{on } \partial\Omega. \tag{11}$$

[1] Here, $\frac{\partial^2u}{\partial\nu^2}:=\sum_{i,j=1}^n\frac{\partial^2u}{\partial x_i\partial x_j}\nu_i\nu_j$ and $\frac{\partial^3u}{\partial\nu^3}:=\sum_{i,j,k=1}^n\frac{\partial^3u}{\partial x_i\partial x_j\partial x_k}\nu_i\nu_j\nu_k$.

In particular, Theorem 4 tells us that minimizers of (7) (if they exist!) are smooth functions. Moreover, since a solution of (9) may be sign-changing, further regularity seems to be false in general, at least for noninteger values of p.

Remark 1. In some cases the boundary conditions may be significantly simplified.

First, assume that $\partial\Omega$ is "somewhere flat," namely that there exists $\Gamma \subset \partial\Omega$ of positive $(n-1)$-dimensional Hausdorff measure such that $\Gamma \subset \{x_n = 0\}$. Then, the second boundary condition in (10) becomes

$$\frac{\partial^3 u}{\partial x_n^3} - 2\frac{\partial \Delta u}{\partial x_n} = 0 \quad \text{on } \Gamma.$$

Second, assume that Ω is the unit ball and that u is radially symmetric, $u = u(|x|)$; note that we have *no condition* which ensures u to be radially symmetric! However, in this case, (10) become $u''(1) = u'''(1) - (n-1)u'(1) = 0$, while (11) become $u'(1) = u'''(1) + (n-1)u''(1) = 0$.

As far as we are aware, biharmonic semilinear elliptic equations as (9) have so far been tackled only in the spaces $H_0^2(\Omega)$ or $H^2 \cap H_0^1(\Omega)$ (where (3) is a norm). In the first case, Dirichlet boundary conditions ($u = \frac{\partial u}{\partial \nu} = 0$) arise, whereas in the second case Navier boundary conditions ($u = \Delta u = 0$) appear. Hence, the present paper is also a first contribution to biharmonic semilinear problems with boundary conditions (10) and (11).

Note that (8) admits the two constant solutions

$$u_0 \equiv 0, \qquad u_1 \equiv a^{1/(p-2)}. \tag{12}$$

It is therefore of some interest to find out whether (8) also admits nonconstant solutions. As a consequence of Theorems 3 and 4, we obtain

Corollary 1. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^n$ ($n \geqslant 5$) and that $2 < p < 2^*$. There exists $\bar{a}(p) \geqslant 0$ such that if $a \geqslant \bar{a}(p)$, then* (8) *admits a $C^{4,\alpha}(\overline{\Omega})$ nonconstant solution.*

Corollary 1 nothing says about *small* values of a. In the second order subcritical case, it is known [9] that for sufficiently small a, the corresponding equation only admits constant solutions. In the critical case, counterexamples in [3] show that a similar result is false in general; nevertheless, it is proved in [4] that the *mountain-pass solution* (or minimal energy solution) is constant for sufficiently small a. These results are obtained by showing that for small a any solution u satisfies $u \equiv \bar{u}$, where $\bar{u}$ is the mean value of u. In turn, this is obtained by using Wirtinger's inequality which enables to estimate $\|u - \bar{u}\|_2$ in terms of $\|\nabla u\|_2$. In the fourth order equation, one would need an a priori estimate of $\|u - \bar{u}\|_2$ in terms of $\|D^2 u\|_2$, which does not hold in general. Hence, the following question naturally arises: is it true that for sufficiently small a, (8) only admits constant solutions or that the mountain-pass solution is constant? We have no answer to this question but we have some reasons to believe that it might be negative.

Another question which arises in connection with Corollary 1 is the *sign* of the solutions of (8). For the corresponding first order minimization problem, it is clear that u is a minimizer if and only if $|u|$ is a minimizer; this shows that a minimizer may be chosen nonnegative (in fact positive by the maximum principle). For the second order minimization problems considered in the present paper, this simple trick fails since $|u|$ may not be in $H^2(\Omega)$. On the other hand, a simple application of the divergence theorem enables us to prove

Theorem 5. *Assume that Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n \geqslant 5)$, that $2 < p \leqslant 2^*$ and that $a = 0$. Let $u \geqslant 0$ be a solution of* (8); *then $u \equiv 0$.*

In view of the above discussion, it is clear that Theorem 5 is not satisfactory. One should also exclude the existence of *sign-changing* solutions.

2. Proof of Theorem 1

2.1. Proof of (i)

Let $B = \{u \in \mathcal{D}^{2,2};\ \|u\|_{2^*} = 1\}$. For any $a > 0$, we have

$$S = \inf_{u\in B} \|D^2u\|_2^2 \leqslant \inf_{u\in B\cap H^2(\mathbb{R}^n)} \|D^2u\|_2^2 \leqslant \inf_{u\in B\cap H^2(\mathbb{R}^n)} \left(\|D^2u\|_2^2 + a\|u\|_2^2\right)$$
$$= \Sigma\left(\mathbb{R}^n, a\right), \tag{13}$$

where the first inequality follows from the (proper) inclusion $H^2(\mathbb{R}^n) \subset \mathcal{D}^{2,2}$.

In order to show the converse inequality, we construct a suitable minimizing sequence. For all $\varepsilon > 0$ consider the radial function

$$v_\varepsilon(r) := u_\varepsilon(r) - u_\varepsilon(1)$$

where $u_\varepsilon(|x|) = u_\varepsilon(x)$ is defined in (2). Let now

$$z_\varepsilon(r) = \begin{cases} v_\varepsilon(r) & \text{if } r \in [0, 1/2], \\ w_\varepsilon(r) & \text{if } r \in [1/2, 1], \\ 0 & \text{if } r \in [1, \infty) \end{cases} \tag{14}$$

where $w_\varepsilon(r) = a_\varepsilon(r-1)^3 + b_\varepsilon(r-1)^2$ and $a_\varepsilon, b_\varepsilon$ are chosen in such a way that $w_\varepsilon(1/2) = v_\varepsilon(1/2)$ and $w'_\varepsilon(1/2) = v'_\varepsilon(1/2)$. Hence, $z_\varepsilon \in H^2(\mathbb{R}^n)$ for all $\varepsilon > 0$; note that if $n > 8$, then $u_\varepsilon \in H^2(\mathbb{R}^n)$ and instead of $\{z_\varepsilon\}$ one can directly take $\{u_\varepsilon\}$.

A simple computation shows that $\lim_{\varepsilon\to 0} a_\varepsilon = \lim_{\varepsilon\to 0} b_\varepsilon = 0$ so that, if we let $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^n} \left|D^2 z_\varepsilon\left(|x|\right)\right|^2 = \int_{B_{1/2}} \left|D^2 u_\varepsilon\left(|x|\right)\right|^2 + o(1) = S^{n/4} + o(1),$$
$$\int_{\mathbb{R}^n} \left|z_\varepsilon\left(|x|\right)\right|^{2^*} = \int_{B_{1/2}} \left|u_\varepsilon\left(|x|\right)\right|^{2^*} + o(1) = S^{n/4} + o(1),$$
$$\int_{\mathbb{R}^n} \left|z_\varepsilon\left(|x|\right)\right|^2 = o(1). \tag{15}$$

Finally, set $Z_\varepsilon := z_\varepsilon/\|z_\varepsilon\|_{2^*}$ so that $Z_\varepsilon \in B \cap H^2(\mathbb{R}^n)$. Therefore, by definition of $\Sigma(\mathbb{R}^n, a)$ and by (15) we have

$$\Sigma\big(\mathbb{R}^n, a\big) \leqslant \big\|D^2 Z_\varepsilon\big\|_2^2 + a\|Z_\varepsilon\|_2^2 \quad \text{for all } \varepsilon > 0,$$
$$\lim_{\varepsilon\to 0}\big(\big\|D^2 Z_\varepsilon\big\|_2^2 + a\|Z_\varepsilon\|_2^2\big) = S. \tag{16}$$

The first part of statement (i) in Theorem 1 follows at once from (13) and (16). The second part is readily obtained by contradiction: if the infimum is achieved, then the minimizer would violate the Sobolev inequality $\|D^2u\|_2^2 \geqslant S$ which holds for all $u \in B$.

2.2. *Proof of (ii)*

Here, we take into account the effect of the curvature of the boundary $\partial\Omega$, following an idea from [2,14]. Since Ω is smooth and bounded, there exists $\bar{x} \in \partial\Omega$ such that in a neighborhood of $\bar{x}$, Ω lies on one side of the tangent hyperplane at $\bar{x}$ and the mean curvature with respect to the unit outward normal at $\bar{x}$ is positive. With a change of coordinates, we may assume that $\bar{x} = 0$ (the origin), that the tangent hyperplane coincides with $x_n = 0$ and that Ω lies locally in $\mathbb{R}^n_+ = \{x = (x', x_n);\ x_n > 0\}$. More precisely, there exists $R > 0$ and a smooth function $\rho : \omega \to \mathbb{R}_+$ (where $\omega = \{x' \in \mathbb{R}^{n-1};\ |x'| < R\}$) such that

$$(x', x_n) \in \Omega \cap B_R \quad \Leftrightarrow \quad x_n > \rho(x'),$$
$$(x', x_n) \in \partial\Omega \cap B_R \quad \Leftrightarrow \quad x_n = \rho(x').$$

Furthermore, since the curvature is positive at 0, there exist λ_i $(i = 1, \ldots, n-1)$ such that

$$\sum_{i=1}^{n-1}\lambda_i > 0 \quad \text{and} \quad \rho(x') = \sum_{i=1}^{n-1}\lambda_i x_i^2 + O\big(|x'|^3\big) \quad \text{as } x' \to 0. \tag{17}$$

Let $\Lambda := \{x \in B_R;\ 0 < x_n < \rho(x')\}$. Fix $\sigma > 0$ sufficiently small so that

$$L := (-\sigma, \sigma)^n \subset B_{R/4}$$

and define

$$\Delta := (-\sigma, \sigma)^{n-1}.$$

Let ϕ be a radial C^∞ function such that $0 \leqslant \phi \leqslant 1$ and

$$\phi(r) = \begin{cases} 1 & \text{if } r \leqslant R/4, \\ 0 & \text{if } r \geqslant R/2 \end{cases}$$

and define the function $U_\varepsilon(x) := \phi(|x|)u_\varepsilon(x)$. For our convenience, we set $\mu_n = \omega_{n-1}\sum_{i=1}^{n-1}\lambda_i$ and $C_n := [(n-4)(n-2)n(n+2)]^{(n-4)/8}$. We claim that, as $\varepsilon \to 0$:

$$\int_\Omega \big|D^2 U_\varepsilon\big|^2 = \frac{S^{n/4}}{2} - \left\{\begin{matrix} 2\sqrt[4]{105}\,\mu_5\varepsilon\log\frac{1}{\varepsilon} & \text{if } n = 5, \\ C_n^2\frac{(n-4)^2}{8(n-1)^2}(n^3+2n^2-9n-2)\mu_n\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-5}{2})}{\Gamma(n-2)}\varepsilon & \text{if } n \geqslant 6 \end{matrix}\right\}$$
$$+ o(\varepsilon), \tag{18}$$

$$\int_\Omega U_\varepsilon^{2^*} = \frac{S^{n/4}}{2} - \frac{C_n^{2^*}\mu_n}{2(n-1)}\frac{\Gamma\big(\frac{n+1}{2}\big)\Gamma\big(\frac{n-1}{2}\big)}{\Gamma(n)}\varepsilon + o(\varepsilon). \tag{19}$$

Postponing the proofs of (18) and (19), we conclude the proof of statement (ii). Note first that as $\varepsilon \to 0$ we have

$$\int_\Omega U_\varepsilon^2 = \int_{\Omega\cap B_{R/2}} U_\varepsilon^2 \leqslant \int_{B_{R/2}} u_\varepsilon^2 = \begin{cases} O(\varepsilon^4) & \text{if } n > 8, \\ O\big(\varepsilon^4 \log \frac{1}{\varepsilon}\big) & \text{if } n = 8, \\ O(\varepsilon^{n-4}) & \text{if } n = 5, 6, 7. \end{cases} \tag{20}$$

By using (18)–(20), we obtain for all $a > 0$:

$$\begin{aligned} &\Sigma(\Omega, a) \\ &\quad \leqslant \frac{\|D^2 U_\varepsilon\|_2^2 + a\|U_\varepsilon\|_2^2}{\|U_\varepsilon\|_{2*}^2} \\ &\quad = \frac{S}{2^{4/n}} - \begin{cases} \mu_n C_n^2 2^{1-4/n} S^{1-n/4} \frac{(n-4)^2(n^2-n-4)}{(n-1)^2} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-5}{2})}{\Gamma(n-2)} \varepsilon + o(\varepsilon) & \text{if } n \geqslant 6, \\ \mu_5 4^{3/5} \sqrt[4]{\frac{105}{S}}\, \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon) & \text{if } n = 5. \end{cases} \end{aligned}$$

Clearly, this last term becomes strictly less than $S/2^{4/n}$ for sufficiently small ε so (ii) follows.

Proof of (18). We split the integral as

$$\int_\Omega \big|D^2 U_\varepsilon\big|^2 = \frac{1}{2}\int_{B_R} + \int_{\Omega\setminus B_R} - \int_{\Lambda\cap L} - \int_{\Lambda\setminus L}. \tag{21}$$

In view of the estimates in [6, (3.23)], we have

$$\frac{1}{2}\int_{B_R} \big|D^2 U_\varepsilon\big|^2 = \frac{1}{2} S^{n/4} + O\big(\varepsilon^{n-4}\big), \qquad \int_{\Omega\setminus B_R} \big|D^2 U_\varepsilon\big|^2 = O\big(\varepsilon^{n-4}\big),$$

$$\int_{\Lambda\setminus L} \big|D^2 U_\varepsilon\big|^2 = O\big(\varepsilon^{n-4}\big).$$

Therefore, (18) will follow if we show that

$$\int_{\Lambda\cap L} \big|D^2 U_\varepsilon\big|^2 \approx \begin{cases} 2\sqrt[4]{105}\, \mu_5 \varepsilon \log \frac{1}{\varepsilon} & \text{if } n = 5, \\ C_n^2 \frac{(n-4)^2}{8(n-1)^2} (n^3 + 2n^2 - 9n - 2) \mu_n \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-5}{2})}{\Gamma(n-2)} \varepsilon & \text{if } n \geqslant 6. \end{cases} \tag{22}$$

By computing $|D^2 u_\varepsilon|^2$, we obtain

$$\begin{aligned} \int_{\Lambda\cap L} \big|D^2 U_\varepsilon\big|^2 = C_n^2 (n-4)^2 \varepsilon^{n-4} \Bigg[&\big(n^2 - 5n + 8\big) \int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2 + |x|^2)^{n-2}} \\ &+ (n-2)^2 \varepsilon^4 \int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2 + |x|^2)^n} \\ &- 2(n-2)(n-3)\varepsilon^2 \int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2 + |x|^2)^{n-1}} \Bigg]. \end{aligned} \tag{23}$$

We now estimate the three integrals in (23). With the change of variables $x_n = \sqrt{\varepsilon^2+|x'|^2}\cdot y_n$ and using $\int_0^s (1+t^2)^b\,dt = s + O(s^3)$ for all $b<0$, we obtain

$$\int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2+|x|^2)^{n-2}} = \int_{\Delta}\int_0^{\rho(x')} \frac{dx_n\,dx'}{(\varepsilon^2+|x'|^2+x_n^2)^{n-2}}$$
$$= \int_{\Delta}\int_0^{\frac{\rho(x')}{\sqrt{\varepsilon^2+|x'|^2}}} \frac{dy_n}{(1+y_n^2)^{n-2}}\frac{dx'}{(\varepsilon^2+|x'|^2)^{n-5/2}}$$
$$= \int_{\Delta} \frac{\rho(x')}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx' + O\biggl(\int_{\Delta}\frac{\rho^3(x')}{(\varepsilon^2+|x'|^2)^{n-1}}\,dx'\biggr). \quad (24)$$

We are so led to estimate

$$\int_{\Delta} \frac{\rho(x')}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx'$$
$$= \sum_{i=1}^{n-1}\lambda_i \int_{\Delta}\frac{x_i^2}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx' + O\biggl(\int_{\Delta}\frac{|x'|^3}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx'\biggr)$$
$$= \frac{\sum_{i=1}^{n-1}\lambda_i}{n-1}\varepsilon^{5-n}\int_{\Delta/\varepsilon}\frac{|y|^2}{(1+|y|^2)^{n-2}}\,dy + O\biggl(\varepsilon^{6-n}\int_{\Delta/\varepsilon}\frac{|y|^3}{(1+|y|^2)^{n-2}}\,dy\biggr) \quad (25)$$

where we used (17), the change of variables $x' = \varepsilon y$ and the following identity:

$$\int_{\Delta}\frac{x_i^2}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx' = \frac{1}{n-1}\int_{\Delta}\frac{|x'|^2}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx'.$$

We now distinguish two cases.

Case $n \geqslant 6$. In this case

$$\lim_{\varepsilon\to 0}\int_{\Delta/\varepsilon}\frac{|y|^2}{(1+|y|^2)^{n-2}}\,dy = \int_{\mathbb{R}^{n-1}}\frac{|y|^2}{(1+|y|^2)^{n-2}}\,dy = \omega_{n-1}\int_0^{\infty}\frac{r^n}{(1+r^2)^{n-2}}\,dr$$
$$= \omega_{n-1}\frac{\Gamma\bigl(\frac{n+1}{2}\bigr)\Gamma\bigl(\frac{n-5}{2}\bigr)}{2\Gamma(n-2)}.$$

Case $n = 5$. In this case $B_\sigma \subset \Delta \subset B_{2\sigma} \subset \mathbb{R}^4$. Moreover,

$$\int_{B_{2\sigma/\varepsilon}\setminus B_{\sigma/\varepsilon}}\frac{|y|^2}{(1+|y|^2)^3}\,dy = \omega_4\int_{\sigma/\varepsilon}^{2\sigma/\varepsilon}\frac{r^5}{(1+r^2)^3}\,dr \leqslant \omega_4\int_{\sigma/\varepsilon}^{2\sigma/\varepsilon}\frac{dr}{r} = \omega_4\log 2.$$

Hence, if $n = 5$, as $\varepsilon \to 0$ we get

$$\int_{\Delta/\varepsilon} \frac{|y|^2}{(1+|y|^2)^3}\,dy = \int_{|y|<\sigma/\varepsilon} \frac{|y|^2}{(1+|y|^2)^3}\,dy + O(1) = \omega_4 \int_0^{\sigma/\varepsilon} \frac{r^5}{(1+r^2)^3}\,dr + O(1)$$
$$= \omega_4 \log\frac{1}{\varepsilon} + O(1).$$

Furthermore, the same arguments enable us to show that

$$\varepsilon^{6-n} \int_{\Delta/\varepsilon} \frac{|y|^3}{(1+|y|^2)^{n-2}}\,dy = \begin{cases} O(\varepsilon^{6-n}) & \text{if } n \geqslant 7,\\ O\big(\log\frac{1}{\varepsilon}\big) & \text{if } n = 6,\\ O(1) & \text{if } n = 5.\end{cases}$$

Recalling the definition of μ_n, and inserting all the above estimates into (25), we obtain

$$\int_{\Delta} \frac{\rho(x')}{(\varepsilon^2+|x'|^2)^{n-2}}\,dx' \approx \frac{\mu_n}{n-1} \begin{cases} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-5}{2})}{2\Gamma(n-2)}\varepsilon^{5-n} & \text{if } n \geqslant 6,\\ \log\frac{1}{\varepsilon} & \text{if } n = 5.\end{cases} \tag{26}$$

Once more, one can use the same arguments as above to show that

$$\int_{\Delta} \frac{\rho^3(x')}{(\varepsilon^2+|x'|^2)^{n-1}}\,dx' = \begin{cases} O(\varepsilon^{7-n}) & \text{if } n \geqslant 8,\\ O\big(\log\frac{1}{\varepsilon}\big) & \text{if } n = 7,\\ O(1) & \text{if } n = 5, 6,\end{cases}$$

which, together with (24) and (26), yields

$$\int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2+|x|^2)^{n-2}} \approx \frac{\mu_n}{n-1} \begin{cases} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n-5}{2})}{2\Gamma(n-2)}\varepsilon^{5-n} & \text{if } n \geqslant 6,\\ \log\frac{1}{\varepsilon} & \text{if } n = 5.\end{cases} \tag{27}$$

In exactly the same way, we may estimate the other integrals in (23) and obtain (as $\varepsilon \to 0$):

$$\varepsilon^4 \int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2+|x|^2)^{n}} \approx \frac{\mu_n}{n-1} \frac{\Gamma\big(\frac{n+1}{2}\big)\Gamma\big(\frac{n-1}{2}\big)}{2\Gamma(n)}\varepsilon^{5-n},$$
$$\varepsilon^2 \int_{\Lambda\cap L} \frac{dx}{(\varepsilon^2+|x|^2)^{n-1}} \approx \frac{\mu_n}{n-1} \frac{\Gamma\big(\frac{n+1}{2}\big)\Gamma\big(\frac{n-3}{2}\big)}{2\Gamma(n-1)}\varepsilon^{5-n}. \tag{28}$$

Taking into account the properties of the gamma function, and inserting (27) and (28) into (23) gives (22) so that the proof of (18) is complete. □

Proof of (19). We also split $\int_\Omega U_\varepsilon^{2^*}$ according to (21) and we use the estimates in [6, (3.25)], so that

$$\frac{1}{2}\int_{B_R} U_\varepsilon^{2^*} = \frac{1}{2}S^{n/4} + O\big(\varepsilon^n\big), \qquad \int_{\Omega\setminus B_R} U_\varepsilon^{2^*} = O\big(\varepsilon^n\big), \qquad \int_{\Lambda\setminus L} U_\varepsilon^{2^*} = O\big(\varepsilon^n\big).$$

Therefore, (19) will follow if we show that

$$\int_{\Lambda\cap L} U_\varepsilon^{2^*} = \frac{C_n^{2^*}\mu_n}{2(n-1)} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n)}\varepsilon + o(\varepsilon). \tag{29}$$

By arguing as for the Hessian norm, we obtain

$$\int_{\Lambda\cap L} U_\varepsilon^{2^*} = \int_{\Lambda\cap L} u_\varepsilon^{2^*} = \frac{C_n^{2^*}\mu_n}{(n-1)}\varepsilon \cdot \int_0^\infty \frac{r^n}{(1+r^2)^n}\,dr + o(\varepsilon)$$

and (29) follows at once. □

3. Proof of Theorem 2

Consider first the case $k = 1$ so that $\mathbb{R}^n_{1+} = \mathbb{R}^n_+$. Consider the functions z_ε introduced in (14). By symmetry and (15) we deduce that $z_\varepsilon \in H^2_\nu(\mathbb{R}^n_+)$ and

$$\int_{\mathbb{R}^n_+} \left|D^2 z_\varepsilon\big(|x|\big)\right|^2 = \frac{S^{n/4}}{2} + o(1), \qquad \int_{\mathbb{R}^n_+} \left|z_\varepsilon\big(|x|\big)\right|^{2^*} = \frac{S^{n/4}}{2} + o(1),$$

$$\int_{\mathbb{R}^n_+} \left|z_\varepsilon\big(|x|\big)\right|^2 = o(1)$$

as $\varepsilon \to 0$. Therefore

$$\lim_{\varepsilon\to 0} \frac{\|D^2 z_\varepsilon\|_2^2 + a\|z_\varepsilon\|_2^2}{\|z_\varepsilon\|_{2^*}^2} = \frac{S}{2^{4/n}}$$

which proves that

$$\Sigma^\nu\big(\mathbb{R}^n_+, a\big) \leqslant \frac{S}{2^{4/n}}. \tag{30}$$

For contradiction, assume that there exists $v \in H^2_\nu(\mathbb{R}^n_+)$ such that

$$\frac{\|D^2 v\|_2^2 + a\|v\|_2^2}{\|v\|_{2^*}^2} \leqslant \frac{S}{2^{4/n}} \tag{31}$$

and set

$$w(x', x_n) = \begin{cases} v(x', x_n) & \text{if } x_n \geqslant 0, \\ v(x', -x_n) & \text{if } x_n < 0. \end{cases} \tag{32}$$

Then, $w \in H^2(\mathbb{R}^n)$ and by doubling the integrals, we obtain

$$\frac{\|D^2 w\|_2^2 + a\|w\|_2^2}{\|w\|_{2^*}^2} \leqslant S$$

which contradicts Theorem 1(i). Therefore, there exists no $v \in H^2_\nu(\mathbb{R}^n_+)$ such that (31) holds. This means that

$$\frac{\|D^2 v\|_2^2 + a\|v\|_2^2}{\|v\|_{2^*}^2} > \frac{S}{2^{4/n}} \quad \text{for all } v \in H^2_\nu(\mathbb{R}^n_+)$$

which, together with (30), shows that $\Sigma^\nu(\mathbb{R}^n_+, a) = \frac{S}{2^{4/n}}$ and that the infimum is not achieved.

The cases $k = 2, \dots, n$ are similar; one should proceed with k iterated reflections as (32).

4. Proof of Theorem 3

We only prove the result for $\Sigma_p(\Omega, a)$, the case $\Sigma_p^\nu(\Omega, a)$ being completely similar. Let $\{u_m\}_{m \geqslant 0} \subset H^2(\Omega)$ be a minimizing sequence for $\Sigma_p(\Omega, a)$ in (7) such that $\|u_m\|_p = 1$. Then, $\{u_m\}_{m \geqslant 0}$ is bounded in $H^2(\Omega)$ and there exists $\bar{u} \in H^2(\Omega)$ such that $u_m \rightharpoonup \bar{u}$ in $H^2(\Omega)$, up to a subsequence. By the compact embedding $H^2(\Omega) \subset L^p(\Omega)$, we deduce $u_m \to \bar{u}$ in $L^p(\Omega)$ so that $\|\bar{u}\|_p = 1$. Moreover, by lower semicontinuity of the norm with respect to weak convergence we infer that

$$\left\| D^2 \bar{u} \right\|_2^2 + a\|\bar{u}\|_2^2 \leqslant \Sigma_p(\Omega, a)$$

and we conclude.

In order to show that the minimizer for $\Sigma_p(\Omega, a)$ is nonconstant for sufficiently large a, we have to rule out u_1, see (12). To this end, for every $u \in H^2(\Omega)$ we define

$$f(u) := \frac{\|D^2 u\|_2^2 + a\|u\|_2^2}{\|u\|_p^2},$$

so that $f(u_1) = a|\Omega|^{(p-2)/p}$. The proof of Theorem 3 will be complete once we find $\bar{a}(p) \geqslant 0$ such that

$$a|\Omega|^{(p-2)/p} > \inf_{u \in H^2(\Omega)} f(u) \quad \text{for all } a \geqslant \bar{a}(p). \tag{33}$$

Without loss of generality, we assume that $0 \in \Omega$; then, we define the function

$$\varphi_a(x) = \begin{cases} (1 - a|x|^4)^2 & \text{for } |x| < 1/\sqrt[4]{a}, \\ 0 & \text{for } |x| \geqslant 1/\sqrt[4]{a} \end{cases}$$

whose support is contained in Ω provided a is sufficiently large, say $a \geqslant \tilde{a}$. For such a, we have $\varphi_a \in H^2(\Omega)$ and we may compute $f(\varphi_a)$; by straightforward computations in radial coordinates one sees that

$$\|\varphi_a\|_p^p = a^{-n/4} K_p \quad \text{and} \quad \left\| D^2 \varphi_a \right\|_2^2 = a^{1-n/4} C_n,$$

where

$$K_p = \omega_n \int_0^1 \left(1 - \rho^4\right)^{2p} \rho^{n-1}\, d\rho \quad \text{and} \quad C_n = 64\omega_n \left[\frac{n+48}{n+12} - \frac{2(n+20)}{n+8} + \frac{n+8}{n+4}\right].$$

Summarizing, for every $2 < p < 2^*$ and every $a \geqslant \tilde{a}$, we have

$$f(\varphi_a) = \frac{C_n + K_2}{K_p^{2/p}} a^{\frac{n}{2p} - \frac{n}{4} + 1}.$$

Therefore, if $a \geqslant \tilde{a}$ (to ensure $\varphi_a \in H^2(\Omega)$) and

$$a > \left(\frac{C_n + K_2}{K_p^{2/p} |\Omega|^{(p-2)/p}}\right)^{4p/n(p-2)} \tag{34}$$

then $f(u_1) > f(\varphi_a)$ so that (33) holds and u_1 is not a minimizer for (7).

Remark 2. Let $p = 2^*$ and suppose that we have proved that the minimum in (7) is achieved. From Theorem 1(ii), we know that

$$\inf_{u \in H^2(\Omega)} f(u) = \Sigma(\Omega, a) < \frac{S}{2^{4/n}}.$$

Hence, (33) would follow if

$$a \geqslant \frac{S}{(2|\Omega|)^{4/n}}.$$

On the other hand, if the minimum in (7) is achieved, then the lower bound (34) also holds for $p = 2^*$. Combining these facts, would show that

$$\bar{a}(2^*) \leqslant \min\left\{\frac{S}{(2|\Omega|)^{4/n}}, \frac{C_n + K_2}{K_{2^*}^{2/2^*} |\Omega|^{4/n}}\right\}.$$

5. Proof of Theorem 4

If $u \in H^2(\Omega)$ (respectively $u \in H^2_\nu(\Omega)$) is a minimizer of (7), then there exists a Lagrange multiplier $\lambda \in \mathbb{R} \setminus \{0\}$ such that u is a critical point of $F(u) := \|D^2 u\|_2^2 + a\|u\|_2^2 + \lambda(\|u\|_p^p - 1)$, namely

$$F'(u)[v] = \int_\Omega \left(2D^2 u D^2 v + 2auv + \lambda p|u|^{p-2}uv\right) = 0 \quad \text{for all } v \in H^2(\Omega)$$
$$\left(\text{respectively } v \in H^2_\nu(\Omega)\right).$$

Then, $\bar{u} := (p\lambda/2)^{1/(p-2)} u$ satisfies (8).

Next, let us recall the following integration by parts formula:

$$\int_\Omega D^2 u\, D^2 v = \int_\Omega \Delta^2 u v - \int_{\partial\Omega} \frac{\partial \Delta u}{\partial \nu} v + \sum_{i,j=1}^n \int_{\partial\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_i} \nu_j,$$
$$\forall u \in H^4(\Omega),\ \forall v \in H^2(\Omega), \tag{35}$$

where $\nu_j = \frac{\partial \nu}{\partial x_j} = \cos(\nu, x_j)$ denotes the jth component of the normal ν. In order to highlight the boundary conditions, we first transform the last boundary integral according to

[10, pp. 75–76]; we introduce a system of local tangential coordinates to $\partial\Omega$, namely $t_k = t_k(x_1, \dots, x_n)$ $(k = 1, \dots, n-1)$ so that $(t_1, \dots, t_{n-1}, \nu)$ is a complete orthonormal system diffeomorphic to $(x_1, \dots, x_n)$. Then, we write

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial x_i} \nu_j = \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial t_k} \frac{\partial t_k}{\partial x_i} \nu_j + \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial \nu} \nu_i \nu_j$$

and integrations by parts with respect to tangential coordinates yield

$$\sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial v}{\partial t_k} \frac{\partial t_k}{\partial x_i} \nu_j = -\sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial t_k}{\partial x_i} \nu_j \right) v$$

so that (35) becomes

$$\int_{\Omega} D^2 u \, D^2 v = \int_{\Omega} \Delta^2 u v - \int_{\partial\Omega} \frac{\partial \Delta u}{\partial \nu} v - \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial t_k}{\partial x_i} \nu_j \right) v + \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} \frac{\partial v}{\partial \nu}, \tag{36}$$

which is precisely (5.19) in [10] with $a = 1$ and $b = 0$. By computing the derivative of the product, we obtain

$$\begin{aligned} &\sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial t_k}{\partial x_i} \nu_j \right) v \\ &\quad = \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \frac{\partial t_k}{\partial x_i} \nu_j v + \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \int_{\partial\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial}{\partial t_k} \left(\frac{\partial t_k}{\partial x_i} \nu_j \right) v. \end{aligned} \tag{37}$$

Next, we remark that

$$\frac{\partial^3 u}{\partial x_i^2 \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) = \sum_{k=1}^{n-1} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \frac{\partial t_k}{\partial x_i} + \frac{\partial}{\partial \nu} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_i \quad \text{for all } i, j$$

and therefore

$$\begin{aligned} \sum_{i,j=1}^{n} \sum_{k=1}^{n-1} \frac{\partial}{\partial t_k} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \frac{\partial t_k}{\partial x_i} \nu_j &= \sum_{i,j=1}^{n} \left[\frac{\partial^3 u}{\partial x_i^2 \partial x_j} - \frac{\partial}{\partial \nu} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right) \nu_i \right] \nu_j \\ &= \frac{\partial \Delta u}{\partial \nu} - \frac{\partial^3 u}{\partial \nu^3}. \end{aligned} \tag{38}$$

Plugging (38) into (37) gives

$$\sum_{i,j=1}^{n}\sum_{k=1}^{n-1}\int_{\partial\Omega}\frac{\partial}{\partial t_k}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial t_k}{\partial x_i}\nu_j\right)v$$
$$=\int_{\partial\Omega}\left(\frac{\partial\Delta u}{\partial\nu}-\frac{\partial^3 u}{\partial\nu^3}+\sum_{i,j=1}^{n}\sum_{k=1}^{n-1}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)\right)v.$$

Finally, inserting this into (36) yields

$$\int_{\Omega}D^2u\,D^2v=\int_{\Omega}\Delta^2uv+\int_{\partial\Omega}\left(\frac{\partial^3 u}{\partial\nu^3}-2\frac{\partial\Delta u}{\partial\nu}-\sum_{i,j=1}^{n}\sum_{k=1}^{n-1}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)\right)v$$
$$+\int_{\partial\Omega}\frac{\partial^2 u}{\partial\nu^2}\frac{\partial v}{\partial\nu}. \tag{39}$$

According to formula (39), if $u\in H^4(\Omega)$ we may rewrite (8) as

$$\int_{\Omega}\left(\Delta^2u+au-|u|^{p-2}u\right)v+\int_{\partial\Omega}\left(\frac{\partial^3 u}{\partial\nu^3}-2\frac{\partial\Delta u}{\partial\nu}-\sum_{i,j=1}^{n}\sum_{k=1}^{n-1}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)\right)v$$
$$+\int_{\partial\Omega}\frac{\partial^2 u}{\partial\nu^2}\frac{\partial v}{\partial\nu}=0 \tag{40}$$

for all $v\in H^2(\Omega)$ (respectively $v\in H^2_\nu(\Omega)$). By taking $v\in C^\infty_c(\Omega)$ in (40), we see that

$$\Delta^2u+au-|u|^{p-2}u=0\quad\text{a.e. in }\Omega. \tag{41}$$

Consider first the case of test functions in $H^2(\Omega)$. For any $\phi\in C^\infty(\partial\Omega)$ let $v\in C^4(\overline{\Omega})$ be the unique solution of

$$\begin{cases}\Delta^2v=0 & \text{in }\Omega,\\ \frac{\partial v}{\partial\nu}=\phi & \text{on }\partial\Omega,\\ v=0 & \text{on }\partial\Omega.\end{cases}$$

Then, plugging v into (40) and recalling (41) entails

$$\int_{\partial\Omega}\frac{\partial^2 u}{\partial\nu^2}\phi=0\quad\text{for all }\phi\in C^\infty(\partial\Omega);$$

this clearly implies that

$$\frac{\partial^2 u}{\partial\nu^2}=0\quad\text{on }\partial\Omega. \tag{42}$$

Inserting (41) and (42) into (40) immediately yields

$$\frac{\partial^3 u}{\partial\nu^3}-2\frac{\partial\Delta u}{\partial\nu}-\sum_{i,j=1}^{n}\sum_{k=1}^{n-1}\frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial}{\partial t_k}\left(\frac{\partial t_k}{\partial x_i}\nu_j\right)=0\quad\text{on }\partial\Omega. \tag{43}$$

We now consider the case of test functions in $H^2_\nu(\Omega)$. Since also $u \in H^2_\nu(\Omega)$, we have

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{44}$$

so that the last term in (40) vanishes. Hence, by (41), we obtain again (43).

We now need to justify the original assumption $u \in H^4(\Omega)$ made in order to obtain (35) and (41). The first step consists in verifying that the boundary conditions satisfy the complementing condition:

Lemma 1. *The boundary conditions* (42)–(43) (*respectively* (44)–(43)) *satisfy the complementing condition.*

Proof. We follow the notations of [5, pp. 625–633]. Alternatively, one may also refer to [11, Definition 8.28] for an equivalent formulation of the complementing condition.

Firstly, we have $L'(\xi) = |\xi|^4$ for all $\xi \in \mathbb{R}^n$. Therefore, if ν denotes the outward normal to $\partial\Omega$ and ξ any vector parallel to $\partial\Omega$, the polynomial $\tau \mapsto L'(\xi + \tau\nu)$ admits $\tau = i|\xi|$ as double root with positive imaginary part. Hence,

$$M^+(\xi, \tau) = \big(\tau - i|\xi|\big)^2.$$

Next, we notice that for the boundary conditions (42)–(43) we have respectively $B'_1(\xi + \tau\nu) = \tau^2$ and $B'_2(\xi + \tau\nu) = 2|\xi|^2\tau + \tau^3$. Therefore,

$$B'_1(\xi + \tau\nu) = |\xi|\big(2i\tau + |\xi|\big) \quad \text{mod}\,\big[M^+(\xi, \tau)\big],$$
$$B'_2(\xi + \tau\nu) = |\xi|^2\big(-\tau + 2i|\xi|\big) \quad \text{mod}\,\big[M^+(\xi, \tau)\big];$$

since $\tau \mapsto |\xi|(2i\tau + |\xi|)$ and $\tau \mapsto |\xi|^2(-\tau + 2i|\xi|)$ are linearly independent (as polynomials of the variable τ), the boundary conditions (42)–(43) satisfy the complementing condition.

For the boundary conditions (44)–(43) only B'_1 changes and becomes $B'_1(\xi + \tau\nu) = \tau$. Therefore,

$$B'_1(\xi + \tau\nu) = \tau \quad \text{mod}\,\big[M^+(\xi, \tau)\big],$$
$$B'_2(\xi + \tau\nu) = |\xi|^2\big(-\tau + 2i|\xi|\big) \quad \text{mod}\,\big[M^+(\xi, \tau)\big];$$

since $\tau \mapsto \tau$ and $\tau \mapsto |\xi|^2(-\tau + 2i|\xi|)$ are linearly independent, also the boundary conditions (44)–(43) satisfy the complementing condition. □

Arguing as in [13], we may now prove:

Lemma 2. *Assume that* $u \in H^2(\Omega)$ (*respectively* $u \in H^2_\nu(\Omega)$) *satisfies* (8). *Then,* $u \in L^q(\Omega)$ *for all* $1 \leqslant q < \infty$.

Proof. Since the proofs are similar, we prove the lemma only in the case where $u \in H^2(\Omega)$. Note first that by Lax–Milgram theorem, for any $w \in L^{2n/(n+4)}(\Omega)$ there exists a unique $u \in H^2(\Omega)$ such that

$$\int_\Omega \big(D^2uD^2v + auv\big) = \int_\Omega wv \quad \text{for all } v \in H^2(\Omega).$$

This defines a Green function $G : L^{2n/(n+4)}(\Omega) \to H^2(\Omega)$ such that $G(w) = u$ where u is the unique solution of the above variational problem; by what we have proved above, u formally satisfies (42)–(43).

Next, let $\alpha(x) := |u(x)|^{p-2}$ (recall $2 < p \leqslant 2^*$). Then, by assumption, $\alpha \in L^{n/4}(\Omega)$ and u satisfies

$$\Delta^2 u + au = \alpha(x)u \quad \text{in } \mathcal{D}'(\Omega), \tag{45}$$

see (35) and (41). By [13, Lemma B2], for any $\varepsilon > 0$ there exist $g_\varepsilon \in L^{n/4}(\Omega)$ and $f_\varepsilon \in L^\infty(\Omega)$ such that $\|g_\varepsilon\|_{n/4} < \varepsilon$ and (45) reads $\Delta^2 u + au = g_\varepsilon(x)u + f_\varepsilon(x)$ or, equivalently,

$$u - G_\varepsilon(u) = h_\varepsilon, \tag{46}$$

where $h_\varepsilon = G(f_\varepsilon)$, $G_\varepsilon(u) = G(g_\varepsilon u)$ and G denotes the above defined Green function for the operator $(\Delta^2 + a)$ with the boundary conditions (42)–(43). Fix any $q \in [1, \infty)$; using the Hardy–Littlewood–Sobolev inequality and repeating the arguments of Step 2, p. 272 in [13], one obtains that $G_\varepsilon : L^q(\Omega) \to L^q(\Omega)$ and that if ε is sufficiently small, then $\|G_\varepsilon\|_{L^q \to L^q} < \frac{1}{2}$. Finally, Step 3, p. 273 in [13] may be repeated with no modifications. This proves the lemma. □

Proof of Theorem 4. We use Schauder estimates and we argue as in the proof of [13, Lemma B3]. More precisely, Lemmas 1 and 2 imply that $u \in W^{4,p}(\Omega)$ for all $1 \leqslant p < \infty$; embedding theorems and smoothness of $\partial\Omega$ then show that $u \in C^{3,\alpha}(\overline{\Omega})$. Hence, $|u|^{p-2}u \in C^{0,\alpha}(\overline{\Omega})$ so that Schauder theory finally yields $u \in C^{4,\alpha}(\overline{\Omega})$. Therefore, u is a classical solution of (9) which satisfies the boundary conditions (42)–(43) or (44)–(43). This completes the proof. □

6. Proof of Theorem 5

Let $a = 0$ and let $u \geqslant 0$ be a solution of (8). By taking $v \equiv 1$ in (8), we infer

$$\int_\Omega u^{p-1} = 0$$

which implies at once that $u \equiv 0$.

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References

[1] R.A. Adams, Sobolev Spaces, Academic Press, 1975.

[2] Adimurthi, G. Mancini, The Neumann problem for elliptic equations with critical non-linearity, in: A. Ambrosetti, A. Marino (Eds.), Nonlinear Analysis: A Tribute in Honour of G. Prodi, in: Quad. Sc. Norm. Sup. Pisa, 1991, pp. 9–25.
[3] Adimurthi, S.L. Yadava, Existence and nonexistence of positive radial solutions of Neumann problems with critical Sobolev exponents, Arch. Ration. Mech. Anal. 115 (1991) 275–296.
[4] Adimurthi, S.L. Yadava, On a conjecture of Lin–Ni for a semilinear Neumann problem, Trans. Amer. Math. Soc. 336 (1993) 631–637.
[5] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure Appl. Math. 12 (1959) 623–727.
[6] D.E. Edmunds, D. Fortunato, E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator, Arch. Ration. Mech. Anal. 112 (1990) 269–289.
[7] F. Gazzola, H.C. Grunau, M. Squassina, Existence and nonexistence results for critical growth biharmonic elliptic equations, Calc. Var. 18 (2003) 117–143.
[8] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, Comment. Math. Helv. 73 (1998) 206–231.
[9] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.
[10] J.L. Lions, Problèmes aux limites en théorie des distributions, Acta Math. 94 (1955) 13–153.
[11] M. Renardy, R.C. Rogers, An Introduction to Partial Differential Equations, Springer, New York, 1993.
[12] C.A. Swanson, The best Sobolev constant, Appl. Anal. 47 (1992) 227–239.
[13] R. van der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$, Differential Integral Equations 6 (1993) 259–276.
[14] X.J. Wang, Neumann problems of semilinear elliptic equations involving critical Sobolev exponents, J. Differential Equations 93 (1991) 283–310.